Extension of Milnor link invariants to welded links

Haruko Aida Miyazawa
Institute for Mathematics and Computer Science, Tsuda University
Kodai Wada
Department of Mathematics, Kobe University
Akira Yasuhara
Faculty of Commerce, Waseda University

1 Introduction

The present article is a summary of the paper [13]. We refer the reader to [13] for more details and full proofs.

In [11, 12], Milnor defined a family of isotopy invariants of classical links in the 3-sphere, called *Milnor $\overline{\mu}$-invariants*. Given an n-component classical link L, the *Milnor number* $\mu_L(I) \in \mathbb{Z}$ of L is specified by a finite sequence I of indices in $\{1, \ldots, n\}$. This integer is only well-defined up to a certain indeterminacy $\Delta_L(I)$, i.e. the residue class $\overline{\mu}_L(I)$ of $\mu_L(I)$ modulo $\Delta_L(I)$ is an invariant of L. It is shown in [12, Theorem 8] that $\overline{\mu}_L(I)$ is invariant under link-homotopy when the sequence I has no repeated indices. Here, *link-homotopy* is an equivalence relation generated by self-crossing changes and isotopies (cf. [11]). In [6], Habegger and Lin defined Milnor numbers for *classical string links* in the 3-ball, and proved that they are integer-valued invariants. In this sense, Milnor numbers are suitable for classical string links rather than classical links. These numbers for classical string links are called *Milnor μ-invariants*.

The notion of welded links, introduced by Fenn, Rimányi, and Rourke in [5], is a diagrammatic generalization of classical links in the 3-sphere. It naturally yields the notion of welded string links. *Welded (string) links* are generalized (string) link diagrams considered up to an extended set of Reidemeister moves. The aim of this article is to give an extension of Milnor $\overline{\mu}$-invariants to welded links in a combinatorial way.

In [4], Dye and Kauffman first tried to extend Milnor link-homotopy $\overline{\mu}$-invariants to welded links. Kotorii pointed out in [7, Remark 4.6] that the extension of Dye and Kauffman is incorrect. In fact, there exists a classical link having two different values of the Dye-Kauffman's $\overline{\mu}$. Hence the Dye-Kauffman's $\overline{\mu}$ is not well-defined even for classical links (see Remark 6.5).

A successful extension is due to Kravchenko and Polyak in [8]. Using Gauss diagrams, they extended Milnor link-homotopy μ-invariants to welded tangles, which are slight generalizations of welded string links. In [7], Kotorii gave an extension of Milnor link-homotopy $\overline{\mu}$-invariants to welded links via the theory of nanowords introduced by

Turaev in [15]. Both extensions are combinatorial, but they are restricted to the case of link-homotopy invariants.

In [1], Audoux, Bellingeri, Meilhan and Wagner defined a 4-dimensional version of Milnor μ-invariants. Combining this version of Milnor μ-invariants with the Tube map, they extended Milnor isotopy μ-invariants to welded string links. Here, the *Tube map* is a map from welded string links to ribbon 2-dimensional string links in the 4-ball (cf. [16, 14]). Recently, Chrisman in [3] defined Milnor $\overline{\mu}$-invariants for welded links with similar ingredients as in [1], and proved that they are *welded concordance* invariants. While Milnor invariants for welded objects are given in [1, 3], their approaches are topological. The authors believe that it is important to consider a combinatorial approach, since the advantage of welded objects is that they are combinatorial.

In [12], Milnor gave an algorithm to compute $\overline{\mu}$-invariants for a classical link based on its diagram. This algorithm can be applied to generalized link diagrams. By the result of Chrisman in [3], the values given by the algorithm are invariants of welded links. Hence, it is theoretically possible to prove that the values are invariant under welded isotopies, from a diagrammatic point of view. In this article, we actually give such a diagrammatic proof. Our approach is purely combinatorial, self contained, and different from [8, 7, 1, 3].

2 Preliminaries

For an integer $n \geq 1$, an *n-component virtual link diagram* is the image of an immersion of n ordered and oriented circles into the plane, whose singularities are only transverse double points. Such double points are divided into *classical crossings* and *virtual crossings* as shown in Figure 2.1.

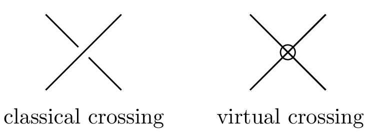

Figure 2.1: Two types of double points

Welded Reidemeister moves consist of Reidemeister moves R1–R3, virtual moves V1–V4 and the over-crossings commute move OC as shown in Figure 2.2. A *welded isotopy* is a finite sequence of welded Reidemeister moves, and an *n-component welded link* is an equivalence class of n-component virtual link diagrams under welded isotopy. We emphasize that all virtual link diagrams and welded links are ordered and oriented.

Let D be an n-component virtual link diagram. Put a *base point* p_i on some arc of each ith component, which is disjoint from all crossings of D ($1 \leq i \leq n$). A *base point system* of D is an ordered n-tuple $\mathbf{p} = (p_1, \ldots, p_n)$ of base points on D. We denote by $(D, \mathbf{p})$ a virtual link diagram D with a base point system $\mathbf{p}$. The classical under-crossings of D and base points $p_1, \ldots, p_n$ divide D into a finite number of segments possibly with classical over-crossings and virtual crossings. We call such a segment an *arc* of $(D, \mathbf{p})$.

As shown in Figure 2.3, let a_{i1} be the outgoing arc from the base point p_i, and let $a_{i2}, \ldots, a_{im_i+1}$ be the other arcs of the ith component in turn with respect to the orienta-

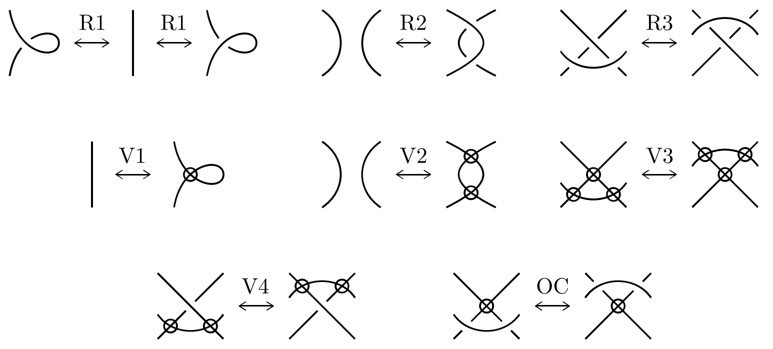

Figure 2.2: Welded Reidemeister moves

tion, where $m_i + 1$ is the number of arcs of the ith component of $(D, \mathbf{p})$ $(1 \leq i \leq n)$. In the figure, $u_{ij} \in \{a_{kl}\}$ denotes the arc which separates a_{ij} and a_{ij+1}. Let $\varepsilon_{ij} \in \{\pm 1\}$ be the sign of the crossing among a_{ij}, u_{ij} and a_{ij+1}, and we put

$$v_{ij} = u_{i1}^{\varepsilon_{i1}} u_{i2}^{\varepsilon_{i2}} \cdots u_{ij}^{\varepsilon_{ij}}$$

for $1 \leq j \leq m_i$. We call the word v_{ij} *a partial longitude* of $(D, \mathbf{p})$.

