Studying knot invariants that count diagrams

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1 Introduction

This survey presents some methods used to study knot invariants defined from configuration space integrals. In Section 2, the main tools are presented on the simple example of the linking number of two knots in $\mathbb{R}^3$. The most important feature of this section is the definition of *propagators*, an example of which was studied by Fukaya [3], and which were defined by Lescop in [4]. A detailed study of the perturbative expansion of Chern-Simons theory based on propagators can be found in [6]. Here, we focus on similar invariants, but for high-dimensional knots $\mathbb{R}^n \hookrightarrow M^\circ$, where n is odd, and M° is a punctured homology $(n+2)$-sphere. In Section 3, we present the definition of generalized Bott-Cattaneo-Rossi (BCR) invariants for such knots, as from our article [10]. These invariants generalize a construction of Bott [1] and of Cattaneo and Rossi [2]. They admit an expression in terms of Alexander polynomial(s), which is the result of [8]. The obtention of such an expression relies on the use of specific propagators associated with the knot. Section 4 contains some insights on possible extensions of these constructions to other interesting objects, which are not knot invariants (i.e. 0-cocycles on the space of knots) anymore, but cochains on the space of knots. We hope that such constructions may lead to interesting cocycles. I thank the organizers of ILDT for the opportunity to present these topics, and especially Tomotada Ohtsuki for the invitation. I also thank Tadayuki Watanabe for his questions when I was in Matsue at Summer 2019, and after my ILDT talk, which are the starting point of some questions of Section 4.

2 Simplest example : the linking number

In this survey, all manifolds are smooth and oriented, and all maps are smooth.

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2.1 General definition

The most straightforward definition for the linking number is the following one.

Definition 2.1. Let X and Y be two chains of an n-manifold M such that $\dim(X) + \dim(Y) + 1 = n$, and assume that X is null-homologous. Let Σ_X be a chain of M with boundary X and assume that Y and Σ_X are transverse. The linking number is the algebraic intersection

$$\mathrm{lk}(X, Y) = \langle \Sigma_X, Y \rangle_M$$

and does not depend of the choice of a surface Σ_X as above.

In this section, we will describe some equivalent definitions of linking number that "count" the diagram of Figure 1.

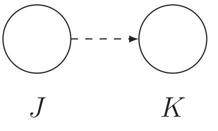


Figure 1: A diagram representing the linking number

2.2 Definition from Gauss map in $\mathbb{R}^3$

Let us now assume that $M = \mathbb{R}^3$ and fix two disjoint knots $J, K \colon \mathbb{S}^1 \hookrightarrow \mathbb{R}^3$. Their linking number is $\mathrm{lk}(J, K) = \mathrm{lk}(J(\mathbb{S}^1), K(\mathbb{S}^1))$.

In this case, define the Gauss map

$$\begin{array}{rccc} G\colon & \mathbb{S}^1 \times \mathbb{S}^1 & \to & \mathbb{S}^2 \\ & (t, u) & \mapsto & \frac{K(u) - J(t)}{||K(u) - J(t)||}. \end{array}$$

Now, we can define the degree of this map G with two methods.

Proposition 2.2. *Let ω be a 2-form on $\mathbb{S}^2$ with total area 1. The number $\int_{\mathbb{S}^1 \times \mathbb{S}^1} G^*(\omega)$ does not depend on ω. It is called the* degree $\deg(G)$ *of G.*

Proposition 2.3. *For any choice of a regular value $x \in \mathbb{S}^2$ of G,*

$$\deg(G) = \sum_{(t,u) \in G^{-1}(\{x\})} \varepsilon(t, u),$$

where $\varepsilon(t, u)$ is the sign of the determinant of the tangent map $T_{(t,u)}G$ in any two oriented bases.

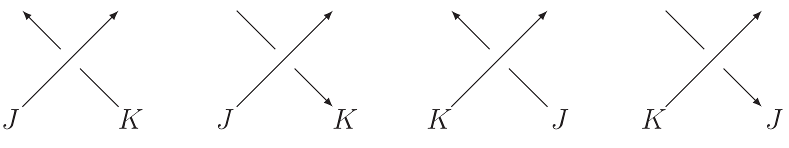
With the notations of the previous subsection, the linking number admits an equivalent definition, as follows.

