Convergence theorems of Picard-Mann iteration for generalized nonexpansive mappings on a geodesic space

測地距離空間における generalized nonexpansive 写像を用いた Picard-Mann 型の収束定理

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Abstract

In this paper, we first prove existence of fixed points for generalized nonexpansive mappings which satisfy the so-called condition (E) in CAT(1) spaces. Then, we prove convergence theorems to them by using Picard-Mann hybrid iteration process.

1 Introduction

Approximating fixed points is one of the main topics in the fixed point theory. In the theory, we have considered some methods to approximate fixed points. For instance, Mann type iterative method has been known as a very popular method to do that. That definition is as follows: Let X be a metric space having some convexity structure. For a mapping $T : X \to X$ and given point $x_1 \in X$, let

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n,$$

for $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a coefficient sequence for convex combination. In 2012, He, Fang, Lopez and Li [3] proved the following Δ-convergence theorem by using Mann type iteration in a complete CAT(κ) space.

Theorem 1 (He, Fang, Lopez and Li [3]). *Let X be a complete* CAT(κ) *space. Let*

$T : X \to X$ be a nonexpansive mapping such that $F(T) \neq \varnothing$. Let $x_1 \in X$ be such that $d(x_1, F(T)) < D_\kappa/4$, where $D_\kappa = \infty$ if $\kappa \leq 0$ and $D_\kappa = \pi/\sqrt{\kappa}$ if $\kappa > 0$. For a sequence of positive real unmbers $\{\alpha_n\} \subset]0,1[$ satisfying $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$, define a sequence $\{x_n\}$ by

$$x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n) T x_n,$$

for $n \in \mathbb{N}$. Then, $\{x_n\}$ is Δ-convergent to some point in $F(T)$.

On the other hand, Picard type iteration has been studied, which is another method to approximate fixed points. That definition is as follows: For a mapping $T : X \to X$ and given point $x_1 \in X$, let

$$x_{n+1} = T x_n,$$

for $n \in \mathbb{N}$. This type of iteration was used in the famous Banach contraction principle. In 2013, Khan introduced Picard-Mann hybrid iterative process defined by the composition of Mann type and Picard type iterations. Its definition is as follows: For a mapping $T : X \to X$ and given point $x_1 \in X$, let

$$\begin{aligned} x_{n+1} &= T y_n, \\ y_n &= (1-\alpha_n) x_n \oplus \alpha_n T x_n, \end{aligned}$$

where $\alpha_n \in]b,c[\subset]0,1[$ for $n \in \mathbb{N}$. In 2017, Ritika and Khan proved Δ-convergence and strong convergence theorems by using it for a generalized nonexpansive mapping in a complete CAT(0) space.

Theorem 2 (Ritika and Khan [6]). *Let X be a complete* CAT(0) *space, C a closed convex subset of X. Let T be a generalized nonexpansive mapping of C into itself and $F(T) \neq \varnothing$. Let $x_1 \in C$. Suppose that $\{x_n\} \subset C$ is a sequence defined by Picard-Mann hybrid iteration. Then, $\{x_n\}$ is Δ-convergent an element of $F(T)$.*

In this paper, we consider Δ-convergence and strong convergence theorems with Picard-Mann hybrid iteration for a generalized nonexpansive mapping in complete CAT(1) spaces.

2 Preliminaries

Let X be a metric space. For $x, y \in X$ and $l \geq 0$, a mapping $c : [0,l] \to X$ is called a geodesic with endpoints $x, y \in X$ if it satisfies $c(0) = x, c(l) = y$, and $d(c(t), c(s)) = |t-s|$ for every $t, s \in [0,l]$. If a geodesic with endpoints x and y exists for all $x, y \in X$, we call X a geodesic space. In this paper, we assume X has a unique geodesic for every $x, y \in X$, which is called a uniquely geodesic space. Then, we denote the image of the geodesic with endpoints $x, y \in X$ by $[x, y]$, which is well defined.

For $x, y, z \in X$ such that $d(x,y)+d(y,z)+d(z,x) < 2\pi$, a geodesic triangle $\triangle(x,y,z)$ with vertices $x, y, z \in X$ is defined as $[x,y] \cup [y,z] \cup [z,x]$. Its comparision triangle

$\triangle(\bar{x}, \bar{y}, \bar{z})$ is defined as the triangle in the 2-dimensional unit sphere $\mathbb{S}^2$ whose length of each corresponding edge is identical with that of the original triangle;

$$d(x, y) = d_{\mathbb{S}^2}(\bar{x}, \bar{y}), \quad d(y, z) = d_{\mathbb{S}^2}(\bar{y}, \bar{z}), \quad d(z, x) = d_{\mathbb{S}^2}(\bar{z}, \bar{x}).$$

A point $\bar{p} \in [\bar{x}, \bar{y}]$ is called a comparison point for $p \in [x, y]$ if $d(x, p) = d_{\mathbb{S}^2}(\bar{x}, \bar{p})$. If for any $p, q \in \triangle(x, y, z)$ and their comparison points $\bar{p}, \bar{q} \in \triangle(\bar{x}, \bar{y}, \bar{z})$, the inequality

$$d(p, q) \leq d_{\mathbb{S}^2}(\bar{p}, \bar{q})$$

holds for all triangles in X, then we call X a CAT(1) space. For $x, y \in X$ and $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that $d(x, z) = (1 - t)d(x, y)$ and $d(z, y) = td(x, y)$. We denote it by $tx \oplus (1 - t)y$. In CAT(1) spaces, the following inequality holds;

$$\cos d(tx \oplus (1-t)y, z) \sin d(x, y) \geq \cos d(x, z) \sin td(x, y) + \cos d(y, z) \sin (1 - t)d(x, y)$$

for $x, y, z \in X$ such that $d(x, y) + d(y, z) + d(z, x) < 2\pi$ and for all $t \in [0, 1]$. This inequality plays an important role in CAT(1) spaces.

