HYPERSEQUENT CALCULI FOR INTERMEDIATE PREDICATE LOGICS

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ABSTRACT. We report on the current status of our on-going project to develop well-behaved hypersequent calculi for intermediate predicate logics, such as the linearity axiom **LIN**: $(\varphi \to \psi) \vee (\psi \to \varphi)$ and the constant domain axiom **CD**: $\forall x (\varphi \vee \psi(x)) \to \varphi \vee \forall x \psi(x)$.

1. INTRODUCTION

Gentzen-style sequent calculus is a proof system for sequents $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite sequences of formulae. Since the arrow symbol $\Rightarrow$ behaves as meta-implication, implicational axioms can be well transformed to inference rules. For example, the $\wedge$-introduction axiom (schema) $\varphi \to \psi \to \varphi \wedge \psi$ can be reformulated as the following rule:

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi} \ (\wedge\text{-R}).$$

On the other hand, sequent calculus does not well manipulate axioms whose outermost logical symbols are not implications such as the linearity axiom **LIN**: $(\varphi \to \psi) \vee (\psi \to \varphi)$, the weak law of excluded middle **WLEM**: $\neg\varphi \vee \neg\neg\varphi$, and the constant domain axiom **CD**: $\forall x (\varphi \vee \psi(x)) \to \varphi \vee \forall x \psi(x)$. Note that these axioms care about disjunctions (and universal quantifiers). See e.g. Kashima [12] for some fundamental problems concerning **CD**.

Hypersequent calculus was first introduced by Avron [1]. A *hypersequent* is a finite sequence $(\Gamma_i \Rightarrow \varphi_i)_{i=1}^n$ of sequents, and is usually denoted as follows:

$$\Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n.$$

The sequents $\Gamma_i \Rightarrow \Delta_i$ are called *components* of the hypersequent. Throughout this paper, we denote hypersequents by meta-symbols $G, H, \ldots$, sequents by $S, T, \ldots$, and formulae by $\varphi, \psi, \ldots$; the concatenation of (possibly empty) hypersequents G and H by $G \mid H$.

The hypersequent calculus **HLK** of classical propositional logic (**CL**) is given by the inference rules listed in Table 1.1.

Fact 1.1. *(1) If* **HLK** $\vdash \Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$*, then* $\bigvee_{i=1}^n (\bigwedge \Gamma_i \to \bigvee \Delta_i)$ *is* **CL***-valid.*
(2) If φ *is* **CL***-valid, then* **HLK** $\vdash\ \Rightarrow \varphi$*.*

We can obtain the hypersequent calculus for intuitionistic propositional logic (**INT**) by mimicking Gentzen's **LJ** or Maehara's **LJ′**. More precisely, **HLJ** is the subsystem of **HLK**, where sequents are restricted to *single-conclusion*; and **HLJ′** is the subsystem of **HLK**, where the rule ($\to$ -R) is restricted to

$$\frac{\varphi, \Gamma \Rightarrow \psi \mid G}{\Gamma \Rightarrow \varphi \to \psi \mid G} \ (\to\text{-R}').$$

Fact 1.2. *(1) If* **HLJ** $\vdash \Gamma_1 \Rightarrow \varphi_1 \mid \cdots \mid \Gamma_n \Rightarrow \varphi_n$*, then* $\bigvee_{i=1}^n (\bigwedge \Gamma_i \to \varphi_i)$ *is* **INT***-valid.*
(2) If **HLJ′** $\vdash \Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$*, then* $\bigvee_{i=1}^n (\bigwedge \Gamma_i \to \bigvee \Delta_i)$ *is* **INT***-valid.*
(3) If φ *is* **INT***-valid, then* **HLJ** $\vdash\ \Rightarrow \varphi$ *and* **HLJ′** $\vdash\ \Rightarrow \varphi$*.*

The pipe symbol | can be interpreted as meta-disjunctions, so hypersequent calculus well manipulates disjunctive axioms. For example, Gödel–Dummett propositional logic **GD** (i.e. **INT** + **LIN**) can be characterised by the following structural rule, called the *communication rule*:

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Axioms

$$\frac{}{\varphi \Rightarrow \varphi}\ (\mathrm{Id}) \qquad \frac{}{\bot \Rightarrow \varphi}\ (\mathrm{Bot})$$

External structural rules

$$\frac{G}{S \mid G}\ (\mathrm{ew})$$

$$\frac{S \mid S \mid G}{S \mid G}\ (\mathrm{ec})$$

$$\frac{G \mid S \mid T \mid H}{G \mid T \mid S \mid H}\ (\mathrm{ee})$$

Internal structural rules

$$\frac{\Gamma \Rightarrow \Delta \mid G}{\varphi, \Gamma \Rightarrow \Delta \mid G}\ (\mathrm{iw\text{-}L}) \qquad \frac{\Gamma \Rightarrow \Delta \mid G}{\Gamma \Rightarrow \Delta, \psi \mid G}\ (\mathrm{iw\text{-}R})$$

$$\frac{\varphi, \varphi, \Gamma \Rightarrow \Delta \mid G}{\varphi, \Gamma \Rightarrow \Delta \mid G}\ (\mathrm{ic\text{-}L}) \qquad \frac{\Gamma \Rightarrow \Delta, \psi, \psi \mid G}{\Gamma \Rightarrow \Delta, \psi \mid G}\ (\mathrm{ic\text{-}R})$$

$$\frac{\Gamma_1, \varphi, \psi, \Gamma_2 \Rightarrow \Delta \mid G}{\Gamma_1, \psi, \psi, \Gamma_2 \Rightarrow \Delta \mid G}\ (\mathrm{ie\text{-}L}) \qquad \frac{\Gamma \Rightarrow \Delta_1, \varphi, \psi, \Delta_2 \mid G}{\Gamma \Rightarrow \Delta_1, \psi, \varphi, \Delta_2 \mid G}\ (\mathrm{ie\text{-}R})$$

Cut

$$\frac{\Gamma_0 \Rightarrow \Delta_0, \delta \mid G \quad \delta, \Gamma_1 \Rightarrow \Delta_1 \mid G}{\Gamma_0, \Gamma_1 \Rightarrow \Delta_0, \Delta_1 \mid G}\ (\mathrm{cut})$$

