Diffeomorphisms of some 4-manifolds constructed by theta graph claspers

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1 Introduction

In this article, manifolds and maps between them are assumed to be C^∞, spaces of mappings between manifolds are equipped with the weak C^∞-topology. For a compact manifold X, let $\mathrm{Diff}_\partial(X)$ denote the group of self-diffeomorphisms of X that restrict to the identity on the boundary. For a compact submanifold $Y \subset X$, let $\mathrm{Emb}_\partial(Y, X)$ denote the space of embeddings $Y \to X$ that agree on ∂Y with the inclusion $Y \subset X$. Let $\mathrm{Diff}(X)$ and $\mathrm{Emb}(Y, X)$ denote the group of self-diffeomorphisms of X and the space of embeddings $Y \to X$, respectively, without the boundary condition. The homotopy types of these spaces have been extensively studied by various methods. We study the following problem.

Problem 1.1. *What is* $\pi_0\mathrm{Diff}_\partial(D^3 \times S^1)$*?*

The group $\pi_0\mathrm{Diff}_\partial(D^3 \times S^1)$ is sometimes called the mapping class group of $D^3 \times S^1$ and is of interest from several viewpoints. Here we shall mention an insight of D. Gabai from the "4-dimensional light bulb problem", about which some results were stated in [Ga] and developed in detail in his joint work with R. Budney [BG]. Gabai pointed out that there is an important application of the study of $\pi_0\mathrm{Diff}_\partial(D^3 \times S^1)$ to the 4-dimensional smooth Schoenflies conjecture ([Ki, Problem 4.32]).

Conjecture 1.2 (4-dimensional smooth Schoenflies conjecture)**.** *Any smooth embedding* $S^3 \to S^4$ *can be extended to a smooth embedding* $D^4 \to S^4$*. In other words, any embedding* $S^3 \to S^4$ *is trivial up to isotopy, or* $\pi_0\mathrm{Emb}(S^3, S^4) = 0$*.*

This is one of the major problems in 4-dimensional topology. It is known that any smooth 3-sphere in S^4 bounds a "topological" 4-disk in S^4 (Mazur [Ma]). Thus, if Conjecture 1.2 would be false, then there would exist a smooth 3-sphere in S^4 which bounds an exotic 4-disk, i.e., a 4-manifold homeomorphic to but not diffeomorphic to the standard 4-disk, contradicting to the 4-dimensional smooth Poincaré conjecture.

In 3-dimension, Hatcher proved that the restriction map $\mathrm{Emb}(D^3, \mathbb{R}^3) \to \mathrm{Emb}(S^2, \mathbb{R}^3)$ is a homotopy equivalence, by proving that the induced maps on homotopy groups are isomorphisms ([Hat]). The result for π_0 is the smooth Schoenflies theorem in 3-dimension (Alexander's theorem, [Hat2]). On the other hand, we have proved that the restriction map $\pi_k\mathrm{Emb}(D^4, \mathbb{R}^4) \to \pi_k\mathrm{Emb}(S^3, \mathbb{R}^4)$ is not surjective for some higher k in [Wa1].

Gabai related $\mathrm{Diff}_\partial(D^3\times S^1)$ with $\mathrm{Emb}_\partial(D^3, D^3\times S^1)$ by applying Cerf–Palais' theorem. We consider D^3 as a submanifold of $D^3\times S^1$ via the standard inclusion $\iota : D^3 \to D^3\times\{*\} \subset D^3 \times S^1$. According to Cerf–Palais theorem ([Ce, Pa]), the restriction map

$$\mathrm{Diff}_\partial(D^3 \times S^1) \to \mathrm{Emb}_\partial(D^3, D^3 \times S^1),$$

given by $g \mapsto g \circ \iota$, is a locally trivial fiber bundle with fiber having the homotopy type of $\mathrm{Diff}_\partial(D^4)$. In [BG], a stronger result is proved. It is proved in [BG] that there is a homotopy equivalence

$$\mathrm{Diff}_\partial(D^3 \times S^1) \simeq \mathrm{Emb}_\partial(D^3, D^3 \times S^1) \times \mathrm{Diff}_\partial(D^4).$$

