An improvement of the coefficient condition for a convergence theorem in a complete geodesic space
完備測地距離空間における収束定理の係数条件の改良

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Abstract

In 2009, Kimura proved a weak convergence theorem in Hilbert spaces. In this paper, we attempt to extend this theorem for convex minimization problems in Hilbert spaces to that in complete CAT(1) spaces. As a result, we obtain a new theorem.

1 Introduction

In this paper, we consider the following result for approximating a zero of an accretive operator.

Theorem 1 (Kimura [1]). *Let H be a Hilbert space, A an m-accretive operator of H satisfying that $A^{-1}0 \neq \emptyset$. Let $\{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $[0,1]$ such that $\sum_{n=1}^{\infty} \beta_n = \infty$ and that both $\{\beta_n\}$ and $\{\gamma_n\}$ converge to 0. Let $\{e_n\}$ be a sequence in H such that $\sum_{n=1}^{\infty} \|e_n\| < \infty$. For an initial point $x_1 \in H$, generate an iterative sequence $\{x_n\}$ as follows:*

$$\begin{aligned} y_n &= (I + A)^{-1} x_n, \\ \alpha_n &\in [\min\{\beta_n, \|x_n - y_n\| - \gamma_n\}, 1] \cap [0,1], \\ x_{n+1} &= (1 - \alpha_n) x_n + \alpha_n y_n + e_n. \end{aligned}$$

Then, $x_n \rightharpoonup x_0 \in A^{-1}0$.

We also know that we can apply this theorem to the convex minimization problem. In this paper, we extend it to the setting of complete CAT(1) spaces. In order to prove our result, the notion of resolvent plays an important role. The definition of resolvents is as follows [2]:

$$J_f x_n = \operatorname*{argmin}_{y\in X}\{f(y) + \tan d(y, x_n) \sin d(y, x_n)\},$$

where $f : X \to]-\infty, \infty]$ is a proper lower semicontinuous function.

2 Preliminaries

Let X be a uniquely geodesic space, and $x, y, z \in X$. We take $\bar{x}, \bar{y}, \bar{z} \in \mathbb{R}^2$ such that $d(x, y) = \|\bar{x} - \bar{y}\|, d(y, z) = \|\bar{y} - \bar{z}\|, d(z, x) = \|\bar{z} - \bar{x}\|$. The sets Δ and $\bar{\Delta}$ are defined by $\Delta = [x, y] \cup [y, z] \cup [z, x]$ and $\bar{\Delta} = [\bar{x}, \bar{y}] \cup [\bar{y}, \bar{z}] \cup [\bar{z}, \bar{x}]$. X is called a CAT(0) space if for all Δ, $p, q \in \Delta$ and their comparison points $\bar{p}, \bar{q} \in \bar{\Delta}$, the inequality

$$d(p, q) \leq \|\bar{p} - \bar{q}\|$$

holds. Moreover, if $d(x, y) + d(y, z) + d(z, x) < 2\pi$, we can take $\bar{\Delta}(\bar{x}, \bar{y}, \bar{z}) \subset \mathbb{S}^2$ such that $d(x, y) = d_{\mathbb{S}^2}(\bar{x}, \bar{y}), d(y, z) = d_{\mathbb{S}^2}(\bar{y}, \bar{z}), d(z, x) = d_{\mathbb{S}^2}(\bar{z}, \bar{x})$. X is called a CAT(1) space if for all such Δ, $p, q \in \Delta$ and their comparison points $\bar{p}, \bar{q} \in \bar{\Delta}$, the inequality

$$d(p, q) \leq d_{\mathbb{S}^2}(\bar{p}, \bar{q})$$

holds. We say that a CAT (1) space X is admissible

$$d(w, w') < \frac{\pi}{2}$$

for all $w, w' \in X$.