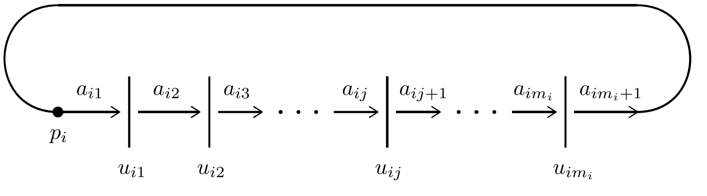

Figure 2.3: A schematic illustration of the ith component

Let $A = \langle \alpha_1, \dots, \alpha_n \rangle$ be the free group of rank n, and let $\overline{A}$ be the free group on the set $\{a_{ij}\}$ of arcs. For an integer $q \geq 1$, a sequence of homomorphisms

$$\eta_q = \eta_q(D, \mathbf{p}) : \overline{A} \longrightarrow A$$

associated with $(D, \mathbf{p})$ is defined inductively by

$$\begin{aligned} &\eta_1(a_{ij}) = \alpha_i, \\ &\eta_{q+1}(a_{i1}) = \alpha_i, \quad \text{and} \quad \eta_{q+1}(a_{ij}) = \eta_q(v_{ij-1}^{-1})\alpha_i \eta_q(v_{ij-1}) \quad (2 \leq j \leq m_i + 1). \end{aligned}$$

Note that our definition of η_q is very similar to the original one in [12], but they are not the same because, in [12], $a_{i1} \cup a_{im_i+1}$ is a single arc. In Section 3, we investigate virtual link diagrams with base point systems up to local moves relative base point system. The difference of the definition of arcs is essential for Theorem 3.1, see Remark 6.6.

Let $\mathbb{Z}\langle\langle X_1, \dots, X_n \rangle\rangle$ be the ring of formal power series in non-commutative variables $X_1, \dots, X_n$ with integer coefficients. The *Magnus expansion* is a homomorphism

$$E : A \longrightarrow \mathbb{Z}\langle\langle X_1, \dots, X_n \rangle\rangle$$

defined, for $1 \leq i \leq n$, by

$$E(\alpha_i) = 1 + X_i \quad \text{and} \quad E(\alpha_i^{-1}) = 1 - X_i + X_i^2 - X_i^3 + \cdots .$$

Remark 2.1 ([9, Corollary 5.7])**.** Let $q \geq 1$ be an integer and A_q the qth term of the lower central series of A. For $x \in A_q$, we have $E(x) = 1 + (\text{terms of degree} \geq q)$.

For each $1 \leq i \leq n$, let w_i be the sum of the signs of all classical self-crossings of the ith component of $(D, \mathbf{p})$. We call the word $l_i = a_{i1}^{-w_i} v_{im_i}$ the *ith preferred longitude* of $(D, \mathbf{p})$.

Definition 2.2. For a sequence $j_1 \ldots j_s i$ $(1 \leq s < q)$ of indices in $\{1, \ldots, n\}$, the *Milnor number* $\mu^{(q)}_{(D,\mathbf{p})}(j_1 \ldots j_s i)$ of $(D, \mathbf{p})$ is the coefficient of $X_{j_1} \cdots X_{j_s}$ in $E(\eta_q(l_i))$.

Remark 2.3. For $1 \leq s < q$, we have $\mu^{(q)}_{(D,\mathbf{p})}(j_1 \ldots j_s i) = \mu^{(q+1)}_{(D,\mathbf{p})}(j_1 \ldots j_s i)$. Therefore, by taking the integer q sufficiently large, we may ignore q and denote $\mu^{(q)}_{(D,\mathbf{p})}(j_1 \ldots j_s i)$ by $\mu_{(D,\mathbf{p})}(j_1 \ldots j_s i)$. In the rest of this article, q is assumed to be a sufficiently large integer.

3 Milnor numbers and welded isotopy relative base point system

A *local move relative base point system* is a local move on a virtual link diagram with a base point system such that it keeps the positions of base points. A *$\overline{w}$-isotopy* is a finite sequence of welded Reidemeister moves relative base point system and a local move as shown in Figure 3.1. We emphasize that in a $\overline{\text{w}}$-isotopy, we do *not* allow to use two local moves as shown in Figure 3.2. We call the two local moves *base-change moves.*

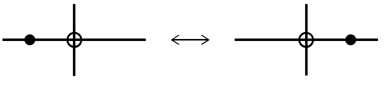

Figure 3.1: A base point passing through a virtual crossing

Figure 3.2: Base-change moves

The following theorem gives the invariance of Milnor numbers under $\overline{\text{w}}$-isotopy.

Theorem 3.1. *Let $(D, \mathbf{p})$ and $(D', \mathbf{p}')$ be virtual link diagrams with base point systems. If $(D, \mathbf{p})$ and $(D', \mathbf{p}')$ are $\overline{w}$-isotopic, then $\mu_{(D,\mathbf{p})}(I) = \mu_{(D',\mathbf{p}')}(I)$ for any sequence I.*

Let l_i and l'_i be the ith preferred longitudes of $(D, \mathbf{p})$ and $(D', \mathbf{p}')$, respectively $(1 \leq i \leq n)$. To show Theorem 3.1, we observe the difference between $\eta_q(D, \mathbf{p})(l_i)$ and $\eta_q(D', \mathbf{p}')(l'_i)$ under $\overline{\text{w}}$-isotopy.

Proposition 3.2. *If* $(D, \mathbf{p})$ *and* $(D', \mathbf{p}')$ *are* $\overline{\text{w}}$*-isotopic, then* $\eta_q(D, \mathbf{p})(l_i) \equiv \eta_q(D', \mathbf{p}')(l_i')$ $(\text{mod } A_q)$.

We admit this proposition and prove that it implies Theorem 3.1.

