Problems on Low-dimensional Topology, 2021

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This is a list of open problems on low-dimensional topology with expositions of their history, background, significance, or importance. This list was made by editing manuscripts written by contributors of open problems to the online conference "Intelligence of Low-dimensional Topology" whose live streaming is distributed from Research Institute for Mathematical Sciences, Kyoto University in May 19–21, 2021.

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1 Invariants of high-dimensional long knots from counting diagrams

(David Leturcq)

Let n be a positive integer. When $n \geq 3$ is odd, Bott [1], and then Cattaneo and Rossi [2] defined an invariant $(Z_k)_{k\in\mathbb{N}\setminus\{0,1\}}$ for *long knots*, which are embeddings $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+2}$ with a constrained behaviour outside the unit ball.

Let us briefly explain their original definition: look at connected oriented graphs $\Gamma = \big(V(\Gamma), E(\Gamma)\big)$ with two kinds of vertices and two kinds of edges, such that any vertex is as in Figure 1.

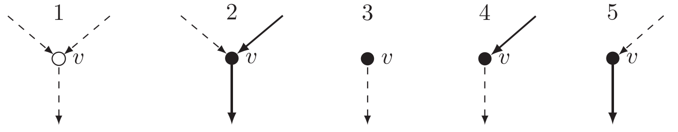

Figure 1: The five possible behaviors near a vertex of a BCR diagram

Filled circles are called *internal vertices*, white circles are called *external vertices*, plain edges are called *internal edges*, and dashed edges are called *external edges*. We denote their respective sets as $V_i(\Gamma)$, $V_e(\Gamma)$, $E_i(\Gamma)$ and $E_e(\Gamma)$. Such graphs have an even number of vertices. The *degree* of Γ is the integer $\deg(\Gamma) = \frac{1}{2}\mathrm{Card}(V(\Gamma))$. Let us denote $n(e)$ the integer $n-1$ if e is an internal edge, and $n+1$ if e is an external edge.

Given a diagram Γ and a long embedding ψ, we can set

$$C_\Gamma(\psi) = \big\{c\colon V(\Gamma) \hookrightarrow \mathbb{R}^{n+2} \;\big|\; c_{|V(\Gamma)} = \psi \circ c_i \text{ for some map } c_i\colon V_i(\Gamma) \hookrightarrow \mathbb{R}^n\big\}.$$

Elements of this space are called *configurations* and are the data of pairwise distinct points of $\mathbb{R}^{n+2}$ for any vertex of Γ, such that the points associated to internal vertices lie in $\psi(\mathbb{R}^n)$. On such a space, for any edge e, we define

$$\begin{array}{rccl} p_e\colon & C_\Gamma(\psi) & \longrightarrow & \mathbb{S}^{n(e)} \\ & c & \longmapsto & \begin{cases} \frac{c(w)-c(v)}{||c(w)-c(v)||} & \text{if } e \text{ is an external edge from } v \text{ to } w, \\ \frac{c_i(w)-c_i(v)}{||c_i(w)-c_i(v)||} & \text{if } e \text{ is an internal edge from } v \text{ to } w. \end{cases} \end{array}$$

The Bott-Cattaneo-Rossi invariant $Z_k(\psi)$ is defined as

$$Z_k(\psi) = \sum_{\Gamma\in\mathcal{G}_k} \frac{1}{\mathrm{Card}(\mathrm{Aut}(\Gamma))} \int_{C_\Gamma(\psi)} \bigwedge_{e\in E(\Gamma)} {p_e}^*(\omega_{n(e)}),$$

where $\omega_{n(e)}$ is the $SO\big(n(e)+1\big)$-invariant form on $\mathbb{S}^{n(e)}$ with total volume one, where $\mathcal{G}_k$ is the set of connected diagrams with degree k such that any trivalent vertex is adjacent to one univalent vertex, and where $\mathrm{Aut}(\Gamma)$ denotes the automorphism group of the oriented graph Γ that map an internal/external edge/vertex to an edge/vertex of same nature.

