On group algebras of R. Thompson's group F

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We introduced the amenability problem on R. Thompson's group F and our approach to solve it last year. In this note, we will proceed with the research. We first see a brief introduction for the amenability problem for Thompson's group F and our approach to solving the problem. After that, we introduce Guba's recent results ([7] and [8]) which include a important information for our approach to solve the problem. Finally, we consider what they suggest for our study.

1 Amenable groups

In this note, we will consider the amenability problem for R. Thompson's group F. We believe that F is non-amenable and have been continuing to show it by knowing a property of the group algebra KF of F over a field K. We begin with the definition of an amenable group.

An amenable group is a group whose subsets admit an invariant finitely additive probability measure. Originally defined von Neumann in 1929. For a set X, we denote the power set of X by $\mathcal{P}(X)$:

$$\mathcal{P}(X) = \{S \mid S \subseteq X\}$$

Definition 1.1. (Amenable) A group G is amenable if for $\mathcal{P}(G)$, there exists $\mu : P(G) \longrightarrow [0,1]$ such that

1. $\mu(G) = 1$,
2. if S and T are disjoint subsets of G, then $\mu(S \cup T) = \mu(S) + \mu(T)$,
3. if $S \in P(G)$ and $g \in G$, then $\mu(gS) = \mu(S)$.

Many equivalent condition for amenability are known. The next is known as Fϕlner condition:

Remark 1.2. A group G is amenable if and only if for any ε and for any finite subset S of G, there exists a finite subset of G such that

$$|sE\Delta E| < \varepsilon |E|$$

for all s in S, where Δ is the symmetric difference of sE and E.

The class of amenable groups contains all finite groups, all abelian groups, and more generally, all solvable groups. It is closed under the operations of taking subgroups, taking quotients, and taking extensions, and taking inductive limits.

On the other hand, if G has a non-abelian free subgroup, then G is not amenable. We can naturally ask whether the converse is true or not. It is known as von Neumann conjecture since Day [4] attributed it to von Neumann and the conjecture was disproved in 1980. Ol'shanskiĭ [11] found the first counterexample, and later Ol'shanskiĭ and Sapir [12] did the first finitely presented example.

Now, R. Thompson's group F does not have any non-abelian free subgroup, but it is not yet known whether F is amenable or not, which is the famous amenability problem for F.

2 R. Thompson's group F and its group algebra

Originally Thompson's groups $F \subseteq T \subseteq V$ were defined by Richard Thompson in 1965 to construct finitely-presented groups with unsolvable word problems [9]. All of F, T and V are finitely generated non-noetherian groups. T and V are simple groups but F is not so. We refer the reader to Cannon, Floyd and Parry [3] for a more detailed discussion of the Thompson's groups F, T and V.

In this note, our main interest is in Thompson's group F. Thompson's group F is defined as a group of piecewise linear maps of the interval $[0, 1]$ as follows:

Definition 2.1. Thompson's group F is the group (under composition) of those homeomorphisms of the interval $[0, 1]$, which satisfy the following conditions:

1. they are piecewise linear and orientation-preserving,
2. in the pieces where the maps are linear, the slope is always a power of 2, and
3. the breakpoints are dyadic, i.e., they belong to the set $D \times D$, where $D = [0, 1] \cap \mathbb{Z}[\frac{1}{2}]$.

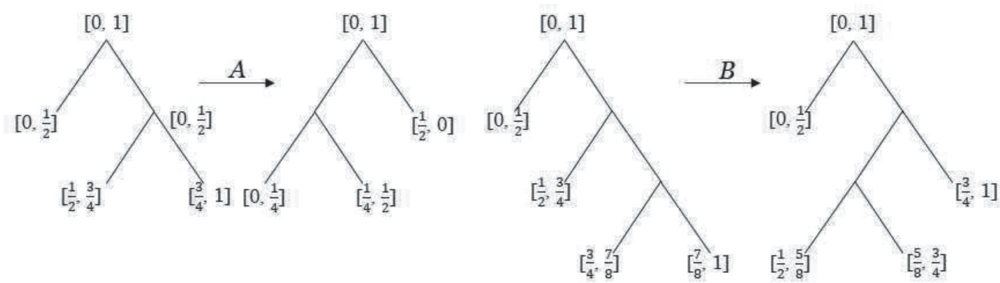
Example 2.2. The following two functions A and B are elements in Thompson's group F.

$$A(x) = \begin{cases} \frac{x}{2} & 0 \leq x \leq \frac{1}{2} \\ x - \frac{1}{4} & \frac{1}{2} \leq x \leq \frac{3}{4} \\ 2x - 1 & \frac{3}{4} \leq x \leq 1 \end{cases} \qquad B(x) = \begin{cases} x & 0 \leq x \leq \frac{1}{2} \\ \frac{x}{2} + \frac{1}{4} & \frac{1}{2} \leq x \leq \frac{3}{4} \\ x - \frac{1}{8} & \frac{3}{4} \leq x \leq \frac{7}{8} \\ 2x - 1 & \frac{7}{8} \leq x \leq 1 \end{cases}$$

An element of F can be represented by a tree pair diagram which is a pair of binary trees with the same number of leaves.

