

Identical Duals

— Gap Function —

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Abstract

We consider *identical duals* of two pairs of minimization (primal) problems and maximization (dual) problems from a view point of *gap function*. The identical dual means that both optimum points of a primal problem and its dual one are identical. An identity

$$(CI) \quad \sum_{k=1}^{n-1} [(x_{k-1} - x_k)\mu_k + x_k(\mu_k - \mu_{k+1})] + (x_{n-1} - x_n)\mu_n + x_n\mu_n = x_0\mu_1$$

is called *complementary* [17]. The complementary identity leads to a gap function. We show that the complementary identity and the gap function play a fundamental part in analyzing an identical duality between primal and dual.

1 Identical Dual 1

As a pair of primal problem and dual problem, we take n -variable optimization problems:

$$(P_1) \quad \begin{array}{ll} \text{minimize} & \sum_{k=1}^{n-1} [(x_{k-1} - x_k)^2 + x_k^2] + (x_{n-1} - x_n)^2 + x_n^2 \\ \text{subject to} & \text{(i) } x \in R^n, \quad \text{(ii) } x_0 = c \end{array}$$

$$(D_1) \quad \begin{array}{ll} \text{Maximize} & 2c\mu_1 - \sum_{k=1}^{n-1} [\mu_k^2 + (\mu_k - \mu_{k+1})^2] - \mu_n^2 - \mu_n^2 \\ \text{subject to} & \text{(i) } \mu \in R^n. \end{array}$$

First we present an identity, which plays a fundamental role in analyzing the pair. Let $x = \{x_k\}_0^n$, $\mu = \{\mu_k\}_1^n$ be any two sequences of real number with $x_0 = c$. Then an identity

$$(C_1) \quad c\mu_1 = \sum_{k=1}^{n-1} [(x_{k-1} - x_k)\mu_k + x_k(\mu_k - \mu_{k+1})] + (x_{n-1} - x_n)\mu_n + x_n\mu_n$$

holds true. This identity is called *complementary*. Furthermore the complementary identity implies that

$$(QI_1) \quad \begin{aligned} & \sum_{k=1}^n [(x_{k-1} - x_k)^2 + x_k^2] + \sum_{k=1}^{n-1} [\mu_k^2 + (\mu_k - \mu_{k+1})^2] + 2\mu_n^2 - 2c\mu_1 \\ &= \sum_{k=1}^{n-1} [(x_{k-1} - x_k - \mu_k)^2 + (x_k - \mu_k + \mu_{k+1})^2] + (x_{n-1} - x_n - \mu_n)^2 + (x_n - \mu_n)^2. \end{aligned}$$

This is an identity on $R^n \times R^n$, which is called *quadratic*.

Now we define three functions $f, g : R^n \rightarrow R^1$, $h : R^n \times R^n \rightarrow R^1$ by

$$\begin{aligned} f(x) &= \sum_{k=1}^n [(x_{k-1} - x_k)^2 + x_k^2] \\ g(\mu) &= 2c\mu_1 - \sum_{k=1}^{n-1} [\mu_k^2 + (\mu_k - \mu_{k+1})^2] - 2\mu_n^2 \\ h(x, \mu) &= \sum_{k=1}^{n-1} [(x_{k-1} - x_k - \mu_k)^2 + (x_k - \mu_k + \mu_{k+1})^2] + (x_{n-1} - x_n - \mu_n)^2 + (x_n - \mu_n)^2. \end{aligned}$$

They are called *primal*, *dual* and *gap* functions, respectively. Then (QI_1) is summarized as follows.

Lemma 1 *It holds that*

$$(QI_1) \quad f(x) - g(\mu) = h(x, \mu).$$

We consider a linear system of $2n$ -equation on $2n$ -variable (x, μ) :

$$(EC_1) \quad \begin{aligned} & c - x_1 = \mu_1, \quad x_1 = \mu_1 - \mu_2 \\ & x_{k-1} - x_k = \mu_k, \quad x_k = \mu_k - \mu_{k+1} \quad 2 \leq k \leq n-1 \\ & x_{n-1} - x_n = \mu_n, \quad x_n = \mu_n. \end{aligned}$$

Lemma 2 *It holds that*

- (i) $h(x, \mu) \geq 0 \quad \forall (x, \mu) \in R^n \times R^n$
- (ii) $h(x, \mu) = 0 \iff (x, \mu) \text{ satisfies } (EC_1).$

Corollary 1 *It holds that*

- (i) $f(x) \geq g(\mu) \quad \forall (x, \mu) \in R^n \times R^n$
- (ii) $f(x) = g(\mu) \iff (x, \mu) \text{ satisfies } (EC_1).$

Definition 1 We say that that (P_1) and (D_1) are *dual to each other* and (EC_1) is an *equality condition* (EC) if Corollary 1 (i), (ii) hold. Then we say that one is *dual* of the other. This definition applies for any triplet such as $\{(P_1), (D_1), (EC_1)\}$.

