On certain algebraic structures of symmetric spaces

ー対称空間に付随した代数構造についてー

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Introduction

この小論ではある type の triple system を与え, そして, その三項系を用いて対称空間を構成することが目的です. 特に複素構造を持つ場合です. 以下歴史的な話 (著者の仕事を中心にして) を踏まえながら new idea を述べたいと考えます.

$Freudenthal --> Kantor --> Okubo\ and\ Kamiya$

と継続する Jordan river の三項系代数（数理代数学）の話が主な話題です.

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From mathematical history's viewpoint, the concept discussed here first appeared with a class of nonassociative algebras, that is commutative Jordan algebras, which was the defining subspace g_{-1} in the Tits-Kantor-Koecher (for short TKK) construction of 3-graded Lie algebras $g = g_{-1} \oplus g_0 \oplus g_1$, such that $[g_i, g_j] \subseteq g_{i+j}$. Nonassociative algebras are rich in algebraic structures, and they provide an important common ground for various branches of mathematics, not only for pure algebra and differential geometry, but also for representation theory and algebraic geometry (for example, [11], [39], [63], [64]). Specially, the concept of nonassociative algebras such as Jordan and Lie (super)algebras plays an important role in many mathematical and physical subjects ([5], [10]-[13], [15], [26], [28], [29], [38], [47], [48], [52], [55], [56], [57], [59]). We also note that the construction and characterization of these algebras can be expressed in terms of the notion of triple systems ([1]-[4], [6]-[8], [20], [23], [24], [33], [38], [43]-[46], [49], [51]) by using the standard embedding method ([22], [41], [42], [50], [54]). In particular, the generalized Jordan triple system of second order, or $(-1,1)$-Freudenthal Kantor triple system (for short $(-1,1)$-FKTS), is a useful concept ([13]-[21], [34]-[37], [40], [53]) for the constructions of simple Lie algebras, while the $(-1,-1)$-FKTS plays the same role ([6], [22], [25], [27], [57], [58]) for the construction of Lie superalgebras, while the δ-Jordan Lie triple systems act similarly for that of Jordan superalgebras ([23], [24], [49]). Specially, we have constructed a model of Lie superalgebras $D(2,1;\alpha)$, $G(3)$ and $F(4)$ ([25]).

As a final comment of this introduction, we provide well-known results as follows; if A is a unital commutative Jordan algebra, then the triple product given by

$$\{xyz\} = (xy)z + x(yz) - y(xz)$$

defines a Jordan triple system, i.e., it satisfies the two relations $\{xy\{abc\}\} = \{\{xya\}bc\} - \{a\{yxb\}c\} + \{ab\{xyc\}\}$ (this relation is often called a fundamental identity), $\{xyz\} = \{zyx\}$ and next the new triple product $[xyz]$ given by

$$[xyz] = \{xyz\} - \{yxz\}$$

defines a Lie triple system. Briefly summarizing this article, we will generalize these results and construct Lie (super)algebras associated with a generalized Jordan triple system. Toward to its applications, we will give a construction of symmmetric spaces with an almost complex structure.

1 Preamble and Definitions

In this paper triple systems have finite dimension being defined over a field Φ of characteristic $\neq 2$ or 3, unless otherwise specified. In order to render the paper as self-contained as possible, we recall first the definition of a generalized Jordan triple system of second order (for short GJTS of 2nd order)([38]).

A vector space V over a field Φ endowed with a trilinear operation $V \times V \times V \to V$, $(x, y, z) \longmapsto (xyz)$ is said to be a *GJTS of 2nd order* if the following conditions are fulfilled:

$$(ab(xyz)) = ((abx)yz) - (x(bay)z) + (xy(abz)), \tag{1}$$

$$K(K(a, b)x, y) - L(y, x)K(a, b) - K(a, b)L(x, y) = 0, \tag{2}$$

where $L(a, b)c := (abc)$ and $K(a, b)c := (acb) - (bca)$.

A *Jordan triple system* (for short JTS) satisfies (1) and the following condition

$$(abc) = (cba), \ i.e., \ K(a, c)b = 0. \tag{3}$$

The JTS is a special case in the GJTS of 2nd order since $K(x, y) \equiv 0$.

We next can generalize the concept of GJTS of 2nd order as follows (see [13], [14], [18], [22], [28], [54] and the earlier references therein).

For $\varepsilon = \pm 1$ and $\delta = \pm 1$, a triple product that satisfies the identities

$$(ab(xyz)) = ((abx)yz) + \varepsilon(x(bay)z) + (xy(abz)), \tag{4}$$

$$K(K(a, b)x, y) - L(y, x)K(a, b) + \varepsilon K(a, b)L(x, y) = 0, \tag{5}$$

where

$$L(a, b)c := (abc), \quad K(a, b)c := (acb) - \delta(bca), \tag{6}$$

is called an $(\varepsilon, \delta)-Freudenthal-Kantor$ *triple system* (for short (ε, δ)-FKTS). An (ε, δ)-FKTS is said to be *unitary* if $Id \in \{K(a, b)\}_{span}$.

A triple system satisfying only the identity (4) is called a *generalized* FKTS (for short GFKTS), while the identity (5) is called the *second order condition.*

Remark From the relation Eq. (6), we note that

$$K(b, a) = -\delta K(a, b). \tag{7}$$

A triple system is called a (α, β, γ) *triple system associated with a bilinear form* if

$$(xyz) = \alpha < x, y > z + \beta < y, z > x + \gamma < z, x > y,$$

where $< x, y >$ is a bilinear form such that $< x, y >= \kappa < y, x >$, $\kappa = \pm 1$, $\alpha, \beta, \gamma \in \Phi$.

From now on we will mainly consider this type of triple system.

An (ε, δ)-FKTS is said to be *balanced* if there is a bilinear form $< x, y >\in \Phi^*$ such that $K(x, y) =< x, y > Id$, that is, $dim\ \{K(x, y)\}_{span} = 1$ holds.

Remark We note that a balanced triple system (i.e., it fulfills $K(x, y) =< x, y > Id$) is unitary, since $Id \in \{K(x, y)\}_{span}$.

Triple products are denoted by (xyz), $\{xyz\}$, $[xyz]$ and $< xyz >$ upon their suitability.

Remark We note that the concept of GJTS of 2nd order coincides with that of $(-1, 1)$-FKTS. Thus we can construct the corresponding Lie algebras by means of the standard embedding method ([6], [13]-[18], [22], [25], [27], [36], [54]).