$\pi_0\mathrm{Emb}_\partial(D^3, D^3 \times S^1)$ and $\pi_0\mathrm{Emb}_\partial(D^3, D^4)$ are related in the following way. There are natural embeddings

$$\mathrm{Emb}_\partial(D^3, D^3 \times (-\varepsilon, \varepsilon)) \xrightarrow{i} \mathrm{Emb}_\partial(D^3, D^3 \times S^1) \xrightarrow{\lambda} \mathrm{Emb}_\partial(D^3, D^3 \times \mathbb{R}),$$

where i is induced by inclusion $(-\varepsilon, \varepsilon) \subset S^1$ and λ is given by the (unique) lift in the infinite cyclic cover of $D^3 \times S^1$. One may see that both $\pi_0\mathrm{Emb}_\partial(D^3, D^3 \times (-\varepsilon, \varepsilon))$ and $\pi_0\mathrm{Emb}_\partial(D^3, D^3 \times \mathbb{R})$ are isomorphic to $\pi_0\mathrm{Emb}_\partial(D^3, D^4)$ as sets. Since $\lambda \circ i$ is homotopic to the identity, the natural map $i_* : \pi_0\mathrm{Emb}_\partial(D^3, D^3 \times (-\varepsilon, \varepsilon)) \to \pi_0\mathrm{Emb}_\partial(D^3, D^3 \times S^1)$ is injective, and the image of i_* is precisely the subset of elements that survives in $\pi_0\mathrm{Emb}_\partial(D^3, D^3 \times \mathbb{R})$.

Theorem 1.3 (Budney–Gabai [BG])**.** *(a) Explicitly constructed a subgroup of the abelian group $\pi_0\mathrm{Diff}_\partial(D^3 \times S^1)$ of infinite rank by their "barbells" (with twists).*

(b) $\pi_0\mathrm{Emb}_\partial(D^3, D^3 \times S^1)$ is naturally an abelian group. The subgroup of (a) has image in $\pi_0\mathrm{Emb}_\partial(D^3, D^3 \times S^1)$ of infinite rank.

(c) The image of the subgroup of (a) in $\pi_0\mathrm{Diff}_0(S^3\times S^1)$ has infinite rank, where $\mathrm{Diff}_0(S^3\times S^1)$ is the subgroup of $\mathrm{Diff}(S^3 \times S^1)$ consisting of diffeomorphisms homotopic to the identity.

(d) The subgroup of $\pi_0\mathrm{Diff}_0(S^3 \times S^1)$ of (a) has image in $\pi_0\mathrm{Emb}_0(S^3, S^3 \times S^1)$ of infinite rank, where $\mathrm{Emb}_0(S^3, S^3 \times S^1)$ is the space of embeddings $S^3 \to S^3 \times S^1$ homotopic to the standard inclusion $S^3 \to S^3 \times \{*\} \subset S^3 \times S^1$.

(e) Every counterexample of the 4-dimensional Schoenflies conjecture can be obtained from $\pi_0\mathrm{Diff}_\partial(D^3 \times S^1)$.

(f) An element δ of $\pi_0\mathrm{Emb}_\partial(D^3, D^3 \times S^1)$ is a counterexample of the 4-dimensional Schoenflies conjecture if and only if $\lambda_*(\delta) \neq 0$, i.e., nontrivial after lifting to cyclic covers.

(g) $\pi_0\mathrm{Emb}_\partial(D^3, D^3 \times S^1) \cong \pi_1\mathrm{Emb}_\partial(D^2, D^4)$.

This is a part of Budney and Gabai's results in [BG]. They proved (a), (b), (c), (d) by utilizing "embedding calculus" for the space of embeddings of circles and arcs in manifolds, which is a homotopy theoretic method. Roughly, a "barbell" introduces a

1-parameter crossing change between two arcs in a 4-manifold, given by spinning an arc along a meridian 2-sphere of another arc. D. Gabai remarked that the geometric criterion (f) had been known by Mazur since many years ago, and Gabai rediscovered the same criterion. They applied the criterion (f) to see that their construction does not give a counterexample to the 4-dimensional smooth Schoenflies conjecture. The barbells with twists and the results (c), (d) have been added in the recently updated version (v3) of [BG].