From Corollary 1, it turns out that both are *dual to each other*, and (EC₁) is an *equality condition*.

Lemma 3 (EC₁) *has a unique solution:*

$$\begin{aligned} x &= (x_1, x_2, \dots, x_k, \dots, x_{n-1}, x_n) \\ &= \frac{c}{F_{2n+1}}(F_{2n-1}, F_{2n-3}, \dots, F_{2n-2k+1}, \dots, F_3, F_1), \end{aligned} \quad (1)$$

$$\begin{aligned} \mu &= (\mu_1, \mu_2, \dots, \mu_k, \dots, \mu_{n-1}, \mu_n) \\ &= \frac{c}{F_{2n+1}}(F_{2n}, F_{2n-2}, \dots, F_{2n-2k}, \dots, F_4, F_2). \end{aligned} \quad (2)$$

Here $\{F_n\}$ is the *Fibonacci sequence*. This is defined as the solution to the second-order linear difference equation

$$x_{n+2} - x_{n+1} - x_n = 0, \quad x_1 = 1, x_0 = 0. \quad (3)$$

n	\dots	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11
F_n	\dots	-1	1	0	1	1	2	3	5	8	13	21	34	55	89

Table 1 Fibonacci sequence $\{F_n\}$

Proof. From (EC₁), we have a pair of linear systems of n -variable on n -equation:

$$\begin{aligned} (EQ_1) \quad \begin{aligned} c &= 3x_1 - x_2 & c &= 2\mu_1 - \mu_2 \\ x_1 &= 3x_2 - x_3 & \mu_1 &= 3\mu_2 - \mu_3 \\ \vdots & & \vdots & \\ x_{n-2} &= 3x_{n-1} - x_n & \mu_{n-2} &= 3\mu_{n-1} - \mu_n \\ x_{n-1} &= 2x_n & \mu_{n-1} &= 3\mu_n. \end{aligned} \end{aligned}$$

The left system has a solution x in (1), while the right has a solution μ in (2). \square

Theorem 1 *The primal (P₁) has a minimum value $m = c(c - \hat{x}_1) = \frac{F_{2n}}{F_{2n+1}}c^2$ at a path*

$$\begin{aligned} \hat{x} &= (\hat{x}_1, \hat{x}_2, \dots, \hat{x}_k, \dots, \hat{x}_{n-1}, \hat{x}_n) \\ &= \frac{c}{F_{2n+1}}(F_{2n-1}, F_{2n-3}, \dots, F_{2n-2k+1}, \dots, F_3, F_1). \end{aligned}$$

The dual (D₁) has a maximum value $M = c\mu_1^ = \frac{F_{2n}}{F_{2n+1}}c^2$ at a path*

$$\begin{aligned} \mu^* &= (\mu_1^*, \mu_2^*, \dots, \mu_k^*, \dots, \mu_{n-1}^*, \mu_n^*) \\ &= \frac{c}{F_{2n+1}}(F_{2n}, F_{2n-2}, \dots, F_{2n-2k}, \dots, F_4, F_2). \end{aligned}$$

Let $x = \{x_k\}_0^n$, $\mu = \{\mu_k\}_1^n$ be any two sequences of real number with $x_0 = c$. Then a complementary identity

$$(C_1) \quad c\mu_1 = \sum_{k=1}^{n-1} [(x_{k-1} - x_k)\mu_k + x_k(\mu_k - \mu_{k+1})] + (x_{n-1} - x_n)\mu_n + x_n\mu_n$$

holds true.

Let us define two sequences $y = \{y_k\}_1^{2n}$, $\nu = \{\nu_k\}_1^{2n}$ from $x = \{x_k\}_0^n$, $\mu = \{\mu_k\}_1^n$ through

$$\begin{aligned} y_1 &= c - x_1, \quad y_2 = x_1, \quad y_3 = x_1 - x_2, \quad y_4 = x_2, \quad y_5 = x_2 - x_3 \\ &\dots, \quad y_{2n-2} = x_{n-1}, \quad y_{2n-1} = x_{n-1} - x_n, \quad y_{2n} = x_n \\ \nu_1 &= \mu_1, \quad \nu_2 = \mu_1 - \mu_2, \quad \nu_3 = \mu_2, \quad \nu_4 = \mu_2 - \mu_3, \quad \nu_5 = \mu_3 \\ &\dots, \quad \nu_{2n-2} = \mu_{n-1} - \mu_n, \quad \nu_{2n-1} = \mu_n, \quad \nu_{2n} = \mu_n \end{aligned} \tag{4}$$