For $\delta = \pm 1$, a triple system $(a, b, c) \mapsto [abc], a, b, c \in V$ is called a δ-*Lie triple system* (for short δ-LTS) if the following three identities are fulfilled

$$\begin{gathered} [abc] = -\delta[bac], \\ [abc] + [bca] + [cab] = 0, \\ [ab[xyz]] = [[abx]yz] + [x[aby]z] + [xy[abz]], \end{gathered} \tag{8}$$

where $a, b, x, y, z \in V$. An 1-LTS is a *LTS* while a -1-LTS is an *anti-LTS*, by ([14]). Note that the set $L(V,V)$ of all left multiplications $L(x,y)$ of V is a Lie subalgebra of $Der\ V$, where we denote by $L(x,y)z = [xyz]$.

Proposition 1.1 ([14], [15], [22]) *Let $(U(\varepsilon,\delta), < xyz >)$ be an (ε,δ)-FKTS. If J is an endomorphism of $U(\varepsilon,\delta)$ such that $J < xyz > = < JxJyJz >$ and $J^2 = -\varepsilon\delta Id$, then $(U(\varepsilon,\delta), [xyz])$ is a LTS (if $\delta = 1$) or an anti-LTS (if $\delta = -1$) with respect to the product*

$$[xyz] := < xJyz > -\delta < yJxz > +\delta < xJzy > - < yJzx > . \tag{9}$$

Corollary ([13]) *Let $U(\varepsilon,\delta)$ be an (ε,δ)-FKTS. Then the vector space $T(\varepsilon,\delta) = U(\varepsilon,\delta) \oplus U(\varepsilon,\delta)$ becomes a LTS (if $\delta = 1$) or an anti-LTS (if $\delta = -1$) with respect to the triple product*

$$\left[\begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix}\begin{pmatrix} e \\ f \end{pmatrix}\right] = \begin{pmatrix} L(a,d) - \delta L(c,b) & \delta K(a,c) \\ -\varepsilon K(b,d) & \varepsilon(L(d,a) - \delta L(b,c)) \end{pmatrix}\begin{pmatrix} e \\ f \end{pmatrix}. \tag{10}$$

Thus we can obtain the standard embedding Lie algebra (if $\delta = 1$) or Lie superalgebra (if $\delta = -1$), $L(U(\varepsilon,\delta)) = D(T(\varepsilon,\delta), T(\varepsilon,\delta)) \oplus T(\varepsilon,\delta)$, associated with $T(\varepsilon,\delta)$ where $D(T(\varepsilon,\delta), T(\varepsilon,\delta))$ is the set of inner derivations of $T(\varepsilon,\delta)$;

$$D(T(\varepsilon,\delta), T(\varepsilon,\delta)) := \left\{\begin{pmatrix} L(a,b) & \delta K(c,d) \\ -\varepsilon K(e,f) & \varepsilon L(b,a) \end{pmatrix}\right\}_{span},$$

$$T(\varepsilon,\delta) := \left\{\begin{pmatrix} x \\ y \end{pmatrix}\middle| x, y \in U(\varepsilon,\delta)\right\}_{span}.$$

We use the following notation:

$$\mathbf{k} := \{K(x,y) \in End\ U(\varepsilon,\delta) | x, y \in U(\varepsilon,\delta)\}\ and$$

$$\{EFG\} := EFG + GFE,\ \forall E, F, G \in \mathbf{k}.$$

Then, we may make the structure of a JTS $\mathbf{k}$ with respect to the triple product $\{EFG\} \in \mathbf{k}$ ([20]). Also the JTS $\mathbf{k}$ is called *nondegenerate* if $K(x,y) = 0$ for any $y \in U(\epsilon,\delta)$ implies $x = 0$. Hence we have the following Propositon.

Proposition 1.2 ([15], [31]) *Let U be a unitary (ε,δ)-FKTS and $L(U)$ be the standard embedding Lie (super)algebra associated with U. Then the following are equivalent:*

(i) U is simple,

(ii) the Lie (super)algebra $L(U)$ is simple,

(iii) the JTS $\mathbf{k} := \{K(a,b)\}_{span}$ is simple and nondegenerate.

Remark We note that $L(U) = L(U(\varepsilon,\delta)) := L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ is the five graded Lie (super)algebra such that $U(\varepsilon,\delta) \oplus U(\varepsilon,\delta) = L_{-1} \oplus L_1 = T(\varepsilon,\delta)$ (δ-LTS), $L_{-2} = \mathbf{k}$ (JTS) and $D(T(\varepsilon,\delta), T(\varepsilon,\delta)) = L_{-2} \oplus L_0 \oplus L_2$ (the derivation of $T(\varepsilon,\delta)$) equipped with $[L_i, L_j] \subseteq L_{i+j}$ and $L_{-1} \oplus L_1 = L(U)/L_{-2} \oplus L_0 \oplus L_2$. In Introduction, we had used the notation $g = g_{-1} \oplus g_0 \oplus g_1$ instead of $L_{-1} \oplus L_0 \oplus L_1$. This Lie (super)algebra construction is one of reasons to study nonassociative algebras and triple systems (for a Lie superalgebra, refer to ([12], [52])).

2 A generalized curvature and torsion tensors

Let $L = L(U(\varepsilon,\delta)) = L(W,W) \oplus W$ be the Lie (super)algebra defined from a δ LTS W as in section one, that is, the δ-LTS $W = T(\varepsilon,\delta) = L_{-1} \oplus L_1$ is induced from $L_{-1} = U(\varepsilon,\delta)$ (as L_{-1} has the structure of a (ε,δ)-FKTS).

We introduce a generalization of covariant derivative $\bigtriangledown$ in differential geometry as follows; $\bigtriangledown : L \to \text{End}\ L$ defined by

$\bigtriangledown_X Y = [X,Y] = -\delta[Y,X]$,
$\bigtriangledown_X [Y,Z] = [YZX] = -\delta[ZYX]$,
$\bigtriangledown_{[X,Y]} Z = -[XYZ] = -\delta[YXZ]$,
$\bigtriangledown_{[X,Y]}[V,Z] = [[V,Z][X,Y]] = -\delta[[X,Y][V,Z]]$,
for any $X,Y,Z,V \in W$.

