Classification of Zeropotent Algebras of Dimension 3 over $\mathbb{R}$ *

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1 Introduction

We address the classification problem of zeropotent algebra of dimension 3. An algebra A (not necessarily associative) is called *zeropotent* if $x^2 = 0$ for all $x \in A$. Lie algebras are typical zeropotent algebras.

We reported the classification results of zeropotent algebras of dimension 3 over the complex number field $\mathbb{C}$ in [1]. In the present article we report the results over the real number field $\mathbb{R}$. We discuss the classification problem of zeropotent algebras of dimension 3, and compare the results over $\mathbb{C}$ and $\mathbb{R}$.

2 Structure matrix

Let A be a zeropotent algebra over K of dimension 3 with a linear basis $\{e_1, e_2, e_3\}$. Because A is zeropotent, $e_1^2 = e_2^2 = e_3^2 = 0$, $e_1e_2 = -e_2e_1$, $e_1e_3 = -e_3e_1$ and $e_2e_3 = -e_3e_2$. Write

$$\begin{cases} e_2e_3 &= a_{11}e_1 + a_{12}e_2 + a_{13}e_3 \\ e_3e_1 &= a_{21}e_1 + a_{22}e_2 + a_{23}e_3 \\ e_1e_2 &= a_{31}e_1 + a_{32}e_2 + a_{33}e_3 \end{cases} \tag{1}$$

with $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33} \in K$. With the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \tag{2}$$

*This is a summary of our results [2] and [3].

we can rewrite (1) as

$$\begin{pmatrix} e_2e_3 \\ e_3e_1 \\ e_1e_2 \end{pmatrix} = A \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

We call (2) the *structure matrix* of the algebra A. We use the same A both for the matrix and for the algebra.

3 Matrix equation for isomorphism

Let A' be another zeropotent algebra on a basis $\{e_1', e_2', e_3'\}$ given by

$$\begin{pmatrix} e_2'e_3' \\ e_3'e_1' \\ e_1'e_2' \end{pmatrix} = A' \begin{pmatrix} e_1' \\ e_2' \\ e_3' \end{pmatrix} \quad \text{with } A' = \begin{pmatrix} a_{11}' & a_{12}' & a_{13}' \\ a_{21}' & a_{22}' & a_{23}' \\ a_{31}' & a_{32}' & a_{33}' \end{pmatrix}. \tag{3}$$

Let $\Phi : A \to A'$ be an isomorphism given by a *transformation matrix*

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix},$$

that is,

$$\begin{pmatrix} \Phi(e_1) \\ \Phi(e_2) \\ \Phi(e_3) \end{pmatrix} = X \begin{pmatrix} e_1' \\ e_2' \\ e_3' \end{pmatrix}.$$

Theorem 3.1. *A and A' are isomorphic if and only if there is a nonsingular transformation matrix X satisfying*

$$A' = \frac{1}{|X|}\, {}^tXAX.$$

Cororally 3.2. *If A and A' are isomorphic, then*
(i) $\operatorname{rank} A = \operatorname{rank} A'$,
(ii) *A is symmetric if and only if A' is symmetric.*

4 Jacobi elements

By Corollary 2.2, the rank and symmetry are invariant under isomorphism of algebras. However, the determinant is not invariant unfortunately, but we have an important invariant called the *Jacobi element*. The Jacobi element $\operatorname{jac}(A)$ of A is defined, with respect to the base $\{e_1, e_2, e_3\}$, by

$$\operatorname{jac}(A) = e_1(e_2e_3) + e_2(e_3e_1) + e_3(e_1e_2).$$

A is a Lie algebra if and only if $\operatorname{jac}(A) = 0$.

