Extension theorems for linear codes *

Tatsuya Maruta †
Department of Mathematical Sciences
Osaka Prefecture University

1 Introduction

We denote by $\mathbb{F}_q$ the field of q elements. A linear code of length n, dimension k over $\mathbb{F}_q$ is a k-dimensional subspace $\mathcal{C}$ of the vector space $\mathbb{F}_q^n$ of n-tuples over $\mathbb{F}_q$. The vectors in $\mathcal{C}$ are called *codewords*. $\mathcal{C}$ is called an $[n,k,d]_q$ code if it has minimum Hamming weight d. A $k \times n$ matrix G whose rows form a basis of $\mathcal{C}$ is a *generator matrix* of $\mathcal{C}$. We only consider *non-degenerate* linear codes whose generator matrices have no all-zero column. The weight distribution of $\mathcal{C}$ is the list of numbers A_i, which is the number of codewords of $\mathcal{C}$ with weight i. The weight distribution with $(A_0, A_d, ...) = (1, \alpha, ...)$ is expressed as $0^1 d^\alpha \cdots$ in this paper. A q-ary linear code $\mathcal{C}$ is *w-weight (mod q)* if $\mathcal{C}$ has exactly w kinds of weights under modulo q for codewords.

For an $[n,k,d]_q$ code $\mathcal{C}$ with a generator matrix G, $\mathcal{C}$ is called *t-extendable* (to $\mathcal{C}'$) if there exist t vectors $h_1, \ldots, h_t \in \mathbb{F}_q^k$ such that the extended matrix $[G, h_1^{\mathrm{T}}, \cdots, h_t^{\mathrm{T}}]$ generates an $[n+t, k, d+t]_q$ code $\mathcal{C}'$. A 1-extendable code is simply called *extendable* and the extended code $\mathcal{C}'$ is called an *extension* of $\mathcal{C}$. It is well known that every $[n,k,d]_2$ code with d odd is extendable [14]. It is also known that every $[n,1,d]_q$ code is extendable and that an $[n,2,d]_q$ code is not extendable if and only if $n = s(q+1)$, $d = sq$ for some $s \in \mathbb{N}$ [15]. So, we assume that $k \geq 3$.

The "optimal linear codes problem" is a fundamental problem in coding theory to find $n_q(k,d)$, the minimum length n for which an $[n,k,d]_q$ code exists for given q, k, d [1, 3]. See [22] for the updated tables of $n_q(k,d)$ for some small q and k. Extension theorems could be applied to the optimal linear codes problem to find new codes from old ones or to prove the nonexistence of linear codes with certain parameters. The aim of this paper is to give a survey of recent progress on the extendability of linear codes and their applications.

2 Extension theorems

The most well-known extension theorem is the following for 2-weight (mod q) codes.

*This paper is a preliminary version and the final version will be submitted to elsewhere.
†Corresponding author. E-mail address: maruta@mi.s.osakafu-u.ac.jp

Theorem 2.1 ([2, 4])**.** *Every $[n,k,d]_q$ code with gcd$(d,q)=1$ satisfying $A_i=0$ for all $i \not\equiv 0, d \pmod q$ is extendable.*

For the case $\gcd(d,q)=2$, the following is known.

Theorem 2.2 ([26])**.** *For $q=2^h$ with $h \geq 3$, every $[n,k,d]_q$ code with gcd$(d,q)=2$ satisfying $A_i=0$ for all $i \not\equiv 0, d \pmod q$ is 2-extendable.*

For an $[n,k,d]_q$ code $\mathcal{C}$, let

$$\Phi_0 = \frac{1}{q-1}\sum_{q|i,i>0} A_i, \quad \Phi_1 = \frac{1}{q-1}\sum_{i \not\equiv 0,d \pmod q} A_i,$$

where the notation $q|i$ means that q is a divisor of i. The pair of integers (Φ_0,Φ_1) is called the *diversity* of $\mathcal{C}$ ([17, 19]). Theorem 2.1 shows that $\mathcal{C}$ is extendable if $\Phi_1=0$. This condition can be weakened as follows.