Furtheremore, a generalized curvature tensor defined by

$$C_\delta(X,Y) = \bigtriangledown_X \bigtriangledown_Y - \delta \bigtriangledown_Y \bigtriangledown_X - \bigtriangledown_{[X,Y]} \tag{11}$$

is identically zero, i.e., $C_\delta(X,Y) = 0$ in L, for any $X,Y \in W$. Indeed, we demonstrate the proof below.

First we calculate

$$\begin{aligned} C_\delta(X,Y)Z &= (\bigtriangledown_X \bigtriangledown_Y - \delta \bigtriangledown_Y \bigtriangledown_X)Z - \bigtriangledown_{[X,Y]} Z \\ &= \bigtriangledown_X [Y,Z] - \delta \bigtriangledown_Y [X,Z] + [XYZ] \\ &= [YZX] - \delta[XZY] + [XYZ] \\ &= [YZX] + [ZXY] + [XYZ] = 0. \end{aligned}$$

Second, it follow

$$\begin{aligned} C_\delta(X,Y)[V,Z] &= (\bigtriangledown_X \bigtriangledown_Y - \delta \bigtriangledown_Y \bigtriangledown_X)[V,Z] - \bigtriangledown_{[X,Y]}[V,Z] \\ &= [X,[VZY]] - \delta[Y,[VZX]] + \delta[[X,Y],[V,Z]] \\ &= [X, L(V,Z)Y] - \delta[Y, L(V,Z)X] - L(V,Z)[X,Y] = 0 \end{aligned}$$

(by $[Y, L(V,Z)X] = -\delta[L(V,Z)X, Y]$ and $[[X,Y],[V,Z]] = -\delta[[V,Z],[X,Y]]$) for any $X,Y,Z,V \in T(\varepsilon,\delta)$.

However a generalized torsion tensor defined by

$$S_\delta(X,Y) = \bigtriangledown_X Y - \delta \bigtriangledown_Y X - [X,Y] \tag{12}$$

is not zero, since it gives $S_\delta(X,Y) = [X,Y] - \delta[Y,X] - [X,Y] = [X,Y]$.

To later section we next define the Nijenhuis operator

$$N(X,Y) = [JX,JY] + J^2[X,Y] - J[JX,Y] - J[X,JY],$$

where J is an almost complex structure on W, this concept (the case of $\delta = 1$) is appeared in [32].

3 Examples of (ε,δ)-JTS

We consider here examples of the particular case when $K(x,y) \equiv 0$ (identically), that is of an (ε,δ)-JTS.

Example 3.1 Let V be a vector space with a symmetric bilinear form $< x,y >$. Then

$$< xyz > = < x,y > z + < y,z > x - < z,x > y$$

defines on V a $(-1,1)$-JTS.
Note that $(-1,1)$-JTS is same as the JTS.

Example 3.2 Let V be a vector space with an anti-symmetric bilinear form $< x,y >$. Then

$$< xyz > = < x,y > z + < y,z > x - < z,x > y$$

defines on V a $(1,-1)$-JTS.

Example 3.3 Let V be a vector space with a symmetric bilinear form $< x,y >$. Then

$$< xyz > = < x,y > z - < y,z > x$$

defines on V a $(-1,-1)$-JTS.

Example 3.4 Let V be a vector space with an anti-symmetric bilinear form $< x, y >$. Then

$$< xyz >=< x, y > z- < y, z > x$$

defines on V a $(1,1)$-JTS.

Example 3.5 Let V be a set of alternative matrix $Asym(n, \Phi) = \{x|^t x = -x\}$. Then

$$< xyz >= x^t yz - \varepsilon z^t yx, \quad where \ \forall \ x, y, z \in V$$

defines on V a $(\varepsilon, -\varepsilon)$ JTS.

Proposition 3.1 *Let $(U, < xyz >)$ be an (ε, δ)-JTS. Then the triple system is a δ-LTS with respect to the new product*

$$[xyz] =< xyz > -\delta < yxz > . \tag{13}$$

Proposition 3.2 *Let $(U, < xyz >)$ be a triple system with $< xyz >=< y, z > x$ and $< x, y >= -\varepsilon < y, x >$. Then this triple system is an (ε, δ)-FKTS.*

Proposition 3.3 ([16], [18]) *Let U be a balanced $(1,1)$-FKTS satisfying $<< xxx >, x >\equiv 0$ (identically) and $< x, y >$ is nondegenerate. Then U has a triple product defined by*

$$< xyz >= \frac{1}{2}(< y, x > z+ < y, z > x+ < x, z > y). \tag{14}$$

On the other hand, note that the balanced $(1,1)$-FKTS induced from an exceptional Jordan algebra is closely related to the 56 dimensional meta symplectic geometry due to H. Freudenthal ([13], [15], [16] and the earlier references therein). Also the correspondence of a quaternionic symmetric space and the balanced (1,1) FKTS has been studied in ([5]).

4 Symmetric spaces associated with (ε, δ)Jordan triple systems

This article (section) means that the symmetric space in sense of Prof. Wolfgang Bertram [Lecture Note in Math. "The geometry of Jordan and Lie structures" Springer, vol.1754 (2000)]. In his book, the following is described;

a) the category of germs of symmetric space with invariant almost complex structure is equivalent to the category of Lie triple systems with invariant complex structure,

b) the category of germs of symmetric space with invariant polarizations is equivalent to the category of Lie triple system with invariant polarization.

Hence from these results, it seems that it is important to construct a Lie triple system from a Jordan triple system (abbreviated JTS). Therefore we construct δ-Lie triple systems associated with a $(\varepsilon, -\varepsilon)$-JTS of a more general case (when $\varepsilon\delta = -1$).

From now on, let V be a (ε, δ)-JTS with $\varepsilon = -\delta$, that is, $\varepsilon\delta = -1$ and $\widehat{W}$ be a subset in the δ-LTS $W = T(\varepsilon, \delta)$ satisfying

$$\widehat{W} := \widehat{W}_+ \cup \widehat{W}_-, \text{ where } \widehat{W}_+ := \left\{ \begin{pmatrix} x \\ x \end{pmatrix} \middle| x \in V \right\}, \ \widehat{W}_- := \left\{ \begin{pmatrix} x \\ -x \end{pmatrix} \middle| x \in V \right\}.$$

Then we set $J = \begin{pmatrix} i & 0 \\ 0 & -1 \end{pmatrix}$, $i = \sqrt{-1}$, and we get $JW_+ = W_-, \ JW_- = W_+$.