The result of Bott, Cattaneo, and Rossi is that such a formula is well-defined (the integrals are convergent), and that Z_k is an isotopy invariant. In [12], Watanabe proved that these invariants are related to Alexander polynomials for long ribbon knots, using the finite type theory defined by Habiro, Kanenobu and Shima in [3]. Because it is obtained from finite type invariant theory methods, this formula contains some indeterminacies.

In [5, 6], we defined some more flexible generalization of these invariants, and we use this flexible setting to compute the invariants Z_k in terms of linking numbers of some surface whose boundary is the knot. Furthermore, this extends the definition to other manifolds, and also to even dimensions and dimension one.

When $n \not\equiv 3 \mod 4$ (and for some class of knots when $n \equiv 3 \mod 4$), this formula yields

$$\sum_{k\geq 2} Z_k(\psi)h^k = (-1)^n \sum_{d=1}^{n} \mathrm{Ln}\big(\Delta_{d,\psi}(e^h)\big) \tag{1}$$

for any long knot $\psi\colon \mathbb{R}^n \hookrightarrow \mathbb{R}^{n+2}$, where $\Delta_{d,\psi}(t)$ is the d-th Alexander polynomial as defined by Levine in [7].

Question 1.1 (D. Leturcq)**.** *Does the above formula extend to all long knots* $\psi\colon \mathbb{R}^n \hookrightarrow \mathbb{R}^{n+2}$ *when* $n \equiv 3 \mod 4$ *?*

We can also try to look to more general diagrams than those of $\mathcal{G}_k$. For simplicity, let us now assume n is odd. For a diagram Γ with its vertices as in Figure 1, a *vertex-orientation* is the data of a cyclic order on the three half-edges adjacent to each trivalent vertex. We represent such an orientation by the counter-clockwise order in the plane. We define the $\mathbb{Q}$-vector space $\mathcal{A}$ spanned by the equivalence classes of vertex-oriented diagrams without loops with vertices as in Figure 1, up to the relations of Figure 2, and the relations $[\Gamma'] = (-1)^{a+b}[\Gamma]$, where $[\Gamma]$ only differ by the vertex-orientation of a vertices, the orientation of b internal edges, and the orientation of any external edges. The diagrams in Figure 1 are vertex-oriented, and the orientation of the internal edges common to all diagrams of a given relation are not depicted.

For integers (k, b) we let $\mathcal{G}_{k,b}$ be the set of connected diagrams with degree k and first Betti number b (which is equivalent to $\mathrm{Card}(E(\Gamma)) = 2k + b - 1$). This set is non-empty if and only if $0 \leq b \leq k$.

The space $\mathcal{A}$ naturally splits in $\mathcal{A} = \bigoplus\limits_{0\leq b\leq k} \mathcal{A}_{k,b}$, where $\mathcal{A}_{k,b}$ is the subspace spanned by the diagrams of $\mathcal{G}_{k,b}$.

Question 1.2 (D. Leturcq)**.** *Can we compute the dimension of* $\mathcal{A}_{k,b}$ *? At least, can we determine exactly when this subspace is non zero ?*

Let $\mathcal{K}$ denote the space of long knots $\psi\colon \mathbb{R}^n \hookrightarrow \mathbb{R}^{n+2}$. Instead of looking to $C_\Gamma(\psi)$ for some specific ψ, we can define a fiber space $C_\Gamma \to \mathcal{K}$ whose fiber above ψ