Theorem 2.3 ([27])**.** *Every $[n,k,d]_q$ code with gcd$(d,q)=1$ is extendable if $\Phi_1 < q^{k-2}$.*

It often happens that an optimal $[n,k,d]_q$ code is necessarily extendable when $d \equiv -1 \pmod q$. For example, $[69-e,5,45-e]_3$ codes are optimal for $e=0,1,2$ and every $[68,5,44]_3$ code is extendable [24]. But it is not known whether every $[67,5,43]_3$ code is extendable or not. The famous Golay $[11,6,5]_3$ code is extendable by the above theorem.

As for 3-weight and 4-weight (mod q) codes, the following theorems are known.

Theorem 2.4 ([17, 29])**.** *Let $\mathcal{C}$ be an $[n,k,d]_q$ code with $q \geq 5$, $d \equiv -2 \pmod q$, satisfying $A_i=0$ for all $i \not\equiv 0,-1,-2 \pmod q$. Then $\mathcal{C}$ is extendable.*

Theorem 2.5 ([26])**.** *For $q=2^h$ with $h \geq 3$, every $[n,k,d]_q$ code with d odd satisfying $A_i=0$ for all $i \not\equiv 0,d \pmod{q/2}$ is extendable.*

As another application, extension theorems are often applied to prove the non-existence of codes with certain parameters by showing that they are extendable to a non-existing code. Actually, the motivation to prove Theorem 2.4 for $q=2^h$ with $h \geq 3$ was to prove the non-existence of $[328,4,286]_8$, $[474,4,414]_8$, $[803,4,702]_8$ and $[858,4,750]_8$ codes all of which attain the Griesmer bound [7]. See also [13] for the extendability of linear codes attaining the Griesmer bound.

We denote by θ_j the number of points in PG(j,q), i.e., $\theta_j := (q^{j+1}-1)/(q-1) = q^j+q^{j-1}+\cdots+q+1$. We set $\theta_0=1$ and $\theta_j=0$ for $j<0$ for convenience. For ternary linear codes ($q=3$), the following is known.

Theorem 2.6 ([19], [23], [27])**.** *Let $\mathcal{C}$ be an $[n,k,d]_3$ code with gcd$(3,d)=1$. Then $\mathcal{C}$ is extendable if one of the following conditions holds:*

(a) $\Phi_0=\theta_{k-3}$, (b) $\Phi_1 < 3^{k-2}$, (c) $\Phi_0+\Phi_1 < \theta_{k-2}+3^{k-2}$,
(d) $\Phi_0+\Phi_1 > \theta_{k-2}+3^{k-2}+2\cdot 3^{k-3}$, (e) $2\Phi_0+\Phi_1 \leq 2\theta_{k-2}$.

Moreover, $\mathcal{C}$ is 2-extendable if $(\Phi_0,\Phi_1) \in \{(\theta_{k-2},0),(\theta_{k-3},2\cdot 3^{k-2}),(\theta_{k-2}+3^{k-2},3^{k-2})\}$ when $d \equiv 1 \pmod 3$.

The condition (c) of Theorem 2.6 and the case $(\Phi_0, \Phi_1) = (\theta_{k-2} + 3^{k-2}, 3^{k-2})$ can be generalized as follows.

Theorem 2.7 ([16]). *Every $[n, k, d]_p$ code with gcd$(d, p) = 1$, p prime, is extendable if $\Phi_0 + \Phi_1 < \theta_{k-2} + p^{k-2}$.*

Theorem 2.8 ([26]). *Let $\mathcal{C}$ be an $[n, k, d]_q$ code with gcd$(3, d) = 1$. Then $\mathcal{C}$ is extendable if $(\Phi_0, \Phi_1) = (\theta_{k-1} - 2q^{k-2}, q^{k-2})$.*

Let $\mathcal{C}$ be an $[n, k, d]_q$ code with $d \not\equiv 0 \pmod q$. The *weight spectrum modulo q* (q-**WS**) is defined as the q-tuple $(w_0, w_1, \ldots, w_{q-1})$ with

$$w_0 = \Phi_0, \quad w_j = \frac{1}{q-1} \sum_{i \equiv j \pmod q} A_i \text{ for } j = 1, 2, \ldots, q-1$$

to investigate the extendability of q-ary linear codes for $q \geq 4$ ([10]). Note that $\Phi_1 = \sum_{j \neq 0, d'} w_j$, where d' is an integer with $d \equiv d' \pmod q$, $1 \leq d' \leq q-1$. For quaternary linear codes ($q = 4$), the following results are known.