Proof of Theorem 3.1. By Proposition 3.2, we have

$$\eta_q(D, \mathbf{p})(l_i) \equiv \eta_q(D', \mathbf{p}')(l_i') \pmod{A_q}.$$

This together with Remark 2.1 implies that

$$E(\eta_q(D, \mathbf{p})(l_i)) - E(\eta_q(D', \mathbf{p}')(l_i')) = (\text{terms of degree} \geq q).$$

Hence, by definition, $\mu_{(D,\mathbf{p})}(j_1 \ldots j_s i) = \mu_{(D',\mathbf{p}')}(j_1 \ldots j_s i)$ for any sequence $j_1 \ldots j_s i$ with $s < q$. ∎

Example 3.3. Consider the 3-component link diagram D and its base point system $\mathbf{p} = (p_1, p_2, p_3)$ in the left of Figure 3.3. Let a_{ij} be the arcs of $(D, \mathbf{p})$. Since $l_1 = a_{21}, l_2 = a_{21}^{-1}(a_{11}a_{23})$, and $l_3 = a_{21}^{-1}a_{22}$, by definition we have

$$\begin{cases} \eta_3(l_1) = \alpha_2, \\ \eta_3(l_2) = \alpha_2^{-1}\alpha_1\alpha_2^{-1}\alpha_1^{-1}\alpha_2^{-1}\alpha_1\alpha_2\alpha_1^{-1}\alpha_2\alpha_1\alpha_2^{-1}\alpha_1^{-1}\alpha_2\alpha_1\alpha_2, \\ \eta_3(l_3) = \alpha_2^{-1}\alpha_1^{-1}\alpha_2\alpha_1. \end{cases}$$

By a direct computation, we have

$$\begin{cases} E(\eta_3(l_1)) = 1 + X_2, \\ E(\eta_3(l_2)) = 1 + X_1 + (\text{terms of degree} \geq 3), \\ E(\eta_3(l_3)) = 1 - X_1X_2 + X_2X_1 + (\text{terms of degree} \geq 3). \end{cases}$$

Hence it follows that

$$\mu_{(D,\mathbf{p})}(21) = 1, \ \mu_{(D,\mathbf{p})}(12) = 1, \ \mu_{(D,\mathbf{p})}(123) = -1, \quad \text{and} \quad \mu_{(D,\mathbf{p})}(213) = 1,$$

and that $\mu_{(D,\mathbf{p})}(I) = 0$ for any sequence I with length ≤ 3 except for $21, 12, 123$, and 213.

Consider another base point system $\mathbf{p}' = (p_1', p_2', p_3')$ of D in the right of Figure 3.3. Then we have $l_1 = a_{22}, l_2 = a_{21}^{-1}(a_{22}a_{11})$, and $l_3 = a_{22}^{-1}a_{21}$, and hence

$$\eta_3(l_1) = \alpha_2, \ \eta_3(l_2) = \alpha_1, \quad \text{and} \quad \eta_3(l_3) = 1.$$

This implies that

$$\mu_{(D,\mathbf{p}')}(21) = 1 \quad \text{and} \quad \mu_{(D,\mathbf{p}')}(12) = 1,$$

and that $\mu_{(D,\mathbf{p}')}(I) = 0$ for any sequence I with length ≤ 3 except for 21 and 12. Therefore, by Theorem 3.1, $(D, \mathbf{p})$ and $(D, \mathbf{p}')$ are not $\overline{\text{w}}$-isotopic.

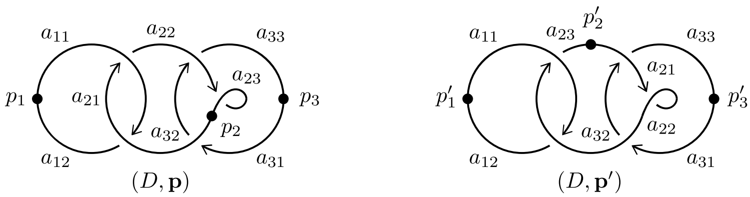

Figure 3.3: A 3-component link diagram D with different base point systems $\mathbf{p} = (p_1, p_2, p_3)$ and $\mathbf{p}' = (p_1', p_2', p_3')$

4 Change of base point system

In this section, we fix an n-component virtual link diagram D, and observe behavior of $\eta_q(l_i)$ under a change of base point system for D (Theorem 4.3).

An *arc* of D is a segment along D which goes from a classical under-crossing to the next one, where classical over-crossings and virtual crossings are ignored. We emphasize that the definition of arcs of D is slightly different from that of arcs of $(D, \mathbf{p})$. For each $1 \leq i \leq n$, we choose one arc of the ith component and denote it by a_{i1}. Let $a_{i2}, \ldots, a_{im_i}$ be the other arcs of the ith component in turn with respect to the orientation, where m_i denotes the number of arcs of the ith component. Throughout this section, we fix these arcs $a_{i1}, \ldots, a_{im_i}$ for D.

Given a base point system $\mathbf{p} = (p_1, \ldots, p_n)$ of D, let $\mathbf{p}(i)$ denote the integer of the second subscript of the arc containing p_i $(1 \leq i \leq n)$. Consider the virtual link diagram D with a base point system $\mathbf{p} = (p_1, \ldots, p_n)$. For each ith component of $(D, \mathbf{p})$, the base point p_i divides the arc $a_{i\mathbf{p}(i)}$ of D into two arcs. We assign the labels $b_i^{\mathbf{p}}$ and $a_{i\mathbf{p}(i)}$ to the two arcs of $(D, \mathbf{p})$ as shown in Figure 4.1. The labels of the other arcs of $(D, \mathbf{p})$ are the same as those of the corresponding arcs of D.

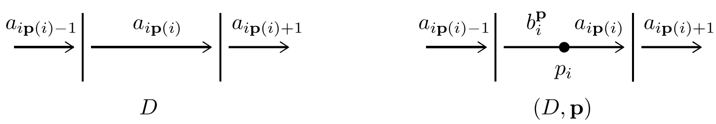

Figure 4.1:

In this setting, the homomorphism $\eta_q(D, \mathbf{p})$ associated with $(D, \mathbf{p})$ is described as follows. We put $\eta_q^{\mathbf{p}} = \eta_q(D, \mathbf{p})$ for short. The domain of $\eta_q^{\mathbf{p}}$ is the free group $\overline{A}$ on $\{a_{ij}\} \cup \{b_i^{\mathbf{p}}\}$. The homomorphism $\eta_q^{\mathbf{p}}$ from $\overline{A}$ into A is given inductively by

$$\begin{aligned}
&\eta_1^{\mathbf{p}}(a_{ij}) = \alpha_i, \ \eta_1^{\mathbf{p}}(b_i^{\mathbf{p}}) = \alpha_i, \\
&\eta_{q+1}^{\mathbf{p}}(a_{i\mathbf{p}(i)}) = \alpha_i, \ \eta_{q+1}^{\mathbf{p}}(a_{ij}) = \eta_q^{\mathbf{p}}((v_{ij-1}^{\mathbf{p}})^{-1})\alpha_i\eta_q^{\mathbf{p}}(v_{ij-1}^{\mathbf{p}}) \quad (j \neq \mathbf{p}(i)), \\
&\text{and} \quad \eta_{q+1}^{\mathbf{p}}(b_i^{\mathbf{p}}) = \eta_q^{\mathbf{p}}((v_{i\mathbf{p}(i)-1}^{\mathbf{p}})^{-1})\alpha_i\eta_q^{\mathbf{p}}(v_{i\mathbf{p}(i)-1}^{\mathbf{p}}),
\end{aligned}$$

