On the square of a uniserial module

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Abstract

A module M is said to be *lifting* if, for any submodule N of M, there exists a direct summand X of M contained in N such that N/X is small in M/X. A module M is said to satisfy the *finite internal exchange property* if, for any direct summand X of M and any finite direct sum decomposition $M = \bigoplus_{i=1}^{n} M_i$, there exists $M'_i \subseteq M_i$ $(i = 1, 2, \ldots, n)$ such that $M = X \oplus (\bigoplus_{i=1}^{n} M'_i)$. In this paper, we solve negatively the open problem "does any lifting module satisfy the finite internal exchange property?" by considering the square of a certain lifting module.

1 Background

In 1953, Eckmann and Schopf proved that any module M over an arbitrary ring is essential in an injective module. Such injective module is called the *injective hull* of M. The dual concept of the injective hull is called the *projective cover*. In general, any module does not necessarily have the projective cover. Thus Bass considered rings whose any (finitely generated) right module has the projective cover, and such rings were named *right perfect (semiperfect)*, in 1960. Any right (or left) artinian ring is right perfect. Right perfect rings and semiperfect rings are characterized by "lifting modules" as follows: a ring R is right perfect if and only if any projective right R-module is lifting, a ring R is semiperfect if and only if the right R-module R is lifting. We think that the research of the structure of lifting modules is important in order to study perfect rings and semiperfect rings.

The fundamental problem "When is the direct sum of lifting modules lifting?" has been unsolved yet. In general, the direct sum of lifting modules is not lifting. For instance, abelian groups $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/8\mathbb{Z}$ are lifting, but $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/8\mathbb{Z})$ is not lifting. Many researchers tried to solve this problem, and several results has been obtained. Baba and Harada proved the following:

Theorem ([2, Theorem 1]) Let $M_1, M_2, \ldots, M_n$ be LE-lifting modules, then $\bigoplus_{i=1}^{n} M_i$ is lifting if and only if M_i is almost M_j-projective for any distinct i and j.

Here, almost projectivity was introduced by Harada and Tozaki in [7] as follows: a module M is called *almost N-projective* for a module N if, for any module X, any homomorphism $f : M \to X$ and any epimorphism $g : N \to X$, one of the following holds:

(i) there exists a homomorphism $h : M \to N$ such that $f = gh$,

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(ii) there exist a nonzero direct summand N' of N and a homomorphism $h' : N' \to M$ such that $g|_{N'} = fh'$.

After that, the property "*FIEP (finite internal exchange property)* " was introduced by Hanada, Kuratomi and Oshiro in [6]. The above theorem was generalized more by Kuratomi as follows:

Theorem ([8, Theorem 3.7]) Let $M_1, M_2, \ldots, M_n$ be lifting modules with FIEP, then $M = \bigoplus_{i=1}^n M_i$ is lifting with FIEP if and only if M_i is generalized M/M_i-projective for any $i = 1, 2, \ldots, n$.

Here, generalized projectivity was introduced by Mohamed and Müller in [11] as follows: a module M is called *generalized N-projective* for a module N if, for any module X, any homomorphism $f : M \to X$ and any epimorphism $g : N \to X$, there exist direct sum decompositions $M = M_1 \oplus M_2$ and $N = N_1 \oplus N_2$, a homomorphism $h_1 : M_1 \to N_1$ and an epimorphism $h_2 : N_2 \to M_2$ such that $f|_{M_1} = gh_1$ and $g|_{N_2} = fh_2$. Now the following problem is raised:

Problem: Does any lifting module satisfy FIEP?

This problem had not been solved since it was mentioned in [9] and [4]. In this paper, we first give a characterization for the square of a certain lifting module to be lifting. After that, we make an example of a lifting module which does not satisfy FIEP, using the above characterization.

2 Preliminaries

Throughout this paper, R is a ring with identity and modules are unitary right R-modules. Let M be a module and N a submodule of M. N is said to be *essential* in M (or an *essential submodule* of M) if $N \cap X$ is nonzero for any nonzero submodule X of M and we denote by $N \subseteq_e M$ in this case. N is said to be *small* in M (or a *small submodule* of M) if $N + X \neq M$ for any proper submodule X of M and we denote by $N \ll M$ in this case. A module X is said to be an *essential extension* of M if M is isomorphic to an essential submodule of X. A module Q is said to be a *small cover* of M if M is isomorphic to a small factor module of Q, that is, there exists an epimorphism $f : Q \to M$ such that $\operatorname{Ker} f \ll Q$. A submodule K of N is said to be a *coessential submodule* of N in M if $N/K \ll M/K$ and we denote $K \subseteq_c^M N$ in this case.