, respectively. Then an identity

$$(C_1^*) \quad c\nu_1 = \sum_{k=1}^{2n} y_k \nu_k$$

holds *under a constraint* – a linear system of $4n$ -variables (y, ν) on $2n$ -equations – :

$$\begin{aligned} (C^{y\nu}) \quad & \begin{array}{ll} c = y_1 + y_2 & \nu_1 = \nu_2 + \nu_3 \\ y_2 = y_3 + y_4 & \nu_3 = \nu_4 + \nu_5 \\ \vdots & \vdots \\ y_{2n-4} = y_{2n-3} + y_{2n-2} & \nu_{2n-3} = \nu_{2n-2} + \nu_{2n-1} \\ y_{2n-2} = y_{2n-1} + y_{2n} & \nu_{2n-1} = \nu_{2n}. \end{array} \end{aligned}$$

An equality (C_1^*) with constraint $(C^{y\nu})$ is called a $2n$ -variable *conditional complementarity*. This is simply written as (C_1^*) under $(C^{y\nu})$.

Now let $y = \{y_k\}_1^{2n}$, $\nu = \{\nu_k\}_1^{2n}$ satisfy $(C_1^{y\nu})$. Then an elementary inequality with equality

$$2xy \leq x^2 + y^2 \quad \text{on } R^2 ; \quad x = y \tag{5}$$

yields

$$2c\nu_1 \leq \sum_{k=1}^{2n} (y_k^2 + \nu_k^2).$$

Thus we have an inequality

$$2c\nu_1 - \sum_{k=1}^{2n} \nu_k^2 \leq \sum_{k=1}^{2n} y_k^2.$$

The sign of equality holds iff

$$(EC_1) \quad y_k = \nu_k \quad 1 \leq k \leq 2n. \quad (6)$$

Hence we have a pair of conditional optimization problems:

$$(P_1^*) \quad \begin{aligned} & \text{minimize} \quad y_1^2 + y_2^2 + \cdots + y_{2n-1}^2 + y_{2n}^2 \\ & \text{subject to} \quad (1) \quad y_1 + y_2 = c \\ & \quad \quad \quad (2) \quad y_3 + y_4 = y_2 \\ & \quad \quad \quad \vdots \\ & \quad \quad \quad (n-1) \quad y_{2n-3} + y_{2n-2} = y_{2n-4} \\ & \quad \quad \quad (n) \quad y_{2n-1} + y_{2n} = y_{2n-2} \\ & \quad \quad \quad (n+1) \quad y \in R^{2n} \end{aligned}$$

$$(D_1^*) \quad \begin{aligned} & \text{Maximize} \quad 2c\nu_1 - (\nu_1^2 + \nu_2^2 + \cdots + \nu_{2n-1}^2 + \nu_{2n}^2) \\ & \text{subject to} \quad [1] \quad \nu_2 + \nu_3 = \nu_1 \\ & \quad \quad \quad [2] \quad \nu_4 + \nu_5 = \nu_3 \\ & \quad \quad \quad \vdots \\ & \quad \quad \quad [n-1] \quad \nu_{2n-2} + \nu_{2n-1} = \nu_{2n-3} \\ & \quad \quad \quad [n] \quad \nu_{2n} = \nu_{2n-1} \\ & \quad \quad \quad [n+1] \quad \nu \in R^{2n}. \end{aligned}$$

Let (AC_1) be an *augmentation* of the system $(C_1^{y\nu})$ with the additional equality condition (EC_1) :

$$(AC_1) \quad \begin{array}{ll} c = y_1 + y_2 & \nu_1 = \nu_2 + \nu_3 \\ y_2 = y_3 + y_4 & \nu_3 = \nu_4 + \nu_5 \\ \vdots & \vdots \\ y_{2n-4} = y_{2n-3} + y_{2n-2} & \nu_{2n-3} = \nu_{2n-2} + \nu_{2n-1} \\ y_{2n-2} = y_{2n-1} + y_{2n} & \nu_{2n-1} = \nu_{2n} \\ y_k = \nu_k & 1 \leq k \leq 2n. \end{array}$$