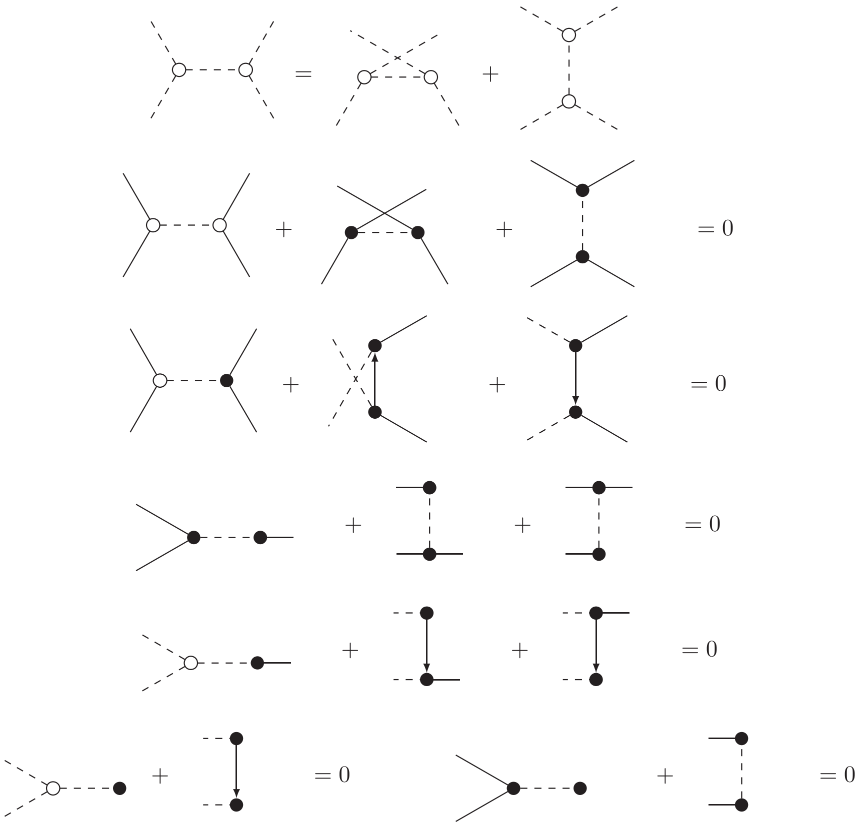

Figure 2: Relations on diagrams

is $C_\Gamma(\psi)$. The maps p_e above extends to this infinite-dimensional space, and we set

$$\Omega_{k,b} = \sum_{\Gamma \in \mathcal{G}'_{k,b}} \frac{1}{\mathrm{Card}(\mathrm{Aut}(\Gamma))} \int_{C_\Gamma(\psi)} \bigwedge_{e \in E(\Gamma)} {p_e}^*(\omega_{n(e)})[\Gamma] \in \mathcal{A}_{k,b},$$

where $[\Gamma]$ denotes the class of Γ in $\mathcal{A}$. This formula still converges, and it defines an element of $\Omega^{(b-1)(n-1)}(\mathcal{K}; \mathcal{A}_{k,b})$.

Note that when $b = 0$, the space $\mathcal{A}_{k,b}$ is isomorphic to $\mathbb{Q}$, and the cochain $\Omega_{k,1}$ identifies with Z_k. The invariance of Z_k corresponds to the fact that $\Omega_{k,1}$ is a cocycle. Following a question of T. Watanabe during our stay in Matsue in 2019, we conjecture the following.

Conjecture 1.3 (D. Leturcq)**.** *When $n \geq 1$ is odd, $\Omega_{k,b}$ is a cocycle on $\mathcal{K}$.*

The proof of this conjecture would rely on usual arguments on annulation of faces of configuration spaces, and the principal faces are ruled out by the relations defining $\mathcal{A}$. For lower values of (k, b), all the other faces vanish. However, for bigger diagrams, we may need to add some relations in the definition of $\mathcal{A}$, or to use appropriate propagators rather than pull-backs of volume forms on the spheres.

In order to determine when this cocycle is non-trivial, it would be necessary to know when $\mathcal{A}_{k,b}$ is non zero in general. The space $\mathcal{A}$ resembles a lot the space $\mathcal{A}_J$ of "Jacobi diagrams" used for the Kontsevich integral or the perturbative expansion of Chern-Simons theory. Applying a linear form w to these diagram-valued invariants yields a numerical invariant and we can recover by this method all the Vassiliev invariants.

Moreover, one can associate a representation ρ of a semi-simple Lie algebra with a linear form $w_\rho \colon \mathcal{A} \to \mathbb{Q}$ to obtain explicit examples. This recovers already known invariants, as the Jones polynomial. It is natural to ask if the high-dimensional analogue $\Omega_{k,b}$ satisfies similar properties.

