Equilibrium problems and the proximal point algorithm on a complete geodesic space
完備測地距離空間における均衡問題と近接点法

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Abstract

We consider an equilibrium problem on a complete geodesic space having a curvature bounded above by one. We deal with a resolvent operator defined by the author, and apply the proximal point algorithm with this operator to approximate a solution to the problem.

1 Introduction

An equilibrium problem is to find a point x_0 in a subset K of a metric space X such that $f(x_0, y) \geq 0$ for every $y \in K$, where $f\colon K \times K \to \mathbb{R}$ is a given bifunction. We know that the class of equilibrium problems includes those of some important nonlinear problems such as convex minimization problems, fixed point problems, variational inequality problems, minimax problems, and others; see [1].

To analyze this problem, the notion of resolvent operators plays a crucial role. The following result shows well-definedness and several important properties of the resolvent operators in the setting of Hilbert spaces.

Theorem 1.1 (Combettes–Hirstoaga [3]). *Let H be a Hilbert space and K a nonempty closed convex subset of H. Let $f\colon K \times K \to \mathbb{R}$ and S_f the set of the solutions to the equilibrium problem for f. Suppose the following conditions:*

- $f(y, y) = 0$ *for all* $y \in K$*;*
- $f(y, z) + f(z, y) \leq 0$ *for all* $y, z \in K$*;*
- $f(y, \cdot)\colon K \to \mathbb{R}$ *is lower semicontinuous and convex for every* $y \in K$*;*
- $f(\cdot, z)\colon K \to \mathbb{R}$ *is upper hemicontinuous for every* $z \in K$*.*

Then the resolvent operator J_f *defined by*

$$J_f(x) = \left\{ z \in K \;\middle|\; \inf_{y \in K} (f(z,y) + \langle z - x, y - z \rangle) \geq 0 \right\}$$

has the following properties:

(i) *The domain of* J_f *is* H*;*
(ii) J_f *is single-valued and firmly nonexpansive;*
(iii) $\operatorname{Fix} J_f = \{z \in K \mid J_f z = z\}$ *coincides with* S_f*;*
(iv) S_f *is closed and convex.*

This result was generalized to the setting of Hadamard spaces by [6], and recently, the author [5] proposed a new definition and properties of the resolvent operators in the setting of CAT(1) spaces; see Theorem 3.1.

In this work, we apply the proximal point algorithm with this new resolvent operator to approximate a solution to the equilibrium problem, and obtain a Δ-convergence theorem of the generated sequence to its solution to the problem defined on a complete CAT(1) space.

2 Preliminaries

Let X be a metric space. For $x, y \in X$ with $l = d(x,y)$, a geodesic joining x and y is a mapping $c\colon [0,l] \to X$ such that $c(0) = x$, $c(l) = y$, and $d(c(s), c(t)) = |s - t|$ for any $s, t \in [0, l]$. If a geodesic joining x and y exists for every $x, y \in X$, then we call X a geodesic space. Further, if such a geodesic is unique for each $x, y \in X$, then X is called a uniquely geodesic space.

For a uniquely geodesic space X, we may define the convex combination between two points in a natural way. Namely, for $x, y \in X$ and $t \in [0,1]$, define $tx \oplus (1-t)y = c((1-t)d(x,y))$. Then it follows that

$$d(tx \oplus (1-t)y, x) = (1-t)d(x,y) \text{ and } d(tx \oplus (1-t)y, y) = td(x,y).$$

A CAT(1) space is usually defined as a geodesic space such that every triangle is thinner than its comparison triangle on a model space which is the two-dimensional unit sphere $\mathbb{S}^2$. One can find the formal definition and its equivalent conditions in [2] for instance. One of them is as follows: A uniquely geodesic space X is a CAT(1) space if and only if

$$\begin{aligned}\cos d(tx \oplus (1-t)y, z) \sin d(x,y) \\ \geq \cos d(x,z) \sin(td(x,y)) + \cos d(y,z) \sin((1-t)d(x,y))\end{aligned}$$

holds for every $x, y, z \in X$ with $d(y,z) + d(z,x) + d(x,y) < 2\pi$ and $t \in [0,1]$. This inequality can be understood as a generalization of the parallelogram laws and is a powerful tool to analyze our problems.