Logical rules

$$\frac{\varphi_i, \Gamma \Rightarrow \Delta \mid G}{\varphi_1 \wedge \varphi_2 \Rightarrow \Delta \mid G}\ (\wedge_i\text{-L}) \qquad \frac{\Gamma \Rightarrow \Delta, \varphi_1 \mid G \quad \Gamma \Rightarrow \Delta, \varphi_2 \mid G}{\Gamma \Rightarrow \Delta, \varphi_1 \wedge \varphi_2 \mid G}\ (\wedge\text{-R})$$

$$\frac{\varphi_1, \Gamma \Rightarrow \Delta \mid G \quad \varphi_2, \Gamma \Rightarrow \Delta \mid G}{\varphi_1 \vee \varphi_2, \Gamma \Rightarrow \Delta \mid G}\ (\vee\text{-L}) \qquad \frac{\Gamma \Rightarrow \Delta, \varphi_i \mid G}{\Gamma \Rightarrow \Delta, \varphi_1 \vee \varphi_2 \mid G}\ (\vee_i\text{-R})$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \mid G \quad \psi, \Gamma \Rightarrow \Delta \mid G}{\varphi \to \psi, \Gamma \Rightarrow \Delta \mid G}\ (\to\text{-L}) \qquad \frac{\varphi, \Gamma \Rightarrow \Delta, \psi \mid G}{\Gamma \Rightarrow \Delta, \varphi \to \psi \mid G}\ (\to\text{-R})$$

TABLE 1.1.

Quantifier rules

$$\frac{[t/x]\varphi, \Gamma \Rightarrow \Delta \mid G}{\forall x\varphi, \Gamma \Rightarrow \Delta \mid G}\ (\forall\text{-L}) \qquad \frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \forall x\varphi}\ (\forall\text{-R}_{ss})$$

provided that x does not freely occur in Γ.

$$\frac{\varphi, \Gamma \Rightarrow \Delta}{\exists x\varphi, \Gamma \Rightarrow \Delta}\ (\exists\text{-L}_s) \qquad \frac{\Gamma \Rightarrow \Delta, [t/x]\psi \mid G}{\Gamma \Rightarrow \Delta, \exists x\psi \mid G}\ (\exists\text{-R})$$

provided that x does not freely occur in Γ, Δ.

Table 1.2.

$$\frac{\Gamma, \Delta \Rightarrow \Theta \mid G \quad \Gamma', \Delta' \Rightarrow \Theta' \mid G}{\Gamma, \Delta' \Rightarrow \Theta \mid \Gamma', \Delta \Rightarrow \Theta' \mid G}\ (\text{com}).$$

This rule is an intermediate between the external (hypersequent-level) structure and the internal (sequent-level) structure.

Fact 1.3 (Avron [2], [3]). *(1) If* **HLJ** + (com) $\vdash \Gamma_1 \Rightarrow \varphi_1 \mid \cdots \mid \Gamma_n \Rightarrow \varphi_n$*, then* $\bigvee_{i=1}^{n} (\bigwedge \Gamma_i \to \varphi_i)$ *is* **GD***-valid.*
(2) If **HLJ**′ + (com) $\vdash \Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$*, then* $\bigvee_{i=1}^{n} (\bigwedge \Gamma_i \to \bigvee \Delta_i)$ *is* **GD***-valid.*
(3) If φ *is* **GD***-valid, then* **HLJ** + (com) $\vdash \Rightarrow\varphi$ *and* **HLJ**′ + (com) $\vdash \Rightarrow\varphi$*.*

Proof. We only recall the proof of **HLJ** + (com) $\vdash \Rightarrow$**LIN**.

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{\varphi, \varnothing \Rightarrow \varphi}\ (\text{Id}) \quad \overline{\psi, \varnothing \Rightarrow \psi}\ (\text{Id})}{\varphi, \varnothing \Rightarrow \psi \mid \psi, \varnothing \Rightarrow \varphi}\ (\text{com})}{\Rightarrow \varphi \to \psi \mid \Rightarrow \psi \to \varphi}\ (\to\text{-R}'), (\text{ee})}{\Rightarrow (\varphi \to \psi) \vee (\psi \to \varphi) \mid \Rightarrow (\varphi \to \psi) \vee (\psi \to \varphi)}\ (\vee\text{-R}), (\text{ee})}{\Rightarrow (\varphi \to \psi) \vee (\psi \to \varphi)}\ (\text{ec})$$

□

Let us move on to predicate logics. The hypersequent calculi for intuitionistic predicate logic ($\forall$**INT**) can be obtained by adding **HLJ** and **HLJ**′ with the quantifier rules (Table 1.2). We refer to the resulting systems as $\forall$**HLJ** and $\forall$**HLJ**′, respectively. As the eigenvariable condition suggests, an (open) hypersequent $(\Gamma_1 \Rightarrow \Delta_1)(\vec{x}) \mid \cdots \mid (\Gamma_n \Rightarrow \Delta_n)(\vec{x})$ with free variables $\vec{x}$ represents a closed formula $\forall \vec{x} \bigvee_{i=1}^{n} (\bigwedge \Gamma_i(\vec{x}) \to \bigvee \Delta_i(\vec{x}))$.