where

$$v_{ij}^{\mathbf{p}} = \begin{cases} u_{i\mathbf{p}(i)}^{\varepsilon_{i\mathbf{p}(i)}} u_{i\mathbf{p}(i)+1}^{\varepsilon_{i\mathbf{p}(i)+1}} \cdots u_{ij}^{\varepsilon_{ij}} & (\mathbf{p}(i) \le j \le m_i), \\ u_{i\mathbf{p}(i)}^{\varepsilon_{i\mathbf{p}(i)}} u_{i\mathbf{p}(i)+1}^{\varepsilon_{i\mathbf{p}(i)+1}} \cdots u_{im_i}^{\varepsilon_{im_i}} u_{i1}^{\varepsilon_{i1}} \cdots u_{ij}^{\varepsilon_{ij}} & (1 \le j \le \mathbf{p}(i) - 1), \end{cases}$$

and $v_{i0}^{\mathbf{p}} = v_{im_i}^{\mathbf{p}}$. Furthermore, the ith preferred longitude $l_i^{\mathbf{p}}$ of $(D, \mathbf{p})$ is given by

$$l_i^{\mathbf{p}} = a_{i\mathbf{p}(i)}^{-w_i} v_{i\mathbf{p}(i)-1}^{\mathbf{p}}.$$

We now define a word $\lambda_i^{\mathbf{p}} \in \overline{A}$ $(1 \le i \le n)$ by

$$\lambda_i^{\mathbf{p}} = \begin{cases} u_{i1}^{\varepsilon_{i1}} u_{i2}^{\varepsilon_{i2}} \cdots u_{i\mathbf{p}(i)-1}^{\varepsilon_{i\mathbf{p}(i)-1}} & (\mathbf{p}(i) \ne 1), \\ 1 & (\mathbf{p}(i) = 1), \end{cases}$$

and a sequence of homomorphisms $\phi_q^{\mathbf{p}} : A \longrightarrow A$ by

$$\begin{aligned} &\phi_1^{\mathbf{p}}(\alpha_i) = \alpha_i \quad \text{and} \\ &\phi_q^{\mathbf{p}}(\alpha_i) = \eta_{q-1}^{\mathbf{p}}(\lambda_i^{\mathbf{p}}) \alpha_i \eta_{q-1}^{\mathbf{p}}((\lambda_i^{\mathbf{p}})^{-1}) \quad (q \ge 2). \end{aligned}$$

Notice that the homomorphism $\phi_q^{\mathbf{p}}$ sends each α_i to some conjugate element.

A *semi-arc* of D is a segment along D which goes from a classical under-/over-crossing to the next one, where virtual crossings are ignored. Let $\mathcal{P}$ be the set of base point systems of D. Let $\mathcal{P}_0 \subset \mathcal{P}$ be the set of all $(p_1, \dots, p_n) \in \mathcal{P}$ such that each p_i lies on a semi-arc which starts at a classical under-crossing. We denote by $\mathbf{p}_* = (p_1^*, \dots, p_n^*) \in \mathcal{P}_0$ the base point system such that each p_i^* lies on the arc a_{i1}. For the homomorphism $\eta_q^{\mathbf{p}_*}$ associated with $(D, \mathbf{p}_*)$, partial longitudes $v_{ij}^{\mathbf{p}_*}$, and preferred longitudes $l_i^{\mathbf{p}_*}$ of $(D, \mathbf{p}_*)$, we simply put $\eta_q = \eta_q^{\mathbf{p}_*}$, $v_{ij} = v_{ij}^{\mathbf{p}_*}$, and $l_i = l_i^{\mathbf{p}_*}$.

Let $M_q^{\mathbf{p}}$ be the normal closure of $\{\phi_q^{\mathbf{p}}([\alpha_i, \eta_q(l_i)]) \mid 1 \le i \le n\}$ in A and let $M_q = \prod_{\mathbf{p} \in \mathcal{P}_0} M_q^{\mathbf{p}}$. Notice that $M_q = \prod_{\mathbf{p} \in \mathcal{P}_0} \phi_q^{\mathbf{p}}(M_q^{\mathbf{p}_*})$.

Proposition 4.1. *Let $\mathbf{p}_0 \in \mathcal{P}_0$. For any $1 \le i \le n$,*

$$\eta_q^{\mathbf{p}_0}(l_i^{\mathbf{p}_0}) \equiv \phi_q^{\mathbf{p}_0}(\eta_q((\lambda_i^{\mathbf{p}_0})^{-1} l_i \lambda_i^{\mathbf{p}_0})) \pmod{A_q M_q^{\mathbf{p}_0}}.$$

Proposition 4.2. *Let $\mathbf{p} \in \mathcal{P}$, and $\mathbf{p}_0 \in \mathcal{P}_0$ with $\mathbf{p}_0(k) = \mathbf{p}(k)$ $(1 \le k \le n)$. For any $1 \le i \le n$, $\eta_q^{\mathbf{p}}(l_i^{\mathbf{p}}) \equiv \eta_q^{\mathbf{p}_0}(l_i^{\mathbf{p}_0}) \pmod{A_q M_q^{\mathbf{p}_0}}$.*

Combining Propositions 4.1 and 4.2, the following is obtained immediately.

Theorem 4.3. *Let $\mathbf{p} \in \mathcal{P}$, and $\mathbf{p}_0 \in \mathcal{P}_0$ with $\mathbf{p}_0(k) = \mathbf{p}(k)$ $(1 \le k \le n)$. For any $1 \le i \le n$, $\eta_q^{\mathbf{p}}(l_i^{\mathbf{p}}) \equiv \phi_q^{\mathbf{p}_0}(\eta_q((\lambda_i^{\mathbf{p}_0})^{-1} l_i \lambda_i^{\mathbf{p}_0})) \pmod{A_q M_q^{\mathbf{p}_0}}$. Hence $\eta_q^{\mathbf{p}}(l_i^{\mathbf{p}}) \equiv \phi_q^{\mathbf{p}_0}(\eta_q((\lambda_i^{\mathbf{p}_0})^{-1} l_i \lambda_i^{\mathbf{p}_0})) \pmod{A_q M_q}$.*

5 Milnor numbers and welded isotopy

Let D be an n-component virtual link diagram of a welded link L, and $\mathbf{p}$ a base point system of D. As shown in Example 3.3, the Milnor number $\mu_{(D,\mathbf{p})}(I)$ depends on the choice of $\mathbf{p}$. Hence it is *not* an invariant of the welded link L. On the other hand, we show in this section that $\mu_{(D,\mathbf{p})}(I)$ modulo a certain indeterminacy is an invariant of L (Theorem 5.1).