A module M is said to satisfy the *finite internal exchange property* (or briefly, *FIEP*) if, for any direct summand X of M and any finite direct sum decomposition $M = \bigoplus_{i=1}^n M_i$, there exists $M_i' \subseteq M_i$ $(i = 1, 2, \ldots, n)$ such that $M = X \oplus (\bigoplus_{i=1}^n M_i')$. This property is naturally considered in the study of direct sum decompositions of a module. In fact, any vector space over a field satisfies FIEP. However not necessarily for a module over a ring. For example, an abelian group $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ have a direct summand $(1, 0)\mathbb{Z}$ and a direct sum decomposition $\mathbb{Z}^2 = (2, 3)\mathbb{Z} \oplus (3, 4)\mathbb{Z}$. Since $(2, 3)\mathbb{Z}$ and $(3, 4)\mathbb{Z}$ are indecomposable and $(1, 0)\mathbb{Z} \oplus (2, 3)\mathbb{Z} \neq \mathbb{Z}^2 \neq (1, 0)\mathbb{Z} \oplus (3, 4)\mathbb{Z}$, we see $\mathbb{Z}^2$ does not satisfy FIEP.

Let $M = A \oplus B$ be a module and $h : A \to B$ a homomorphism. Then $\{a + h(a) \mid a \in A\}$ is called a *graph* of h and denoted by $\langle h \rangle$. It is clear that $M = \langle h \rangle \oplus B$, $M = A + \langle h \rangle$ if h is an epimorphism, and $A \cap \langle h \rangle = \operatorname{Ker} h$.

A module M is said to be *lifting* if, for any submodule N of M, there exists a direct summand X of M such that $X \subseteq_c^M N$. An indecomposable lifting module is called *hollow*.

It is well-known that a module M is hollow if and only if any proper submodule of M is small. A module M is called *uniform* if any nonzero submodule of M is essential. It is well-known that uniform modules (hollow modules, resp.) are closed under nonzero submodules and essential extensions (nonzero factor modules and small covers, resp.). A module M is said to be *uniserial* if its submodules are linearly ordered by inclusion. Clearly, any uniserial module is hollow and uniform. However the converse is not true. We consider

$$R = \begin{pmatrix} K & K & K & K \\ 0 & K & 0 & K \\ 0 & 0 & K & K \\ 0 & 0 & 0 & K \end{pmatrix}, \quad M_R = (K, K, K, K)$$

where K is a field. Then M has only 6 submodules

$$M,\ (0, K, K, K),\ (0, K, 0, K),\ (0, 0, K, K),\ (0, 0, 0, K),\ 0.$$

Hence M is hollow and uniform but not uniserial.

For undefined terminologies, the reader is referred to [1], [3], [4], [10] and [12].

3 Main results

Lemma 3.1 *Let A and B be modules and put $M = A \oplus B$. For any nonzero proper direct summand X of M, the following holds:*

(1) If A and B are hollow, then so is X.

(2) If A and B are uniform, then so is X.

Proof Let $p : M = A \oplus B \to A$ and $q : M = A \oplus B \to B$ be canonical projections.

(1) Since A and B are hollow and X is non-small, X satisfies either $p(X) = A$ or $q(X) = B$. Without loss of generality, we can take X with $p(X) = A$. By $X \neq M$, we see $X \cap B \ll B$ because B is hollow. Since X is a proper direct summand of M, we obtain $\operatorname{Ker} p|_X = X \cap B \ll X$. As $p|_X : X \to A$ is a small epimorphism, X is a small cover of A. Therefore X is hollow.

(2) Since A and B are uniform and X is non-essential, X satisfies either $X \cap A = 0$ or $X \cap B = 0$. Without loss of generality, we can take X with $X \cap A = 0$. Then $q|_X : X \to B$ is a nonzero monomorphism. Therefore X is uniform because it is isomorphic to a submodule of a uniform module B. □

Here we give a key lemma in this paper.

Lemma 3.2 *Let U be a hollow and uniform module (e.g., a uniserial module) and put $M = U^2$, $U_1 = U \times 0$ and $U_2 = 0 \times U$. Then for any submodule N_1 of U_1 and any epimorphism h_1 from N_1 to U_2, $\langle h_1 \rangle$ is a direct summand of M.*

Proof If $N_1 = U_1$ or $\operatorname{Ker} h_1 = 0$, it is clear that $M = \langle h_1 \rangle \oplus U_2$ or $M = \langle h_1 \rangle \oplus U_1$. We assume $N_1 \neq U_1$ and $\operatorname{Ker} h_1 \neq 0$, and take a submodule N_2 of U_2 which is a natural isomorphic image of N_1 and an epimorphism h_2 from N_2 to U_1. Now we prove $M = \langle h_1 \rangle \oplus \langle h_2 \rangle$.