Theorem 2 (Kimura and Kohsaka [2])**.** *Let X be an admissible complete* CAT(1) *space, $f : X \to]-\infty, \infty]$ a proper convex lower semicontinuous function. For $\eta > 0$, let $J_{\eta f}$ be the resolvent of ηf. Let $\{x_n\}$ be a sequence defined by $x_1 \in X$ and*

$$x_{n+1} = \alpha_n x_n \oplus (1 - \alpha_n) J_{\lambda_n f} x_n$$

for $n = 1, 2, \ldots,$ where $\{\alpha_n\}$ is a sequence in $[0, 1[$ and $\{\lambda_n\}$ is a sequence of positive real numbers such that $\sum_{n=1}^{\infty}(1 - \alpha_n)\lambda_n = \infty$. Then the following hold.

(i) *The set* $\operatorname{argmin}_X f$ *is nonempty if and only if $\{J_{\lambda_n f} x_n\}$ is spherically bounded and* $\sup_n d(J_{\lambda_n f} x_n, x_n) < \pi/2$;

(ii) *if* $\operatorname{argmin}_X f$ *is nonempty and* $\sup_n \alpha_n < 1$, *then both $\{x_n\}$ and $\{J_{\lambda_n f} x_n\}$ are Δ-convergent to an element x_∞ of* $\operatorname{argmin}_X f$.

3 Main result

The following theorem is the main result of this paper.

Theorem 3. *Let X be an admissible complete* CAT(1) *space. Let $f : X \to]-\infty, \infty]$ be a proper convex lower semicontinuous function and suppose that* $\operatorname{argmin} f \neq \emptyset$. *Let $\{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $[0,1]$ such that $\sum_{n=1}^{\infty} \beta_n = \infty$ and that both $\{\beta_n\}$ and $\{\gamma_n\}$ converge to 0. For an initial point $x_1 \in X$, generate a sequence $\{x_n\}$ as follows:*

$$\begin{aligned}
&y_n = J_f x_n, \\
&\alpha_n \in [\min\{\beta_n, d(x_n, y_n) - \gamma_n\}, 1] \cap [0,1], \\
&x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n y_n.
\end{aligned}$$

Suppose that one of the following conditions holds:

- $\inf_{n\in\mathbb{N}} \alpha_n > 0$;
- $\sum_{n=1}^{\infty} \alpha_n < \infty$.

Then, $x_n \xrightarrow{\Delta} x_0 \in \operatorname{argmin} f$.

Proof. If $\inf_{n\in\mathbb{N}} \alpha_n > 0$, it is already shown by Theorem 2. So, we consider the case that $\sum_{n=1}^{\infty} \alpha_n < \infty$. We show $x_n \xrightarrow{\Delta} x_0$. Put $M = \sup_{j\in\mathbb{N}} d(y_j, x_j)$. Then

$$\begin{aligned}
d(x_{n+1}, x_n) &= d((1-\alpha_n)x_n \oplus \alpha_n y_n, x_n) \\
&= \alpha_n d(y_n, x_n) \\
&\leq \alpha_n M.
\end{aligned}$$

For $m, n \in \mathbb{N}$ such that $m < n$, we have

$$\begin{aligned}
d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \\
&= \sum_{j=m}^{n-1} d(x_{j+1}, x_j) \leq \sum_{j=m}^{n-1} \alpha_j M \leq \sum_{j=m}^{\infty} \alpha_j M.
\end{aligned}$$

Since $\sum_{n=1}^{\infty} \alpha_n$ is finite, $\{x_n\}$ is a Cauchy sequence. Therefore, $x_n \to x_0$, and hence $x_n \xrightarrow{\Delta} x_0$.