Theorem 2.9 ([12, 16, 18, 25]). *Let $\mathcal{C}$ be an $[n, k, d]_4$ code with $k \geq 3$, d odd. Then $\mathcal{C}$ is extendable if one of the following conditions holds:*

(a) $w_0 = \theta_{k-4}$, (b) $\Phi_1 = w_2$, (c) $w_2 = 0$,
(d) $w_0 + w_2 < \theta_{k-2} + 4^{k-2}$, (e) $w_0 + w_2 > \theta_{k-2} + 2 \cdot 4^{k-2} - 4$.

Theorem 2.10 ([12]). *Let $\mathcal{C}$ be an $[n, k, d]_4$ code with $\theta_{k-4} < w_0 \leq \theta_{k-3}$, $k \geq 4$, $d \not\equiv 0 \pmod 4$. Then, $w_0 = \theta_{k-3}$ and $\mathcal{C}$ is extendable. Moreover, $\mathcal{C}$ is 2-extendable when d is even.*

Theorem 2.11 ([28]). *Let $\mathcal{C}$ be an $[n, k, d]_4$ code with $\Phi_1 = 0$, $k \geq 3$, $d \equiv 2 \pmod 4$. Then $\mathcal{C}$ is 2-extendable if $w_0 < \theta_{k-2} + 2 \cdot 4^{k-3}$ or $w_0 > \theta_{k-2} + 2 \cdot 4^{k-2} - 4$.*

Theorem 2.12 ([12]). *Let $\mathcal{C}$ be an $[n, k, d]_4$ code with $\Phi_1 = 4^{k-1}$, $k \geq 3$, $d \equiv 2 \pmod 4$. Then $\mathcal{C}$ is 2-extendable if $w_0 < \theta_{k-3} + 2 \cdot 4^{k-4}$ or $w_0 > \theta_{k-3} + 2 \cdot 4^{k-3} - 4$.*

Theorem 2.13 ([12]). *Let $\mathcal{C}$ be an $[n, k, d]_4$ code with $k \geq 3$, $d \equiv 2 \pmod 4$. Then $\mathcal{C}$ is 2-extendable if one of the following conditions holds:*

(a) $w_1 = 0$ *and* $w_3 > 0$, (b) $w_1 > 0$ *and* $w_3 = 0$.

Theorem 2.14 ([12]). *Let $\mathcal{C}$ be an $[n, k, d]_4$ code with diversity (Φ_0, Φ_1), $k \geq 4$, d odd. If $\mathcal{C}$ is not extendable, then $\Phi_0 \geq \theta_{k-3} + 2$.*

Theorem 2.15 ([12, 10]). *Let $\mathcal{C}$ be an $[n, k, d]_4$ code with $k \geq 3$, $d \equiv 1 \pmod 4$. Then $\mathcal{C}$ is 3-extendable if one of the following conditions holds:*

(a) $w_0 = \theta_{k-4}$, (b) $w_0 = \theta_{k-3}$ *and* $w_2 = 3 \cdot 4^{k-2}$, (c) $w_j = 0$ *for* $j = 2$ *or* 3,
(d) $(w_0, w_1, w_2) = (\theta_{k-3}, 6 \cdot 4^{k-3}, 4^{k-2})$.

Theorem 2.16 ([10]). *Let $\mathcal{C}$ be an $[n, k, d]_4$ code with $w_0 + w_2 = \theta_{k-2}$, $k \geq 3$, $d \equiv 1 \pmod 4$. Then $\mathcal{C}$ is 3-extendable if either $w_1 - w_0 < 4^{k-2} + 4 - \theta_{k-3}$ or $w_1 - w_0 > 10 \cdot 4^{k-3} - \theta_{k-3}$.*

Recently, Kanda [8, 9] found a new type of extension theorems for ternary and quaternary linear codes to prove the non-existence of $[512, 6, 340]_3$, $[383, 5, 286]_4$ and $[447, 5, 334]_4$ codes.