The main result of this article is another approach to the statement (a) of Theorem 1.3, obtained independently of [BG]. Before stating the result, we recall the fact that $\mathrm{Diff}_\partial(X)$ has a classifying space $B\mathrm{Diff}_\partial(X)$ and that the homotopy group $\pi_k B\mathrm{Diff}_\partial(X)$ is identified with the set of isomorphism classes of X-bundles $\pi : E \to S^k$ that is standard near the fiber of the base point $\pi^{-1}(*)$ and near ∂X. Also, there is a natural isomorphism $\pi_k \mathrm{Diff}_\partial(X) \cong \pi_{k+1} B\mathrm{Diff}_\partial(X)$ of groups.

Theorem 1.4 ([Wa2]). *Surgery on "Θ-graph claspers" generates a subgroup of* $\pi_1 B\mathrm{Diff}_\partial(D^3\times S^1)$ *of infinite rank, where by a Θ-graph we mean a graph consisting of two vertices with three edges connecting them.*

Roughly, a Θ-graph clasper surgery is defined by twisting a small neighborhood of an embedded Θ-graph in $D^3 \times S^1$ in some complicated way. This is a higher dimensional analogue of the theory of graph claspers due to Goussarov and Habiro ([Gu, Hab]). We detect nontrivial elements by using a version of "configuration space integrals", which is a higher dimensional analogue of invariants of Marché ([Ma]) and Lescop ([Les]) defined for knots and 3-manifolds. The general method of configuration space integrals for families of manifolds is originally due to Kontsevich ([Ko]).

We emphasize that there seems to be no difficulty to generalize this theorem to arbitrary trivalent graphs by an analogue of Lescop's method on equivariant perturbative invariant ([Les]), which fits to our graph clasper construction. General trivalent graphs would then give many nontrivial elements of $\pi_k B\mathrm{Diff}_\partial(D^3 \times S^1)$ for all $k \geq 1$.

It would be natural to ask how different the two constructions "barbells" and "Θ-graph claspers" in $\pi_0\mathrm{Diff}_\partial(D^3 \times S^1)$ are. We checked that the two constructions are equivalent, in the sense of the following proposition.

Proposition 1.5 ([Wa3]). *In* $\pi_1 B\mathrm{Diff}_\partial(D^3 \times S^1)$*, we have the following.*

(i) A "Θ-graph clasper" can be written by one "strict barbell".

(ii) A "strict barbell" can be written by a sum of "Θ-graph claspers".

By using this proposition and a result of [BG] saying that some strict barbells generates $\pi_1\mathrm{Emb}^{\mathrm{fr}}(S^1, S^3 \times S^1)_{f_0}$, where $\mathrm{Emb}^{\mathrm{fr}}(S^1, S^3 \times S^1)_{f_0}$ is the component of the inclusion $f_0 : S^1 \xrightarrow{=} S^1 \times \{*\} \subset S^1 \times S^3$ in the space of framed embeddings $S^1 \to S^3 \times S^1$, we see that Θ-graph claspers give infinitely many nontrivial elements of $\pi_0\mathrm{Diff}_0(S^3\times S^1)$, which survive in $\pi_0\mathrm{Emb}_0(S^3, S^3 \times S^1)$. This can also be obtained just by combining Proposition 1.5 and Budney–Gabai's stronger Theorem 1.3 (a), (c) instead of our Theorem 1.4. We will explain more about this corollary later.

Remark 1.6. We have seen that some more elements constructed by collections of basic claspers can be written by sums of Θ-graphs. D. Gay used Cerf theory to describe general handle structures for elements of $\pi_0\mathrm{Diff}_\partial(D^4)$ ([Gay]). By considering similarly for $D^3 \times S^1$, it could be seen that 1-parameter families generated by collections of basic claspers are in the 1,2-handle pairs subgroup of $\pi_0\mathrm{Diff}_\partial(D^3 \times S^1)$ (or of π_0 of the group of pseudoisotopies of $D^3 \times S^1$). These observations suggest that Θ-graphs generate a large part of $\pi_1 B\mathrm{Diff}_\partial(D^3 \times S^1)$ and that Budney–Gabai's invariant W_3 detects that part since Proposition 1.5 and Theorem 1.3(a) imply that W_3 detects Θ-graphs.