Formally, a tree pair diagram is an ordered pair (R, S) of τ-trees such that R and S have the same number of leaves, where τ is defined as follows. The vertices of τ are the standard dyadic intervals in $[0, 1]$. An edge of τ is pair (I, J) of standard dyadic intervals I and J such that either I is the left half of J, in which case (I, J) is a left edge, or I is the right half of J, in which case (I, J) is a right edge.

For example, A and B described above are as follows:



We will skip the detail about tree diagram. What we should here note is that some other representations for F are also known and an approach to see its properties is often depend on the representation.

In our approach, we use the following presentation:

$$F = \langle x_0, x_1, x_2, \cdots x_n, \cdots , \mid x_i^{-1} x_j x_i = x_{j+1}, \text{ for } i < j \rangle.$$

For the above presentation, every non-trivial element of F can be expressed in unique normal form

$$x_0^{\alpha_0} x_1^{\alpha_1} \cdots x_n^{\alpha_n} x_n^{-\beta_n} \cdots x_1^{-\beta_1} x_0^{-\beta_0},$$

where n, $\alpha_0, \ldots, \alpha_n, \beta_0, \cdots, \beta_n$ are non-negative integers such that

1. exactly one of α_n and β_n is non-zero and
2. if $\alpha_k > 0$ and $\beta_k > 0$ for some integer k with $0 \leq k < n$, then $\alpha_{k+1} > 0$ or $\beta_{k+1} > 0$.

In the above form, although α_i and β_i need not to be non-zero, actually, we often omit zero parts when we write a concrete element. If $W \in F$ and the normal form is $W = x_0^{\alpha_0} x_1^{\alpha_1} \cdots x_n^{\alpha_n} x_n^{-\beta_n} \cdots x_1^{-\beta_1} x_0^{-\beta_0}$, then we call the sum $\sum_{i=0}^{n}(\beta_i + \alpha_i)$ the degree of W, and it is denoted by $deg(W)$.

For example, for $W = x_0^3 x_7^2 x_4^{-1} x_6^{-1} x_3^2 x_0^{-4} \in F$, we have

$$\begin{aligned} W &= x_0^3 x_7^2 x_4^{-1} x_6^{-1} x_3^2 x_0^{-4} &&= x_0^3 x_3^2 x_9^2 x_6^{-1} x_8^{-1} x_0^{-4} &&= x_0^3 x_3^2 x_9^2 x_9^{-1} x_6^{-1} x_0^{-4} \\ &= x_0^3 x_3^2 x_9 x_6^{-1} x_0^{-4} &&= x_0 x_1^2 x_7 x_4^{-1} x_0^{-2} \ ; && deg(W) = 7. \end{aligned}$$

As is mentioned above, F is finitely generated and finitely presented. In addition, it is also known that F is torsion free, orderable and has no non-abelian free subgroups. On the other hand, unlike T and V, F has no non-abelian free subgroups, which leads to a well known question of whether F is amenable or not.

Our direction to know the answer of this question is to study a property of group algebras of Thompson's group F. The next beautiful theorem make it possible for us to do it.

Before seeing the theorem, recall that a domain (i.e., it is a ring with no nonzero divisors) R is a (right) Ore domain provided that for each $A, B \in R$ with $B \neq 0$, there exist $X, Y \in R$ with $Y \neq 0$ such that $AY = BX$. As is well known, if R is a (right) Ore domain then R has the (right) classical ring of quotients which is a division ring (a noncommutative field).

Theorem 2.3. (Tamari [13], 1954, Kielak [2], 2019) *Let G be a group and K a field. Suppose that the group algebra KG is a domain. Then G is amenable if and only if KG is an Ore domain.*

In the above theorem, the necessity for amenability has been well known for a long time by V. S. Guba's manuscript [6] which is a list of open problem about Thompson's groups. We should here note that this implication is true, which is seen by using the property of amenability in Remark 1.2, even if KG is not domain. In [6], Guba asked whether the converse was true or not. Theorem 2.3 gave a positive answer for his question. Since Thompson's group F is orderable, the group algebra KF is a domain for any field K, and so Theorem 2.3 translates amenability of F into the Ore condition of KF.