First we show $M = \langle h_1 \rangle + \langle h_2 \rangle$. Let $\iota_i : h_i^{-1}(N_j) \to U_i$ $(i \neq j)$ be the inclusion mapping. Then $\operatorname{Im} \iota_i = h_i^{-1}(N_j) \subseteq h_i^{-1}(U_j) = N_i \subsetneq U_i$ $(i \neq j)$. We define a homomorphism h_i' from

$h_i^{-1}(N_j)$ to U_i by $h'_i(x) = h_j h_i(x)$ for $x \in h_i^{-1}(N_j)$ $(i \neq j)$. Then h'_i is onto $(i = 1, 2)$. Since U_i is hollow, we obtain that $\iota_i - h'_i : h_i^{-1}(N_j) \to U_i$ is onto $(i \neq j)$. For any element $u_1 + u_2$ of M $(u_i \in U_i)$, there exists an element x_i of $h_i^{-1}(N_j)$ such that $(\iota_i - h'_i)(x_i) = u_i$ $(i \neq j)$. Hence

$$\begin{aligned} u_1 + u_2 &= ((x_1 - h_2(x_2)) + h_1(x_1 - h_2(x_2))) + ((x_2 - h_1(x_1)) + h_2(x_2 - h_1(x_1))) \\ &\in \langle h_1 \rangle + \langle h_2 \rangle. \end{aligned}$$

Therefore $M = \langle h_1 \rangle + \langle h_2 \rangle$.

Next we show $\langle h_1 \rangle \cap \langle h_2 \rangle = 0$. We see

$$(\langle h_1 \rangle \cap \langle h_2 \rangle) \cap \operatorname{Ker} h_1 = (\langle h_1 \rangle \cap \langle h_2 \rangle) \cap (\langle h_1 \rangle \cap N_1) \subseteq \langle h_2 \rangle \cap N_1 = 0.$$

Since $\langle h_1 \rangle \cong N_1$ is uniform and $\operatorname{Ker} h_1 \neq 0$, we obtain $\langle h_1 \rangle \cap \langle h_2 \rangle = 0$. □

The following is one of our main results.

Theorem 3.3 *Let U be a hollow and uniform module and put $M = U^2$, $U_1 = U \times 0$ and $U_2 = 0 \times U$. Then the following conditions are equivalent:*

(a) M is lifting,

(b) for any module X, any homomorphism $f : U_1 \to X$ and any epimorphism $g : U_2 \to X$, one of the following holds:

(i) there exists a homomorphism $h : U_1 \to U_2$ such that $f = gh$,

(ii) there exist a submodule N of U_2 and an epimorphism $h : N \to U_1$ such that $g|_N = fh$,

(c) for any module X, any homomorphism $f : U_1 \to X$ and any epimorphism $g : U_2 \to X$, one of the following holds:

(i) there exists a homomorphism $h : U_1 \to U_2$ such that $f = gh$,

(ii) there exist a submodule K of $\operatorname{Ker} g$ *and a monomorphism $h : U_1 \to U_2/K$ such that $g'h = f$, where $g' : U_2/K \to X$ is defined by $g'(\overline{u}) = g(u)$ for $\overline{u} \in U_2/K$.*

Proof Let $p_i : M = U_1 \oplus U_2 \to U_i$ be the canonical projection $(i = 1, 2)$.

(a) ⇒ (b): Let $f : U_1 \to X$ be a nonzero homomorphism and $g : U_2 \to X$ an epimorphism. We define a homomorphism $\varphi : M \to X$ by $\varphi(u_1 + u_2) = f(u_1) - g(u_2)$ for $u_i \in U_i$ $(i = 1, 2)$. Since M is lifting, there exists a direct summand A of M such that $A \subseteq_c^M \operatorname{Ker} \varphi$. Then $M = \operatorname{Ker} \varphi + U_2 = A + U_2$ because g is onto. So $p_1(A) = U_1$.

If $A \cap U_2 = 0$, we can define a homomorphism $h : U_1 = p_1(A) \to U_2$ by $h(p_1(a)) = p_2(a)$ for $a \in A$, and h satisfies $f = gh$. Therefore (i) holds.

Otherwise we see $A \cap U_1 = 0$ since U is uniform. Hence we can define an epimorphism $h : p_2(A) \to p_1(A) = U_1$ by $h(p_2(a)) = p_1(a)$ for $a \in A$, and h satisfies $g|_{p_2(A)} = fh$. Therefore (ii) holds.