Question 1.4 (D. Leturcq)**.** *Do we know what cocycles are obtained after applying linear forms on $\mathcal{A}$ to $\Omega_{k,b}$?*

Is there any natural algebraic structure that can yield some (non-trivial) linear maps $w \colon \mathcal{A} \to \mathbb{Q}$? If yes, can we identify the obtained cocycle $w \circ \Omega_{k,b} \in H^{(b-1)(n-1)}(\mathcal{K}; \mathbb{Q})$?

Question 1.5 (D. Leturcq)**.** *Can we compute the cocycles $\Omega_{k,b}$ using appropriate propagators, in order to get a formula similar to Formula* (1) *?*

When $b = 1$, Sakai and Watanabe [10] studied such diagrams in order to define cocycles on the space of long embeddings $\mathbb{R}^j \hookrightarrow \mathbb{R}^n$, when $n - j \geq 2$. The relations between diagrams depend on the parity of the dimensions j and n.

Problem 1.6 (D. Leturcq)**.** *Define an analogue of $\mathcal{A}$ when n is even, and $j \neq n-2$. Extend Conjecture 1.3, Questions 1.4 and 1.5 to long embeddings $\mathbb{R}^j \hookrightarrow \mathbb{R}^n$.*

2 The representations of stated skein algebras on surfaces

(Julien Korinman)[2]

For a *marked surface* $\mathbf{\Sigma} = (\Sigma, \mathcal{A})$ and a complex number $A^{1/2} \in \mathbb{C}^*$, the (Kauffman-bracket) *stated skein algebra* $\mathcal{S}_A(\mathbf{\Sigma})$ was introduced by Bonahon-Wong and Lê and is a generalisation of Przytycki-Turaev's skein algebra. A reduced version $\mathcal{S}_A^{red}(\mathbf{\Sigma})$ was also introduced by Costantino-Lê. Skein algebras appear in Topological Quantum Field Theories through their finite dimensional representations. Such a representation exists if and only if the parameter A is a root of unity. We state here a list of open questions/problems towards the resolution of the following:

Problem 2.1 (J. Korinman)**.** *Classify all finite dimensional weight representations of stated skein algebras and their reduced versions when A is a root of unity of odd order.*

Here a weight representation means a representation which is semi-simple as a module over the center of $\mathcal{S}_A(\mathbf{\Sigma})$. The two conditions of been "weight" and that the order of A is odd are taken here for simplicity. For now on, we fix a root of unity $A^{1/2}$ such that its square A has odd order N.

Let $\mathcal{Z}$ denote the center of $\mathcal{S}_A(\mathbf{\Sigma})$ and write $\widehat{\mathcal{X}}(\mathbf{\Sigma}) := \mathrm{Specm}\,(\mathcal{Z})$. The Chebyshev-Frobenius morphism $Ch_A : \mathcal{S}_{+1}(\mathbf{\Sigma}) \to \mathcal{Z}$ is finite and induces a finite branched covering $\pi : \widehat{\mathcal{X}}(\mathbf{\Sigma}) \to \mathrm{Specm}(\mathcal{S}_{+1}(\mathbf{\Sigma})) \cong \mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma})$ over the relative SL_2 character variety. An indecomposable weight representation $\rho : \mathcal{S}_A(\mathbf{\Sigma}) \to \mathrm{End}(V)$ sends central elements to scalar operators, so induces maximal ideals $\hat{\mathfrak{m}}_\rho \in \widehat{\mathcal{X}}(\mathbf{\Sigma})$ and $\mathfrak{m}_\rho = \pi(\hat{\mathfrak{m}}_\rho) \in \mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma})$. $\mathfrak{m}_\rho$ is called the *classical shadow* of ρ which factorizes through the finite dimensional algebras:

$$\mathcal{S}_A(\mathbf{\Sigma})_{\mathfrak{m}_\rho} := \mathcal{S}_A(\mathbf{\Sigma}) \Big/ Ch_A(\mathfrak{m}_\rho)\mathcal{S}_A(\mathbf{\Sigma}) \quad \text{and} \quad \mathcal{S}_A(\mathbf{\Sigma})_{\hat{\mathfrak{m}}_\rho} := \mathcal{S}_A(\mathbf{\Sigma}) \Big/ \hat{\mathfrak{m}}_\rho \mathcal{S}_A(\mathbf{\Sigma}).$$

Drozd classified finite dimensional $\mathbb{C}$ algebras into three families: the algebras with finite, tame and wild representation type. For an algebra A with wild representation type, the problem of classifying all indecomposable A-module is undecidable (the word problem for finite presentation groups can be embedded into that problem), so Problem 2.1 might be undecidable as well (it is the case for the bigon) and we need to be less ambitious: let us try to classify all finite dimensional indecomposable representation ρ whose classical shadow is such that $\mathcal{S}_A(\mathbf{\Sigma})_{\mathfrak{m}_\rho}$ has not wild type representation.