2 Configuration space integrals

We use a version of configuration space integrals to detect nontrivial elements of $\pi_1 B\mathrm{Diff}_\partial(D^3 \times S^1)$. We have the following result.

Proposition 2.1. *"Configuration space integrals" gives a homomorphism*

$$Z_\Theta : \pi_1 B\mathrm{Diff}_\partial(D^3 \times S^1) \to \bigwedge^3 \mathbb{Q}[t^{\pm 1}]/\sim,$$

where the equivalence relation $\sim$ is generated by the relation $f \wedge g \wedge h \sim t^n f \wedge t^n g \wedge t^n h$ for $n \in \mathbb{Z}$.

This is an analogue of the invariant for knots and 3-manifolds with trivial Alexander polynomial by Marché and Lescop ([Ma, Les]), valued in $\mathrm{Sym}^3\,\mathbb{Q}[t^{\pm 1}]/\sim$.

The invariant Z_Θ is defined roughly as follows. Instead of $D^3 \times S^1$, we take $X = \mathbb{R}^3 \times S^1$ for a technical reason and we consider $S^3 = \mathbb{R}^3 \cup \{\infty\}$. An element of $\pi_1 B\mathrm{Diff}_\partial(D^3 \times S^1)$ can be represented by an X-bundle

$$X \to E \xrightarrow{\pi} S^1$$

that is standard near infinity and the base point of S^1, with structure group $\mathrm{Diff}_c(X)$, the group of diffeomorphisms with compact support. Note that there is a homotopy equivalence $\mathrm{Diff}_\partial(D^3 \times S^1) \simeq \mathrm{Diff}_c(X)$. Then we consider the associated bundle to π with fiber a compactified configuration space of two labeled points in X:

$$\overline{\mathrm{Conf}}_2(X) \to E\overline{\mathrm{Conf}}_2(\pi) \to S^1.$$

Here, $\overline{\mathrm{Conf}}_2(X)$ is a compactification of the configuration space $\mathrm{Conf}_2(X) = X \times X \setminus \Delta_X$, which is obtained from $(S^3 \times S^1) \times (S^3 \times S^1)$ by blowing-up strata involving the diagonal $\Delta_{S^3 \times S^1}$ and $\infty \times S^1$. The interior of $\overline{\mathrm{Conf}}_2(X)$ is canonically identified with $\mathrm{Conf}_2(X)$ and the inclusion $\mathrm{Conf}_2(X) \to \overline{\mathrm{Conf}}_2(X)$ is a homotopy equivalence.

We take three generic chains P_1, P_2, P_3 of the total space $E\overline{\mathrm{Conf}}_2(\pi)$ which represent the same element of $H_6(E\overline{\mathrm{Conf}}_2(\pi), \partial E\overline{\mathrm{Conf}}_2(\pi); \mathbb{Q}[t^{\pm 1}])$ and which are "standard" near the diagonal and infinity. The boundary condition is a bit complicated, but the chains with the boundary condition can be found by rather a straightforward analogue of [Les, Proposition 12.2], which is based on a computation of twisted homology of the configuration space. Then Z_Θ is defined by the triple intersection in $E\overline{\mathrm{Conf}}_2(\pi)$ for generic chains:

$$Z_\Theta = \frac{1}{6}\langle P_1, P_2, P_3 \rangle.$$

Remark 2.2. In [Wa2], a detailed proof of the well-definedness of Z_Θ is given in a slightly different manner. The detailed proof given in [Wa2] of the corresponding fact is for the closed manifold $\Sigma \times S^1$, where Σ is the Poincaré homology 3-sphere. Then we show that the image of

$$\pi_1 B\mathrm{Diff}_\partial(D^3 \times S^1) \to \pi_1 B\mathrm{Diff}_0(\Sigma \times S^1)$$

is nontrivial. The boundary condition for P_i is simpler for closed manifold. Nevertheless, the computation of the twisted homology of the configuration space is not different between $\Sigma \times S^1$ and $D^3 \times S^1$.