The linear system (AC_1) is of $4n$ -variables on $4n$ -equations. Let (y, ν) satisfy (AC_1) . Then both sides become a common value with five expressions:

$$(5V_1) \quad \begin{aligned} & y_1^2 + y_2^2 + \cdots + y_{2n}^2 \\ & = cy_1 \\ & = 2c\nu_1 - (\nu_1^2 + \nu_2^2 + \cdots + \nu_{2n}^2) \\ & = \nu_1^2 + \nu_2^2 + \cdots + \nu_{2n}^2 \\ & = c\nu_1. \end{aligned}$$

The system (AC₁) has indeed a unique common solution:

$$\begin{aligned}
 y &= (y_1, y_2, \dots, y_k, \dots, y_{2n-1}, y_{2n}) \\
 &= \frac{c}{F_{2n+1}}(F_{2n}, F_{2n-1}, \dots, F_{2n-k+1}, \dots, F_2, F_1), \\
 \nu &= (\nu_1, \nu_2, \dots, \nu_k, \dots, \nu_{2n-1}, \nu_{2n}) \\
 &= \frac{c}{F_{2n+1}}(F_{2n}, F_{2n-1}, \dots, F_{2n-k+1}, \dots, F_2, F_1).
 \end{aligned}$$

Theorem 2 *The primal (P₁) has a minimum value $m = \frac{F_{2n}}{F_{2n+1}}c^2$ at a path*

$$\begin{aligned}
 \hat{y} &= (\hat{y}_1, \hat{y}_2, \dots, \hat{y}_k, \dots, \hat{y}_{2n-1}, \hat{y}_{2n}) \\
 &= \frac{c}{F_{2n+1}}(F_{2n}, F_{2n-1}, \dots, F_{2n-k+1}, \dots, F_2, F_1).
 \end{aligned}$$

The dual (D₁) has a maximum value $M = \frac{F_{2n}}{F_{2n+1}}c^2$ at a path

$$\begin{aligned}
 \nu^* &= (\nu_1^*, \nu_2^*, \dots, \nu_k^*, \dots, \nu_{2n-1}^*, \nu_{2n}^*) \\
 &= \frac{c}{F_{2n+1}}(F_{2n}, F_{2n-1}, \dots, F_{2n-k+1}, \dots, F_2, F_1).
 \end{aligned}$$

Both optimal solutions (point and value) are identical:

$$\hat{x} = \mu^*, \quad m = M.$$

Further both are Fibonacci.

Thus *Fibonacci Identical Duality* (FID) holds between (P₁) and (D₁) [15–17].

We remark that the $2n$ -variable pair is a transliteration from n -variable one (P₁), (D₁).

2 Identical Dual 2

Next we consider the following pair

$$\begin{aligned}
 (P_m) \quad & \text{minimize} \quad \sum_{k=1}^{n-1} [(x_{k-1} - x_k)^2 + x_k^2] + (x_{n-1} - x_n)^2 + \frac{F_{m+1}}{F_m} x_n^2 \\
 & \text{subject to} \quad \text{(i) } x \in R^n, \quad \text{(ii) } x_0 = c \\
 (D_m) \quad & \text{Maximize} \quad 2c\mu_1 - \sum_{k=1}^{n-1} [\mu_k^2 + (\mu_k - \mu_{k+1})^2] - \mu_n^2 - \frac{F_m}{F_{m+1}} \mu_n^2 \\
 & \text{subject to} \quad \text{(i) } \mu \in R^n,
 \end{aligned}$$

where $\{F_n\}$ is the *Fibonacci sequence*. The identity (C_1) is enhanced to

$$(C_m) \quad c\mu_1 = \sum_{k=1}^{n-1} [(x_{k-1} - x_k)\mu_k + x_k(\mu_k - \mu_{k+1})] + (x_{n-1} - x_n)\mu_n + \sqrt{\frac{F_{m+1}}{F_m}} x_n \sqrt{\frac{F_m}{F_{m+1}}} \mu_n$$

where $m \geq 1$. This identity is called F_m -complementary.

Furthermore the complementary identity implies that

$$\begin{aligned} & \sum_{k=1}^{n-1} [(x_{k-1} - x_k)^2 + x_k^2] + (x_{n-1} - x_n)^2 + \frac{F_{m+1}}{F_m} x_n^2 \\ & + \sum_{k=1}^{n-1} [\mu_k^2 + (\mu_k - \mu_{k+1})^2] + \mu_n^2 + \frac{F_m}{F_{m+1}} \mu_n^2 - 2c\mu_1 \\ (QI_m) \quad & = \sum_{k=1}^{n-1} [(x_{k-1} - x_k - \mu_k)^2 + (x_k - \mu_k + \mu_{k+1})^2] \\ & + (x_{n-1} - x_n - \mu_n)^2 + \left(\sqrt{\frac{F_{m+1}}{F_m}} x_n - \sqrt{\frac{F_m}{F_{m+1}}} \mu_n \right)^2. \end{aligned}$$

This is an identity on $R^n \times R^n$, which is called *quadratic*.