Let D denote the PI-dimension of $\mathcal{S}_A(\mathbf{\Sigma})$. The *Azumaya locus* of $\mathcal{S}_A(\mathbf{\Sigma})$ is

$$\mathcal{AL} = \left\{ \hat{x} \in \widehat{\mathcal{X}}(\mathbf{\Sigma}) \;\middle|\; \mathcal{S}_A(\mathbf{\Sigma})_{\hat{x}} \cong Mat_D(\mathbb{C}) \right\}.$$

The *fully Azumaya locus* is the image $\mathcal{FAL} := \pi(\mathcal{AL}) \subset \mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma})$. An important result is the

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Unicity representation theorem: The Azumaya locus is dense in $\widehat{\mathcal{X}}(\mathbf{\Sigma})$. Therefore the fully Azumaya locus is dense as well. All the previous discussion extends word-by-word to reduced stated skein algebras.

Problem 2.2 (J. Korinman)**.** *Compute the fully Azumaya loci of $\mathcal{S}_A(\mathbf{\Sigma})$ and $\mathcal{S}_A^{red}(\mathbf{\Sigma})$.*

This problem has been solved by Brown-Goodearl for the bigon and by Ganev-Jordan-Safranov for the marked surface $\mathbf{\Sigma}_{g,0}^0$ made of a genus g surface with one boundary component and exactly one boundary arc. It remains open for other marked surfaces. When $\mathfrak{m}$ belongs to the fully Azumaya locus, a theorem of Brown-Gordon permits to determine $\mathcal{S}_A(\mathbf{\Sigma})_{\mathfrak{m}}$ explicitly, thus the classification of indecomposable weight representations over the fully Azumaya locus is easy, once we are able to compute it. A second powerful tool is Brown-Gordon's Poisson orders theory: it implies that if $\mathfrak{m}$ and $\mathfrak{m}'$ belong to the same symplectic leaf of $\mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma})$, then $\mathcal{S}_A(\mathbf{\Sigma})_{\mathfrak{m}} \cong \mathcal{S}_A(\mathbf{\Sigma})_{\mathfrak{m}'}$. We can do better: the group $(\mathbb{C}^*)^{\mathcal{A}}$ acts on $\mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma})$, thus on the symplectic leaves. Call *equivariant symplectic leaves* the $(\mathbb{C}^*)^{\mathcal{A}}$-orbits of the symplectic leaves. If $\mathfrak{m}$ and $\mathfrak{m}'$ belong to the same equivariant symplectic leaf, then $\mathcal{S}_A(\mathbf{\Sigma})_{\mathfrak{m}} \cong \mathcal{S}_A(\mathbf{\Sigma})_{\mathfrak{m}'}$.

Problem 2.3 (J. Korinman)**.**

(1) *Classify the equivariant symplectic leaves of $\mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma})$.*

(2) *For each leaf $\mathcal{F}$, choose a representative $\mathfrak{m} \in \mathcal{F}$ and determine the representation type of $\mathcal{S}_A(\mathbf{\Sigma})_{\mathfrak{m}}$. If it is not wild, classify all its finite dimensional indecomposable representations.*

