Classification of Zeropotent Algebras of Dimension 3 over $\mathbb{R}$ (Logic, Language, Algebraic system and Related Areas in Computer Science)

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Classification of Zeropotent Algebras of Dimension 3 over $\mathbb{R}$ *

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1 Introduction

We address the classification problem of zeropotent algebra of dimension 3. An algebra $A$ (not necessarily associative) is called zeropotent if $x^2 = 0$ for all $x \in A$. Lie algebras are typical zeropotent algebras.

We reported the classification results of zeropotent algebras of dimension 3 over the complex number field $\mathbb{C}$ in [1]. In the present article we report the results over the real number field $\mathbb{R}$. We discuss the classification problem of zeropotent algebras of dimension 3, and compare the results over $\mathbb{C}$ and $\mathbb{R}$.

2 Structure matrix

Let $A$ be a zeropotent algebra over $K$ of dimension 3 with a linear basis $\{e_1, e_2, e_3\}$. Because $A$ is zeropotent, $e_1^2 = e_2^2 = e_3^2 = 0$, $e_1 e_2 = -e_2 e_1$, $e_1 e_3 = -e_3 e_1$ and $e_2 e_3 = -e_3 e_2$. Write

\[
\begin{align*}
  e_2 e_3 &= a_{11} e_1 + a_{12} e_2 + a_{13} e_3 \\
  e_3 e_1 &= a_{21} e_1 + a_{22} e_2 + a_{23} e_3 \\
  e_1 e_2 &= a_{31} e_1 + a_{32} e_2 + a_{33} e_3
\end{align*}
\]  

(1)

with $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33} \in K$. With the matrix

\[
A = \begin{pmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{pmatrix}
\]  

(2)

*This is a summary of our results [2] and [3].
we can rewrite (1) as
\[
\begin{pmatrix}
e_2e_3 \\
e_3e_1 \\
e_1e_2
\end{pmatrix}
= A
\begin{pmatrix}
e_1 \\
e_2 \\
e_3
\end{pmatrix}.
\]

We call (2) the \textit{structure matrix} of the algebra \( A \). We use the same \( A \) both for the matrix and for the algebra.

3 Matrix equation for isomorphism

Let \( A' \) be another zeropotent algebra on a basis \( \{e'_1, e'_2, e'_3\} \) given by
\[
\begin{pmatrix}
e'_2e'_3 \\
e'_3e'_1 \\
e'_1e'_2
\end{pmatrix}
= A'
\begin{pmatrix}
e'_1 \\
e'_2 \\
e'_3
\end{pmatrix}
\text{ with } A' = \begin{pmatrix}
a'_{11} & a'_{12} & a'_{13} \\
a'_{21} & a'_{22} & a'_{23} \\
a'_{31} & a'_{32} & a'_{33}
\end{pmatrix}.
\]

Let \( \Phi : A \to A' \) be an isomorphism given by a transformation matrix
\[
X = \begin{pmatrix}
x_{11} & x_{12} & x_{13} \\
x_{21} & x_{22} & x_{23} \\
x_{31} & x_{32} & x_{33}
\end{pmatrix},
\]
that is,
\[
\begin{pmatrix}
\Phi(e_1) \\
\Phi(e_2) \\
\Phi(e_3)
\end{pmatrix}
= X
\begin{pmatrix}
e'_1 \\
e'_2 \\
e'_3
\end{pmatrix}.
\]

\textbf{Theorem 3.1.} \( A \) and \( A' \) are isomorphic if and only if there is a nonsingular transformation matrix \( X \) satisfying
\[
A' = \frac{1}{|X|} X A X.
\]

\textbf{Corollary 3.2.} If \( A \) and \( A' \) are isomorphic, then
\begin{enumerate}
\item [(i)] \( \text{rank } A = \text{rank } A' \),
\item [(ii)] \( A \) is symmetric if and only if \( A' \) is symmetric.
\end{enumerate}

4 Jacobi elements

By Corollary 2.2, the rank and symmetry are invariant under isomorphism of algebras. However, the determinant is not invariant unfortunately, but we have an important invariant called the \textit{Jacobi element}. The Jacobi element \( \text{jac}(A) \) of \( A \) is defined, with respect to the base \( \{e_1, e_2, e_3\} \), by
\[
\text{jac}(A) = e_1(e_2e_3) + e_2(e_3e_1) + e_3(e_1e_2).
\]
\( A \) is a Lie algebra if and only if \( \text{jac}(A) = 0 \).
For algebras $A$ and $A'$ with structure matrices in (2) and (3) respectively, let
\[
\text{Jac}(A) = a_1 e_1 + a_2 e_2 + a_3 e_3 \quad \text{and} \quad \text{Jac}(A') = a'_1 e'_1 + a'_2 e'_2 + a'_3 e'_3.
\]
Then, we have

**Theorem 4.1.** If $A$ and $A'$ are isomorphic with a transformation matrix $X$, then

\[
(a_1, a_2, a_3)X = |X|(a'_1, a'_2, a'_3).
\]

### 5 Classification

**Theorem 5.1.** Zeropotent algebras over $\mathbb{C}$ of dimension 3 are classified, up to isomorphism, into 10 families

\[
A_0, A_1, A_2, A_3, A_5, A_6, A_8, A_9, \{A_4(a)\}_{a \in \mathbb{H}}, \{A_7(a)\}_{a \in \mathbb{H}}
\]
defined by

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}, \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}, \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}, \begin{pmatrix}
a & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

where

\[\mathbb{H} = \{z \in \mathbb{C} | -\pi/2 < \arg(z) \leq \pi/2\}\]
is the complex half plane.

Over $\mathbb{R}$, we have the algebras defined by the same matrices

\[
A_0, A_1, A_2, A_3, A_5, A_6, A_8, A_9
\]
as above, while the family $A_4(a)$ ($a \geq 0$) is split to two families $\{A_4^a(a)\}_{a \geq 0}$ and $\{A_4^b(a)\}_{a \geq 0}$ defined by

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & a \\
0 & 0 & 1
\end{pmatrix}, \begin{pmatrix}
0 & 0 & 0 \\
0 & -1 & a \\
0 & 0 & 1
\end{pmatrix}
\]
respectively, and the family $A_7(a)$ is split into three families $\{A_7^a(a)\}_{a \geq 0}$, $\{A_7^b(a)\}_{a \geq 0}$ and $\{A_7^c(a)\}_{0 < a \leq 2}$ defined by

\[
\begin{pmatrix}
a & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \begin{pmatrix}
a & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \begin{pmatrix}
-1 & a & 0 \\
0 & -1 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]
respectively.
6 Transformation

Over $\mathbb{C}$ we have an isomorphism

$$A^{\mathbb{C}}_d(a) \cong A^{\mathbb{C}}_d(-(1+i)a)$$

with transformation matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 1+i \\ 0 & -1 & -1 \end{pmatrix},$$

In addition, we have isomorphisms

$$A^{\mathbb{C}}_d(a) \cong A^{\mathbb{C}}_d(-(1+i)a)$$

with transformation matrix

$$\begin{pmatrix} 1 & 1+i & 0 \\ -1 & -i & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$A^{\mathbb{C}}_d(a) \cong A^{\mathbb{C}}_d(a)$$

with transformation matrix

$$\begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

They are not isomorphic over $\mathbb{R}$ and form different families of non-isomorphic algebras.

In general, $\mathbb{C}$ can be an arbitrary algebraically closed field, and $\mathbb{R}$ can be a real closed field, that is, $K(\sqrt{-1})$ is an algebraically closed field of characteristic not equal to 0.

References