We say a CAT(1) space is admissible if $d(x, y) < \pi/2$ for every $x, y \in X$. If X is an admissible CAT(1) space, then the closed convex hull $\operatorname{cl}\operatorname{co} A$ of a subset A of X is defined as the intersection of all closed convex subsets of X including A. We say an admissible CAT(1) space X has the convex hull finite property [8] if every continuous selfmapping on $\operatorname{cl}\operatorname{co} E$ has a fixed point for every finite subset E of X.

Let $\{x_n\}$ be a sequence in a metric space X. A point $x_0 \in X$ is called an asymptotic center of $\{x_n\}$ if it satisifies that

$$\limsup_{n\to\infty} d(x_n, x_0) = \inf_{y\in X} \limsup_{n\to\infty} d(x_n, y).$$

Suppose X is an admissible complete CAT(1) space. We say that $\{x_n\}$ is spherically bounded if

$$\inf_{y\in X} \limsup_{n\to\infty} d(x_n, y) < \frac{\pi}{2}$$

holds. We say that $\{x_n\}$ is Δ-convergent to $x_0 \in X$ if x_0 is an asymptotic center of every subsequence $\{x_{n_i}\}$ of $\{x_n\}$. It is known that an asymptotic center of a spherically bounded sequence is always unique, and every spherically bounded sequence in admissible complete CAT(1) space has a Δ-convergent subsequence.

Let X be an admissible CAT(1) space and $T\colon X \to X$. The set of fixed points of T is denoted by $\operatorname{Fix} T$, that is,

$$\operatorname{Fix} T = \{z \in T \mid z = Tz\}.$$

A mapping $T\colon X \to X$ is said to be spherically nonspreading of sum type if it satisfies

$$\cos d(x, Ty) + \cos d(Tx, y) \le 2\cos d(Tx, Ty)$$

for every $x, y \in X$. We remark that the spherical nonspreadingness of mappings was originally defined in [7], and later, another type of spherical nonspreadingness was proposed in [4]. To distinguish them, in [4], the former one is said to be spherically nonspreading of product type, and latter is that of sum type.

We know that if a spherically nonspreading mapping T of sum type has a fixed point, then it is quasinonexpansive in the sense that $\operatorname{Fix} T \ne \emptyset$ and $d(Tx, z) \le d(x, z)$ for every $x \in X$ and $z \in \operatorname{Fix} T$.

Let C be a nonempty closed convex subset of an admissible complete CAT(1) space X. For $x \in X$, there exists a unique point $p_x \in C$ which is nearest to x, that is, it satisfies $d(x, p_x) = \inf_{z\in C} d(x, z)$. Using this fact, we define a mapping $P_C\colon X \to C$ by $P_C x = p_x$ for each $x \in X$ and we call it a metric projection onto C. We know that P_C is a quasinonexpansive mapping such that $\operatorname{Fix} P_C = C$.

Let X be a CAT(1) space and $g\colon X \to \mathbb{R}$. We say g is convex if

$$g(tx \oplus (1-t)y) \le tg(x) + (1-t)g(y)$$

for any $x, y \in X$ and $t \in]0, 1[$. A function g is said to be lower semicontinuous if $g(x_0) \le \liminf_{n\to\infty} g(x_n)$ whenever a sequence $\{x_n\}$ converges to x_0. We also say g is upper hemicontinuous if $\limsup_{t\to+0} g(tx \oplus (1-t)y) \le g(y)$ for every $x, y \in X$.

Theorem 2.1. *Let X be an admissible complete* CAT(1) *space and $\{x_n\}$ a spherically bounded sequence in X. Then, a unique asymptotic center belongs to* $\bigcap_{k\in\mathbb{N}} \operatorname{cl}\operatorname{co}\{x_k, x_{k+1}, x_{k+2}, \dots\}$.