Now we define three functions $f, g : R^n \rightarrow R^1, h : R^n \times R^n \rightarrow R^1$ by

$$\begin{aligned} f(x) &= \sum_{k=1}^{n-1} [(x_{k-1} - x_k)^2 + x_k^2] + (x_{n-1} - x_n)^2 + \frac{F_{m+1}}{F_m} x_n^2 \\ g(\mu) &= 2c\mu_1 - \sum_{k=1}^{n-1} [\mu_k^2 + (\mu_k - \mu_{k+1})^2] - \mu_n^2 - \frac{F_m}{F_{m+1}} \mu_n^2 \\ h(x, \mu) &= \sum_{k=1}^{n-1} [(x_{k-1} - x_k - \mu_k)^2 + (x_k - \mu_k + \mu_{k+1})^2] \\ &+ (x_{n-1} - x_n - \mu_n)^2 + \left(\sqrt{\frac{F_{m+1}}{F_m}} x_n - \sqrt{\frac{F_m}{F_{m+1}}} \mu_n \right)^2. \end{aligned}$$

They are called *primal*, *dual* and *gap* functions, respectively. Then (QI_m) is summarized as follows.

Lemma 4 *It holds that*

$$(QI_m) \quad f(x) - g(\mu) = h(x, \mu).$$

We consider a linear system of $2n$ -equation on $2n$ -variable (x, μ) :

$$\begin{aligned} (EC_m) \quad & c - x_1 = \mu_1, \quad x_1 = \mu_1 - \mu_2 \\ & x_{k-1} - x_k = \mu_k, \quad x_k = \mu_k - \mu_{k+1} \quad 2 \leq k \leq n-1 \\ & x_{n-1} - x_n = \mu_n, \quad \frac{F_{m+1}}{F_m} x_n = \mu_n. \end{aligned}$$

Lemma 5 *It holds that*

- (i) $h(x, \mu) \geq 0 \quad \forall (x, \mu) \in R^n \times R^n$
- (ii) $h(x, \mu) = 0 \iff (x, \mu) \text{ satisfies } (EC_m).$

Corollary 2 *It holds that*

- (i) $f(x) \geq g(\mu) \quad \forall (x, \mu) \in R^n \times R^n$
- (ii) $f(x) = g(\mu) \iff (x, \mu) \text{ satisfies } (EC_m).$

From Corollary 2, it turns out that (P_m) and (D_m) are dual to each other, and (EC_m) is an equality condition. The equality condition (EC_m) is a linear system of $2n$ -equations on $2n$ -variables (x, μ) .

Lemma 6 *Let (x, μ) satisfy (EC_m) . Then both sides become a common value with five expressions:*

$$\begin{aligned} (5V_m) \quad & f(x) = c(c - x_1) = g(\mu) \\ & = \sum_{k=1}^{n-1} [\mu_k^2 + (\mu_k - \mu_{k+1})^2] + \mu_n^2 + \frac{F_m}{F_{m+1}} \mu_n^2 = c\mu_1. \end{aligned}$$

The primal (P_m) has a minimum value

$$m = f(x) = c(c - x_1)$$

at x , while the dual (D_m) has a maximum value

$$M = g(\mu) = \sum_{k=1}^{n-1} [\mu_k^2 + (\mu_k - \mu_{k+1})^2] + \mu_n^2 + \frac{F_m}{F_{m+1}} \mu_n^2 = c\mu_1$$

at μ .

Lemma 7 (EC_m) *has indeed a unique solution:*

$$\begin{aligned} x &= (x_1, x_2, \dots, x_k, \dots, x_{n-1}, x_n) \\ &= \frac{c}{F_{m+2n}} (F_{m+2n-2}, F_{m+2n-4}, \dots, F_{m+2n-2k}, \dots, F_{m+2}, F_m), \end{aligned} \quad (7)$$

$$\begin{aligned} \mu &= (\mu_1, \mu_2, \dots, \mu_k, \dots, \mu_{n-1}, \mu_n) \\ &= \frac{c}{F_{m+2n}} (F_{m+2n-1}, F_{m+2n-3}, \dots, F_{m+2n-2k+1}, \dots, F_{m+3}, F_{m+1}). \end{aligned} \quad (8)$$