Theorem 2.17 ([8])**.** *Let $\mathcal{C}$ be an $[n, k, d]_3$ code with gcd$(d, 3) = 1$ satisfying that $A_i = 0$ for all $i \not\equiv 0, -1, -2 \pmod 9$. Then $\mathcal{C}$ is extendable.*

Theorem 2.18 ([9])**.** *Let $\mathcal{C}$ be an $[n, k, d]_4$ code with $k \geq 3$, $d \equiv -2 \pmod{16}$ satisfying $A_i = 0$ for all $i \not\equiv 0, -2 \pmod{16}$. Then $\mathcal{C}$ is extendable.*

For this type of extension theorems for quaternary linear codes, we give some new results from [11].

Theorem 2.19. *Let $\mathcal{C}$ be an $[n, k, d]_4$ code with $k \geq 3$, $d \not\equiv 0 \pmod{16}$ satisfying $A_i = 0$ for all $i \not\equiv 0, 1 + 4s, 2 + 4t, 3 + 4(s + t + 1) \pmod{16}$ with $s, t \in \{0, 1, 2, 3\}$. Then, $\mathcal{C}$ is extendable if one of the following conditions holds:*

(a) $d \equiv 1 + 4s \pmod{16}$,

(b) $d \equiv 2 + 4t \pmod{16}$ *and* $(w_0, w_1) \neq (\theta_{k-4}, 9 \cdot 4^{k-3})$, $(\theta_{k-4} + 3 \cdot 4^{k-3}, 2 \cdot 4^{k-3})$,

(c) $d \equiv 3 + 4(s + t + 1) \pmod{16}$ *and* $w_0 \neq \theta_{k-3} + 4^{k-3}$.

Theorem 2.20. *Let $\mathcal{C}$ be an $[n, k, d]_4$ code with $k \geq 3$, $d \not\equiv 0 \pmod{16}$ satisfying $A_i = 0$ for all $i \not\equiv 0, 1 + 4s, 2 + 4t, 3 + 4(s + t + 3) \pmod{16}$ with $s, t \in \{0, 1, 2, 3\}$. Then, $\mathcal{C}$ is extendable if one of the following conditions holds:*

(a) $d \equiv 1 + 4s \pmod{16}$ *and* $(w_0, w_1) \neq (\theta_{k-2} + 4^{k-3}, 11 \cdot 4^{k-3})$,

(b) $d \equiv 2 + 4t \pmod{16}$ *and* $(w_0, w_1) \neq (\theta_{k-4} + 3 \cdot 4^{k-3}, 2 \cdot 4^{k-3})$,

(c) $d \equiv 3 + 4(s + t + 3) \pmod{16}$ *and* $(w_0, w_1) \neq (\theta_{k-2} + 4^{k-3}, 3 \cdot 4^{k-3})$.

Theorem 2.21. *Let $\mathcal{C}$ be an $[n, k, d]_4$ code with $k \geq 3$, $d \not\equiv 0 \pmod{16}$ satisfying $A_i = 0$ for all $i \not\equiv 0, 1 + 4s, 2 + 4t, 3 + 4(s - t) \pmod{16}$ with $s, t \in \{0, 1, 2, 3\}$. Then, $\mathcal{C}$ is extendable if one of the following conditions holds:*

(a) $d \equiv 1 + 4s \pmod{16}$ *and* $w_0 \neq \theta_{k-4} + 2 \cdot 4^{k-3}$,

(b) $d \equiv 2 + 4t \pmod{16}$ *and* $(w_0, w_1) \neq (\theta_{k-4}, 3 \cdot 4^{k-3})$, $(\theta_{k-4} + 3 \cdot 4^{k-3}, 6 \cdot 4^{k-3})$,

(c) $d \equiv 3 + 4(s - t) \pmod{16}$.