Fact 1.4. *(1) If* $\forall$**HLJ** $\vdash \Gamma_1 \Rightarrow \varphi_1 \mid \cdots \mid \Gamma_n \Rightarrow \varphi_n$*, the universal closure of* $\bigvee_{i=1}^{n} (\bigwedge \Gamma_i \to \varphi_i)$ *is* $\forall$**INT***-valid.*
(2) If $\forall$**HLJ**′ $\vdash \Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$*, the universal closure of* $\bigvee_{i=1}^{n} (\bigwedge \Gamma_i \to \bigvee \Delta_i)$ *is* $\forall$**INT***-valid.*
(3) If φ *is* $\forall$**INT***-valid, then* $\forall$**HLJ** $\vdash \Rightarrow\varphi$ *and* $\forall$**HLJ**′ $\vdash \Rightarrow\varphi$*.*

We also consider the multi-component single-conclusioned $\forall$-right rule:

$$\frac{\Gamma \Rightarrow \varphi \mid G}{\Gamma \Rightarrow \forall x\varphi \mid G}\ (\forall\text{-R}_{ms}),$$

where the variable x does not freely occur in the lower hypersequent. Apparently the rule ($\forall$-R$_{ms}$) asserts that $\forall \vec{w} \forall x ((\gamma(\vec{w}) \to \varphi(x, \vec{w})) \vee \psi(\vec{w}))$ implies $\forall \vec{w} ((\gamma(\vec{w}) \to \forall x \varphi(x, (\vec{w}))) \vee \psi((\vec{w})))$, a form of **CD**. However, to extract **CD** from ($\forall$-R$_{ms}$), we need the communication rule (com). The combination of ($\forall$-R$_{ms}$) and (com) characterises Gödel–Dummett predicate logic $\forall$**GD** := $\forall$**INT** + **LIN** + **CD**.

Fact 1.5 (Baaz and Zach [4]). *(1) If* $\forall$**HLJ** + ($\forall$-R$_{ms}$) + (com) $\vdash \Gamma_1 \Rightarrow \varphi_1 \mid \cdots \mid \Gamma_n \Rightarrow \varphi_n$*, the universal closure of* $\bigvee_{i=1}^{n} (\bigwedge \Gamma_i \to \varphi_i)$ *is* $\forall$**GD***-valid.*
(2) If $\forall$**HLJ**′ + ($\forall$-R$_{ms}$) + (com) $\vdash \Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$*, the universal closure of* $\bigvee_{i=1}^{n} (\bigwedge \Gamma_i \to \bigvee \Delta_i)$ *is* $\forall$**GD***-valid.*
(3) If φ *is* $\forall$**GD***-valid, then* $\forall$**HLJ** + ($\forall$-R$_{ms}$) + (com) $\vdash \Rightarrow\varphi$ *and* $\forall$**HLJ**′ + ($\forall_{ms}$-R) + (com) $\vdash \Rightarrow\varphi$*.*

Proof. We only recall the proof of $\forall\mathbf{HLJ} + (\forall\text{-R}_{\mathrm{ms}}) + (\mathrm{com}) \vdash\ \Rightarrow\mathbf{CD}$.

$$\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,(\mathrm{Id})}{\varphi \Rightarrow \varphi \mid \varphi \Rightarrow \psi(x)}\,(\mathrm{ew})\,(\mathrm{ee}) \qquad \dfrac{\dfrac{}{\psi(x) \Rightarrow \psi(x)}\,(\mathrm{Id}) \quad \dfrac{}{\varphi \Rightarrow \varphi}\,(\mathrm{Id})}{\psi(x) \Rightarrow \varphi \mid \varphi \Rightarrow \psi(x)}\,(\mathrm{com})$$

$$\dfrac{\varphi \Rightarrow \varphi \mid \varphi \Rightarrow \psi(x) \qquad \psi(x) \Rightarrow \varphi \mid \varphi \Rightarrow \psi(x)}{\dfrac{\varphi \vee \psi(x) \Rightarrow \varphi \mid \varphi \Rightarrow \psi(x)}{\varphi \Rightarrow \psi(x) \mid \varphi \vee \psi(x) \Rightarrow \varphi}\,(\mathrm{ee})}\,(\vee\text{-L})$$

$$\dfrac{\dfrac{}{\psi(x) \Rightarrow \psi(x)}\,(\mathrm{Id})}{\psi(x) \Rightarrow \psi(x) \mid \varphi \vee \psi(x) \Rightarrow \varphi}\,(\mathrm{ew})\,(\mathrm{ee})$$

$$\dfrac{\varphi \Rightarrow \psi(x) \mid \varphi \vee \psi(x) \Rightarrow \varphi \qquad \psi(x) \Rightarrow \psi(x) \mid \varphi \vee \psi(x) \Rightarrow \varphi}{\varphi \vee \psi(x) \Rightarrow \psi(x) \mid \varphi \vee \psi(x) \Rightarrow \varphi}\,(\vee\text{-L})$$

$$\dfrac{\varphi \vee \psi(x) \Rightarrow \psi(x) \mid \varphi \vee \psi(x) \Rightarrow \varphi}{\dfrac{\forall x\,(\varphi \vee \psi(x)) \Rightarrow \psi(x) \mid \forall x\,(\varphi \vee \psi(x)) \Rightarrow \varphi}{\dfrac{\forall x\,(\varphi \vee \psi(x)) \Rightarrow \forall x \psi(x) \mid \forall x\,(\varphi \vee \psi(x)) \Rightarrow \varphi}{\dfrac{\forall x\,(\varphi \vee \psi(x)) \Rightarrow \varphi \vee \forall x \psi(x) \mid \forall x\,(\varphi \vee \psi(x)) \Rightarrow \varphi \vee \forall x \psi(x)}{\dfrac{\forall x\,(\varphi \vee \psi(x)) \Rightarrow \varphi \vee \forall x \psi(x)}{\Rightarrow \forall x\,(\varphi \vee \psi(x)) \to \varphi \vee \forall x \psi(x)}\,(\to\text{-R}')}\,(\mathrm{ec})}\,(\vee\text{-R}),(\mathrm{ee})}\,(\forall\text{-R}_{\mathrm{ms}})}\,(\forall\text{-L}),(\mathrm{ee})$$

□

Similarly, the multi-component $\exists$-left rule

$$\dfrac{\varphi, \Gamma \Rightarrow \Delta \mid G}{\exists x \varphi, \Gamma \Rightarrow \Delta \mid G}\,(\exists\text{-L}_{\mathrm{m}})$$

asserts that $\forall \vec{w} \forall x\,((\varphi(x, \vec{w}) \wedge \gamma(\vec{w}) \to \delta(\vec{w})) \vee \psi(\vec{w}))$ implies $\forall \vec{w}\,((\exists x \varphi(x, \vec{w}) \wedge \gamma(\vec{w}) \to \delta(\vec{w})) \vee \psi(\vec{w}))$, and depends on **CD**. Recall the (informal) proof of this assertion: suppose $\forall x\,((\varphi(x) \wedge \gamma \to \delta) \vee \psi)$. Applying the axiom schema **CD**, we have $\forall x\,(\varphi(x) \wedge \gamma \to \delta) \vee \psi$. Since $\forall x\,(\varphi(x) \wedge \gamma \to \delta) \to (\exists x \varphi(x) \wedge \gamma \to \delta)$ is $\forall$**INT**-valid, we obtain the desired conclusion $(\exists x \varphi(x) \wedge \gamma \to \delta) \vee \psi$. The system $\forall\mathbf{HLJ}' + (\exists\text{-L}_{\mathrm{m}}) + (\mathrm{com})$ is therefore sound with respect to $\forall$**GD**.