Proof. From (EC_m) , we have a pair of linear systems of n -variable on n -equation:

$$(EQ_m) \quad \begin{array}{ll} c = 3x_1 - x_2 & c = 2\mu_1 - \mu_2 \\ x_1 = 3x_2 - x_3 & \mu_1 = 3\mu_2 - \mu_3 \\ \vdots & \vdots \\ x_{n-2} = 3x_{n-1} - x_n & \mu_{n-2} = 3\mu_{n-1} - \mu_n \\ x_{n-1} = \frac{F_{m+2}}{F_m} x_n & \mu_{n-1} = \frac{F_{m+3}}{F_{m+1}} \mu_n. \end{array}$$

The left system has a solution x in (7), while the right has a solution μ in (8). \square

Let us define two sequences $y = \{y_k\}_1^{2n}$, $\nu = \{\nu_k\}_1^{2n}$ from $x = \{x_k\}_0^n$, $\mu = \{\mu_k\}_1^n$ through

$$\begin{aligned} y_1 &= c - x_1, \quad y_2 = x_1, \quad y_3 = x_1 - x_2, \quad y_4 = x_2, \quad y_5 = x_2 - x_3 \\ &\dots, \quad y_{2n-2} = x_{n-1}, \quad y_{2n-1} = x_{n-1} - x_n, \quad y_{2n} = x_n \\ \nu_1 &= \mu_1, \quad \nu_2 = \mu_1 - \mu_2, \quad \nu_3 = \mu_2, \quad \nu_4 = \mu_2 - \mu_3, \quad \nu_5 = \mu_3 \\ &\dots, \quad \nu_{2n-2} = \mu_{n-1} - \mu_n, \quad \nu_{2n-1} = \mu_n, \quad \nu_{2n} = \mu_n \end{aligned} \quad (9)$$

, respectively. Then an identity

$$(C_m^*) \quad c\nu_1 = \sum_{k=1}^{2n-1} y_k \nu_k + \sqrt{\frac{F_{m+1}}{F_m}} y_{2n} \sqrt{\frac{F_m}{F_{m+1}}} \nu_{2n}$$

holds *under a constraint* – a linear system of $4n$ -variables (y, ν) on $2n$ -equations – :

$$(C^{y\nu}) \quad \begin{array}{ll} c = y_1 + y_2 & \nu_1 = \nu_2 + \nu_3 \\ y_2 = y_3 + y_4 & \nu_3 = \nu_4 + \nu_5 \\ \vdots & \vdots \\ y_{2n-4} = y_{2n-3} + y_{2n-2} & \nu_{2n-3} = \nu_{2n-2} + \nu_{2n-1} \\ y_{2n-2} = y_{2n-1} + y_{2n} & \nu_{2n-1} = \nu_{2n}. \end{array}$$

An equality (C_m^*) with constraint $(C^{y\nu})$ is called a $2n$ -variable *conditional complementarity*. This is simply written as (C_m^*) under $(C^{y\nu})$.

Now let $y = \{y_k\}_1^{2n}$, $\nu = \{\nu_k\}_1^{2n}$ satisfy $(C^{y\nu})$. Then the elementary inequality with equality yields

$$2c\nu_1 \leq \sum_{k=1}^{2n-1} (y_k^2 + \nu_k^2) + \frac{F_{m+1}}{F_m} y_{2n}^2 + \frac{F_m}{F_{m+1}} \nu_{2n}^2.$$

Thus we have an inequality

$$2c\nu_1 - \sum_{k=1}^{2n-1} \nu_k^2 - \frac{F_m}{F_{m+1}} \nu_{2n}^2 \leq \sum_{k=1}^{2n-1} y_k^2 + \frac{F_{m+1}}{F_m} y_{2n}^2.$$

The sign of equality holds iff

$$(EC_m) \quad y_k = \nu_k \quad 1 \leq k \leq 2n-1, \quad F_{m+1} y_{2n} = F_m \nu_{2n}. \quad (10)$$

We remark that an equivalence

$$\sqrt{\frac{F_{m+1}}{F_m}} y_{2n} = \sqrt{\frac{F_m}{F_{m+1}}} \nu_{2n} \iff \frac{F_{m+1}}{F_m} y_{2n} = \nu_{2n}$$

yields the last equality.