Theorem 2.22. *Let $\mathcal{C}$ be an $[n, k, d]_4$ code with $k \geq 3$, $d \not\equiv 0 \pmod{16}$ satisfying $A_i = 0$ for all $i \not\equiv 0, 1 + 4s, 2 + 4t, 3 + 4(s - t + 2) \pmod{16}$ with $s, t \in \{0, 1, 2, 3\}$. Then, $\mathcal{C}$ is extendable if one of the following conditions holds:*

(a) $d \equiv 1 + 4s \pmod{16}$ *and* $(w_0, w_1) \neq (\theta_{k-3} + 5 \cdot 4^{k-3}, 9 \cdot 4^{k-3})$,

(b) $d \equiv 2 + 4t \pmod{16}$ *and* $(w_0, w_1) \neq (\theta_{k-3} + 2 \cdot 4^{k-3}, 6 \cdot 4^{k-3})$,

(c) $d \equiv 3 + 4(s - t + 2) \pmod{16}$ *and* $(w_0, w_1) \neq (\theta_{k-3} + 5 \cdot 4^{k-3}, 4^{k-3})$.

Setting $(n, m) = (3, 3)$ in Theorem 2.19 and $(n, m) = (0, 3)$ in Theorem 2.22, we get the following corollaries.

Corollary 2.23. *Let $\mathcal{C}$ be an $[n, k, d]_4$ code with $k \geq 3$, $d \not\equiv 0 \pmod{16}$ satisfying $A_i = 0$ for all $i \not\equiv 0, -1, -2, -3 \pmod{16}$. Then, $\mathcal{C}$ is extendable if either*

(a) $d \equiv -3 \pmod{16}$,

(b) $d \equiv -2 \pmod{16}$ *and* $(w_0, w_1) \neq (\theta_{k-4}, 9 \cdot 4^{k-3})$, $(\theta_{k-4} + 3 \cdot 4^{k-3}, 2 \cdot 4^{k-3})$, *or*

(c) $d \equiv -1 \pmod{16}$ *and* $w_0 \neq \theta_{k-3} + 4^{k-3}$.

Corollary 2.24. *Let $\mathcal{C}$ be an $[n, k, d]_4$ code with $k \geq 3$, $d \not\equiv 0 \pmod{16}$ satisfying $A_i = 0$ for all $i \not\equiv 1, 0, -1, -2 \pmod{16}$. Then, $\mathcal{C}$ is extendable if either*

(a) $d \equiv -1 \pmod{16}$ *and* $(w_0, w_1) \neq (\theta_{k-3} + 5 \cdot 4^{k-3}, 4^{k-3})$,

(b) $d \equiv -2 \pmod{16}$ *and* $(w_0, w_1) \neq (\theta_{k-3} + 2 \cdot 4^{k-3}, 6 \cdot 4^{k-3})$, *or*

(c) $d \equiv 1 \pmod{16}$ *and* $(w_0, w_1) \neq (\theta_{k-3} + 5 \cdot 4^{k-3}, 9 \cdot 4^{k-3})$.

Example 2.1. Let $\mathcal{C}_1$ be the $[18, 3, 13]_4$ code with generator matrix

$$G_1 = \begin{bmatrix} 1\,0\,0\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,0\,0\,0 \\ 0\,1\,0\,1\,1\,1\,2\,2\,2\,3\,3\,3\,0\,0\,0\,1\,1\,1 \\ 0\,0\,1\,2\,3\,0\,2\,3\,0\,1\,3\,0\,1\,2\,3\,1\,2\,3 \end{bmatrix}.$$

Then, $\mathcal{C}_1$ has weight distribution $0^1 13^{27} 14^{27} 15^9$ with 4-WS $(0, 9, 9, 3)$. $\mathcal{C}_1$ is extendable by Corollary 2.23. Actually, we get a $[19, 3, 14]_4$ code $\mathcal{C}_1'$ with weight distribution $0^1 14^{36} 15^{24} 16^3$ by adding the column $(1, 2, 1)^{\mathrm{T}}$ to G_1. $\mathcal{C}_1'$ is 2-extendable by Theorem 2.13. Hence, $\mathcal{C}_1$ is 3-extendable.

Example 2.2. Let $\mathcal{C}_2$ be the $[20, 3, 14]_4$ code with generator matrix

$$G_2 = \begin{bmatrix} 1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,0\,0\,0\,0\,0\,0 \\ 1\,1\,1\,2\,2\,2\,3\,3\,3\,3\,0\,0\,0\,0\,1\,1\,0\,0\,1\,1 \\ 2\,3\,0\,2\,3\,0\,0\,1\,2\,3\,0\,1\,2\,3\,0\,0\,1\,1\,3\,3 \end{bmatrix}.$$

Then, $\mathcal{C}_2$ has weight distribution $0^1 14^{24} 15^{12} 16^{15} 17^{12}$ with 4-WS $(5, 4, 8, 4)$, which is extendable by Corollary 2.24. Actually, we get a $[22, 3, 16]_4$ code with weight distribution $0^1 16^{39} 18^{24}$ by adding the columns $(1, 1, 1)^{\mathrm{T}}$ and $(1, 2, 1)^{\mathrm{T}}$ to G_2.