For a sequence $i_1 \dots i_r$ of indices in $\{1,\dots,n\}$, the *indeterminacy* $\Delta_{(D,\mathbf{p})}(i_1 \dots i_r)$ of $(D,\mathbf{p})$ is the greatest common divisor of all $\mu_{(D,\mathbf{p})}(j_1 \dots j_s)$, where $j_1 \dots j_s$ $(2 \le s < r)$ is obtained from $i_1 \dots i_r$ by removing at least one index and permuting the remaining indices cyclicly. In particular, we set $\Delta_{(D,\mathbf{p})}(i_1 i_2) = 0$.

Theorem 5.1. *Let D and D' be virtual diagrams of a welded link. Let $\mathbf{p}$ and $\mathbf{p}'$ be base point systems of D and D', respectively. Then $\mu_{(D,\mathbf{p})}(I) \equiv \mu_{(D',\mathbf{p}')}(I) \pmod{\Delta_{(D,\mathbf{p})}(I)}$ and $\Delta_{(D,\mathbf{p})}(I) = \Delta_{(D',\mathbf{p}')}(I)$ for any sequence I.*

This theorem guarantees the well-definedness of the following definition.

Definition 5.2. Let L be an n-component welded link. For a sequence I of indices in $\{1,\dots,n\}$, the *Milnor $\overline{\mu}$-invariant* $\overline{\mu}_L(I)$ of L is the residue class of $\mu_{(D,\mathbf{p})}(I)$ modulo $\Delta_{(D,\mathbf{p})}(I)$ for any virtual diagram D of L and any base point system $\mathbf{p}$ of D.

Remark 5.3. The Milnor $\overline{\mu}$-invariant of welded links, defined above, coincides with the extension of Chrisman in [3] for any sequence. In particular, for classical links, the invariant coincides with the original one in [12].

In the remainder of this section, we fix D and its arcs a_{ij} $(1 \le i \le n, 1 \le j \le m_i)$, and use the same notation as in Section 4. In this setting, the Milnor number $\mu_{(D,\mathbf{p})}(j_1 \dots j_s i)$ of $(D,\mathbf{p})$ is given by the coefficient of $X_{j_1} \cdots X_{j_s}$ in $E(\eta_q^{\mathbf{p}}(l_i^{\mathbf{p}}))$. For short, we put $\mu_{\mathbf{p}}(I) = \mu_{(D,\mathbf{p})}(I)$ and $\Delta_{\mathbf{p}}(I) = \Delta_{(D,\mathbf{p})}(I)$. In particular, we put $\mu(I) = \mu_{(D,\mathbf{p}_*)}(I)$ and $\Delta(I) = \Delta_{(D,\mathbf{p}_*)}(I)$.

For each $1 \le i \le n$, we define a subset $\mathcal{D}_i$ of $\mathbb{Z}\langle\langle X_1,\dots,X_n\rangle\rangle$ to be

$$\left\{ \sum \nu(j_1 \dots j_s) X_{j_1} \cdots X_{j_s} \;\middle|\; \begin{array}{lll} \nu(j_1 \dots j_s) \equiv 0 & \pmod{\Delta(j_1 \dots j_s i)} & (s < q), \\ \nu(j_1 \dots j_s) \in \mathbb{Z} & & (s \ge q). \end{array} \right\}.$$

Lemma 5.4 (cf. [12, (12)–(15) on page 292])**.** *Let $x, y \in A$ and $\mathbf{p} \in \mathcal{P}$. For any $1 \le i \le n$, the following hold.*

(1) $E(x^{-1}\eta_q(l_i)x) - E(\eta_q(l_i)) \in \mathcal{D}_i$.

(2) $E(\phi_q^{\mathbf{p}}(\eta_q(l_i))) - E(\eta_q(l_i)) \in \mathcal{D}_i$.

(3) *If $x \equiv y \pmod{A_q M_q}$, then $E(x) - E(y) \in \mathcal{D}_i$.*

We admit this lemma and prove that it, together with Theorem 4.3, implies the following proposition.

Proposition 5.5. *For any $\mathbf{p} \in \mathcal{P}$, the following hold.*

(1) $\mu_{\mathbf{p}}(I) \equiv \mu(I) \pmod{\Delta(I)}$ *for any sequence I.*

(2) $\Delta_{\mathbf{p}}(I) = \Delta(I)$ *for any sequence I.*

Proof. (1) For any $1 \le i \le n$, it is enough to show that

$$E(\eta_q^{\mathbf{p}}(l_i^{\mathbf{p}})) - E(\eta_q(l_i)) \equiv 0 \pmod{\mathcal{D}_i}.$$

Let $\mathbf{p}_0 \in \mathcal{P}_0$ with $\mathbf{p}_0(k) = \mathbf{p}(k)$ $(1 \leq k \leq n)$. By Theorem 4.3, we have

$$\eta_q^{\mathbf{p}}(l_i^{\mathbf{p}}) \equiv \phi_q^{\mathbf{p}_0}(\eta_q((\lambda_i^{\mathbf{p}_0})^{-1} l_i \lambda_i^{\mathbf{p}_0})) \pmod{A_q M_q}.$$

Put $x = \phi_q^{\mathbf{p}_0}(\eta_q(\lambda_i^{\mathbf{p}_0})) \in A$. Then by Lemma 5.4 it follows that

$$\begin{aligned} E(\eta_q^{\mathbf{p}}(l_i^{\mathbf{p}})) - E(\eta_q(l_i)) &\equiv E(x^{-1}\phi_q^{\mathbf{p}_0}(\eta_q(l_i))x) - E(\eta_q(l_i)) \pmod{\mathcal{D}_i} \\ &\equiv E(x^{-1}\phi_q^{\mathbf{p}_0}(\eta_q(l_i))x) - E(x^{-1}\eta_q(l_i)x) \pmod{\mathcal{D}_i} \\ &= E(x^{-1})\left(E(\phi_q^{\mathbf{p}_0}(\eta_q(l_i))) - E(\eta_q(l_i))\right)E(x) \\ &\equiv 0 \pmod{\mathcal{D}_i}. \end{aligned}$$

Since we may assume that q is sufficiently large,

$$\mu_{\mathbf{p}}(j_1 \dots j_s i) - \mu(j_1 \dots j_s i) \equiv 0 \pmod{\Delta(j_1 \dots j_s i)}$$

for any sequence $j_1 \dots j_s i$.