This problem was solved for the bigon by Brown-Gordon and for the algebra $\mathcal{S}_A^{red}(\mathbb{D}_1)$ by the author and remains open for every other marked surfaces. The computation of the symplectic leaves of $\mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma}_{g,0}^0)$ was done by Ganev-Jordan-Safranov who found that one leaf is open dense. The computation of the symplectic leaves of $\mathcal{X}_{\mathrm{SL}_2}(\Sigma_g, \emptyset)$ for a closed genus $g \geq 2$ surface is simple: the smooth locus made of the classes of irreducible representations $r : \pi_1(\Sigma_g, v) \to \mathrm{SL}_2$ is symplectic, the locus made of the classes of diagonal representations which are not scalars is symplectic and each singleton $\{r_0\}$, for $r_0 : \pi_1(\Sigma_g, v) \to \pm\mathbb{1}_2$ scalar, is a symplectic leaf. Note that when a symplectic leaf is dense, then it is included in the fully Azumaya locus, therefore (Ganev-Jordan-Safranov) the smooth locus of $\mathcal{X}_{\mathrm{SL}_2}(\Sigma_g, \emptyset)$ and the open dense leaf of $\mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma}_{g,0}^0)$ both are included in the fully Azumaya loci (which is equal to the Azumaya loci in these cases). An important remaining question is the

Question 2.4 (J. Korinman)**.** *For a closed genus $g \geq 2$ surface, is the locus of diagonal (non scalar) representations included in the Azumaya locus ?*

Note that if the class of one such diagonal representation is in the Azumaya locus, then all of them are. In addition to these very general theorems, there exist three concrete families of representations for stated skein algebras which are:

(1) The *Witten-Reshetikhin-Turaev* representations ρ^{WRT} coming from modular TQFTs at odd roots of unity. They are representations of skein algebras of

unmarked surfaces and are irreducible. For closed surfaces, they have classical shadow the class of a central representation and their dimension is strictly smaller than the PI-dimension N^{3g-3}. We can deduce from their existence that the scalar representations do not belong to the Azumaya locus of $\mathcal{S}_A(\Sigma_g, \emptyset)$.

(2) The *Blanchet-Costantino-Geer-Patureau-Mirand* representations ρ^{BCGP} coming from non semi-simple TQFTs at odd roots of unity. They are representations of skein algebras of unmarked surfaces and have their dimension equal to the PI-dimension of the skein algebra. For closed surfaces, their classical shadows are the class of diagonal and scalar representations.

(3) The *Bonahon-Wong* or *quantum Teichmüller* representations ρ^{BW} defined using the quantum trace. They are representations of the reduced stated skein algebras of arbitrary marked surfaces and their dimension coincide with the PI-dimension of the corresponding reduced stated skein algebra, except maybe for closed surfaces and for scalar classical shadows in which case it is only known that their dimension is $\leq N^{3g-3}$. For non-closed surfaces, the set of their classical shadows is dense in $\mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma})$ and it is equal to $\mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma})$ for closed surfaces.

The quantum Teichmüller representations are defined using quantum traces starting from irreducible representations of quantum tori and there might be several such representations inducing the same character over the center of $\mathcal{S}_A(\mathbf{\Sigma})$ without been isomorphic.

Question 2.5 (J. Korinman)**.**

(1) *Are the representations ρ^{BCGP} with non scalar classical shadow irreducible ? Indecomposable ? Projective ?*

(2) *Are the representation ρ^{BW} with non scalar classical shadow irreducible ? Indecomposable ? Projective ? Are they isomorphic to the representations ρ^{BCGP} which has the same shadow ?*

(3) *Given ρ^{BW}, ρ'^{BW} two quantum Teichmüller representations which induce the same character over the center of $\mathcal{S}_A(\mathbf{\Sigma})$, are they isomorphic ?*

(4) *For $\rho^{WRT}, \rho^{BW}, \rho^{BCGP}$ representations of the skein algebra of a genus $g \geq 2$ closed surface all having the same classical shadow which is a scalar representation, are these three representations related ? Are ρ^{BW} and ρ^{BCGP} isomorphic ? Is ρ^{WRT} a sub-representation of one of them ? What is the dimension of ρ^{BW} ?*

The third item of Question 2.5 was proved to be true when $\mathbf{\Sigma} = \mathbb{D}_n$ is a genus 0 surface with $n+1$ boundary components and two boundary arcs in one component. The author deduced from this fact families of projective representation of the braid groups related to the ADO and Kashaev invariants. If, as expected, it is true in general, then one would obtain families of finite dimensional projective representations of the mapping class groups and the Torelli groups. Concerning the first and second item, note that if one finds a representation ρ^{BW} or ρ^{BCGP} with diagonal classical shadow which is irreducible, then we would have proved that all diagonal representations are in the Azumaya locus (so we would have solved Question 2.4) and that two representations ρ^{BW} and ρ^{BCGP} with the same diagonal shadow are isomorphic.