Hence we have a pair of conditional optimization problems:

$$\begin{aligned}
 & \text{minimize} \quad y_1^2 + y_2^2 + \cdots + y_{2n-1}^2 + \frac{F_{m+1}}{F_m} y_{2n}^2 \\
 & \text{subject to} \quad (1) \quad y_1 + y_2 = c \\
 & \quad \quad \quad (2) \quad y_3 + y_4 = y_2 \\
 & \quad \quad \quad \vdots \\
 & \quad \quad \quad (n-1) \quad y_{2n-3} + y_{2n-2} = y_{2n-4} \\
 & \quad \quad \quad (n) \quad y_{2n-1} + y_{2n} = y_{2n-2} \\
 & \quad \quad \quad (n+1) \quad y \in R^{2n}
 \end{aligned}
 \tag{P_m^*}$$

$$\begin{aligned}
 & \text{Maximize} \quad 2c\nu_1 - \left(\nu_1^2 + \nu_2^2 + \cdots + \nu_{2n-1}^2 + \frac{F_m}{F_{m+1}} \nu_{2n}^2 \right) \\
 & \text{subject to} \quad [1] \quad \nu_2 + \nu_3 = \nu_1 \\
 & \quad \quad \quad [2] \quad \nu_4 + \nu_5 = \nu_3 \\
 & \quad \quad \quad \vdots \\
 & \quad \quad \quad [n-1] \quad \nu_{2n-2} + \nu_{2n-1} = \nu_{2n-3} \\
 & \quad \quad \quad [n] \quad \nu_{2n} = \nu_{2n-1} \\
 & \quad \quad \quad [n+1] \quad \nu \in R^{2n}.
 \end{aligned}
 \tag{D_m^*}$$

Let (AC_m) be an *augmentation* of the system $(C_m^{y\nu})$ with the additional equality condition (EC_m) :

$$\begin{aligned}
 & c = y_1 + y_2 & \nu_1 &= \nu_2 + \nu_3 \\
 & y_2 = y_3 + y_4 & \nu_3 &= \nu_4 + \nu_5 \\
 & \vdots & \vdots & \\
 & (AC_m) \quad y_{2n-4} = y_{2n-3} + y_{2n-2} & \nu_{2n-3} &= \nu_{2n-2} + \nu_{2n-1} \\
 & y_{2n-2} = y_{2n-1} + y_{2n} & \nu_{2n-1} &= \nu_{2n} \\
 & y_k = \nu_k \quad 1 \leq k \leq 2n-1, \quad F_{m+1} y_{2n} = F_m \nu_{2n}.
 \end{aligned}$$

The linear system (AC_m) is of $4n$ -variables on $4n$ -equations. Let (y, ν) satisfy (AC_m) .

The system (AC_m) has indeed a unique solution:

$$\begin{aligned}
 & y = (y_1, y_2, \dots, y_k, \dots, y_{2n-2}, y_{2n-1}, y_{2n}) \\
 &= \frac{c}{F_{m+2n}} (F_{m+2n-1}, F_{m+2n-2}, \dots, F_{m+2n-k}, \dots, F_{m+2}, F_{m+1}, \underline{F_m}), \\
 & \nu = (\nu_1, \nu_2, \dots, \nu_k, \dots, \nu_{2n-2}, \nu_{2n-1}, \nu_{2n}) \\
 &= \frac{c}{F_{m+2n}} (F_{m+2n-1}, F_{m+2n-2}, \dots, F_{m+2n-k}, \dots, F_{m+2}, F_{m+1}, \underline{F_{m+1}}).
 \end{aligned}$$

Note that only the last elements are different, as underlined. However, in Case $m = 1$, both solutions are identical:

$$\begin{aligned} y &= (y_1, y_2, \dots, y_k, \dots, y_{2n-2}, y_{2n-1}, y_{2n}) \\ &= \nu = (\nu_1, \nu_2, \dots, \nu_k, \dots, \nu_{2n-2}, \nu_{2n-1}, \nu_{2n}) \\ &= \frac{c}{F_{2n+1}}(F_{2n}, F_{2n-1}, \dots, F_{2n-k+1}, \dots, F_3, F_2, \underline{F_1}). \end{aligned}$$

We note that $F_2 = F_1 = 1$.