(2) This is proved by induction on the length k of I. For $k = 2$, we have $\Delta_{\mathbf{p}}(I) = \Delta(I) = 0$ by definition. Assume that $k \geq 2$. Let $\mathcal{J}_1(I)$ (resp. $\mathcal{J}_{\geq 1}(I)$) be the set of all sequences obtained from I by removing exactly one index (resp. at least one index) and permuting the remaining indices cyclicly. For any $J \in \mathcal{J}_1(I)$, we have $\Delta_{\mathbf{p}}(J) = \Delta(J)$ by the induction hypothesis. Then it follows that

$$\begin{aligned} \Delta_{\mathbf{p}}(I) &= \gcd\{\mu_{\mathbf{p}}(J) \mid J \in \mathcal{J}_{\geq 1}(I)\} \\ &= \gcd\left(\bigcup_{J \in \mathcal{J}_1(I)} (\{\mu_{\mathbf{p}}(J)\} \cup \{\mu_{\mathbf{p}}(J') \mid J' \in \mathcal{J}_{\geq 1}(J)\})\right) \\ &= \gcd\left(\bigcup_{J \in \mathcal{J}_1(I)} (\{\mu_{\mathbf{p}}(J)\} \cup \{\Delta_{\mathbf{p}}(J)\})\right) \\ &= \gcd\left(\bigcup_{J \in \mathcal{J}_1(I)} (\{\mu_{\mathbf{p}}(J)\} \cup \{\Delta(J)\})\right). \end{aligned}$$

By (1) we have $\mu_{\mathbf{p}}(J) \equiv \mu(J) \pmod{\Delta(J)}$. This implies that $\Delta_{\mathbf{p}}(I) = \Delta(I)$. □

Proof of Theorem 5.1. Since $(D, \mathbf{p})$ and $(D', \mathbf{p}')$ are related by $\overline{\mathrm{w}}$-isotopies and base-change moves in Figure 3.2, this follows from Theorem 3.1 and Proposition 5.5. □

6 Self-crossing virtualization

A *self-crossing virtualization* is a local move on virtual link diagrams as shown in Figure 6.1, which replaces a classical crossing involving two strands of a single component with a virtual one. In this section, we show the following theorem as a generalization of [12, Theorem 8].

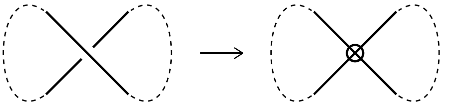

Figure 6.1: Self-crossing virtualization

Theorem 6.1. *Let L and L' be welded links, and let D and D' be virtual link diagrams of L and L', respectively. If D and D' are related by a finite sequence of self-crossing virtualizations and welded isotopies, then $\overline{\mu}_L(I) = \overline{\mu}_{L'}(I)$ for any non-repeated sequence I.*

Remark 6.2. In [2], Audoux and Meilhan proved that two virtual link diagrams are related by a finite sequence of self-crossing virtualizations and welded isotopies if and only if they have equivalent *reduced peripheral systems*. This result together with Theorem 6.1 implies that for welded links, the reduced peripheral system determines Milnor $\overline{\mu}$-invariants for non-repeated sequences.

Let $(D, \mathbf{p})$ be an n-component virtual link diagram with a base point system, and let a_{ij} $(1 \leq i \leq n, 1 \leq j \leq m_i + 1)$ be the arcs of $(D, \mathbf{p})$ as given in Section 2. Recall that $A = \langle \alpha_1, \ldots, \alpha_n \rangle$ denotes the free group of rank n, and $\overline{A}$ denotes the free group on $\{a_{ij}\}$. For $1 \leq k \leq n$, let $A^{(k)} = \langle \alpha_1, \ldots, \alpha_{k-1}, \alpha_{k+1}, \ldots, \alpha_n \rangle$ be the free group of rank $n-1$. We define a homomorphism $\rho_k : A \longrightarrow A^{(k)}$ by

$$\rho_k(\alpha_i) = \begin{cases} \alpha_i & (i \neq k), \\ 1 & (i = k), \end{cases}$$

and denote by $\eta_q^{(k)} = \eta_q^{(k)}(D, \mathbf{p})$ the composition $\rho_k \circ \eta_q : \overline{A} \longrightarrow A^{(k)}$.

Let R be the normal closure of $\{[\alpha_i, g^{-1}\alpha_i g] \mid g \in A, 1 \leq i \leq n\}$ in A, and let $R^{(k)}$ be the normal closure of $\{[\alpha_i, g^{-1}\alpha_i g] \mid g \in A^{(k)}, 1 \leq i \neq k \leq n\}$ in $A^{(k)}$. Note that $[g^{-1}\alpha_i g, h^{-1}\alpha_i h] \in R$ for any $g, h \in A$. In particular, $\eta_q([a_{is}^{\varepsilon}, a_{it}^{\delta}]) \in R$ for any s, t and any $\varepsilon, \delta \in \{\pm 1\}$. Let $A_q^{(k)}$ be the qth term of the lower central series of $A^{(k)}$.

Proposition 6.3. *Let $(D, \mathbf{p})$ and $(D', \mathbf{p}')$ be n-component virtual link diagrams with base point systems. For an integer $k \in \{1, \ldots, n\}$, let l_k and l'_k be the kth preferred longitudes of $(D, \mathbf{p})$ and $(D', \mathbf{p}')$, respectively. If $(D, \mathbf{p})$ and $(D', \mathbf{p}')$ are related by a self-crossing virtualization, then $\eta_q^{(k)}(D, \mathbf{p})(l_k) \equiv \eta_q^{(k)}(D', \mathbf{p}')(l'_k) \pmod{A_q^{(k)} R^{(k)}}$.*

For a sequence $j_1 \ldots j_s i$ $(1 \leq s < q)$ of indices in $\{1, \ldots, n\}$, we denote by $\mu_{(D,\mathbf{p})}^{(q,k)}(j_1 \ldots j_s i)$ the coefficient of $X_{j_1} \cdots X_{j_s}$ in $E(\eta_q^{(k)}(l_i))$. By Remark 2.3, we have

$$\mu_{(D,\mathbf{p})}^{(q,k)}(j_1 \ldots j_s i) = \mu_{(D,\mathbf{p})}^{(q+1,k)}(j_1 \ldots j_s i).$$

Furthermore, if the sequence $j_1 \ldots j_s$ involves the index k, then $\mu_{(D,\mathbf{p})}^{(q,k)}(j_1 \ldots j_s i) = 0$. On the other hand, if $j_1 \ldots j_s$ does not involve k, then $\mu_{(D,\mathbf{p})}^{(q,k)}(j_1 \ldots j_s i) = \mu_{(D,\mathbf{p})}^{(q)}(j_1 \ldots j_s i)$ $(= \mu_{(D,\mathbf{p})}(j_1 \ldots j_s i))$.

Theorem 6.4. *Let $(D,\mathbf{p})$ and $(D',\mathbf{p}')$ be virtual link diagrams with base point systems. If $(D,\mathbf{p})$ and $(D',\mathbf{p}')$ are related by a self-crossing virtualization, then $\mu_{(D,\mathbf{p})}(I)=\mu_{(D',\mathbf{p}')}(I)$ for any non-repeated sequence I.*

Proof. Let k be the last index of a non-repeated sequence I. Then we may put $I=Jk$. Since J does not involve k, we have $\mu_{(D,\mathbf{p})}(Jk)=\mu^{(q,k)}_{(D,\mathbf{p})}(Jk)$ and $\mu_{(D',\mathbf{p}')}(Jk)=\mu^{(q,k)}_{(D',\mathbf{p}')}(Jk)$. To complete the proof, we will show that $\mu^{(q,k)}_{(D,\mathbf{p})}(Jk)=\mu^{(q,k)}_{(D',\mathbf{p}')}(Jk)$.