In Theorem 2.3, if R is the group algebra KG of a group G over a field K, then any element X in KG is expressed as the liner combination $X = \sum_{x\in S_X} \alpha_x x$, where $\alpha_x \in K \setminus \{0\}$ and $S_X = Supp(X)$ is the set of supports of X. Therefore KG satisfies the Ore condition if and only if for each finite subsets A and B in G with $B \neq \emptyset$, and for each α_a and β_b in K $(a \in A, b \in B)$, there exist finite subsets X, Y in G with $Y \neq \emptyset$, γ_x and δ_y in in K $(x \in X, y \in Y)$ such that

$$\sum_{a\in A}\sum_{y\in Y} \alpha_a \delta_y a y = \sum_{b\in B}\sum_{x\in X} \beta_b \gamma_x b x.$$

By our graph theoretical approach (see [1], [10]), we can then get the following statement:

Theorem 2.4. *Let KG be the group algebra of a group G over a field K, and $S_1 = \{s_{11}, \ldots s_{1m}\}$ and $S_2 = \{s_{21}, \ldots, s_{2n}\}$ be non-empty finite subsets in G. If KG satisfies the Ore condition, then S_1 and S_2 satisfies the following property* (P)*:*

(P) *there exist elements $w_1, u_1 \ldots, w_\ell, u_\ell$ in $S_1 \cup S_2$ such that $w_1 u_1^{-1} \cdots w_\ell u_\ell^{-1} = 1$ and (w_i, u_i)'s satisfy the condition: if $w_i = s_{jk}$ then $u_i = s_{j(k+1)}$ and $w_{i+1} \neq s_{j(k+1)}$,*

where $i \in \{1, \ldots, \ell - 1\}$, $(j, k) \in \{(1,1), \ldots, (1,m), (2,1), \ldots, (2,n)\}$ with $(1, m+1) = (1,1)$ and $(2, n+1) = (2,1)$.

Combining Theorem 2.4 with Theorem 2.3, we have

Theorem 2.5. *Let G be a group. If there exist non-empty finite subsets S_1 and S_2 in G such that they fail to satisfy the property* (P)*, then G is non-amenable.*

If G has a subgroup freely generated by two elements, then it has also a free subgroup $\langle a_1, a_2, b_1, b_2 \rangle$ of the rank 4. We can then easily see that $S_1 = \{a_1, a_2\}$ and $S_2 = \{b_1, b_2\}$ fail to satisfy the property (P), and thus G is non-amenable.

3 What Guba's recent results suggest

Very recently, V. S. Guba posted two manuscripts [7] and [8] to arXiv. They are interesting and exiting ones for us. Because his approach is basically same as us; to use Theorem 2.3 and to use the representation $\langle x_0, x_1, \cdots \mid x_i^{-1} x_j x_i = x_{j+1},\ \text{for } i < j \rangle$ of F.

In his paper [7], Guba considers the set of linear combinations of elements $\mathbb{Z}^+ M$ of a positive monoid M of F with positive integer coefficients, and answers Donnelly's question in [5].

We can easily see that (P) in Theorem 2.4 can be replaced with the following (P)′ for $\mathbb{Z}^+ M$:

(P)′ there exist elements $w_1, u_1 \ldots, w_\ell, u_\ell$ in $S_1 \cup S_2$ such that $w_1 u_1^{-1} \cdots w_\ell u_\ell^{-1} = 1$ and (w_i, u_i)'s satisfy the condition: if $w_i = s_{jk}$ then $u_i = s_{j(k+1)}$ and $w_{i+1} \in S_{j'}$ with $j' \neq j$

Hence we have

Theorem 3.1. *Let $\mathbb{Z}^+ M$ be the set of linear combinations of elements of a positive monoid M of F with positive integer coefficients. If there exist non-empty finite subsets S_1 and S_2 in M such that they fail to satisfy the property* (P)′*, then $\mathbb{Z}^+ M$ does not satisfy Ore condition.*

Now, Guba's one of main results in the first manuscript [7] is as follows:

Theorem 3.2. (Guba [7], 2021) *Let M be a positive monoid of $F = \langle x_0, x_1, \cdots \mid x_i^{-1} x_j x_i = x_{j+1},\ \text{for } i < j \rangle$ and $\mathbb{Z}^+ M$ the set of linear combinations of elements of M with positive integer coefficients. Then $\mathbb{Z}^+ M$ does not satisfy the Ore condition.*

In our context, Guba proves that $S_1 = \{1, x_0\}$ and $S_2 = \{1, x_1\}$ fail to satisfy the property (P)′.

If we can replace the positive integer coefficients of $\mathbb{Z}^{+}M$ with the integer coefficients $\mathbb{Z}$ in Theorem 3.2, the negative answer of amenability of F almost has been given. But this result does not mean it. So Guba proceeds with his study on the monoid ring KM in [8].

He first shows that KF satisfies Ore condition if and only if so does KM and also that KM satisfies Ore condition, provided for any homogeneous elements $a, b \in KM$ of same degree, there exist $x, y \in KM$ such that $ax = by$. After that, he gives a partial solution to equations appeared in Ore condition for KM ([8, Theorem 4, Theorem 5]). That is, in our context, his results say that if $S = S_1 \cup S_2 \subset M$ (see Theorem 2.5) is a set in which either all elements are of degree 1 or all elements are of degree 2 consisted x_0, x_1 and x_2, then S satisfies the property (P).

His result suggests that if we would like to find non-empty finite subsets $S_1, S_2 \subset F$ which fail to satisfy the property (P), these sets include an element of degree > 2. We have been trying to show that for some $S_1, S_2 \subset F$ which consist of elements of degree around 20, they fail to satisfy the property (P).

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