From a special case of section one, for δ- LTS W with $K(x, y) \equiv 0$, we have

$$\left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right] = \begin{pmatrix} (L(x_1, y_2) - \delta L(y_1, x_2))z_1 \\ (\varepsilon L(y_2, x_1) - \varepsilon\delta L(x_2, y_1))z_2 \end{pmatrix}.$$

Thus for $\widehat{W}_+$ and $\widehat{W}_-$ respectively, we obtain

$$\left[\begin{pmatrix} x \\ x \end{pmatrix}\begin{pmatrix} y \\ y \end{pmatrix}\begin{pmatrix} z \\ z \end{pmatrix}\right] = \begin{pmatrix} L(x,y)z - \delta L(y,x)z \\ \varepsilon L(y,x)z - \varepsilon\delta L(x,y)z \end{pmatrix} \in \widehat{W}_+,$$

$$\left[\begin{pmatrix} x \\ -x \end{pmatrix}\begin{pmatrix} y \\ -y \end{pmatrix}\begin{pmatrix} z \\ -z \end{pmatrix}\right] = \begin{pmatrix} -L(x,y)z + \delta L(y,x)z \\ \varepsilon L(y,x)z - \varepsilon\delta L(x,y)z \end{pmatrix} \in \widehat{W}_-.$$

Also, we have $[\widehat{W}_-\widehat{W}_-\widehat{W}_+] \subset \widehat{W}_+$ and $[\widehat{W}_+\widehat{W}_+\widehat{W}_-] \subset \widehat{W}_-$ etc.

Here we define for $X,Y,Z \in \widehat{W}_+$ or $X,Y,Z \in \widehat{W}_-$,

$$R(X,Y)Z = -[X,Y,Z] = -[[X,Y],Z],$$

$$T(X,Y)Z = T(X,Y,Z) = -\frac{1}{2}(R(X,Y)Z - JR(X,J^{-1}Y)Z).$$

Hence, for example $T(X,Y)Z = \begin{pmatrix} -L(x,y)z \\ L(x,y)z \end{pmatrix}$ for any $X,Y,Z \in \widehat{W}_-$.

Theorem 4.1. *Under the above assumption and $\sqrt{-1} \in \Phi$ (base field), we have,*

i) $\widehat{W}_+$ and $\widehat{W}_-$ are a δ-LTS with respect the product $[XYZ]$,

ii) $\widehat{W}_+$ and $\widehat{W}_-$ are a $(\varepsilon,-\varepsilon)$-JTS with respect to the product $T(X,Y,Z)$,

iii) $\widehat{W}$ is twisted ($[JXYZ] = -[XJYZ]$) and almost complex with J,

iv) $N(X,Y)$ is vanished, where $N(X,Y) := [JX,JY] + J^2[X,Y] - J[JX,Y]$ $-J[X,JY]$, (Nijenhuis operator)*,*

Note that $JT(X,Y,Z) = T(JX,Y,Z) = -T(X,JY,Z)$ hold for $X,Y,Z \in \widehat{W}_\pm$, i.e., this is hermite in the sense of W. Bertram.

Remark Note that there is no the addition on $\widehat{W}_+ + \widehat{W}_-$, but there is the multiplication $[XYZ]$ on $\widehat{W}$ and the addition on $\widehat{W}_+$ or $\widehat{W}_-$.

Similar if $J = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$. Then we have $\widehat{W}$ *is twisted and polarized, i.e., $J^2 = Id$ on $\widehat{W}$.*

If $J = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$, then it is straight with $T(X,Y) = 0$,

if $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then it is straight with $T(X,Y) = 0$ i.e., $[JXYZ] = [XJYZ]$.

Thus we note that "there are the concept of invariant ($J[XYZ] = [XYJZ]$), automorphism and derivation on $\widehat{W}_\pm$ ".

Note that

if $J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \Longrightarrow N(X,Y) = 0$ (identically zero),

if $J = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \Longrightarrow N(X,Y) \neq 0$,

if $J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \Longrightarrow$ auto, but $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is not auto,

if $J = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ is auto, but $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is not auto.

Sumarizing these results, for the structure J of $\widehat{W}$ we have the table as follows, ($\varepsilon = -1, \delta = 1$)

	almost complex	polarized
twisted auto	$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ *(type I)*	$\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$
twisted anti-auto	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ *(type II)*	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
straight	$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ or $\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$ $T(X,Y) \equiv 0$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $T(X,Y) \equiv 0$

where the anti-automorphism is denoted by $[JXJYJZ] = -J[XYZ]$ on $\widehat{W}$. Note that derivation $_{iff} \Longleftrightarrow$ twisted and invariant.

5 Examples of Lie (super)algebras associated with (ε, δ) Freudenthal-Kantor triple systems

5.1 Examples of simple Lie superalgebra

Example (a) $C(n+1)$ type is of dimension $\dim C(n+1) = 2n^2 + 5n + 1$.

Let U be the set of matrices $M(1, 2n; \Phi)$. Then, by Example 3.2, it follows that the triple product

$$L(x,y)z =< xyz >:=< x,y > z+ < y,z > x- < z,x > y$$

such that the bilinear form fulfills $< x,y >= - < y,x >$, is a $(1,-1)$-JTS, since $K(x,y) \equiv 0$ (identically). Furthermore, the standard embedding Lie superalgebra is 3-graded and of $C(n+1)$ type. For the extended Dynkin diagram, we obtain

$$L_{-1} \oplus L_0 \oplus L_1 := \left\{ \left(\begin{array}{cc} L(a,b) & 0 \\ 0 & \varepsilon L(b,a) \end{array} \right) \middle| \varepsilon = 1 = -\delta \right\}_{span} \oplus \left\{ \left(\begin{array}{c} e \\ f \end{array} \right) \right\}_{span} \cong$$

$$\begin{array}{l} \otimes\ \alpha_1 \quad \alpha_2 \quad \alpha_3 \qquad\qquad \alpha_n \quad \alpha_{n+1} \\ \ \ \| \quad > \quad \circ - \circ - - - - - - \circ <= \circ \\ \otimes\ \alpha_0 \end{array}$$

$$= C(n+1)\ type\ (\alpha_1\ \otimes\ deleted).$$

Also, we obtain

$$L_0 := \left\{ \left(\begin{array}{cc} L(a,b) & 0 \\ 0 & \varepsilon L(b,a) \end{array} \right) \middle| \varepsilon = 1 = -\delta \right\}_{span} \cong$$

$$\begin{array}{l} \alpha_2 \quad \alpha_3 \qquad\qquad \alpha_n \quad \alpha_{n+1} \\ \circ - \circ - - - - - - \circ <= \circ \end{array}$$

$$= C_n \oplus \Phi Id\ (\alpha_1\ \otimes\ and\ \alpha_0\ \otimes\ deleted).$$

Thus the last diagram is obtained from the extended Dynkin diagram of $C(n+1)$ type by deleting $\alpha_1 \otimes$ and $\alpha_0 \otimes$.