For $x\in A_q^{(k)}R^{(k)}$, we put

$$E(x)=1+\sum\nu(j_1\dots j_s)X_{j_1}\cdots X_{j_s}.$$

By Proposition 6.3, it is enough to show that $\nu(j_1\dots j_s)=0$ for any non-repeated sequence $j_1\dots j_s$ with $s<q$.

If $x\in A_q^{(k)}$, then we have $\nu(j_1\dots j_s)=0$ by Remark 2.1. If $x\in R^{(k)}$, then we only need to consider the case $x=[\alpha_i,g^{-1}\alpha_i g]$ $(g\in A^{(k)},1\le i\ne k\le n)$. Then it follows that

$$\begin{aligned} E(x)-1 &= E([\alpha_i,g^{-1}\alpha_i g])-1\\ &= \big(E(\alpha_i g^{-1}\alpha_i g)-E(g^{-1}\alpha_i g\alpha_i)\big)\,E(\alpha_i^{-1}g^{-1}\alpha_i^{-1}g). \end{aligned}$$

Here we observe that

$$\begin{aligned} &E(\alpha_i g^{-1}\alpha_i g)-E(g^{-1}\alpha_i g\alpha_i)\\ &=(1+X_i)E(g^{-1})(1+X_i)E(g)-E(g^{-1})(1+X_i)E(g)(1+X_i)\\ &=X_iE(g^{-1})X_iE(g)-E(g^{-1})X_iE(g)X_i. \end{aligned}$$

This implies that each term of $E(x)-1$ contains X_i at least twice. Hence we have $\nu(j_1\dots j_s)=0$ for any non-repeated sequence $j_1\dots j_s$. $\square$

Proof of Theorem 6.1. Let $\mathbf{p}$ and $\mathbf{p}'$ be base point systems of D and D', respectively. Then $(D,\mathbf{p})$ and $(D',\mathbf{p}')$ are related by a finite sequence of self-crossing virtualizations, $\overline{\text{w}}$-isotopies and base-change moves. If $(D,\mathbf{p})$ and $(D',\mathbf{p}')$ are related by a self-crossing virtualization, then by Theorem 6.4 $\mu_{(D,\mathbf{p})}(I)=\mu_{(D',\mathbf{p}')}(I)$ for any non-repeated sequence I. This implies that $\Delta_{(D,\mathbf{p})}(I)=\Delta_{(D',\mathbf{p}')}(I)$. If $(D,\mathbf{p})$ and $(D',\mathbf{p}')$ are related by a $\overline{\text{w}}$-isotopy or base-change moves, then it follows from Theorems 3.1 and 5.1 that $\mu_{(D,\mathbf{p})}(I)\equiv\mu_{(D',\mathbf{p}')}(I) \pmod{\Delta_{(D,\mathbf{p})}(I)}$ and $\Delta_{(D,\mathbf{p})}(I)=\Delta_{(D',\mathbf{p}')}(I)$. This completes the proof. $\square$

Remark 6.5. It is suggested in [7, 1] that Dye and Kauffman in [4] failed to define Milnor-type "invariants". We clarify why Dye and Kauffman's construction/definition is incorrect. In [4], Dye and Kauffman defined a residue class $\overline{\mu}^{\mathrm{DK}}$ of Milnor numbers μ for virtual link diagrams with base point systems. Their construction follows Milnor's original work [12] but a different indeterminacy $\Delta^{\mathrm{DK}}(j_1\dots j_r i)$, which is defined as the greatest common divisor of all $\mu(k_1\dots k_s i)$, where $k_1\dots k_s$ is a proper "subset" of $j_1\dots j_r$, see [4, page 945]. (Here, "subset" should rather be "subsequence".) We stress that $\Delta^{\mathrm{DK}}(j_1\dots j_r i)$ is determined by Milnor numbers for sequences with the last index i. It is stated in [4, Section 4] that $\overline{\mu}^{\mathrm{DK}}$ does not depend on the choice of base point system, and moreover that it is an invariant of virtual links. However, this is *wrong*. More precisely,

$\overline{\mu}^{\mathrm{DK}}$ is *not* well-defined even for classical link diagrams. In the following, we will show that $\overline{\mu}^{\mathrm{DK}}$ does depend on both Reidemeister moves and the choice of base point system: Let $(D,\mathbf{p})$, $(D,\mathbf{p}')$ and $(D',\mathbf{p})$ be the 3-component link diagrams as in Figure 6.2. (We remark that the definition of arcs of a diagram in [4] coincides with the original one in [12].) Note that $(D,\mathbf{p})$ and $(D,\mathbf{p}')$ have the same diagram and different base point systems, and that $(D,\mathbf{p})$ and $(D',\mathbf{p})$ are related by a single R1 move relative base point system. Let l, l' and l'' be the 3rd longitudes of $(D,\mathbf{p})$, $(D,\mathbf{p}')$ and $(D',\mathbf{p})$, respectively. Then by the definition of η_q in [12, 4], $\eta_3(l) = \alpha_2^{-1}\alpha_1^{-1}\alpha_2\alpha_1$, $\eta_3(l') = \eta_3(l'') = 1$, and hence $E(\eta_3(l)) = 1 + X_2X_1 - X_1X_2 + (\text{terms of degree } \geq 3)$ and $E(\eta_3(l')) = E(\eta_3(l'')) = 1$. Since $\Delta^{\mathrm{DK}}_{(D,\mathbf{p})}(123) = \gcd\big(\mu_{(D,\mathbf{p})}(13), \mu_{(D,\mathbf{p})}(23)\big) = 0$, we have $\overline{\mu}^{\mathrm{DK}}_{(D,\mathbf{p})}(123) = -1$, while $\overline{\mu}^{\mathrm{DK}}_{(D,\mathbf{p}')}(123) = \overline{\mu}^{\mathrm{DK}}_{(D',\mathbf{p})}(123) = 0$.