Proposition 2.4. *For any disjoint knots J and K of $\mathbb{R}^3$,*

$$\mathrm{lk}(J, K) = \deg(G).$$

In particular, Propositions 2.3 and 2.4 yield the following proposition, which may be the most known definition of linking number for knots in $\mathbb{R}^3$.

Proposition 2.5. *Let J and K be two disjoint knots of $\mathbb{R}^3$, and let fix a diagram of the link $J \sqcup K$. Let n_i ($i \in \{1, \ldots, 4\}$) denote the number of crossings between one strand of J and one strand of K as in the i-th picture below.*



The linking number is

$$\mathrm{lk}(J, K) = n_3 - n_4 = n_2 - n_1.$$

The above property is the reason why such an invariant is considered as a "diagram count". Indeed, the formula allows us to interpret $\mathrm{lk}(J, K)$ as a signed count of elements of $J(\mathbb{S}^1) \times K(\mathbb{S}^1)$ with some constraint on the direction of the Gauss map. Here, the constraint on the direction is very easy to visualize, but it is interesting to allow more flexible (and less visualizable) choices on the constraints than the direction of the vector. This greater flexibility will serve two main goals: dealing with other manifolds than the Euclidean spaces $\mathbb{R}^n$ (where the notion of "direction" makes sense), and, most importantly, obtaining some formulas for invariants using more appropriated constraints.

2.3 Definition in terms of propagators

Now, M is a general closed n-manifold, with the rational homology of an n-sphere. Fix a point ∞ of M, and define M° as the manifold $M \setminus \{\infty\}$. We identify a punctured neighborhood of ∞ in M with the complement B°_∞ of the unit ball in $\mathbb{R}^n$ so that M° reads as $B^\circ_\infty \cup B(M)$, where $B(M)$ replaces the unit ball. Such a M° is called an *asymptotic homology* $\mathbb{R}^n$. The typical example is $(M, M^\circ) = (\mathbb{S}^n, \mathbb{R}^n)$.

Definition 2.6. A *parallelization* τ of M° is a trivialization $\tau\colon M^\circ \times \mathbb{R}^n \to TM^\circ$ that coincides on B°_∞ with the canonical trivialization of $\mathbb{R}^n$. For such a parallelization and $x \in M^\circ$, τ_x denotes the isomorphism $\tau(x, \cdot)\colon \mathbb{R}^n \to T_xM^\circ$.

Let $C_2^0(M^\circ)$ denote the (non-compact) manifold $M^\circ \times M^\circ \setminus \mathrm{diag} = \{(x,y) \in M^\circ \times M^\circ \mid x \neq y\}$.

Proposition 2.7. *There exists a compact smooth manifold with boundary and edges $C_2(M^\circ)$ such that*

- *The interior of $C_2(M^\circ)$ identifies canonically with $C_2^0(M^\circ)$.*
- *Any parallelization τ yields a smooth map $G_\tau\colon \partial C_2(M^\circ) \to \mathbb{S}^{n-1}$.*

We refer to [5, Section 2.2] for more details on this construction. However, one can get a rough idea of the construction from the following (partial) remarks.

- The codimension 1 boundary of the compactified configuration space $C_2(M^\circ)$ consists of three kinds of configurations : those where the two points x and y coincide, but where we remember the local direction from x to y (which is an element u of the unitary tangent bundle U_xM°) ; those where one of the points x and y is at ∞, but where we remembered in which direction the point at the infinity "escaped" (this is an element of $U_\infty M$, i.e. of $\mathbb{S}^{n-1}$); and those where x and y both escaped to infinity, that we won't detail here.
- The Gauss map G_τ maps a configuration (x,x) with direction $u \in U_xM$ to $\frac{{\tau_x}^{-1}(u)}{||{\tau_x}^{-1}(u)||} \in \mathbb{S}^{n-1}$. It maps a configuration with x at the infinity in the direction u_x (and $y \in M^\circ$) to $-u_x$, and it maps a configuration with y at the infinity in the direction u_y (and $x \in M^\circ$) to $+u_y$. Here, we do not give the details when $(x,y) = (\infty,\infty)$ (See [5, Proposition 2.3].)

Now, we give two possible definitions of *propagators*.

Definition 2.8. A *propagating form* of (M°, τ) is a closed $(n-1)$-form β on $C_2(M^\circ)$ such that $\beta_{|\partial C_2(M^\circ)}$ is ${G_\tau}^*(\omega)$ for some $(n-1)$-form ω on $\mathbb{S}^{n-1}$ with total volume 1 such that $(-\mathrm{Id}_{\mathbb{S}^{n-1}})^*(\omega) = (-1)^n\omega$.