Problem 1.6. $\forall\mathbf{HLJ}' + (\exists\text{-L}_{\mathrm{m}}) + (\mathrm{com}) \vdash\ \Rightarrow\mathbf{CD}$?

Remark 1.7. One can obtain the proof figure of $\forall\mathbf{HLJ} + (\exists\text{-L}_{\mathrm{m}}) + (\mathrm{com}) \vdash\ \Rightarrow \varphi \wedge \exists x \psi(x) \to \exists x\,(\varphi \wedge \psi(x))$ as the dual of the proof figure in Fact 1.5:

$$\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,(\mathrm{Id})}{\varphi \Rightarrow \varphi \mid \psi(x) \Rightarrow \varphi}\,(\mathrm{ew})\,(\mathrm{ee}) \qquad \dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,(\mathrm{Id}) \quad \dfrac{}{\psi(x) \Rightarrow \psi(x)}\,(\mathrm{Id})}{\varphi \Rightarrow \psi(x) \mid \psi(x) \Rightarrow \varphi}\,(\mathrm{com})$$

$$\dfrac{\varphi \Rightarrow \varphi \mid \psi(x) \Rightarrow \varphi \qquad \varphi \Rightarrow \psi(x) \mid \psi(x) \Rightarrow \varphi}{\dfrac{\varphi \Rightarrow \varphi \wedge \psi(x) \mid \psi(x) \Rightarrow \varphi}{\psi(x) \Rightarrow \varphi \mid \varphi \Rightarrow \varphi \wedge \psi(x)}\,(\mathrm{ee})}\,(\wedge\text{-R})$$

$$\dfrac{\dfrac{}{\psi(x) \Rightarrow \psi(x)}\,(\mathrm{Id})}{\psi(x) \Rightarrow \psi(x) \mid \varphi \Rightarrow \varphi \wedge \psi(x)}\,(\mathrm{ew})\,(\mathrm{ee})$$

$$\dfrac{\psi(x) \Rightarrow \varphi \mid \varphi \Rightarrow \varphi \wedge \psi(x) \qquad \psi(x) \Rightarrow \psi(x) \mid \varphi \Rightarrow \varphi \wedge \psi(x)}{\psi(x) \Rightarrow \varphi \wedge \psi(x) \mid \varphi \Rightarrow \varphi \wedge \psi(x)}\,(\wedge\text{-R})$$

$$\dfrac{\psi(x) \Rightarrow \varphi \wedge \psi(x) \mid \varphi \Rightarrow \varphi \wedge \psi(x)}{\dfrac{\psi(x) \Rightarrow \exists x\,(\varphi \wedge \psi(x)) \mid \varphi \Rightarrow \exists x\,(\varphi \wedge \psi(x))}{\dfrac{\exists x \psi(x) \Rightarrow \exists x\,(\varphi \wedge \psi(x)) \mid \varphi \Rightarrow \exists x\,(\varphi \wedge \psi(x))}{\dfrac{\varphi \wedge \exists x \psi(x) \Rightarrow \exists x\,(\varphi \wedge \psi(x)) \mid \varphi \wedge \exists x \psi(x) \Rightarrow \exists x\,(\varphi \wedge \psi(x))}{\dfrac{\varphi \wedge \exists x \psi(x) \Rightarrow \exists x\,(\varphi \wedge \psi(x))}{\Rightarrow \varphi \wedge \exists x \psi(x) \to \exists x\,(\varphi \wedge \psi(x))}\,(\to\text{-R}')}\,(\mathrm{ec})}\,(\wedge\text{-L}),(\mathrm{ee})}\,(\exists\text{-L}_{\mathrm{m}})}\,(\exists\text{-R}),(\mathrm{ee})$$

Evidently the meta-formula $\varphi \wedge \exists x \psi(x) \to \exists x\,(\varphi \wedge \psi(x))$ is $\forall$**INT**-valid.

Hypersequent calculi for intermediate logics such as **GD**, $\forall$**GD** and $\forall\mathbf{INT} + \mathbf{LIN}$ have been extensively studied. See e.g. [2, 3, 4, 5, 6, 7, 8, 14].

We aims to develop well-behaved proof systems for $\forall\mathbf{INT}+\mathbf{CD}$ and $\forall\mathbf{INT}+\mathbf{LIN}$ via hypersequent calculus. For our purpose, it is beneficial to specify the sources of **CD** and **LIN** in $\forall\mathbf{HLJ} + (\forall_{\mathrm{ms}}\text{-R}) + (\mathrm{com})$. In Section 2, we introduce the right split rule (rs) and the left split rule (ls) to clarify the communication rule (com). We prove that

1. $\mathbf{HLJ}' + (\mathrm{rs})$ and $\mathbf{HLJ}' + (\mathrm{ls})$ are sound and complete with respect to **GD**;
2. $\forall\mathbf{HLJ}' + (\forall\text{-R}_{\mathrm{ms}}) + (\mathrm{rs})$ are sound and complete with respect to $\forall$**GD**; and
3. $\forall\mathbf{HLJ}' + (\forall\text{-R}_{\mathrm{ms}}) + (\mathrm{ls})$ is equivalent to or stronger than $\forall\mathbf{INT} + \mathbf{LIN}$.