Theorem 3 *The primal (P_m) has a minimum value $m = \frac{F_{m+2n-1}}{F_{m+2n}}c^2$ at a path*

$$\begin{aligned} \hat{y} &= (\hat{y}_1, \hat{y}_2, \dots, \hat{y}_k, \dots, \hat{y}_{2n-2}, \hat{y}_{2n-1}, \hat{y}_{2n}) \\ &= \frac{c}{F_{m+2n}}(F_{m+2n-1}, F_{m+2n-2}, \dots, F_{m+2n-k}, \dots, F_{m+2}, F_{m+1}, \underline{F_m}). \end{aligned}$$

The dual (D_m) has a maximum value $M = \frac{F_{m+2n-1}}{F_{m+2n}}c^2$ at a path

$$\begin{aligned} \nu^* &= (\nu_1^*, \nu_2^*, \dots, \nu_k^*, \dots, \nu_{2n-2}^*, \nu_{2n-1}^*, \nu_{2n}^*) \\ &= \frac{c}{F_{m+2n}}(F_{m+2n-1}, F_{m+2n-2}, \dots, F_{m+2n-k}, \dots, F_{m+2}, F_{m+1}, \underline{F_{m+1}}). \end{aligned}$$

Both optimal solutions (point and value) are identical except for the last element:

$$\hat{y}_k = \nu_k^* \quad 1 \leq k \leq 2n-1, \quad m = M.$$

Further both are Fibonacci:

$$\begin{aligned} \hat{y}_k = \nu_k^* &= \frac{F_{m+2n-k}}{F_{m+2n}}c \quad 1 \leq k \leq 2n-1, \quad \hat{y}_{2n} = \frac{F_m}{F_{m+2n}}c, \quad \nu_{2n}^* = \frac{F_{m+1}}{F_{m+2n}}c \\ m = M &= \frac{F_{m+2n-1}}{F_{m+2n}}c^2. \end{aligned}$$

Thus *Fibonacci Identical¹ Duality* (FID) holds between (P_m) and (D_m) [15–17].

References

- [1] E.F. Beckenbach and R.E. Bellman, *Inequalities*, Springer-Verlag, Ergebnisse **30**, 1961.
- [2] R.E. Bellman, *Dynamic Programming*, Princeton Univ. Press, NJ, 1957.
- [3] R.E. Bellman, *Introduction to the Mathematical Theory of Control Processes, Vol.I, Linear Equations and Quadratic Criteria*, Academic Press, NY, 1967.

¹*Identical means identical except for the last element.*

- [4] R.E. Bellman, *Methods of Nonlinear Analysis, Vol.I, Nonlinear Processes*, Academic Press, NY, 1969.
- [5] R.E. Bellman, *Introduction to the Mathematical Theory of Control Processes, Vol.II, Nonlinear Processes*, Academic Press, NY, 1971.
- [6] R.E. Bellman, *Methods of Nonlinear Analysis, Vol.II, Nonlinear Processes*, Academic Press, NY, 1972.
- [7] R.E. Bellman, *Introduction to Matrix Analysis*, McGraw-Hill, New York, NY, 1970 (Second Edition is a SIAM edition 1997).
- [8] R.E. Bellman, *Eye of the Hurricane: an Autobiography*, World Scientific, Singapore, 1984.
- [9] R.E. Bellman and Wm. Karush, On a new functional transform in analysis : the maximum transform, Bull. Amer. Math. Soc. 67(1961), 501-503.
- [10] R.E. Bellman and Wm. Karush, Mathematical programming and the maximum transform, J. SIAM Appl. Math. 10(1962), 550-567.
- [11] R.E. Bellman and Wm. Karush, On the maximum transform and semigroups of transformations, Bull. Amer. Math. Soc. 68(1962), 516-518.
- [12] R.E. Bellman and Wm. Karush, Functional equations in the theory of dynamic programming-XII : an application of the maximum transform, J. Math. Anal. Appl. 6(1963), 155-157.
- [13] R.E. Bellman and Wm. Karush, On the maximum transform, J. Math. Anal. Appl. 6(1963), 57-74.
- [14] S. Iwamoto, *Theory of Dynamic Program*, Kyushu Univ. Press, Fukuoka, 1987 (*in Japanese*).
- [15] S. Iwamoto, *Mathematics for Optimization II Bellman Equation*, Chisen Shokan, Tokyo, 2013 (*in Japanese*).
- [16] S. Iwamoto and Y. Kimura, Semi-Fibonacci programming – odd-variable – , RIMS Kokyuroku, Vol.2158, pp.30–37, 2020.
- [17] S. Iwamoto, Y. Kimura and T. Fujita, On complementary duals – both fixed points – , Bull. Kyushu Inst. Tech, Pure Appl. Math., No.67, pp.1–28, 2020.
- [18] E.S. Lee, *Quasilinearization and Invariant Imbedding*, Academic Press, New York, 1968.
- [19] R.T. Rockafeller, *Conjugate Duality and Optimization*, SIAM, Philadelphia, 1974.
- [20] M. Sniedovich, *Dynamic Programming: foundations and principles*, 2nd ed., CRC Press 2010.