A *propagating chain* of (M°, τ) is a rational $(n+1)$-chain B of $C_2(M^\circ)$ such that ∂B reads $\frac{1}{2}{G_\tau}^{-1}(\{-x,+x\})$ for some $x \in \mathbb{S}^{n-1}$.

For any parallelized asymptotic homology $\mathbb{R}^n$, propagating chains and forms as above exist.

Example 2.9. When $M^\circ = \mathbb{R}^n$, the map $G\colon (x,y) \in C_2^0(\mathbb{R}^n) \mapsto \frac{y-x}{||y-x||} \in \mathbb{S}^{n-1}$ extends to $C_2(\mathbb{R}^n)$, and its restriction to $\partial C_2(\mathbb{R}^n)$ coincides with the Gauss map associated with the canonical parellelization ([5, Lemma 2.2]). This provides canonical examples of propagators : the form $G^*(\omega)$ for the $SO(n)$-invariant $(n-1)$-form ω on $\mathbb{S}^{n-1}$ with total volume 1, and the chains $\frac{1}{2}G^{-1}(\{-x,+x\})$ for $x \in \mathbb{S}^{n-1}$.

Propagators allow a more flexible definition of the linking number as follows. ([8, Lemma 2.22])

Proposition 2.10. *Let* (M°, τ) *be a parallelized asymptotic homology* $\mathbb{R}^n$*. Let* β *(resp.* B*) be a propagating form (resp. chain) of* (M°, τ)*. Let* X *and* Y *be two disjoint cycles with* $\dim(X) + \dim(Y) = n - 1$*.*

$$\mathrm{lk}(X, Y) = \int_{X \times Y} \beta_{|X \times Y} = \langle X \times Y, B \rangle_{C_2(M^\circ)}.$$

2.4 Last notes

The goal of the above section was to illustrate the following idea: starting with an embedding (here, two disjoint knots) and a diagram (here, an edge between two points), we can define a configuration space (here, the product of the two knots). Using propagating forms, we can define a form on the configuration space (here, simply by restriction) and thus a configuration space integral. Equivalently, from propagating chains, we can define an algebraic diagram count (here, the intersection of the propagating chain with the product of knots). In the above example, these two methods yield an invariant, which is the linking number. In general, things are more complicated, and we have to combine more diagrams to get some invariants. The interested reader can refer to [6] for a detailed study of the general 3-dimensional invariant obtained from all these diagram counts, which is valued in some Hopf algebra of diagrams. In the remaining of this survey, we will focus on similar (but numerical) invariants for high-dimensional knots.

3 High-dimensional invariants from diagram counts

3.1 BCR diagrams

In this section, a *diagram* is an oriented graph Γ without looped edge[1] such that the set $V(\Gamma)$ of vertices (resp. $E(\Gamma)$ of edges) is decomposed as $V(\Gamma) = V_i(\Gamma) \sqcup V_e(\Gamma)$ (resp. $E(\Gamma) = E_i(\Gamma) \sqcup E_e(\Gamma)$). The elements of $V_i(\Gamma)$ are called internal vertices, and those of $V_e(\Gamma)$ are called external vertices. Similarly, elements of $E_i(\Gamma)$ are called internal edges, and elements of $E_e(\Gamma)$ are called external edges. On the figures, internal vertices are full red dots, external vertices are empty blue dots (i.e. blue circles), internal edges are solid red arrows, and external edges and dashed blue arrows.

Definition 3.1. A *BCR diagram* is a diagram as above such that

[1] A looped edge is an edge from one vertex to itself.

- Γ is connected,
- any trivalent vertex is adjacent to one univalent vertex,
- any vertex is as in Figure 2.

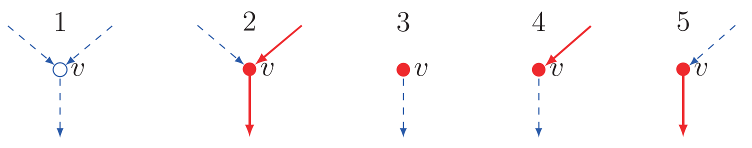


Figure 2: The five possible behaviors near a vertex of a BCR diagram

The definition of BCR diagrams implies that $\mathrm{Card}(E(\Gamma)) = \mathrm{Card}(V(\Gamma))$, and thus that their first Betti number is one. The *degree* of a BCR diagram is half its number of vertices, and is indeed an integer.