In Section 3, we show that

1. $\forall\mathbf{HLJ} + (\mathrm{com})$ and $\forall\mathbf{HLJ}' + (\mathrm{com})$ are sound and complete with respect to $\forall\mathbf{INT} + \mathbf{LIN}$; and
2. $\forall\mathbf{HLJ} + (\forall\text{-R}_{\mathrm{ms}}) + (\exists\text{-L}_{\mathrm{m}})$ and $\forall\mathbf{HLJ}' + (\forall\text{-R}_{\mathrm{ms}}) + (\exists\text{-L}_{\mathrm{m}})$ are sound and complete with respect to $\forall$**INT**.

In Section 4, we conclude the paper with some future research directions.

2. Splitting rules

We analyse the communication rule by dividing it into two rules. We first consider the *right split rule*:

$$\dfrac{\Gamma \Rightarrow \Delta_1, \Delta_2 \mid G}{\Gamma \Rightarrow \Delta_1 \mid \Gamma \Rightarrow \Delta_2 \mid G}\ (\mathrm{rs}).$$

In the algebraic point of view, this rule corresponds to the inequality $(\gamma \to \delta_1 \vee \delta_2) \leq (\gamma \to \delta_1) \vee (\gamma \to \delta_2)$, which is known to be equivalent to **LIN** (see Diener and McKubre-Jordens [9, Proposition 2]).

Theorem 2.1. *(1)* $\mathbf{HLJ}' + (\mathrm{rs})$ *proves* **LIN**.
(2) $\forall\mathbf{HLJ}' + (\forall\text{-R}_{\mathrm{ms}}) + (\mathrm{rs})$ *proves* **CD**.

Proof. The linearity axiom:

$$\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,(\mathrm{Id})}{\varphi \Rightarrow \varphi \vee \psi}\,(\vee_1\text{-R}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\psi \Rightarrow \psi}\,(\mathrm{Id})}{\psi \Rightarrow \varphi \vee \psi}\,(\vee_2\text{-R}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,(\mathrm{Id})}{\varphi \Rightarrow \varphi, \psi}\,(\text{iw-R}) \quad \dfrac{\dfrac{}{\psi \Rightarrow \psi}\,(\mathrm{Id})}{\psi \Rightarrow \varphi, \psi}\,(\text{iw-R})\,(\text{ie-R})}{\varphi \vee \psi \Rightarrow \varphi, \psi}\,(\vee\text{-L})}{\varphi \vee \psi \Rightarrow \varphi \mid \varphi \vee \psi \Rightarrow \psi}\,(\mathrm{rs})}{\psi \Rightarrow \varphi \mid \varphi \vee \psi \Rightarrow \psi}\,(\mathrm{cut})}{\varphi \vee \psi \Rightarrow \psi \mid \psi \Rightarrow \varphi}\,(\mathrm{ee})}{\dfrac{\dfrac{\varphi \Rightarrow \psi \mid \psi \Rightarrow \varphi}{\Rightarrow (\varphi \to \psi) \vee (\psi \to \varphi) \mid \Rightarrow (\varphi \to \psi) \vee (\psi \to \varphi)}\,(\to\text{-R}'), (\vee\text{-R}), (\mathrm{ee})}{\Rightarrow (\varphi \to \psi) \vee (\psi \to \varphi)}\,(\mathrm{ec})}\,(\mathrm{cut})$$

The constant domain axiom:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi}\,(\mathrm{Id})}{\varphi \Rightarrow \varphi, \psi(x)}\,(\text{iw-R}) \quad \dfrac{\dfrac{}{\psi(x) \Rightarrow \psi(x)}\,(\mathrm{Id})}{\psi(x) \Rightarrow \varphi, \psi(x)}\,(\text{iw-R})\,(\text{ie-R})}{\varphi \vee \psi(x) \Rightarrow \varphi, \psi(x)}\,(\vee\text{-L})}{\forall x\,(\varphi \vee \psi(x)) \Rightarrow \varphi, \psi(x)}\,(\forall\text{-L})}{\forall x\,(\varphi \vee \psi(x)) \Rightarrow \varphi \mid \forall x\,(\varphi \vee \psi(x)) \Rightarrow \psi(x)}\,(\mathrm{rs})}{\forall x\,(\varphi \vee \psi(x)) \Rightarrow \varphi \mid \forall x\,(\varphi \vee \psi(x)) \Rightarrow \forall x \psi(x)}\,(\forall\text{-R}_{\mathrm{ms}})}{\forall x\,(\varphi \vee \psi(x)) \Rightarrow \varphi \vee \forall x \psi(x) \mid \forall x\,(\varphi \vee \psi(x)) \Rightarrow \varphi \vee \forall x \psi(x)}\,(\vee\text{-R}), (\mathrm{ee})}{\forall x\,(\varphi \vee \psi(x)) \Rightarrow \varphi \vee \forall x \psi(x)}\,(\mathrm{ec})}{\Rightarrow \forall x\,(\varphi \vee \psi(x)) \to \varphi \vee \forall x \psi(x)}\,(\to\text{-R}')}{}$$

□

Corollary 2.2 (Completeness). *(1) If* φ *is* **GD***-valid, then* $\mathbf{HLJ}' + (\mathrm{rs}) \vdash\ \Rightarrow \varphi$.
(2) If φ *is* $\forall$**GD***-valid, then* $\forall\mathbf{HLJ}' + (\forall\text{-R}_{\mathrm{ms}}) + (\mathrm{rs}) \vdash\ \Rightarrow \varphi$.