Example (b) $B(n,1)$ and $D(n,1)$ type are of dimension $\dim B(n,1) = 2n^2 + 5n + 5$ and $\dim D(n,1) = 2n^2 + 3n + 3$, respectively.

Let U be the set of matrices $M(1, l; \Phi)$. Then, by straihtfoward calculations, it follows that the triple product

$$L(x,y)z =< xyz >:= \frac{1}{2}(< x,y > z- < y,z > x+ < z,x > y)$$

such that the bilinear form fulfills $< x,y >=< y,x >$ is a $(-1,-1)$-FKTS. Furthermore, the standard embedding Lie superalgebra is 5-graded and of $B(n,1)$ type if $l = 2n+1$, or of $D(n,1)$ type if $l = 2n$. For the extended Dynkin diagram, we obtain from the results of § 1 the following.

For the case of $B(n,1)$ type we have

$$L_{-2} \oplus L_0 \oplus L_2 := D(T(-1,-1), T(-1,-1)) =$$

$$\left\{ \left(\begin{array}{cc} L(a,b) & \delta K(c,d) \\ -\varepsilon K(e,f) & \varepsilon L(b,a) \end{array} \right) \middle| \varepsilon = -1 = \delta \right\}_{span} \cong$$

$$\alpha_0 \quad \alpha_1 \quad \alpha_2 \qquad\qquad \alpha_n \quad \alpha_{n+1}$$

$$\circ => \otimes - \circ - - - - - - \circ => \circ$$

$$= A_1 \oplus B_n \; type \; (\alpha_1 \;\; \otimes \;\; deleted).$$

Also, we obtain

$$L_0 := \left\{ \left(\begin{array}{cc} L(a,b) & 0 \\ 0 & \varepsilon L(b,a) \end{array} \right) \middle| \varepsilon = -1 = \delta \right\}_{span} \cong$$

$$\begin{array}{c} \alpha_2 \quad \alpha_3 \qquad\qquad \alpha_n \quad \alpha_{n+1} \\ \circ - \circ - - - - - - \circ => \circ \end{array}$$

$$= B_n \oplus \Phi Id \; (\alpha_1 \;\; \otimes \;\; and \; \alpha_0 \;\; \circ \;\; deleted).$$

Thus the last diagram is obtained from the extended Dynkin diagram of $B(n,1)$ type by deleting α_1 $\otimes$ and α_0 $\circ$.

Similarly, for the case of $D(n,1)$ type we have $L_{-2} \oplus L_0 \oplus L_2 \cong A_1 \oplus D_n, \; L_0 \cong D_n \oplus \Phi Id$. We note that this triple system is balanced and with a complex structure of type II since $K(x,y) =< x,y > Id = L(x,y) + L(y,x)$ (c.f. [32]).

In final of this section we note that the case of balanced is discussed in ([18], [28]). On the other hand, for the construction of simple exceptional Lie algebras $G_2, \; F_4, \; E_6, \; E_7, \; E_8$, refer to ([16], [18], [21]). Also, for the construction of simple Lie superalgebras $G(3), \; F(4), \; D(2,1,\alpha), P(n), Q(n), H(n), \; S(n)$ and $W(n)$, refer to ([22], [25], [27]). Of course, these construction are created from the concept of triple systems without using systems of roots.

5.2 Constructions of B_3-type Lie algebra

In this subsection, we will consider the constructions of simple B_3-type Lie algebra associated with several triple systems (the case of $\varepsilon = -1$ and $\delta = 1$), more easily. That is, we will give several examples; (c) the case of a JTS (i.e., $(-1,1)$-FKTS with $K(x,y) \equiv 0$), (d) the case of a GJTS of 2nd order (i.e., $(-1,1)$-FKTS with $dim\{K(x,y)\}_{span} = 1$), (e) the case of a GJTS of 2nd order (i.e, $(-1,1)$-FKTS with $dim\{K(x,y)\}_{span} = 3$), (f) the case of a derivation induced from a JTS (i.e., a subalgebra of B_4-type).

c) First, we study the case of $g_{-1} = U = Mat(1,5;\Phi)$. Hereafter in this subsection, as a reason of traditional notation, we often would like to denote by g_i instead of $L_i, (i = 0, \pm 1, \pm 2)$ and by $\{xyz\}$ instead of $< xyz >$.

In this case, g_{-1} is a JTS with respect to the product

$$\{xyz\} = x^t yz + y^t zx - z^t xy, \;\; \forall \; x,y,z \in g_{-1}$$

where ${}^t x$ denotes the transpose matrix of x.

By straightforward calculations, the standard embedding Lie algebra $L(U) = g$ can be shown to be a 3-graded B_3-type Lie algebra with $g = g_{-1} \oplus g_0 \oplus g_1$ and a LTS $T(U) = g_{-1} \oplus g_1$. Thus, we have

$$g_0 = Der \; U \oplus Anti - Der \; U \cong B_2 \oplus \Phi H, \; where \; H = \left(\begin{array}{cc} Id & 0 \\ 0 & -Id \end{array} \right).$$

Here in view of the relations $[S(x,y), L(a,b)] = L(S(x,y)a,b) + L(a, S(x,y)b)$, and $[A(x,y), L(a,b)] = L(A(x,y)a,b) \; -L(a, A(x,y)b)$ for all $L(a,b) \in End \; U$, when $\varepsilon = -1, \delta = 1$, we use the following notations;

$$Der \; U := \{L(x,y) - L(y,x)\}_{span},$$

$$Anti - Der \; U := \{L(x,y) + L(y,x)\}_{span},$$

$$g_0 = \left\{ \left(\begin{array}{cc} L(x,y) & 0 \\ 0 & -L(y,x) \end{array} \right) \right\}_{span} = \left\{ S(x,y) + A(x,y) \right\}_{span}$$

where $S(x,y) := L(x,y) - L(y,x) \in Der\ U$, $A(x,y) := L(x,y) + L(y,x) \in Anti - Der\ U$.