Proof. Let $C_k = \operatorname{cl}\operatorname{co}\{x_k, x_{k+1}, x_{k+2}, \dots\}$ for arbitrarily fixed $k \in \mathbb{N}$, and $x_0 \in X$ an asymptotic center of $\{x_n\}$. Then, since the metric projection $P_{C_k}\colon X \to C_k$ is quasinonexpansive and $x_j \in C_k = \operatorname{Fix} P_{C_k}$ for $j \in \{k, k+1, k+2, \dots\}$, we have

$$d(P_{C_k} x_0, x_j) \le d(x_0, x_j)$$

for all $j \in \{k, k+1, k+2, \dots\}$. It follows that

$$\limsup_{n\to\infty} d(P_{C_k} x_0, x_n) \le \limsup_{n\to\infty} d(x_0, x_n).$$

From the uniqueness of the asymptotic center of $\{x_n\}$, we get $x_0 = P_{C_k} x_0 \in C_k$. Since $k \in \mathbb{N}$ is arbitrary, we obtain $x_0 \in \bigcap_{k\in\mathbb{N}} C_k$, the desired result. □

Using this theorem, we get the following result:

Theorem 2.2. *Let X be an admissble* CAT(1) *space and $g\colon X \to \mathbb{R}$ a lower semincontinuous convex function. Then, for a spherically bounded sequence $\{x_n\} \subset X$ with its aymptotic center $x_0 \in X$,*

$$g(x_0) \le \limsup_{n\to\infty} g(x_n).$$

Proof. For $k \in \mathbb{N}$, let

$$C_k = \left\{ x \in X \;\middle|\; g(x) \le \sup_{n\ge k} g(x_n) \right\}.$$

Then, from the convexity and lower semicontinuity of g, we have C_k is closed and convex. Since $\{x_k, x_{k+1}, x_{k+2}, \dots\} \subset C_k$, by Theorem 2.1 we have

$$x_0 \in \operatorname{cl}\operatorname{co}\{x_k, x_{k+1}, x_{k+2}, \dots\} \subset C_k$$

and thus $g(x_0) \le \sup_{n\ge k} g(x_n)$. Since $k \in \mathbb{N}$ is arbitrary, we obtain

$$g(x_0) \le \inf_{k\in\mathbb{N}} \sup_{n\ge k} g(x_n) = \limsup_{n\to\infty} g(x_n),$$

which is the desired result. □

3 Proximal point algorithm for equilibrium problems

Let X be an admissible complete CAT(1) space. For a bifunction $f\colon K \times K \to \mathbb{R}$ defined on a nonempty closed convex subset K of X, we consider an equilibrium problem. This problem is to find a point $z_0 \in K$ such that $f(z_0, y) \geq 0$ for every $y \in K$. The set of its solutions is denoted by $\operatorname{Equil} f$;

$$\operatorname{Equil} f = \left\{ z \in K \;\middle|\; \inf_{y \in K} f(z, y) \geq 0 \right\}.$$

To deal with this problem, we often assume the following conditions for f.

(E1) $f(y, y) = 0$ for every $y \in K$;
(E2) $f(y, z) + f(z, y) \leq 0$ for every $y, z \in K$;
(E3) $f(y, \cdot) \colon K \to \mathbb{R}$ is lower semincontinuous and convex for every $y \in K$;
(E4) $f(\cdot, z) \colon K \to \mathbb{R}$ is upper hemicontinuous for every $z \in K$.

For the equilibruim problems on CAT(1) spaces, the resolvent operator $R_f \colon X \to K$ plays an important role, which was proposed by the author. Its definition and some useful properties are as follows:

Theorem 3.1 (Kimura [5]). *Let X be an admissible complete* CAT(1) *space having the convex hull finite property, K a nonempty closed convex subset of X. Suppose that $f \colon K \times K \to \mathbb{R}$ satisfies the conditions* (E1)–(E4). *For each $x \in X$, define a subset $R_f x$ of K by*

$$R_f x = \left\{ z \in K \;\middle|\; \inf_{y \in K} (f(z, y) - \log \cos d(x, y) + \log \cos d(x, z)) \geq 0 \right\}.$$

Then, the following hold:

(i) *$R_f x$ consists of one point for every $x \in X$, and therefore $R_f \colon X \to K$ is defined as a single-valued mapping;*
(ii) *R_f satisfies the following inequality for any $x_1, x_2 \in X$:*

$$\frac{\cos d(x_1, R_f x_2)}{\cos d(x_1, R_f x_1)} + \frac{\cos d(x_2, R_f x_1)}{\cos d(x_2, R_f x_2)} \leq 2 \cos d(R_f x_1, R_f x_2),$$

and thus R_f is spherically nonspreading of sum type.
(iii) $\operatorname{Fix} R_f = \operatorname{Equil} f$ *and it is closed and convex.*

In the proof of Theorem 3.1, the following properties are also obtained.

Theorem 3.2. *Let X, K, f, and R_f are the same as the previous theorem. Then, the following hold:*

(i) *For $x \in X$ and $w \in K$ with $w \neq R_f x$,*

$$0 \leq f(R_f x, w) + \frac{d(w, R_f x)}{\sin d(w, R_f x)} \left(\cos d(w, R_f x) - \frac{\cos d(x, w)}{\cos d(x, R_f x)} \right);$$

(ii) *for $x \in K$ and $z \in \operatorname{Equil} f$,*

$$\cos d(x, R_f x) \cos d(z, R_f x) \geq \cos d(x, z).$$

For the sake of completeness, we give the proof.

Proof. (i) Since $tw \oplus (1-t)R_f x \in K$ for $t \in]0,1[$, we have

$$\begin{aligned}0 &\le f(R_f x, tw \oplus (1-t)R_f x)\\ &\quad - \log\cos d(x, tw \oplus (1-t)R_f x) + \log\cos d(x, R_f x)\\ &\le tf(R_f x, w) + (1-t)f(R_f x, R_f x)\\ &\quad - \log(\cos d(x,w)\sin(td(w,R_f x)) + \cos d(x, R_f x)\sin((1-t)d(w, R_f x)))\\ &\quad + \log\sin d(w, R_f x) + \log\cos d(x, R_f x)\\ &= tf(R_f x, w) + L(t),\end{aligned}$$

where

$$\begin{aligned}L(t) = &- \log(\cos d(x,w)\sin(td(w,R_f x)) + \cos d(x, R_f x)\sin((1-t)d(w, R_f x)))\\ &+ \log(\cos d(x, R_f x)\sin d(w, R_f x)).\end{aligned}$$

Notice that $L(t) \to 0$ as $t \to +0$. By l'Hospital's rule, we have

$$0 \le f(R_f x, w) + \lim_{t\to+0}\frac{L(t)}{t} = f(R_f x, w) + \lim_{t\to+0}\frac{dL(t)}{dt}.$$

Since

$$\begin{aligned}&\frac{dL(t)}{dt}\\ &= \frac{d(w,R_f x)(-\cos d(x,w)\cos(td(w,R_f x)) + \cos d(x,R_f x)\cos((1-t)d(w,R_f x)))}{\cos d(x,w)\sin(td(w,R_f x)) + \cos d(x,R_f x)\sin((1-t)d(w,R_f x))}\\ &\to \frac{d(w,R_f x)(-\cos d(x,w) + \cos d(x,R_f x)\cos d(w,R_f x))}{\cos d(x,R_f x)\sin d(w,R_f x)}\\ &= \frac{d(w,R_f x)}{\sin d(w,R_f x)}\left(\cos d(w,R_f x) - \frac{\cos d(x,w)}{\cos d(x,R_f x)}\right)\end{aligned}$$

as $t \to +0$, we have

$$0 \le f(R_f x, w) + \frac{d(w,R_f x)}{\sin d(w,R_f x)}\left(\cos d(w,R_f x) - \frac{\cos d(x,w)}{\cos d(x,R_f x)}\right),$$

which is the desired inequality.