Kanda [9] proved the nonexistence of a $[383, 5, 286]_4$ code. As another application of Corollary 2.23, we can prove the following, which is a new result.

Theorem 2.25. *There exists no $[382, 5, 285]_4$ code.*

3 Geometric method

For an integer $k \geq 3$, let $\Sigma = \mathrm{PG}(k-1, q)$ be the projective geometry of dimension $k-1$ over $\mathbb{F}_q$. A *j-flat* is a projective subspace of dimension j in Σ. The 0-flats, 1-flats, 2-flats, 3-flats, $(k-3)$-flats and $(k-2)$-flats in Σ are called *points, lines, planes, solids, secundum* and *hyperplanes*, respectively. We refer to [5] and [6] for geometric terminologies. For $j < 0$, a j-flat is the empty set as the usual convention. In this section, we give the geometric method to investigate linear codes over $\mathbb{F}_q$ through projective geometry.

Let $\mathcal{C}$ be an $[n, k, d]_q$ code with a generator matrix G and let g_i be the i-th row of G $(1 \leq i \leq k)$. For $P = \mathbf{P}(p_1, \ldots, p_k) \in \Sigma$, *the weight of P with respect to G*, denoted by $w_G(P)$, is defined as

$$w_G(P) = \left| \left\{ j \;\middle|\; \sum_{i=1}^{k} p_i g_{ij} \neq 0 \right\} \right| = wt(\sum_{i=1}^{k} p_i g_i),$$

which is the weight of a codeword. A hyperplane H of Σ is defined by a non-zero vector $h = (h_1, \ldots, h_k) \in \mathbb{F}_q^k$ as $H = \{\mathbf{P}(p_1, \ldots, p_k) \in \Sigma \mid h_1 p_1 + \cdots + h_k p_k = 0\}$, where h is called the *defining vector of H*. Let

$$F_d = \{P \in \Sigma \mid w_G(P) = d\}.$$

The following lemma is well-known, see [20, 21].

Lemma 3.1. *An $[n, k, d]_q$ code $\mathcal{C}$ is extendable if and only if there exists a hyperplane H of Σ such that $F_d \cap H = \emptyset$. Moreover, the extended matrix of G by adding the defining vector of H as a column generates an extension of $\mathcal{C}$.*

Now, let $\mathcal{C}$ be an $[n, k, d]_q$ code with q-WS $(w_0, w_1, \ldots, w_{q-1})$ and assume $d \not\equiv 0 \pmod q$. Let

$$\begin{aligned} M_i &= \{P \in \Sigma \mid w_G(P) \equiv i \pmod q\}, \\ M &= \Sigma \setminus M_d. \end{aligned}$$

Then we have $w_i = |M_i|$ for $0 \leq i \leq q-1$. Note that $F_d \cap M_0 = \emptyset$ and $F_d \subset M_d$. As a corollary of Lemma 3.1, we get the following.

Corollary 3.2. *$\mathcal{C}$ is extendable if M contains a hyperplane of Σ.*

Most of extension theorems can be proved applying Corollary 3.2. For example, if an $[n, k, d]_q$ code $\mathcal{C}$ satisfies the condition of Theorem 2.1, one can take a hyperplane H contained in M as $H = M_0$. But finding such a hyperplane is not so easy in general.

Acknowledgment

This work was partially supported by JSPS KAKENHI Grant Number 20K03722.

References

[1] R. Hill, Optimal linear codes, in: Mitchell C. (ed.) Cryptography and Coding II, pp. 75–104. Oxford Univ. Press, Oxford, 1992.

[2] R. Hill, An extension theorem for linear codes, Des. Codes Cryptogr. **17** (1999) 151–157.