d) Second, we study the case of $g_{-1} = U = Mat(2,3;\Phi)$. In this case, g_{-1} is a GJTS of 2nd order (i.e., $(-1,1)$-FKTS) with $dim\ \{K(x,y)\}_{span} = 1$ with respect to the product

$$\{xyz\} = x^t yz + z^t yx - z^t xy,\ \forall\ x,y,z \in g_{-1}.$$

By straightforward calculations, it can be shown that the standard embedding Lie algebra $L(U) = g$ is a 5-graded B_3-type Lie algebra with $g = g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2$ and $dim\ g_{-2} = dim\ g_2 = dim\ \{K(x,y)\}_{span} = 1$. Thus, we have

$$g_0 = Der\ U \oplus Anti - Der\ U \cong A_1 \oplus A_1 \oplus \Phi H,\ where\ H = \begin{pmatrix} Id & 0 \\ 0 & -Id \end{pmatrix}.$$

Furthermore, we obtain a LTS $T(U)$ of $dim\ T(U) = dim\ (g_{-1} \oplus g_1) = 12$,

$$Der(g_{-1} \oplus g_1) = g_{-2} \oplus g_0 \oplus g_2 = A_1 \oplus A_1 \oplus A_1 \cong Der\ T(U).$$

Also, in this case, we note that $T(U) = L(U)/Der\ T(U) = g/(g_{-2} \oplus g_0 \oplus g_2)(= g_{-1} \oplus g_1)$ is the tangent space of a quaternion symmetric space of dimension 12, since $T(U)$ is a Lie triple system associated with g_{-1}.

e) Third, we study the case of $g_{-1} = U = Mat(1,3;\Phi)$. In this case, g_{-1} is a GJTS of 2nd order (i.e., $(-1,1)$-FKTS) with respect to the product

$$\{xyz\} = x^t yz + z^t yx - y^t xz, K(x,y)z = \{xzy\} - \{yzx\},\ \forall\ x,y,z \in g_{-1}.$$

By straightforward calculations, the standard embedding Lie algebra $L(U) = g$ can be shown to be a 5-graded B_3-type Lie algebra with $g = g_{-2} \oplus \cdots \oplus g_2$ and $dim\ g_{-2} = dim\ g_2 = 3$. Thus, we have

$$g_0 = Der\ U \oplus Anti - Der\ U \cong A_2 \oplus \Phi H,\ g_{-2} = \{K(x,y)\}_{span} = Alt(3,3;\Phi).$$

Furthermore, we obtain a LTS $T(U)$ of $dim\ T(U) = dim\ (g_{-1} \oplus g_1) = 6$,

$$Der(g_{-1} \oplus g_1) = g_{-2} \oplus g_0 \oplus g_2 = A_3 \cong Der\ T(U).$$

This case $g_{-2} = \{K(x,y)\}_{span} = \mathbf{k}$ has the structure of a JTS (cf. section 2).
Remark We remark that the case (b) (resp. (e)) is $\delta = -1$ (resp. $\delta = 1$).

f) Finally, we study the case of $g_{-1} = U = Mat(1,7;\Phi)$. In this case, g_{-1} is a JTS (i.e, $(-1,1)$-FKTS with $K(x,y) \equiv 0$) with respect to the product

$$\{xyz\} = x^t yz + y^t zx - z^t xy,\ \ \forall\ x,y,z \in g_{-1}.$$

By straightforward calculations, the standard embedding Lie algebra $L(U) = g$ can be shown to be a 3-graded B_4-type Lie algebra with $g = g_{-1} \oplus g_0 \oplus g_1$ and $dim\ g_{-2} = dim\ g_2 = 0$. Thus, we have

$$g_0 = Der\ U \oplus Anti - Der\ U \cong B_3 \oplus \Phi H,$$

$$Der\ U = \{L(x,y) - L(y,x)\}_{span} = Alt(7,7;\Phi) \cong B_3$$

$$Anti - Der\ U = \{L(x,y) + L(y,x)\}_{span} \cong \Phi H.$$

This case is obtained from $Der\ U$ such that $U = Mat(1,7;\Phi)$ with the JTS structure and so this derivation $Der\ U$ is a subalgebra of the B_4 -type Lie algebra associated with $g_{-1} = U = Mat(1,7;\Phi)$.

6 Appendix (a generalization case of (ε, δ)JTS)

Let $U(\varepsilon,\delta)(=V)$ be a (ε,δ)-FKTS and $J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $i = \sqrt{-1}$. This concept is a generalization of (ε,δ)-JTS in section 4. Then we set $\widehat{W} := \widehat{W}_+ \cup \widehat{W}_-$ satisfying

$\widehat{W}_+ = \left\{ \begin{pmatrix} x \\ x \end{pmatrix} \middle| x \in V \right\}, \widehat{W}_- = \left\{ \begin{pmatrix} x \\ -x \end{pmatrix} \middle| x \in V \right\}$. Hence $\widehat{W} \subset W = L_{-1} \oplus L_1 \subset L = L(W,W) \oplus W$ $(the\ five\ graded\ Lie\ (super)algebra)$.

Theorem 6.1 *For the above $\widehat{W}$ and $\varepsilon\delta = -1$, $\sqrt{-1} \in \Phi$, we have*
i) $\widehat{W}_\pm$ is a δ-LTS with respect the product $[XYZ]$ for any $X, Y, Z \in \widehat{W}_\pm$,
ii) $T(X,Y,Z) = -\frac{1}{2}(R(X,Y)Z - JR(X,J^{-1}Y)Z)$ is a (ε,δ)-FKTS, where $R(X,Y)Z = -[XYZ]$, i.e., $(U(\varepsilon,\delta), L(x,y)z) \leftrightarrow (\widehat{W}_\pm, T(X,Y,Z))$,
iii) $N(X,Y)$ is vanished.

Indeed from the relation (10) in section one, for example, for $\widehat{W}_-$,

$$\left[\begin{pmatrix} x \\ -x \end{pmatrix} \begin{pmatrix} y \\ -y \end{pmatrix} \begin{pmatrix} z \\ -z \end{pmatrix} \right] = \begin{pmatrix} -L(x,y)z + \delta L(y,x)z - \delta K(x,y)z \\ \varepsilon L(y,x)z - \varepsilon\delta L(x,y)z - \varepsilon K(x,y)z \end{pmatrix},$$

$[XYZ] \in \widehat{W}_-$ and $JT(X,Y,Z) \in \widehat{W}_+$ for any $X, Y, Z \in \widehat{W}_-$, we obtain the reuslts.