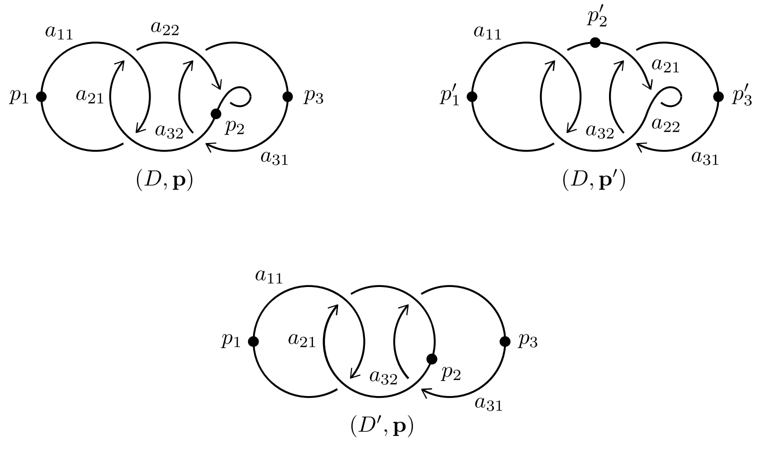

Figure 6.2:

Remark 6.6. In Remark 6.5, for the original definition of arcs in [12], we see that $\mu_{(D,\mathbf{p})}(123) \neq \mu_{(D',\mathbf{p})}(123)$, while $(D,\mathbf{p})$ and $(D',\mathbf{p})$ are related by a single R1 move relative base point system. This implies that Theorem 3.1 does not hold for the original definition of arcs.

7 Welded string links

In the previous sections, we have studied Milnor invariants of welded *links*. Now we address the case of welded *string links*.

Fix n distinct points $0 < x_1 < \cdots < x_n < 1$ in the unit interval $[0,1]$. Let $[0,1]_1, \ldots, [0,1]_n$ be n copies of $[0,1]$. An *n-component virtual string link diagram* is the image of an immersion

$$\bigsqcup_{i=1}^{n} [0,1]_i \longrightarrow [0,1] \times [0,1]$$

such that the image of each $[0,1]_i$ runs from $(x_i, 0)$ to $(x_i, 1)$, and the singularities are only classical and virtual crossings. The n-component virtual string link diagram $\{x_1, \dots, x_n\} \times [0,1]$ in $[0,1] \times [0,1]$ is called the *trivial n-component string link diagram.* An *n-component welded string link* is an equivalence class of n-component virtual string link diagrams under welded isotopy.

Let $\pi : [0,1] \times [0,1] \longrightarrow [0,1]$ be the projection onto the first coordinate. Given an n-component virtual string link diagram S, an n-component virtual link diagram with a base point system is uniquely obtained by identifying points on the boundary of $[0,1] \times [0,1]$ with their images under the projection π. We denote it by $(D_S, \mathbf{p}_S)$, where $\mathbf{p}_S = (\pi(x_1, 0), \dots, \pi(x_n, 0)) = (\pi(x_1, 1), \dots, \pi(x_n, 1))$. We see that if two virtual string link diagrams S and S' are welded isotopic, then $(D_S, \mathbf{p}_S)$ and $(D_{S'}, \mathbf{p}_{S'})$ are $\overline{\text{w}}$-isotopic.

For a sequence I of indices in $\{1, \dots, n\}$, the *Milnor number* $\mu_S(I)$ of S is defined to be $\mu_{(D_S, \mathbf{p}_S)}(I)$. Theorem 3.1 implies the following directly.

Corollary 7.1. *Let S and S' be virtual diagrams of a welded string link. Then $\mu_S(I) = \mu_{S'}(I)$ for any sequence I.*

Combining Theorems 3.1 and 6.4, the following result is obtained immediately.

Corollary 7.2 ([10, Lemma 9.1])**.** *If two virtual string link diagrams S and S' are related by a finite sequence of self-crossing virtualizations and welded isotopies, then $\mu_S(I) = \mu_{S'}(I)$ for any non-repeated sequence I.*

Remark 7.3. The converse of Corollary 7.2 is also true. In fact, it is shown in [1, 10] that Milnor numbers for non-repeated sequences classify virtual string link diagrams up to self-crossing virtualizations and welded isotopies.

We conclude this article with a classification result of virtual link diagrams with base point systems up to an equivalence relation generated by self-crossing virtualizations and $\overline{\text{w}}$-isotopies.

Theorem 7.4. *Let $(D, \mathbf{p})$ and $(D', \mathbf{p}')$ be virtual link diagrams with base point systems. Then the following are equivalent.*

(1) *$(D, \mathbf{p})$ and $(D', \mathbf{p}')$ are related by a finite sequence of self-crossing virtualizations and $\overline{\text{w}}$-isotopies.*

(2) *$\mu_{(D,\mathbf{p})}(I) = \mu_{(D',\mathbf{p}')}(I)$ for any non-repeated sequence I.*

Proof. (1) $\Rightarrow$ (2): This follows from Theorems 3.1 and 6.4 directly.

(2) $\Rightarrow$ (1): For a small disk δ which is disjoint from $(D, \mathbf{p})$ (or $(D', \mathbf{p}')$), by applying VR2 relative base point system and the local move in Figure 3.1 repeatedly, we can deform $(D, \mathbf{p})$ (or $(D', \mathbf{p}')$) such that the intersection between the disk δ and the deformed diagram is the trivial string link diagram whose each component contains the base point. Hence, $D \setminus \delta$ and $D' \setminus \delta$ can be regarded as string link diagrams S and S', respectively. Since $(D_S, \mathbf{p}_S)$ and $(D_{S'}, \mathbf{p}_{S'})$ are $\overline{\text{w}}$-isotopic to $(D, \mathbf{p})$ and $(D', \mathbf{p}')$, respectively, it follows from Theorem 3.1 that

$$\mu_S(I) = \mu_{(D,\mathbf{p})}(I) \quad \text{and} \quad \mu_{S'}(I) = \mu_{(D',\mathbf{p}')}(I)$$

for any non-repeated sequence I. Hence we have $\mu_S(I) = \mu_{S'}(I)$ by assumption. Then, by Remark 7.3, S and S' are related by a finite sequence of self-crossing virtualizations and welded isotopies. This implies that $(D_S, \mathbf{p}_S)$ and $(D_{S'}, \mathbf{p}_{S'})$ are related by a finite sequence of self-crossing virtualizations and $\overline{\mathrm{w}}$-isotopies. □

Remark 7.5. By Theorem 7.4, the two virtual link diagrams with base point systems $(D, \mathbf{p})$ and $(D, \mathbf{p}')$ given in Example 3.3 are not related by a finite sequence of self-crossing virtualizations and $\overline{\mathrm{w}}$-isotopies.

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Institute for Mathematics and Computer Science
Tsuda University
Tokyo 187-8577
JAPAN
E-mail address: `aida@tsuda.ac.jp`

津田塾大学数学・計算機科学研究所　宮澤 治子

Department of Mathematics
Kobe University
Kobe 657-8501
JAPAN
E-mail address: `wada@math.kobe-u.ac.jp`

神戸大学大学院理学研究科　和田 康載

Faculty of Commerce
Waseda University
Tokyo 169-8050
JAPAN
E-mail address: `yasuhara@waseda.jp`

早稲田大学商学学術院　安原 晃