(ii) It is obvious if $z = R_f x$. Otherwise, from (i) we have

$$\cos d(z, R_f x) - \frac{\cos d(x,z)}{\cos d(x, R_f x)} \ge -f(R_f x, z)\frac{\sin d(z, R_f x)}{d(z, R_f x)}.$$

On the other hand, since $z \in \operatorname{Equil} f$, using (E2) we have

$$-f(R_f x, z) \ge f(z, R_f x) \ge 0.$$

Hence we have

$$\cos d(z, R_f x) - \frac{\cos d(x,z)}{\cos d(x, R_f x)} \geq 0,$$

which implies the desired inequality. □

If f satisfies (E1)–(E4) as in the theorem above, then λf also satisfies the same conditions for $\lambda \in]0, \infty[$. Thus we can define the resolvent for λf.

Using the properties of the resolvent operator of bifunction, we obtain the following Δ-convergence theorem of the proximal point algorithm for equilibrium problems on admissible complete CAT(1) spaces.

Theorem 3.3. *Let X be an admissible complete* CAT(1) *space having the convex hull finite property, and K a nonempty closed convex subset of X. Let $f\colon K \times K \to \mathbb{R}$ satisfy the conditions* (E1)–(E4) *mentioned above, and suppose that* $\operatorname{Equil} f \neq \emptyset$. *Let $\{\lambda_n\}$ be a positive real sequence such that $\inf_{n\in\mathbb{N}} \lambda_n > 0$. For $n \in \mathbb{N}$, an operator $R_{\lambda_n f}\colon X \to K$ is defined by*

$$R_{\lambda_n f} x = \left\{ z \in K \;\middle|\; \inf_{y\in K} (\lambda_n f(z,y) - \log\cos d(x,y) + \log\cos d(x,z)) \geq 0 \right\}$$

for $x \in X$. Let $\{x_n\} \subset X$ be a sequence generated by $x_1 \in X$ and

$$x_{n+1} = R_{\lambda_n f} x_n$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ is Δ-convergent to $z_0 \in \operatorname{Equil} f$.

Proof. Let $z \in \operatorname{Equil} f$. By Theorem 3.1 (ii), each $R_{\lambda_n f}$ is spherically nonspreading of sum type, and thus it is quasinonexpansive. Since $z \in \operatorname{Equil} f = \operatorname{Fix} R_{\lambda_n f}$ by Theorem 3.1 (iii), we have

$$0 \leq d(x_{n+1}, z) = d(R_{\lambda_n f} x_n, z) \leq d(x_n, z)$$

for all $n \in \mathbb{N}$. It follows that, for every $z \in \operatorname{Equil} f$, a nonnegative real sequence $\{d(x_n, z)\}$ is nonincreasing and bounded below, and hence it has a limit $c_z \in [0, \infty[$. We also have $\{x_n\}$ is spherically bounded since $c_z = \lim_{n\to\infty} d(x_n, z) \leq d(x_1, z) < \pi/2$. Using Theorem 3.2 (ii), for $z \in \operatorname{Equil} f$ we have

$$\begin{aligned} 1 \geq \cos d(x_n, x_{n+1}) &= \cos d(x_n, R_{\lambda_n f} x_n) \\ &\geq \frac{\cos d(x_n, z)}{\cos d(z, R_{\lambda_n f} x_n)} \\ &= \frac{\cos d(x_n, z)}{\cos d(x_{n+1}, z)} \to \frac{c_z}{c_z} = 1 \end{aligned}$$

as $n \to \infty$. It implies that $\lim_{n\to\infty} \cos d(x_n, x_{n+1}) = 1$ and $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.

Fix $w \in K$ arbitrarily. For $n \in \mathbb{N}$, we first assume that $w \neq x_{n+1} = R_{\lambda_n f} x_n$. Since $\inf_{n\in\mathbb{N}} \lambda_n > 0$ and $d/\sin d$ is bounded on $d \in]0, \pi/2[$, there exists $M \in [0, \infty[$ such that