Next, we show that there exists $\{n_i\}$ such that $d(x_{n_i}, y_{n_i}) \to 0$. We focus on the range of α_n. Put $P = \{n \in \mathbb{N} \mid \alpha_n \in [d(x_n, y_n) - \gamma_n, 1] \cap [0,1]\}$ and $Q = \{n \in \mathbb{N} \mid \alpha_n \in [\beta_n, 1]\}$. Assume that there exists $n_0 \in \mathbb{N}$ such that $n \in Q$ for all $n \geq n_0$. Then we have

$$\infty = \sum_{n=n_0}^{\infty} \beta_n \leq \sum_{n=n_0}^{\infty} \alpha_n < \infty.$$

This is a contradiction. Therefore for all $n_0 \in \mathbb{N}$, there exists $n \geq n_0$ such that $n \in P$. So, there exists $\{n_i\} \subset P$ such that $n_i \geq i$ for all $i \in \mathbb{N}$. Then $\alpha_{n_i} \in [d(x_{n_i}, y_{n_i}) - \gamma_{n_i}, 1]$, and we get $d(x_{n_i}, y_{n_i}) - \gamma_{n_i} \leq \alpha_{n_i} \leq 1$. We know that $\sum_{n=1}^{\infty} \alpha_n < \infty$, and this implies $\alpha_n \to 0$. Hence we get $\lim_{i\to\infty} d(x_{n_i}, y_{n_i}) \leq 0$. Therefore $d(x_{n_i}, y_{n_i}) \to 0$. We also get $y_{n_i} \to x_0$ since $x_n \to x_0$ and $d(x_{n_i}, y_{n_i}) \to 0$.

Next we show $x_0 \in \operatorname{argmin} f$. From the property of resolvents defined by

$$J_f x_n = \operatorname*{argmin}_{y\in X}\{f(y) + \tan d(y, x_n) \sin d(y, x_n)\},$$

for all $y \in X$, we have

$$f(y_{n_i}) + \tan d(y_{n_i}, x_{n_i}) \sin d(y_{n_i}, x_{n_i}) \leq f(y) + \tan d(y, x_{n_i}) \sin d(y, x_{n_i}).$$

Put $t \in \,]0,1[\,$, $w \in X$, and $y = ty_{n_i} \oplus (1-t)w$. Then,

$$\begin{aligned}
&f(y_{n_i}) + \tan d(y_{n_i}, x_{n_i}) \sin d(y_{n_i}, x_{n_i}) \\
&\leq f(ty_{n_i} \oplus (1-t)w) + \tan d(ty_{n_i} \oplus (1-t)w, x_{n_i}) \sin d(ty_{n_i} \oplus (1-t)w, x_{n_i}) \\
&\leq tf(y_{n_i}) + (1-t)f(w) + \frac{1}{\cos d(ty_{n_i} \oplus (1-t)w, x_{n_i})} \\
&\quad - \cos d(ty_{n_i} \oplus (1-t)w, x_{n_i}) \\
&\leq tf(y_{n_i}) + (1-t)f(w) \\
&\quad + \frac{\sin d(y_{n_i}, w)}{\cos d(y_{n_i}, x_{n_i}) \sin(td(y_{n_i}, w)) + \cos d(w, x_{n_i}) \sin((1-t)d(y_{n_i}, w))} \\
&\quad - \frac{\cos d(y_{n_i}, x_{n_i}) \sin(td(y_{n_i}, w)) + \cos d(w, x_{n_i}) \sin((1-t)d(y_{n_i}, w))}{\sin d(y_{n_i}, w)}
\end{aligned}$$

Putting $A_i = d(y_{n_i}, x_{n_i})$, $B_i = d(y_{n_i}, w)$, and $C_i = d(w, x_{n_i})$, we get,

$$\begin{aligned}
f(y_{n_i}) + \frac{1}{1-t}\Bigg(&\tan A_i \sin A_i + \frac{\cos A_i \sin tB_i + \cos C_i \sin(1-t)B_i}{\sin B_i} \\
&- \frac{\sin B_i}{\cos A_i \sin tB_i + \cos C_i \sin(1-t)B_i}\Bigg) \leq f(w).
\end{aligned}$$