A Θ-graph clasper surgery is defined for each embedding $\varphi : (\Theta\text{-graph}) \to X$ with some choice of edge orientations and labels on edges. The homotopy class of φ can be described by a triple (a, b, c) of integers modulo the relation $(a, b, c) \sim (a+n, b+n, c+n)$.

Proposition 2.3 ([Wa2])**.**

$$Z_\Theta\big(\textit{surgery on } \varphi\big) = [t^a \wedge t^b \wedge t^c] + [t^{-a} \wedge t^{-b} \wedge t^{-c}].$$

The computation can be done by an analogue of [Les, KT] for fiber bundles. The integers a, b, c can be arbitrary, and it can be checked that the image of Z_Θ is of infinite rank. This proves Theorem 1.4.

3 Claspers and barbells

Budney and Gabai considered another fiber sequence

$$\mathrm{Diff}_\partial(D^3 \times S^1) \to \mathrm{Diff}_0(S^3 \times S^1) \to \mathrm{Emb}^{\mathrm{fr}}(S^1, S^3 \times S^1)_{f_0}$$

obtained by Cerf–Palais' theorem. The horizontal sequence in the following diagram is exact by the long exact sequence of a fibration:

$$\begin{tikzcd} \pi_1\mathrm{Emb}^{\mathrm{fr}}(S^1, S^3 \times S^1)_{f_0} \arrow[r, "\partial"] & \pi_1 B\mathrm{Diff}_\partial(D^3 \times S^1) \arrow[r, "i"] \arrow[d, "Z_\Theta"] & \pi_1 B\mathrm{Diff}_0(S^3 \times S^1) \\ & \bigwedge^3 \mathbb{Q}[t^{\pm 1}]/\!\sim & \end{tikzcd} \qquad (1)$$

where ∂ is given by taking the (family of) exteriors and i is induced by the inclusion.

We shall only explain how a special case of Proposition 1.5 shows that the Θ-graph clasper surgeries give nontrivial elements of $\pi_0\mathrm{Diff}_0(S^3 \times S^1)$. Budney and Gabai computed the group $\pi_1\mathrm{Emb}^{\mathrm{fr}}(S^1, S^3 \times S^1)_{f_0}$, following a method of Arone–Szymik ([AS]), and constructed a generating set by barbells. Let L_{knot} be the image of $Z_\Theta \circ \partial$ in the above diagram, and let L_Θ be the image of Z_Θ of the subgroup generated by Θ-graph clasper surgeries. We proved the following lemma by using a 4-dimensional analogue of Habiro's clasper moves.

Lemma 3.1. $L_{\mathrm{knot}} \subset 2L_\Theta$.

We remark that the subgroup generated by Θ-graph clasper surgeries is included in that of general barbell with twists of [BG]. As L_Θ has infinite rank, it follows that ∂ has nontrivial cokernel, which is infinitely generated and isomorphic to the image in $\pi_1 B\mathrm{Diff}_0(S^3 \times S^1)$.

Remark 3.2. The result we have obtained for $\mathrm{Diff}_0(S^3 \times S^1)$ is weaker than Budney–Gabai's (Theorem 1.3(c)). We only found infinitely many nontrivial elements in $\pi_0\mathrm{Diff}_0(S^3 \times S^1)$, whereas Budney and Gabai found a subgroup of infinite rank.

4 Concluding remark

Problem 4.1. *What is the correct analogue of the content of §3 in 3-dimension? Does it lead to any nontrivial result?*

The analogues of the groups in the sequence in (1) would be related to finite type invariants of knots in homology $S^2 \times S^1$, homology $D^2 \times S^1$, and homology $S^2 \times S^1$, respectively, of degree 1. This might be used to prove the existence of new finite type invariant of homology $S^2 \times S^1$. One could also study higher odd dimensions first: one could study the exact sequence

$$\pi_{2n-2}\mathrm{Emb}^{\mathrm{fr}}(S^1, S^{2n} \times S^1)_{f_0} \xrightarrow{\partial} \pi_{2n-2}B\mathrm{Diff}_\partial(D^{2n} \times S^1) \xrightarrow{i} \pi_{2n-2}B\mathrm{Diff}_0(S^{2n} \times S^1)$$

for $2n+1 \geq 5$. Budney and Gabai also have some results for the first two groups in higher odd dimensions ([BG]).

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