$$d(w, R_{\lambda_n f} x_n) \leq M \lambda_n \sin d(w, R_{\lambda_n f} x_n)$$

for every $n \in \mathbb{N}$. Using Theorem 3.1 (i) with (E2), we have

$$\begin{aligned}
0 &\leq f(R_{\lambda_n f} x_n, w) + \frac{d(w, R_{\lambda_n f} x_n)}{\lambda_n \sin d(w, R_{\lambda_n f} x_n)} \left(\cos d(w, R_{\lambda_n f} x_n) - \frac{\cos d(x_n, w)}{\cos d(x_n, R_{\lambda_n f} x_n)} \right) \\
&\leq -f(w, R_{\lambda_n f} x_n) + M \left| \cos d(w, R_{\lambda_n f} x_n) - \frac{\cos d(x_n, w)}{\cos d(x_n, R_{\lambda_n f} x_n)} \right| \\
&= -f(w, x_{n+1}) + M \left| \cos d(w, x_{n+1}) - \frac{\cos d(x_n, w)}{\cos d(x_n, x_{n+1})} \right|,
\end{aligned}$$

and thus

$$f(w, x_{n+1}) \leq M \left| \cos d(w, x_{n+1}) - \frac{\cos d(x_n, w)}{\cos d(x_n, x_{n+1})} \right|.$$

Notice that this inequality holds for every $n \in \mathbb{N}$ since it is obviously true if $w = x_{n+1}$. We also have

$$\begin{aligned}
&\left| \cos d(w, x_{n+1}) - \frac{\cos d(x_n, w)}{\cos d(x_n, x_{n+1})} \right| \\
&\leq |\cos d(w, x_{n+1}) - \cos d(x_n, w)| + \cos d(x_n, w) \left| 1 - \frac{1}{\cos d(x_n, x_{n+1})} \right| \\
&= \left| -2 \sin \frac{d(w, x_{n+1}) + d(x_n, w)}{2} \sin \frac{d(w, x_{n+1}) - d(x_n, w)}{2} \right| \\
&\quad + \cos d(x_n, w) \left| 1 - \frac{1}{\cos d(x_n, x_{n+1})} \right| \\
&\leq 2 \sin \frac{d(x_{n+1}, x_n)}{2} + \left| 1 - \frac{1}{\cos d(x_n, x_{n+1})} \right| \to 0
\end{aligned}$$

as $n \to \infty$. Therefore we have

$$\limsup_{n\to\infty} f(w, x_n) \leq 0$$

for all $w \in K$. Let $\{x_{n_i}\}$ be an arbitrary subsequence of $\{x_n\}$ with its asymptotic center $x_0 \in K$. Then, x_0 belongs to $\operatorname{Equil} f$. Indeed, by Theorem 2.1, we have

$$f(w, x_0) \leq \limsup_{i\to\infty} d(w, x_{n_i}) \leq 0$$

for every $w \in K$. For arbitrarily fixed $v \in K$ and for $t \in]0, 1[$, let $w_t = tv \oplus (1-t)x_0 \in K$. Then, by (E1) and (E3), we have

$$0 = f(w_t, w_t) = f(w_t, tv \oplus (1-t)x_0) \leq t f(w_t, v) + (1-t) f(w_t, x_0) \leq t f(w_t, v),$$

which implies $f(tv \oplus (1-t)x_0, v) = f(w_t, v) \geq 0$. Letting $t \to +0$, by (E4) we have

$$f(x_0, v) \geq \limsup_{t \to +0} f(tv \oplus (1-t)x_0, v) \geq 0.$$

Since $v \in K$ is arbitrary, we obtain $x_0 \in \operatorname{Equil} f$.

Let z_0 be an asymptotic center of $\{x_n\}$. We show $z_0 = x_0$. Since $x_0 \in \operatorname{Equil} f$, a positive real sequence $\{d(x_n, x_0)\}$ has a limit c_{x_0}. From the definition of the asymptotic center, we have

$$\limsup_{n \to \infty} d(x_n, x_0) = c_{x_0} = \lim_{i \to \infty} d(x_{n_i}, x_0) \leq \limsup_{i \to \infty} d(x_{n_i}, z_0) \leq \limsup_{n \to \infty} d(x_n, z_0).$$

From the uniqueness of the asymptotic center of $\{x_n\}$, we get $z_0 = x_0$. Since an asymptotic center of every subsequence of $\{x_n\}$ is identical to z_0, we obtain $\{x_n\}$ is Δ-convergent to $z_0 \in \operatorname{Equil} f$. □

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