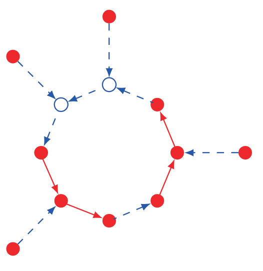


Figure 3: A BCR diagram of degree 6

3.2 Configuration spaces

Let M° be a fixed asymptotic homology $\mathbb{R}^{n+2}$. A *long knot* of M° is an embedding $\psi\colon \mathbb{R}^n \hookrightarrow M^\circ$ such that, for any $x \in \mathbb{R}^n$, if $||x|| \geq 1$, then $\psi(x) = (0, 0, x) \in B^\circ_\infty \subset M^\circ$. Fix a long knot ψ. For any diagram Γ, define the *configuration space*

$$C_\Gamma(\psi) = \{c\colon V(\Gamma) \hookrightarrow M^\circ \mid \text{There exists a map } c_i\colon V_i(\Gamma) \hookrightarrow \mathbb{R}^n, c_{|V_i(\Gamma)} = \psi \circ c_i\}.$$

The above configuration space does not depend on the edges of Γ, but only on the set $V(\Gamma)$ and its partition into internal and external vertices. Edges correspond to maps from this configuration space to two-point configuration spaces as follows. Let e be an edge of Γ from v to w. If e is internal, set

$$\begin{array}{rccc} p_e\colon & C_\Gamma(\psi) & \to & C_2(\mathbb{R}^n) \\ & c & \mapsto & (c_i(v), c_i(w)) \end{array}$$

and, if e is external, set

$$\begin{array}{rlcl} p_e\colon & C_\Gamma(\psi) & \to & C_2(M^\circ) \\ & c & \mapsto & (c(v), c(w)). \end{array}$$

3.3 Configuration space integrals

Assume n is odd. Fix

- an integer $k \geq 2$,
- a parallelization τ of M° as in Definition 2.6,
- a family $F = (\alpha_i, \beta_i)_{1\leq i\leq 2k}$ such that for any i, α_i is a propagating form of $\mathbb{R}^n$ with its canonical parallelization, and β_i is a propagating form of (M°, τ),
- a degree k BCR diagram Γ,
- a bijection σ between $E(\Gamma)$ and $\{1, \ldots, 2k\}$, (Such a σ is called a *numbering* of Γ.).

Define for any edge the form

$$\omega_F(\Gamma, \sigma, e) = \begin{cases} {p_e}^*(\alpha_{\sigma(e)}) & \text{if } e \text{ is internal,} \\ {p_e}^*(\beta_{\sigma(e)}) & \text{if } e \text{ is external,} \end{cases}$$

and set $\omega_F(\Gamma, \sigma) = \bigwedge\limits_{e\in E(\Gamma)} \omega_F(\Gamma, \sigma, e)$. The latter is a form on $C_\Gamma(\psi)$.

The definition of BCR diagrams implies that $\deg(\omega_F(\Gamma, \sigma)) = \dim(C_\Gamma(\psi))$. The integral $\int_{C_\Gamma(\psi)} \omega_F(\Gamma, \sigma)$ converges.[2] Set

$$\langle \Gamma, \sigma\rangle_{F,\psi} = \int_{C_\Gamma(\psi)} \omega_F(\Gamma, \sigma).$$

3.4 BCR invariants

Let $\widetilde{\mathcal{G}_k}$ denote the set of degree k BCR diagrams Γ together with a numbering σ, up to isomorphisms that preserve the nature of edges and vertices and the numberings. Set

$$Z_k^F(\psi) = \frac{1}{(2k)!} \sum_{(\Gamma,\sigma)\in\widetilde{\mathcal{G}_k}} \langle \Gamma, \sigma\rangle_{F,\psi}.$$

[2] Details on orientation of $C_\Gamma(\psi)$ can be found in [10, Section 2.4].