[3] R. Hill, E. Kolev, A survey of recent results on optimal linear codes, in: Holroyd F.C. et al (ed.) Combinatorial Designs and their Applications, pp.127–152. Chapman and Hall/CRC Press Research Notes in Mathematics CRC Press. Boca Raton, 1999.

[4] R. Hill, P. Lizak, Extensions of linear codes, Proc. IEEE Int. Symposium on Inform. Theory, Whistler, Canada (1995) pp. 345.

[5] J.W.P. Hirschfeld, Finite Projective Spaces of Three Dimensions, Clarendon Press, Oxford, 1985.

[6] J.W.P. Hirschfeld, Projective Geometries over Finite Fields 2nd ed., Clarendon Press, Oxford, 1998.

[7] R. Kanazawa, T. Maruta, On optimal linear codes over $\mathbb{F}_8$, Electronic J. Combin. **18** (2011), #P34, 27pp.

[8] H. Kanda, A new extension theorem for ternary linear codes and its application, *Finite Fields Appl.* **2** (2020) 101711.

[9] H. Kanda, The non-existence of $[383, 5, 286]$ and $[447, 5, 334]$ quaternary linear codes, Serdica Mathematical Journal, to appear.

[10] H. Kanda, T. Maruta, On the 3-extendability of quaternary linear codes, *Finite Fields Appl.* **52** (2018) 126–136.

[11] H. Kanda, M. Shirouzu, K. Maehara, T. Maruta, On the extendability of quaternary linear codes with four weights modulo 16, in preparation.

[12] H. Kanda, T. Tanaka, T. Maruta, On the l-extendability of quaternary linear codes, *Finite Fields Appl.* **35** (2015) 159–171.

[13] I.N. Landjev, A. Rousseva, L. Storme, On the extendability of quasidivisible Griesmer arcs, Des. Codes Cryptogr. **79** (2016) 535–547.

[14] F.J. MacWilliams, N.J.A. Sloane, The Theory of Error-Correcting Codes, North-Holland, 1977.

[15] T. Maruta, On the extendability of linear codes, Finite Fields Appl. **7** (2001) 350–354.

[16] T. Maruta, Extendability of linear codes over GF(q) with minimum distance d, $\gcd(d, q) = 1$, Discrete Math. **266** (2003) 377–385.

[17] T. Maruta, A new extension theorem for linear codes, Finite Fields Appl. **10** (2004) 674–685.

[18] T. Maruta, Extendability of quaternary linear codes, Discrete Math. **293** (2005) 195–203.

[19] T. Maruta, Extendability of ternary linear codes, Des. Codes Cryptogr. **35** (2005) 175–190.

[20] T. Maruta, Extendability of linear codes over $\mathbb{F}_q$, Proc. 11th International Workshop on Algebraic and Combinatorial Coding Theory (ACCT), Pamporovo, Bulgaria, 2008, 203–209.

[21] T. Maruta, Extension theorems for linear codes over finite fields, J. Geometry **101** (2011) 173–183.

[22] T. Maruta, Griesmer bound for linear codes over finite fields, `http://mars39.lomo.jp/opu/griesmer.htm`

[23] T. Maruta, K. Okamoto, Some improvements to the extendability of ternary linear codes, Finite Fields Appl. **13** (2007) 259–280.

[24] T. Maruta, Y. Oya, On the minimum length of ternary linear codes, *Des. Codes Cryptogr.* **68** (2013) 407–425.

[25] T. Maruta, M. Takeda, K. Kawakami, New sufficient conditions for the extendability of quaternary linear codes, Finite Fields Appl. **14** (2008) 615–634.

[26] T. Maruta, T. Tanaka, H. Kanda, Some generalizations of extension theorems for linear codes over finite fields, Australas. J. Combin. **60** (2014) 150–157.

[27] T. Maruta, Y. Yoshida, A generalized extension theorem for linear codes, Des. Codes Cryptogr. **62** (2012) 121–130.

[28] T. Tanaka, T. Maruta, A characterization of some odd sets in projective space of order 4 and the extendability of quaternary linear codes, J. Geometry **105** (2014) 79–86.

[29] Y. Yoshida, T. Maruta, An extension theorem for $[n, k, d]_q$ codes with $\gcd(d, q) = 2$, Australas. J. Combin. **48** (2010) 117–131.