3 On the additivity of geometric invariants under 1-connected sum of handlebody-knots

(Tomo Murao)[3]

A *handlebody-knot* is a handlebody embedded in the 3-sphere S^3. A handlebody-knot is *trivial* if its exterior is a handlebody. Let B_1 and B_2 be 3-balls in S^3 such that $B_1 \cup B_2 = S^3$ and $B_1 \cap B_2 = \partial B_1 = \partial B_2$. Let H_i be a genus g_i handlebody-knot in B_i for $i = 1, 2$. If $H_1 \cap H_2$ is a disk, then $H_1 \cup H_2$ is a genus $g_1 + g_2$ handlebody-knot in S^3. We call it the *1-connected sum* of H_1 and H_2 and denote it by $H_1 \#_1 H_2$ (see Figure 3). The handlebody-knot $H_1 \#_1 H_2$ depends only on the handlebody-knots H_1 and H_2. A *diagram* of a handlebody-knot is a diagram of a spatial trivalent graph whose regular neighborhood is the handlebody-knot, where a spatial trivalent graph is a finite trivalent graph embedded in S^3. In this definition, a trivalent graph may be a circle.

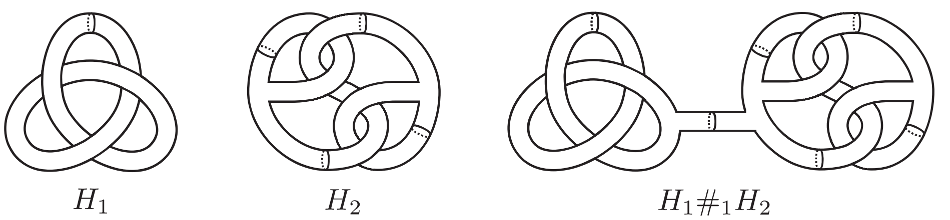

Figure 3: 1-connected sum of handlebody-knots

We introduce some geometric invariants of handlebody-knots. Let H be a handlebody-knot. The *crossing number* $c(H)$ of H is the minimal number of crossings in all diagrams of H. The *unknotting number* $u(H)$ of H is the minimal number of crossing changes which convert H into the trivial handlebody-knot [4]. The *tunnel number* $t(H)$ of H is the minimal number of mutually disjoint arcs $\alpha_1, \ldots, \alpha_n$ properly embedded in $E(H)$ such that $E(H \cup \alpha_1 \cup \cdots \cup \alpha_n)$ is homeomorphic to a handlebody, where $E(\cdot)$ denotes its exterior. The *cutting number* $\mathrm{cut}(H)$ of H is the minimal number of mutually disjoint meridian disks $\Delta_1, \ldots, \Delta_n$ of H such that $E(H - \bigcup_{i=1}^n N(\Delta_i))$ is homeomorphic to a handlebody, where $N(\cdot)$ denotes its regular neighborhood [8].

Remark. It is known that the additivity of tunnel number under 1-connected sum of

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handlebody-knots holds. That is, for any handlebody-knots H_1 and H_2, it follows $t(H_1 \#_1 H_2) = t(H_1) + t(H_2)$.

Remark. It is known that the additivity of unknotting number under 1-connected sum of handlebody-knots does not hold. In particular, for any positive integer n, there exist handlebody-knots H_1 and H_2 such that $u(H_1 \#_1 H_2) = u(H_1) + u(H_2) - n$.