For algebras A and A' with structure matrices in (2) and (3) respectively, let

$$\mathrm{jac}(A) = a_1e_1 + a_2e_2 + a_3e_3 \quad \text{and} \quad \mathrm{jac}(A') = a'_1e'_1 + a'_2e'_2 + a'_3e'_3.$$

Then, we have

Theorem 4.1. *If A and A' are isomorphic with a transformation matrix X, then*

$$(a_1, a_2, a_3)X = |X|(a'_1, a'_2, a'_3).$$

5 Classification

Theorem 5.1. *Zeropotent algebras over $\mathbb{C}$ of dimension 3 are classified, up to isomorphism, into 10 families*

$$A_0, A_1, A_2, A_3, A_5, A_6, A_8, A_9, \{A_4(a)\}_{a\in\mathcal{H}}, \{A_7(a)\}_{a\in\mathcal{H}}$$

defined by

$$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&0&1\\0&0&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&1&0\\0&0&0\\0&0&1\end{pmatrix},$$

$$\begin{pmatrix}0&1&1\\0&0&1\\0&0&1\end{pmatrix}, \begin{pmatrix}1&2&2\\0&1&2\\0&0&1\end{pmatrix}, \begin{pmatrix}1&3&3\\0&1&3\\0&0&1\end{pmatrix}, \begin{pmatrix}0&0&0\\0&1&a\\0&0&1\end{pmatrix}, \begin{pmatrix}1&a&0\\0&1&0\\0&0&1\end{pmatrix},$$

where

$$\mathcal{H} = \{z \in \mathbb{C} \mid -\pi/2 < \arg(z) \leq \pi/2\}$$

is the complex half plane.

Over $\mathbb{R}$, we have the algebras defined by the same matrices

$$A_0, A_1, A_2, A_3, A_5, A_6, A_8, A_9$$

as above, while the family $A_4(a)$ $(a \geq 0)$ is split to two families $\{A_4^\alpha(a)\}_{a\geq 0}$ and $\{A_4^\beta(a)\}_{a\geq 0}$ defined by

$$\begin{pmatrix}0&0&0\\0&1&a\\0&0&1\end{pmatrix}, \begin{pmatrix}0&0&0\\0&-1&a\\0&0&1\end{pmatrix}$$

respectively, and the family $A_7(a)$ is split into three families $\{A_7^\alpha(a)\}_{a\geq 0}$, $\{A_7^\beta(a)\}_{a\geq 0}$ and $\{A_7^\gamma(a)\}_{0<a\leq 2}$ defined by

$$\begin{pmatrix}1&a&0\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}1&a&0\\0&-1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}-1&a&0\\0&-1&0\\0&0&0\end{pmatrix}$$

respectively.

6 Transformation

Over $\mathbb{C}$ we have an isomorphism

$$A_4^{\beta}(a) \cong A_4^{\alpha}(-(1+i)a)$$

with transformation matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 1+i \\ 0 & -1 & -1 \end{pmatrix},$$

In addition, we have isomorphisms

$$A_7^{\beta}(a) \cong A_7^{\alpha}(-(1+i)a)$$

with transformation matrix

$$\begin{pmatrix} 1 & 1+i & 0 \\ -1 & -i & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$A_7^{\gamma}(a) \cong A_7^{\alpha}(a)$$

with transformation matrix

$$\begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

They are not isomorphic over $\mathbb{R}$ and form different families of non-isomorphic algebras.

In general, $\mathbb{C}$ can be an arbitrary algebraically closed field, and $\mathbb{R}$ can be a real closed field, that is, $K(\sqrt{-1})$ is an algebraically closed field of characteristic not equal to 0.

References

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[2] Y. Kobayashi, K. Shirayanagi, S.-E. Takahasi and M. Tsukada, *Classification of three-dimensional zeropotent algebras over an algebraically closed field*, Comm. Algebra, Vol. 45 (12), 5037–5052, 2017.

[3] K. Shirayanagi, S.-E. Takahasi, M. Tsukada and Y. Kobayashi, *Classification of three-dimensional zeropotent algebras over the real number field*, Comm. Algebra, Vol. 46 (11), 4665–4681, 2018.