Theorem 3.2. *The real number $Z_k^F(\psi)$ depend neither on the choice of the family of propagators F, nor of the choice of the parallelization τ. It is invariant under ambient diffeomorphism : if Φ is a diffeomorphism of M° that fixes B_∞° pointwise, then $Z_k^F(\Phi \circ \psi) = Z_k^F(\psi)$.*

We call $Z_k(\psi) = Z_k^F(\psi)$ the degree k generalized BCR invariant *of ψ.*

The above result is [10, Theorem 2.10] for $n \geq 3$, and follows from [7, Corollary 2.15] for $n = 1$. The above definition of (generalized) BCR invariant directly generalizes the original definitions of Bott [1] for $k = 2$, and of Cattaneo and Rossi [2], which are the case where M° is $\mathbb{R}^{n+2}$ with its canonical parallelization and where propagators are given by the forms of Example 2.9.

3.5 BCR invariants from diagram counts

Fix the same setting as in the previous sections, but replace F with a family $F = (A_i, B_i)_{1 \leq i \leq 2k}$, where for any i, A_i is a propagating chain of $\mathbb{R}^n$ with its canonical parallelization and B_i is a propagating chain of (M°, τ).

Now, for any edge e, set

$$D_F(\Gamma, \sigma, e) = \begin{cases} {p_e}^{-1}(A_{\sigma(e)}) & \text{if } e \text{ is internal,} \\ {p_e}^{-1}(B_{\sigma(e)}) & \text{if } e \text{ is external,} \end{cases}$$

which defines a chain of $C_\Gamma(\psi)$.

For a generic[3] choice of F, the chains $(D_F(\Gamma, \sigma, e))_{e \in E(\Gamma)}$ are transverse. The *diagram count* of (Γ, σ) for F is their algebraic intersection number $I_F(\Gamma, \sigma)$.

The following theorem is derived from the previous one by duality, as explained in [10, Section 4].

Theorem 3.3. *For a generic choice of propagating chains F, the diagram counts $I_F(\Gamma, \sigma)$ are well-defined for any degree k numbered BCR diagram (Γ, σ), and the generalized BCR invariant of Theorem 3.2 is*

$$Z_k(\psi) = \frac{1}{(2k)!} \sum_{(\Gamma, \sigma) \in \widetilde{\mathcal{G}_k}} I_F(\Gamma, \sigma).$$

3.6 Computation of BCR invariants

One of the strength of the previous construction is that we can compute BCR invariants using *any* set of propagators. In [8, Section 6], we describe a specific construction of propagating chains associated with some long knots.

[3] See [10, Section 4] for details.

Definition 3.4. A long knot ψ of M° is *rectifiable* if we can choose a parallelization τ of M° such that for any $x, u \in \mathbb{R}^n$, $\tau_{\psi(x)}(0,0,u) = T_x\psi(u)$.

[8, Theorem 2.33] gives the following formula for BCR invariants of rectifiable knots, in terms of Alexander polynomials [4].

Theorem 3.5. *Let ψ be a rectifiable long knot, then*

$$\sum_{k=2}^{\infty} Z_k(\psi)h^k = \sum_{d=1}^{n}(-1)^d \mathrm{Ln}(\Delta_{d,\psi}(e^h)),$$

where $\Delta_{d,\psi}$ denotes the d-th Alexander polynomial of 1-dimensional knots.

The above formula extends a result of Watanabe [13] for the particular class of ribbon long knots, and lifts some indeterminacies that remained in the relation between Alexander polynomials and BCR invariants. In [8, Section 5], we prove that all long knots are rectifiable up to connected sum with a finite number of copies of themselves when $n \equiv 1 \mod 4$. This implies that the above formula extends to all long knots when $n \equiv 1 \mod 4$. For $n = 1$, the result extends as follows, where asymptotic rational homology $\mathbb{R}^3$ are punctured rational homology 3-spheres.

Corollary 3.6. *When $n = 1$, for any null-homologous long knot ψ of a asymptotic rational homology $\mathbb{R}^3$,*

$$\sum_{k=2}^{\infty} Z_k(\psi)h^k = -\mathrm{Ln}(\Delta(e^h)),$$

where Δ_ψ denotes the Alexander polynomial of 1-dimensional knots.

4 Insights

Many open questions naturally arise from the above construction. The case of even-dimensional knots will be soon covered by [9]: the invariants are well-defined, when we restrict to parallelizable asymptotic homology $\mathbb{R}^{n+2}$, and only use parallelizations such that $\tau_{\psi(x)}(0,0,u) = T_x\psi(u)$ for any $x, u \in \mathbb{R}^n$. The formula in terms of Alexander polynomials of Theorem 3.5 still holds, up to some signs. In this last section, we give some ideas on a possible extension of the previous construction to more general diagrams, which are still a work in progress.