(b) ⇒ (a): Let X be a submodule of M. We may assume that X is a proper non-small submodule of M. Since U_1 and U_2 are hollow with $U_1 \cong U_2$, we only consider the case $p_1(X) = U_1$. Then $M = X + U_2$. Let $\pi : M \to M/X$ be the natural epimorphism. Since $\pi|_{U_2}$ is onto, one of the following (i) or (ii) holds:

(i) there exists a homomorphism $h : U_1 \to U_2$ such that $\pi|_{U_1} = \pi|_{U_2}h$,

(ii) there exist a submodule N of U_2 and an epimorphism $h : N \to U_1$ such that $\pi|_N = \pi|_{U_1}h$.

In either case, we see $\langle -h \rangle$ is a direct summand of M by Lemma 3.2, and $\langle -h \rangle \subseteq X$. Put $M = \langle -h \rangle \oplus T$ using a direct summand T of M. Since T is hollow by Lemma 3.1, we obtain $T \cap X \ll T$. By [4, 3.2 (6)], $\langle -h \rangle \subseteq_c^M X$. Therefore M is lifting.

(b) $\Rightarrow$ (c): It is enough to show (b)(ii) $\Rightarrow$ (c)(ii). For any homomorphism $f : U_1 \to X$ and any epimorphism $g : U_2 \to X$, we assume that there exist a submodule N of U_2 and an epimorphism $h : N \to U_1$ such that $g|_N = fh$. Then $\operatorname{Ker} h \subseteq \operatorname{Ker} g$, hence we can define an epimorphism $g' : U_2/\operatorname{Ker} h \to X$ by $g'(\overline{u}) = g(u)$ for $\overline{u} \in U_2/\operatorname{Ker} h$. Let $\overline{h} : N/\operatorname{Ker} h \to U_1$ be the natural isomorphism and $\iota : N/\operatorname{Ker} h \to U_2/\operatorname{Ker} h$ the inclusion mapping, and put $h' = \iota\overline{h}^{-1}$. Clearly, h' is a monomorphism and $g'h' = f$.

(c) $\Rightarrow$ (b): We show (c)(ii) $\Rightarrow$ (b)(ii). For any homomorphism $f : U_1 \to X$ and any epimorphism $g : U_2 \to X$, we assume that there exist a submodule K of $\operatorname{Ker} g$ and a monomorphism $h : U_1 \to U_2/K$ such that $f = g'h$, where $g' : U_2/K \to X$ is defined by $g'(\overline{u}) = g(u)$ for $\overline{u} \in U_2/\operatorname{Ker} h$. We express $\operatorname{Im} h = N/K$. Let $\varphi : N/K \to U_1$ be the inverse map of h and $\pi : N \to N/K$ the natural epimorphism, and put $h' = \varphi\pi$. Then h' is onto and $g|_N = fh'$. □

Lifting modules do not necessarily satisfy FIEP. We can make an example of a lifting module without FIEP, using Theorem 3.3.

Example 3.4 *Let $\mathbb{Z}_{(p)}$ and $\mathbb{Z}_{(q)}$ be the localizations of $\mathbb{Z}$ at two distinct prime numbers p and q respectively. We consider a semiperfect ring $R = \begin{pmatrix} \mathbb{Z}_{(p)} & \mathbb{Q} \\ 0 & \mathbb{Z}_{(q)} \end{pmatrix}$ and its right ideal $L = \begin{pmatrix} 0 & \mathbb{Z}_{(q)} \\ 0 & \mathbb{Z}_{(q)} \end{pmatrix}$, and put $U_R = R/L$. Then U is uniserial whose the endomorphism ring is not local (see, [5]). According to [1, Proposition 12.10], U^2 does not satisfy FIEP. We show U^2 is lifting. For any nonzero homomorphism $f : U \to U/X$ where X is a submodule of U, we can take*

$$f(\overline{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}) = \overline{\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}} + X \quad (x \in \mathbb{Z}_{(p)})$$

If $x \in \mathbb{Z}_{(q)}$, we can define a homomorphism $h : U \to U$ with $h(\overline{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}) = \overline{\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}}$, and h satisfies $\pi h = f$, where π is the natural epimorphism from U to U/X. If $x \notin \mathbb{Z}_{(q)}$, we can express $x = p^m \frac{1}{q^n}\frac{t}{s}$, where $m \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$ and $s, t \in \mathbb{Z} \setminus (p\mathbb{Z} \cup q\mathbb{Z})$. Put $N = \overline{\begin{pmatrix} p^m & 0 \\ 0 & 0 \end{pmatrix}}R$. We can define an epimorphism $h : N \to U$ with $h(\overline{\begin{pmatrix} p^m & 0 \\ 0 & 0 \end{pmatrix}}) = \overline{\begin{pmatrix} q^n\frac{s}{t} & 0 \\ 0 & 0 \end{pmatrix}}$, and h satisfies $fh = \pi|_N$, where π is the natural epimorphism from U to U/X. Therefore U^2 is lifting by Theorem 3.3.

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