Theorem 2.3 (Soundness). *(1) If* $\mathbf{HLJ}' + (\mathrm{rs}) \vdash \Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$*, then* $\bigvee_{i=1}^{n} (\bigwedge \Gamma_i \to \bigvee \Delta_i)$ *is* **GD***-valid.*
(2) If $\forall\mathbf{HLJ}' + (\forall\text{-R}_{\mathrm{ms}}) + (\mathrm{rs}) \vdash \Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$*, then the universal closure of* $\bigvee_{i=1}^{n} (\bigwedge \Gamma_i \to \bigvee \Delta_i)$ *is* $\forall$**GD***-valid.*

Proof. It suffices to show that $(\forall)\,\mathbf{NJ} + \mathbf{LIN} \vdash (\gamma \to \delta_1 \vee \delta_2) \to (\gamma \to \delta_1) \vee (\gamma \to \delta_2)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\delta_2}\,1 \quad \dfrac{\dfrac{}{\delta_1 \to \delta_2}\,7 \quad \dfrac{}{\delta_1}\,2}{\delta_2} \quad \dfrac{\dfrac{}{\gamma \to \delta_1 \vee \delta_2}\,9 \quad \dfrac{}{\gamma}\,3}{\delta_1 \vee \delta_2}}{\dfrac{\delta_1}{\gamma \to \delta_2}\,3}\,1,2}{(\gamma \to \delta_1) \vee (\gamma \to \delta_2)} \qquad \dfrac{\dfrac{\dfrac{}{\delta_1}\,4 \quad \dfrac{\dfrac{}{\delta_2 \to \delta_1}\,8 \quad \dfrac{}{\delta_2}\,5}{\delta_1} \quad \dfrac{\dfrac{}{\gamma \to \delta_1 \vee \delta_2}\,9 \quad \dfrac{}{\gamma}\,6}{\delta_1 \vee \delta_2}}{\dfrac{\delta_1}{\gamma \to \delta_1}\,6}\,4,5}{(\gamma \to \delta_1) \vee (\gamma \to \delta_2)} \qquad \dfrac{}{(\delta_1 \to \delta_2) \vee (\delta_2 \to \delta_1)}\,(\mathbf{LIN})}{\dfrac{(\gamma \to \delta_1) \vee (\gamma \to \delta_2)}{(\gamma \to \delta_1 \vee \delta_2) \to (\gamma \to \delta_1) \vee (\gamma \to \delta_2)}\,9}\,7,8$$

□

As the dual form of the right split, one can consider the *left split rule*:

$$\dfrac{\Gamma_1, \Gamma_2 \Rightarrow \Delta \mid G}{\Gamma_1 \Rightarrow \Delta \mid \Gamma_2 \Rightarrow \Delta \mid G}\ (\mathrm{ls}).$$

This rule corresponds to the inequality $(\gamma_1 \wedge \gamma_2) \to \delta \leq (\gamma_1 \to \delta) \vee (\gamma_2 \to \delta)$, which is equivalent to **LIN** (see Diener and McKubre-Jordens [9, Proposition 2]).

Theorem 2.4. **HLJ**′ + (ls) *proves the generalised De Morgan's law* **GDM**: $((\gamma_1 \wedge \gamma_2) \to \delta) \to (\gamma_1 \to \delta) \vee (\gamma_2 \to \delta)$.

Proof.

$$\dfrac{\dfrac{\overline{\gamma_1 \Rightarrow \gamma_1}\,(\mathrm{Id})}{\gamma_1, \gamma_2 \Rightarrow \gamma_1}\,(\text{iw-L})\,(\text{ie-L}) \qquad \dfrac{\overline{\gamma_2 \Rightarrow \gamma_2}\,(\mathrm{Id})}{\gamma_1, \gamma_2 \Rightarrow \gamma_2}\,(\text{iw-L})}{\gamma_1, \gamma_2 \Rightarrow \gamma_1 \wedge \gamma_2}\,(\wedge\text{-R}) \qquad \dfrac{\overline{\delta \Rightarrow \delta}\,(\mathrm{Id})}{\delta, \gamma_1, \gamma_2 \Rightarrow \delta}\,(\text{iw-L})\,(\text{ie-L})$$

$$\dfrac{\gamma_1, \gamma_2 \Rightarrow \gamma_1 \wedge \gamma_2 \qquad \delta, \gamma_1, \gamma_2 \Rightarrow \delta}{\gamma_1 \wedge \gamma_2 \to \delta, \gamma_1, \gamma_2 \Rightarrow \delta}\,(\to\text{-L})$$

$$\dfrac{\gamma_1 \wedge \gamma_2 \to \delta, \gamma_1, \gamma_2 \Rightarrow \delta}{\gamma_1, \gamma_1 \wedge \gamma_2 \to \delta, \gamma_2, \gamma_1 \wedge \gamma_2 \to \delta \Rightarrow \delta}\,(\text{iw-L})\,(\text{ie-L})$$

$$\dfrac{\gamma_1, \gamma_1 \wedge \gamma_2 \to \delta, \gamma_2, \gamma_1 \wedge \gamma_2 \to \delta \Rightarrow \delta}{\gamma_1, \gamma_1 \wedge \gamma_2 \to \delta \Rightarrow \delta \mid \gamma_2, \gamma_1 \wedge \gamma_2 \to \delta \Rightarrow \delta}\,(\mathrm{ls})$$

$$\dfrac{\gamma_1, \gamma_1 \wedge \gamma_2 \to \delta \Rightarrow \delta \mid \gamma_2, \gamma_1 \wedge \gamma_2 \to \delta \Rightarrow \delta}{\gamma_1 \wedge \gamma_2 \to \delta \Rightarrow \gamma_1 \to \delta \mid \gamma_1 \wedge \gamma_2 \to \delta \Rightarrow \gamma_2 \to \delta}\,(\to\text{-R}'), (\mathrm{ee})$$

$$\dfrac{\gamma_1 \wedge \gamma_2 \to \delta \Rightarrow \gamma_1 \to \delta \mid \gamma_1 \wedge \gamma_2 \to \delta \Rightarrow \gamma_2 \to \delta}{\gamma_1 \wedge \gamma_2 \to \delta \Rightarrow (\gamma_1 \to \delta) \vee (\gamma_2 \to \delta) \mid \gamma_1 \wedge \gamma_2 \to \delta \Rightarrow (\gamma_1 \to \delta) \vee (\gamma_2 \to \delta)}\,(\vee\text{-R}), (\mathrm{ee})$$

$$\dfrac{\dfrac{\gamma_1 \wedge \gamma_2 \to \delta \Rightarrow (\gamma_1 \to \delta) \vee (\gamma_2 \to \delta) \mid \gamma_1 \wedge \gamma_2 \to \delta \Rightarrow (\gamma_1 \to \delta) \vee (\gamma_2 \to \delta)}{\gamma_1 \wedge \gamma_2 \to \delta \Rightarrow (\gamma_1 \to \delta) \vee (\gamma_2 \to \delta)}\,(\mathrm{ec})}{\Rightarrow (\gamma_1 \wedge \gamma_2 \to \delta) \to (\gamma_1 \to \delta) \vee (\gamma_2 \to \delta)}\,(\to\text{-R}')$$

□

Corollary 2.5 (Completeness). *(1) If φ is* **GD***-valid, then* **HLJ**′ + (ls) $\vdash \Rightarrow\varphi$.
(2) If φ is ∀**INT** + **LIN***-valid, then* ∀**HLJ**′ + (∀-R$_{\mathrm{ms}}$) + (ls) $\vdash \Rightarrow\varphi$.