Let Γ be a diagram with vertices as in Figure 2. We do not assume anymore that Γ is connected, nor that trivalent vertices always have one univalent neighbor.

[4] These Alexander polynomials are defined using Alexander invariants from Levine [11]. For $n = 1$, $\Delta_{1,\psi}$ is the usual Alexander polynomial of 1-dimensional knots.

Let σ be a bijection $E(\Gamma) \to \{1, \dots, \mathrm{Card}(E(\Gamma))\}$. Let $F = (\alpha_i, \beta_i)_{1 \leq i \leq \mathrm{Card}(E(\Gamma))}$ be a family of propagating forms as in Section 3.3. If the first Betti number of Γ is greater than one, then the differential form $\omega_F(\Gamma, \sigma)$ have degree greater than the dimension of $C_\Gamma(\psi)$. More precisely,

$$\deg(\omega_F(\Gamma, \sigma)) = \dim(C_\Gamma(\psi)) + (n-1)(b_1(\Gamma) - 1).$$

Thus, the configuration space integral is a cochain $I(\Gamma, \sigma)$ of degree $(n-1)(b_1(\Gamma) - 1)$ on the space $\mathcal{K}$ of long embeddings $\mathbb{R}^n \hookrightarrow M^\circ$. Let $\widetilde{\mathcal{G}}_{k,b}$ denote the set of numbered diagrams (Γ, σ) with $\deg(\Gamma) = k$ and $b_1(\Gamma) = b$, up to numbered diagram isomorphisms. Define the cochain

$$Z_{k,b} = \sum_{(\Gamma,\sigma) \in \widetilde{\mathcal{G}}_{k,b}} I(\Gamma, \sigma)[\Gamma],$$

which takes its values inside the vector space $\mathcal{D}_{k,b}$ spanned by degree k diagrams with first Betti number b. Given a quotient $\pi \colon \mathcal{D}_{k,b} \to \mathcal{A}_{k,b}$ of $\mathcal{D}_{k,b}$, we can form the cochain $\pi \circ Z_{k,b}$.

Note that $Z_{k,1}$ is a 0-cochain, and there exists a quotient $\pi_{k,1} \colon \mathcal{D}_{k,1} \to \mathcal{A}_{k,1}$ such that $\mathcal{A}_{k,1}$ is 1-dimensional, and such that $\pi_{k,1} \circ Z_{k,1}$ identifies with Z_k. Since the last expression is a knot invariant, the 0-cochain $\pi \circ Z_{k,1}$ is a 0-cocycle. Thus, it is natural to ask the following questions.

Questions 4.1. For any (k, b), does there exist a non-trivial quotient $\pi_{k,b} \colon \mathcal{D}_{k,b} \to \mathcal{A}_{k,b}$ such that $\pi_{k,b} \circ Z_{k,b}$ is an $(n-1)(b-1)$-cocycle on the space $\mathcal{K}$? (Such a quotient could be defined from linear relations between similar diagrams.)

Does these cocycles yield any non-trivial class in $H^{(n-1)(b-1)}(\mathcal{K})$?

Does the obtained class in cohomology depend on the choice of propagating forms ?

Can we describe the obtained cocycles in terms of simpler objects on the space of knots ?

Following a study of Sakai and Watanabe [12], we can also extend the study of such invariants to long embeddings $\mathbb{R}^j \hookrightarrow \mathbb{R}^n$ without the "codimension 2" hypothesis. In their article, they proved the existence of relations on diagrams with $b_1(\Gamma) = 1$ that yield a quotient $\pi \colon \mathcal{D}_{k,1} \to \mathcal{A}_{k,1}$ such that $z_k = \pi \circ Z_{k,1}$ is indeed a $(n-j-2)k$-coycle of the space of long embeddings $\mathbb{R}^j \hookrightarrow \mathbb{R}^n$, when one of the three following properties holds:

- n is odd,
- n is even, j is odd, and $k \leq 4$
- n is even, $n \geq 12$, $j = 3$.

Moreover, they proved that some of the obtained cocycles were non-trivial. Following a question of Watanabe after the ILDT talk that preceded this proceeding, we can look for interesting formulas for this cocycle z_k that could be derived from a method similar to the one that allowed us to compute the BCR invariants in Theorem 3.5.

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