Question 3.1 (T. Murao). *Does the equality $c(H_1 \#_1 H_2) = c(H_1) + c(H_2)$ hold for any handlebody-knots H_1 and H_2?*

Question 3.2 (T. Murao). *Does the equality* $\mathrm{cut}(H_1 \#_1 H_2) = \mathrm{cut}(H_1) + \mathrm{cut}(H_2)$ *hold for any handlebody-knots H_1 and H_2?*

4 Quantum character variety of knots

(Jun Murakmai)

Question 4.1 (J. Murakmai). *Does the quantum character variety always split into abelian factor(s) and non-abelian factor(s) ?*

The quantum character varieties of the trefoil knot, the figure eight knot and Whitehead link are all split into two factors. One corresponds to the abelian factor and another one corresponds to the non-abelian factor of the character variety of the classical case.

Question 4.2 (J. Murakmai). *Is there some knot who has more than two factors of the quantum character variety?*

In classical case, knots with such property are given by Ohtsuki-Riley-Sakuma [9]. In classical case, such example is obtained by finding a epimorphism between 2-bridge link groups. Here the fundamental group is extended to the bottom tangle of free arcs, and it is a problem that the epimorphism between link groups can be extended to this free arcs case, or not.

Question 4.3 (J. Murakmai). *Can we construct the quantum A-polynomial of a knot from the quantum character variety?*

Explain the longitude and its parallels in terms of the generators of the skein algebra of the punctured disk, and eliminate the traces corresponding to the products of meridians, then we may get the relation between longitude and meridian. After obtaining such relation, substitute $M + M^{-1}$ and $L + L^{-1}$ for the meridian and longitude, where M and L are the generators of the quantum torus, then it must be a multiple of quantum A-polynomial and the recurrence polynomial of the colored Jones polynomial. But the polynomial obtained from the quantum character variety may has some extra factors.

Question 4.4 (J. Murakmai). *What is the geometry of the quantum character variety?*

For some knots, geometric description of the character variety is explained, for example, in [11]. In the case of the figure eight knot, the character variety is given by a commutative algebra, but the boundary of the knot complement has a structure of torus, and this structure is generalized to quantum torus in skein theory. So the quantum character variety may have some good structure concerning with this quantum torus.

References

[1] Bott, R., *Configuration spaces and imbedding invariants*, Turkish J. Math. **20** (1996) 1–17.

[2] Cattaneo, A. S., Rossi, C. A., *Wilson surfaces and higher dimensional knot invariants*, Comm. Math. Phys. **256** (2005) 513–537.

[3] Habiro, K., Kanenobu, T., Shima, A., *Finite type invariants of ribbon 2-knots*, Low-dimensional topology (Funchal, 1998), 187–196, Contemp. Math. **233**, Amer. Math. Soc., Providence, RI, 1999.

[4] Iwakiri, M., *Unknotting numbers for handlebody-knots and Alexander quandle colorings*, J. Knot Theory Ramifications **24** (2015), no. 14, 1550059, 13 pp.

[5] Leturcq, D., *Generalized Bott-Cattaneo-Rossi invariants of high-dimensional knots*, to appear in J. Math. Soc. Japan, arXiv:1312.2566

[6] Leturcq, D., *Generalized Bott-Cattaneo-Rossi invariants in terms of Alexander polynomials*, arXiv:2003.01007, 2020.

[7] Levine, J. *Polynomial invariants of knots of codimension two*, Ann. of Math. (2) **84** (1966) 537–554.

[8] Murao, T., *On the tunnel number and the cutting number with constituent handlebody-knots*, Topology Appl. **292** (2021), 107632, 14 pp.

[9] Ohtsuki, O., Riley, R., Sakuma, M., *Epimorphisms between 2-bridge link groups*, The Zieschang Gedenkschrift, 417–450, Geom. Topol. Monogr. **14**, Geom. Topol. Publ., Coventry, 2008.

[10] Sakai, K., Watanabe, T., *1-loop graphs and configuration space integral for embedding spaces*, Math. Proc. Cambridge Philos. Soc. **152** (2012) 497–533.

[11] Porti, J., *Character varieties and knot symmetries*, Winter Braids Lecture Notes, Tome **4** (2017) , Exposé no. 2, 21 p.
`https://wbln.centre-mersenne.org/item/10.5802/wbln.18.pdf`

[12] Watanabe, T., *Configuration space integral for long n-knots and the Alexander polynomial*, Algebr. Geom. Topol. **7** (2007) 47–92.