Proof. Trivial. Note that **GDM** implies **LIN** in (∀) **INT**. □

Theorem 2.6 (Soundness). *(1) If* **HLJ**′ + (ls) $\vdash \Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$*, then* $\bigvee_{i=1}^{n} (\bigwedge \Gamma_i \to \bigvee \Delta_i)$ *is* **GD***-valid.*
(2) If ∀**HLJ**′ + (∀-R$_{\mathrm{ms}}$) + (ls) $\vdash \Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$*, the universal closure of* $\bigvee_{i=1}^{n} (\bigwedge \Gamma_i \to \bigvee \Delta_i)$ *is* ∀**GD***-valid.*

Proof. Obvious from the well-known fact that (∀) **NJ** + **LIN** ⊢ **GDM**. □

We have shown that **HLJ**′ + (ls) = **GD** and ∀**INT** + **LIN** ≤ ∀**HLJ**′ + (∀-R$_{\mathrm{ms}}$) + (ls) ≤ ∀**GD**.

Problem 2.7. Decide the exact strength of ∀**HLJ**′ + (∀-R$_{\mathrm{ms}}$) + (ls).

3. Restriction of ∀-right and ∃-left

Recall that the proof of **CD** in ∀**HLJ** + (∀-R$_{\mathrm{ms}}$) + (com) (Fact 1.5) essentially uses the multi-component single-conclusioned ∀-right rule (∀-R$_{\mathrm{ms}}$).

Theorem 3.1 (Completeness). *If φ is* ∀**INT** + **LIN***-valid, then* ∀**HLJ** + (com) $\vdash \Rightarrow\varphi$ *and* ∀**HLJ**′ + (com) $\vdash \Rightarrow\varphi$.

Proof. Obvious from **HLJ** + (com) ⊢ **LIN** (Corollary 1.3) and **HLJ** ⊆ ∀**HLJ** ⊆ ∀**HLJ**′. □

Theorem 3.2 (Soundness). *(1) If* ∀**HLJ** + (com) $\vdash \Gamma_1 \Rightarrow \varphi_1 \mid \cdots \mid \Gamma_n \Rightarrow \varphi_n$*, the universal closure of* $\bigvee_{i=1}^{n} (\bigwedge \Gamma_i \to \varphi_i)$ *is* ∀**INT** + **LIN***-valid.*
(2) If ∀**HLJ**′ + (com) $\vdash \Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$*, the universal closure of* $\bigvee_{i=1}^{n} (\bigwedge \Gamma_i \to \bigvee \Delta_i)$ *is* ∀**INT** + **LIN***-valid.*

Proof. One can verify that all the inference rules are ∀**INT** + **LIN**-valid. □

Corollary 3.3. *The rule* (∀-R$_{\mathrm{ms}}$) *is not derived from* ∀**HLJ** *or* ∀**HLJ**′.

Proof. Otherwise, ∀**HLJ** (∀**HLJ**′) with (com) proves **CD** by Fact 1.5, a contradiction. □

The principal source of **CD** is the rule (∀-R$_{\mathrm{ms}}$); however, the rule (∀-R$_{\mathrm{ms}}$) does not imply **CD** solely.

Theorem 3.4 (Soundness). *(1) If* ∀**HLJ** + (∀-R$_{\mathrm{ms}}$) + (∃-L$_{\mathrm{m}}$) $\vdash \Gamma_1 \Rightarrow \varphi_1 \mid \cdots \mid \Gamma_n \Rightarrow \varphi_n$*, the universal closure of* $\bigwedge \Gamma_i \to \varphi_i$ *is* ∀**INT***-valid for some i.*
(2) If ∀**HLJ**′ + (∀-R$_{\mathrm{ms}}$) + (∃-L$_{\mathrm{m}}$) $\vdash \Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$*, the universal closure of* $\bigwedge \Gamma_i \to \bigvee \Delta_i$ *is* ∀**INT***-valid for some i.*

Proof. We only need to show that ∀**HLJ**′ + (∀-R$_{\mathrm{ms}}$) $\vdash S_1 \mid \cdots \mid S_n$ implies ∀**LJ**′ $\vdash S_i$ for some i. Given a proof figure Π of ∀**HLJ**′ + (∀-R$_{\mathrm{ms}}$) $\vdash S_1 \mid \cdots \mid S_n$, we construct a proof figure Π' of ∀**LJ**′ $\vdash S_i$ for some i by induction on the structure of Π.

Case 1. If Π is $\overline{\varphi \Rightarrow \varphi}$ (Id) or $\overline{\bot \Rightarrow \varphi}$ (Bot), then it is a proof figure of $\forall\mathbf{LJ}'$ at the same time.

Case 2. The last inference rule is one of the external structural rules. For example, if the last rule is the external weakening rule

$$\dfrac{\begin{array}{c}\vdots\\ G\end{array}}{S_1 \mid G}\ (\text{ew})$$

then we have constructed a proof figure of $\forall\mathbf{LJ}' \vdash S_i$ for some $S_i \in G$ by the induction hypothesis. The same applies to external exchange and external contraction.

Case 3. The last inference rule is the cut rule

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma_0 \Rightarrow \Delta_0, \delta \mid G\end{array} \qquad \begin{array}{c}\vdots\\ \delta, \Gamma_1 \Rightarrow \Delta_1 \mid G\end{array}}{\Gamma_0, \Gamma_1 \Rightarrow \Delta_0, \Delta_1 \mid G}\ (\text{cut})$$

then one of the following cases holds by the induction hypothesis.

Case i. There exists a proof figure of $\forall\mathbf{LJ}' \vdash S$ for some $S \in G$ as desired.