Remark. This generalized concept means that there is a symmetric (super)space associated with the (ε,δ)-FKTS, as same methods in section 4. However the details (type I and II) will be discussed in forthcoming paper.

7 Concluding Remarks

In this section, we give several references of mathematical physics in our works. In final comments of this paper, we note that there are applications toward the Yang-Baxter equations associated with triple systems ([26], [50], [62]) and also toward the field theory associated with Hermitian triple systems ([60], [61]).

References

[1] Allison B.N., A class of nonassociative algebras with involution containing the class of Jordan algebras, *Math. Ann.* **237** (1978), no. 2, 133-156.

[2] Allison B.N., Models of isotropic simple Lie algebras, *Comm. Algebra* **7** (1979), no. 17, 1835-1875.

[3] Asano H.; Yamaguti K., A construction of Lie algebras by generalized Jordan triple systems of second order, *Nederl. Akad. Wetensch. (Indag. Math.)* **42** (1980), no. 3, 249-253.

[4] Asano H., Classification of non-compact real simple generalized Jordan triple systems of the second kind, *Hiroshima Math. J.* **21** (1991), 463-489.

[5] Bertram W., Complex and quaternionic structures on symmetric spaces - correspondence with Freudenthal-Kantor triple systems, Theory of Lie Groups and Manifolds, ed. R. Miyaoka and H. Tamaru, *Sophia Kokyuroku in Math.* **45** (2002), 61-80.

[6] Elduque A.; Kamiya N.; Okubo S., Simple $(-1,-1)$ balanced Freudenthal Kantor triple systems, *Glasg. Math. J.* **11** (2003), no. 2, 353-372.

[7] Elduque A.; Kamiya N.; Okubo S., $(-1,-1)$ balanced Freudenthal Kantor triple systems and noncommutative Jordan algebras, *J. Algebra* **294** (2005), no. 1, 19-40.

[8] Faulkner J.R., Structurable triples, Lie triples, and symmetric spaces, *Forum Math.* **6** (1994), 637-650.

[9] Frappat L.; Sciarrino A.; Sorba P., *Dictionary on Lie Algebras and Superalgebras*, Academic Press, San Diego, California 92101-4495, USA, 2000, xxii+410 pp.

[10] Jacobson N., Lie and Jordan triple systems, *Amer. J. Math.* **71** (1949), 149-170.

[11] Jacobson N., *Structure and representations of Jordan algebras.* Amer. Math. Soc. Colloq. Publ., Vol. XXXIX Amer. Math. Soc., Providence, R.I. 1968, x+453 pp.

[12] Kac V.G., Lie superalgebras, *Advances in Math.* **26** (1977), no. 1, 8-96.

[13] Kamiya N., A structure theory of Freudenthal-Kantor triple systems, *J. Algebra* **110** (1987), no. 1, 108-123.

[14] Kamiya N., A construction of anti-Lie triple systems from a class of triple systems, *Mem. Fac. Sci. Shimane Univ.* **22** (1988), 51-62.

[15] Kamiya N., A structure theory of Freudenthal-Kantor triple systems II, *Comment. Math. Univ. St. Paul.* **38** (1989), no. 1, 41-60.

[16] Kamiya N., A structure theory of Freudenthal-Kantor triple systems III, *Mem. Fac. Sci. Shimane Univ.* **23** (1989), 33-51.

[17] Kamiya N., On (ϵ, δ)-Freudenthal-Kantor triple systems, *Nonassociative algebras and related topics* (Hiroshima, 1990), 65-75, World Sci. Publ., River Edge, NJ, 1991.

[18] Kamiya N. The construction of all simple Lie algebras over **C** from balanced Freudenthal-Kantor triple systems, *Contributions to general algebra*, **7** (Vienna, 1990), 205-213, Hölder-Pichler-Tempsky, Vienna, 1991.

[19] Kamiya N. On Freudenthal-Kantor triple systems and generalized structurable algebras, *Non-associative algebra and its applications* (Oviedo, 1993), 198-203, *Math. Appl.,* **303**, Kluwer Acad. Publ., Dordrecht, 1994.

[20] Kamiya N., On the Peirce decompositions for Freudenthal-Kantor triple systems, *Comm. Algebra* **25** (1997), no. 6, 1833-1844.

[21] Kamiya N., On a realization of the exceptional simple graded Lie algebras of the second kind and Freudenthal-Kantor triple systems, *Bull. Polish Acad. Sci. Math.* **46** (1998), no. 1, 55-65.

[22] Kamiya N.; Okubo S., On δ-Lie supertriple systems associated with $(\epsilon, \delta)-$ Freudenthal-Kantor supertriple systems, *Proc. Edinburgh Math. Soc.* **43** (2000), no. 2, 243-260.

[23] Kamiya N.; Okubo S., A construction of Jordan superalgebras from Jordan-Lie triple systems, *Lecture Notes in pure and appl. math.* **211**, Nonass. alg. and its appl. (ed. Costa, Peresi, etc.), (2002), 171-176, Marcel Dekker Inc.

[24] Kamiya N.; Okubo S., A construction of simple Jordan superalgebra of F type from a Jordan-Lie triple system, *Ann. Mate. Pura Appl.* **181** (2002), 339-348.

[25] Kamiya N.; Okubo S., Construction of Lie superalgebras $D(2,1;\alpha), G(3)$ and $F(4)$ from some triple systems, *Proc. Edinburgh Math. Soc.* **46** (2003), no. 1, 87-98.

[26] Kamiya N.; Okubo S., On generalized Freudenthal-Kantor triple systems and Yang-Baxter equations, *Proc. XXIV Inter. Colloq. Group Theoretical Methods in Physics*, IPCS, vol. **173** (2003), 815-818.

[27] Kamiya N.; Okubo S., A construction of simple Lie superalgebras of certain types from triple systems, *Bull. Austral. Math. Soc.* **69** (2004), no. 1, 113-123.

[28] Kamiya N., Examples of Peirce decomposition of generalized Jordan triple system of second order-Balanced cases-, *Noncommutative geometry and representation theory in mathematical physics*, 157-165, Contemp. Math., **391**, AMS, Providence, RI, 2005.

[29] Kamiya N.; Okubo S., Composition, quadratic, and some triple systems, *Non-associative algebra and its applications*, 205-231, Lect. Notes Pure Appl. Math., **246**, Chapman & Hall/CRC, Boca Raton, FL, 2006.