Letting $t \to 1$, we have

$$\begin{aligned}
&\lim_{t\to 1}\Bigg(\frac{1}{1-t}(\tan A_i \sin A_i + \frac{\cos A_i \sin tB_i + \cos C_i \sin(1-t)B_i}{\sin B_i} \\
&\quad - \frac{\sin B_i}{\cos A_i \sin tB_i + \cos C_i \sin(1-t)B_i})\Bigg) \\
&= -\lim_{t\to 1} \frac{d}{dt}\Bigg(\tan A_i \sin A_i + \frac{\cos A_i \sin tB_i + \cos C_i \sin(1-t)B_i}{\sin B_i} \\
&\quad - \frac{\sin B_i}{\cos A_i \sin tB_i + \cos C_i \sin(1-t)B_i}\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&= -\lim_{t\to 1}\biggl(\frac{\cos A_i \cos tB_i \cdot B_i + \cos C_i \cos(1-t)B_i \cdot (-B_i)}{\sin B_i} \\
&\qquad + \frac{\sin B_i(\cos A_i \cos tB_i \cdot B_i + \cos C_i \cos(1-t)B_i \cdot (-B_i))}{(\cos A_i \sin tB_i + \cos C_i \sin(1-t)B_i)^2}\biggr) \\
&= -\biggl(\frac{\cos A_i \cos B_i \cdot B_i + \cos C_i \cdot (-B_i)}{\sin B_i} + \frac{\sin B_i(\cos A_i \cos B_i \cdot B_i + \cos C_i \cdot (-B_i))}{(\cos A_i \sin B_i)^2}\biggr) \\
&= -\biggl(\frac{\cos A_i \cos B_i \cdot B_i + \cos C_i \cdot (-B_i)}{\sin B_i} + \frac{\cos A_i \cos B_i \cdot B_i + \cos C_i \cdot (-B_i)}{\cos^2 A_i \sin B_i}\biggr) \\
&= -\frac{B_i}{\sin B_i}\biggl(\cos A_i \cos B_i - \cos C_i + \frac{\cos A_i \cos B_i - \cos C_i}{\cos^2 A_i}\biggr) \\
&= \frac{B_i}{\sin B_i}\biggl(\cos C_i - \cos A_i \cos B_i + \frac{\cos C_i - \cos A_i \cos B_i}{\cos^2 A_i}\biggr) \\
&= \frac{B_i}{\sin B_i}(\cos C_i - \cos A_i \cos B_i)\biggl(1 + \frac{1}{\cos^2 A_i}\biggr) \\
&= \frac{B_i}{\sin B_i}(\cos C_i - \cos B_i + \cos B_i(1 - \cos A_i))\biggl(1 + \frac{1}{\cos^2 A_i}\biggr).
\end{aligned}
$$

It is obvious that $\cos C_i - \cos B_i \to 0$. Letting $i \to \infty$, we have

$$
\begin{aligned}
&\frac{B_i}{\sin B_i}(\cos C_i - \cos B_i + \cos B_i(1 - \cos A_i))\biggl(1 + \frac{1}{\cos^2 A_i}\biggr) \\
\to\ &\frac{d(x_0, w)}{\sin d(x_0, w)}(0 + 0)\biggl(1 + \frac{1}{1}\biggr) \\
=\ &0.
\end{aligned}
$$

Hence we get

$$
f(x_0) \le \liminf_{i\to\infty} f(y_{n_i}) \le f(w).
$$

This inequality implies $x_0 \in \operatorname{argmin} f$. □

References

[1] Y. Kimura, *Further improvement of a coefficient condition for a weakly convergent iterative scheme*, Nonlinear Anal. **71** (2009), e2023–e2027.

[2] Y. Kimura and F. Kohsaka, *Two modified proximal point algorithms in geodesic spaces with curvature bounded above*, Rend. Circ. Mat. Palermo, II. Ser **68** (2019), 83–104.