Case ii. There exist proof figures Σ_0 and Σ_1 of $\forall\mathbf{LJ}' \vdash \Gamma_0 \Rightarrow \Delta_0, \delta$ and $\forall\mathbf{LJ}' \vdash \delta, \Gamma_1 \Rightarrow \Delta_1$, respectively. The desired proof figure of $\forall\mathbf{LJ}' \vdash \Gamma_0, \Gamma_1 \Rightarrow \Delta_0, \Delta_1$ is obtained as follows:

$$\dfrac{\Sigma_0 \quad \Sigma_1}{\Gamma_0, \Gamma_1 \Rightarrow \Delta_0, \Delta_1}\ (\text{cut}).$$

Case 4. The last inference rule is one of the internal structural rules, the logical rules and the quantifier rules. The same argument works well. For example, if the last rule is the quantifier rule

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \varphi \mid G\end{array}}{\Gamma \Rightarrow \forall x\varphi \mid G}\ (\forall\text{-R}_{\text{ms}})$$

then one of the following cases holds by the induction hypothesis.

Case i. There exists a proof figure of $\forall\mathbf{LJ}' \vdash S$ for some $S \in G$.

Case ii. There exists a proof figure Σ of $\forall\mathbf{LJ}' \vdash \Gamma \Rightarrow \varphi$. We obtain the desired proof figure of $\forall\mathbf{LJ}' \vdash \Gamma \Rightarrow \forall x\varphi$:

$$\dfrac{\Sigma}{\Gamma \Rightarrow \forall x\varphi}\ (\forall\text{-R}).$$

Note that this procedure does not increase the complexity of the proofs (such as the number of symbols, formulae and steps).

□

Remark 3.5. The hypersequent calculi **HLK**, **HLJ**, **HLJ**′ and their predicate versions have the strong soundness property in the sense of Theorem 3.4. On the other hand, the hypersequent calculi with the communication rule (or its variations such as the right split rule) does not possess the strong soundness property. For example, the hypersequent $\varphi \Rightarrow \psi \mid \psi \Rightarrow \varphi$ is provable in such a system, but is neither $\varphi \Rightarrow \psi$ nor $\psi \Rightarrow \varphi$.

4. Future work

A hypersequent calculus of $\forall\mathbf{INT} + \mathbf{CD}$ can be obtained by adding either the multi-component multi-conclusioned $\forall$-right rule

$$\dfrac{\Gamma \Rightarrow \Delta, \varphi \mid G}{\Gamma \Rightarrow \Delta, \forall x\varphi \mid G}\ (\forall\text{-R}_{\text{mm}})$$

or the single-component multi-conclusioned $\forall$-right rule

$$\dfrac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \forall x\varphi}\ (\forall\text{-R}_{\text{sm}}).$$

This however makes no progress on proof theory of $\forall\mathbf{INT} + \mathbf{CD}$ beyond the Gentzen-style proof system. In fact, Maehara's $\forall\mathbf{LJ}'$ with $(\forall\text{-R}_{\text{sm}})$ gives a sequent calculus for $\forall\mathbf{INT} + \mathbf{CD}$ (see e.g. Kashima and Shimura [13]). The system $\forall\mathbf{HLJ}' + (\forall\text{-R}_{\text{mm}}) + (\exists\text{-L}_{\text{m}})$ is merely a hypersequent version of $\forall\mathbf{LJ}' + (\forall\text{-R}_{\text{sm}})$.

Problem 4.1. Find a (well-behaved) hypersequent calculus for $\forall\mathbf{INT} + \mathbf{CD}$, where **CD** is formulated as a *structural rule*. Establish the cut-elimination theorem and the Craig interpolation theorem for such a system.

$$\frac{S \mid G}{\left[x^{\text{global}}/x^{\text{local}}\right] S \mid G} \text{ (share)}$$

where x^{global} does not freely occur in S, G.

$$\frac{S \mid G}{\left[x^{\text{local}}/x^{\text{global}}\right] S \mid G} \text{ (unshare)}$$

where x^{global} does not freely occur in G and x^{local} does not freely occur in S.

Table 4.1.

A hypersequent $(\Gamma_1 \Rightarrow \Delta_1)(\vec{x}) \mid \cdots \mid (\Gamma_n \Rightarrow \Delta_n)(\vec{x})$ with free variables $\vec{x}$ can be translated to closed formulae in two different ways:

$$\forall \vec{x} \bigvee_{i=1}^{n} \left(\bigwedge \Gamma_i(\vec{x}) \to \bigvee \Delta_i(\vec{x}) \right), \quad \bigvee_{i=1}^{n} \forall \vec{x} \left(\bigwedge \Gamma_i(\vec{x}) \to \bigvee \Delta_i(\vec{x}) \right).$$

In the first case, the free variables are considered to be *shared* with all components. In the second case, the free variables are considered not to be shared. In order to manipulate these two translations explicitly, one can introduce two kinds of variables, *global variables* and *local variables*. We immediately observe that the global-to-local conversion rule

$$\frac{\Rightarrow \varphi \mid \Rightarrow \psi\left(x^{\text{global}}\right)}{\Rightarrow \varphi \mid \Rightarrow \psi\left(x^{\text{local}}\right)}$$

corresponds to **CD**: $\forall x\,(\varphi \vee \psi(x)) \to \varphi \vee \forall x \psi(x)$. It might be fruitful to investigate the sharing/unsharing rules (see Table 4.1).

Problem 4.2. Develop hypersequent calculi with the distinction of global and local variables.

Hirai [10, 11] proposed hyper-lambda calculi, models of concurrent computation. Simply typed hyper-lambda calculi correspond to various propositional hypersequent calculi. Notably, the asynchronous hyper-lambda calculus λ-GD corresponds to Avron's system **HLJ** + (com) of **GD**. Through the Curry–Howard correspondence, we can shed light on the computational content of the linearity axiom **LIN**. Naturally, it is expected that the computational content of **CD** can be revealed by considering an appropriate dependently typed hyper-lambda calculus.

Problem 4.3. Develop a dependently typed hyper-lambda calculus corresponding to $\forall$**INT** + **CD**.

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