[30] Kamiya N.; Mondoc D., A new class of nonassociative algebras with involution, *Proc. Japan Acad. Ser. A* **84** (2008), no. 5, 68-72.

[31] Kamiya N.; Mondoc D., Okubo S., A structure theory of $(-1,-1)$-Freudenthal Kantor triple systems, *Bull. Austral. Math. Soc.* **81** (2010), 132-155.

[32] Kamiya N.; Okubo S., Symmetry of Lie algebras associated with $(\epsilon.\delta)$ Freudenthal-Kantor triple systems, *Proc. Edinb. Math. Soc.* **89** (2016), 169-192.

[33] Kaneyuki S.; Asano H., Graded Lie algebras and generalized Jordan triple systems, *Nagoya Math. J.* **112** (1988), 81-115.

[34] Kantor I.L., Graded Lie algebras, *Trudy Sem. Vect. Tens. Anal.* **15** (1970), 227-266.

[35] Kantor I.L., Some generalizations of Jordan algebras, *Trudy Sem. Vect. Tens. Anal.* **16** (1972), 407-499.

[36] Kantor I.L. Models of exceptional Lie algebras, *Soviet Math. Dokl.* Vol. **14** (1973), no. 1, 254-258.

[37] Kantor I.L. A generalization of the Jordan approach to symmetric Riemannian spaces. *The Monster and Lie algebras* (Columbus, OH, 1996), 221-234, *Ohio State Univ. Math. Res. Inst. Publ.,* **7**, de Gruyter, Berlin, 1998.

[38] Kantor I.L.; Kamiya N., A Peirce decomposition for generalized Jordan triple systems of second order, *Comm. Algebra* **31** (2003), no. 12, 5875-5913.

[39] Loos O., *Symmetric spaces. I: General theory.* W. A. Benjamin, Inc., New York-Amsterdam 1969, viii+198 pp.

[40] Koecher M., Embedding of Jordan algebras into Lie algebras I. II., *Amer. J. Math* **89** (1967), 787-816 and **90** (1968), 476-510.

[41] Lister W.G., A structure theory of Lie triple systems, *Trans. Amer. Math. Soc.* **72** (1952), 217-242.

[42] Meyberg K., Lectures on algebras and triple systems, *Lecture Notes*, The University of Virginia, Charlottesville, Va., 1972, v+226 pp.

[43] Mondoc D., Models of compact simple Kantor triple systems defined on a class of structurable algebras of skew-dimension one, *Comm. Algebra* **34** (2006), no. 10, 3801-3815.

[44] Mondoc D., On compact realifications of exceptional simple Kantor triple systems, *J. Gen. Lie Theory Appl.* **1** (2007), no. 1, 29-40.

[45] Mondoc D., Compact realifications of exceptional simple Kantor triple systems defined on tensor products of composition algebras, *J. Algebra* **307** (2007), no. 2, 917-929.

[46] Mondoc D., Compact exceptional simple Kantor triple systems defined on tensor products of composition algebras, *Comm. Algebra* **35** (2007), no. 11, 3699-3712.

[47] Neher E., *Jordan triple systems by the grid approach, Lecture Notes in Mathematics*, **1280**, Springer-Verlag, Berlin, 1987. xii+193 pp.

[48] Okubo S., *Introduction to octonion and other non-associative algebras in physics, Montroll Memorial Lecture Series in Mathematical Physics, 2.* Cambridge University Press, Cambridge, 1995, xii+136 pp.

[49] Okubo S.; Kamiya N., Jordan-Lie superalgebra and Jordan-Lie triple system, *J. Algebra* **198** (1997), no. 2, 388-411.

[50] Okubo S.; Kamiya N., Quasi-classical Lie superalgebras and Lie supertriple systems, *Comm. Algebra* **30** (2002), no. 8, 3825-3850.

[51] Okubo S., Symmetric triality relations and structurable algebras, *Linear Algebra Appl.* **396** (2005), 189-222.

[52] Scheunert M., *The theory of Lie superalgebras. An introduction, Lecture Notes in Mathematics*, **716**. Springer, Berlin, 1979, x+271 pp.

[53] Tits J., Une classe d'algèbres de Lie en relation avec les algèbres de Jordan, *Nederl. Acad. Wetensch. Proc. Ser. A* **65** (1962), 530-535.

[54] Yamaguti K.; Ono A. On representations of Freudenthal-Kantor triple systems $U(\epsilon, \delta)$, *Bull. Fac. School Ed. Hiroshima Univ.,* **7** (1984), no. II, 43-51.

[55] Zhevlakov K.A.; Slinko A.M.; Shestakov I.P.; Shirshov A.I., *Rings that are Nearly Associative*, *Academic Press, Inc., New York-London*, 1982, xi+371 pp.

[56] Zelmanov E., Primary Jordan triple systems, *Sibersk. Mat. Zh.* (1983), no. 4, 23-37.

Added References

[57] Kamiya N.; Mondoc D.; Okubo S., A review of Peirce decomposition for unitary (-1,-1)-Freudenthal-Kantor triple systems, Springer, *Proc. in Math. and Stastics,* vol. **85**, (2014) 145-155.

[58] Kamiya N.; Mondoc D.; Okubo S., A characterization of (-1,-1)-Freudenthal-Kantor triple systems, *Glasgow Math. J.,* **53** (2011) 727-738.

[59] Kamiya N.; Okubo S., On triality of structurable and pre-structurable algebras, *J. Algebra* vol. **416** (2014) 58-88.

[60] Kamiya N.; Sato M., Hermitian (ε, δ)-Freudenthal-Kantor triple systems and certain applications of *-Generalized Jordan triple systems to field theory, *Advanced High Energy Physics,* (2014) Doi.org/10.1155/2014/310264.

[61] Kamiya N.; Sato M., Hermitian triple systems associated with bilinear forms and certain applications to field theory, *Hadronic Journal,* vol. **37** nu.2 (2014), p131-147.

[62] Kamiya N.; Shibukawa Y., Dynamical Yang-Baxter maps and weak Hopf algebras associated with quandles, *Proc. Meeting for Study of Number theory, Hopf Algebras and Related topics, ed. Yamane, Kogiso, Koga and Kimura,* (2019) p1-23, Yokohama Publishers.

[63] Satake I.; *Algebraic structures of symmetric domains,* Princeton University, 1980.

[64] Springer T.; *Jordan algebras and Algebraic Groups*, Springer, 1970.

These references are mainly papers for our study fields (as a survey article in these fields, we have a lot of references).

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