For a bounded sequence $\{x_n\}$ in X, let $r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n)$ for $x \in X$, and define the asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ by

$$r(\{x_n\}) = \inf_{x \in X} r(x, \{x_n\}).$$

The asymptotic center $\mathrm{AC}(\{x_n\})$ of $\{x_n\}$ is defined by

$$\mathrm{AC}(\{x_n\}) = \{p \in X \mid r(p, \{x_n\}) = r(\{x_n\})\}.$$

We say $\{x_n\}$ is Δ-convergent to $x_0 \in X$ if x_0 is the unique asymptotic center of any subsuquence of $\{x_n\}$.

Let X be a CAT(1) space. If any $p, q \in X$ satisfy $d(p, q) < \frac{\pi}{2}$, we say X is admissible. Suppose $\{x_n\} \subset X$. If there exists $u \in X$ such that

$$\limsup_{n\to\infty} d(u, x_n) < \frac{\pi}{2},$$

$\{x_n\}$ is said to be spherically bounded. We know the following theorem regarding spherically bounded sequences in CAT(1) spaces.

Theorem 3 ([1])**.** *Let X be a complete* CAT(1) *space. Suppose $\{x_n\} \subset X$ is spherically bounded. Then,* $\mathrm{AC}(\{x_n\})$ *is singleton and there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x_0 \in X$ such that $x_{n_k} \overset{\Delta}{\rightharpoonup} x_0$.*

For the basic properties of CAT(1) spaces, see [1].

We also know [3] if $\{x_n\}$ is a Δ-convergent sequence in CAT(1) space with a Δ-limit z, then

$$d(z, y) \leq \liminf_{n\to\infty} d(x_n, y)$$

for every $y \in X$. For more details of Δ-convergence, see [4, 5].

Let X be a metric space and $T : X \to X$ a mapping. T is a nonexpansive mapping if

$$d(Tx, Ty) \leq d(x, y)$$

for $x, y \in X$. Let $T : X \to X$ be a mapping and $F(T) = \{z \in X : Tz = z\} \neq \varnothing$. T is a quasi-nonexpansive mapping if

$$d(x, Tp) \leq d(x, p)$$

for $x \in X$ and $p \in F(T)$. Let $T : X \to X$ be a mapping. T is a generalized nonexpansive mapping in the sense of [2] if there exists $\mu \geq 1$ such that

$$d(x, Ty) \leq \mu d(x, Tx) + d(x, y)$$

for $x, y \in X$. If $T : X \to X$ is a generalized nonexpansive mapping and $F(T) \neq \varnothing$, then for $p \in F(T)$,

$$\begin{aligned} d(p, Tx) &\leq \mu d(p, Tp) + d(p, x) \\ &\leq d(p, x). \end{aligned}$$

for every $x \in X$. This shows T is a quasi-nonexpansive mapping. If T is nonexpansive, then we have

$$d(x, Ty) - d(x, Tx) \leq d(Tx, Ty) \leq d(x, y).$$

Thus, there exists $\mu \geq 1$ such that

$$\begin{aligned} d(x, Ty) &\leq d(x, Tx) + d(x, y) \\ &\leq \mu d(x, Tx) + d(x, y). \end{aligned}$$

From this inequality, T is a generalized nonexpansive mapping. Therefore, the class of generalized nonexpansive mapping includes the class of nonexpansive mapping. However, the converse is not true. See [6].

Let X be a CAT(1) space and $T : X \to X$ a mapping. Suppose $x_0 \in X$ and $\{x_n\} \subset X$. T is Δ-demiclosed if it satisfies the following conditions:

$$x_n \stackrel{\Delta}{\rightharpoonup} x_0 \text{ and } d(Tx_n, x_n) \to 0 \Longrightarrow x_0 \in F(T).$$

3 Main results

In this section, we show some convergence theorems in complete CAT(1) spaces. The following results are used to prove the main theorem.

Theorem 4. *Let X be a complete* CAT(1) *space and $\{t_n\} \subset [b, 1-b] \subset]0,1[$ where $0 < b < \frac{1}{2}$. Let $\{x_n\}, \{y_n\} \subset X$ and $w_n = t_n x_n \oplus (1-t_n) y_n$. Suppose $x \in X$ and there exists $r \in]0, \frac{\pi}{2}[$ such that*

$$\begin{aligned} \limsup_{n\to\infty} d(x_n, x) &\le r, \\ \limsup_{n\to\infty} d(y_n, x) &\le r, \\ \lim_{n\to\infty} d(w_n, x) &= r. \end{aligned}$$

Then, $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Proof. We may assume $x_n \neq y_n$ for all $n \in \mathbb{N}$. From the following inequality

$$\cos d(w_n, x) \sin d(x_n, y_n) \ge \cos d(x_n, x) \sin t_n d(x_n, y_n) + \cos d(y_n, x) \sin (1-t_n) d(x_n, y_n),$$

we have

$$\cos d(w_n, x) \ \ge \cos d(x_n, x) \frac{\sin t_n d(x_n, y_n)}{\sin d(x_n, y_n)} + \cos d(y_n, x) \frac{\sin(1-t_n) d(x_n, y_n)}{\sin d(x_n, y_n)}.$$

Let $\varepsilon > 0$ with $\epsilon < \frac{\pi}{2} - r$. From the assumptions, there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} d(x_n, x) &< r + \varepsilon, \\ d(y_n, x) &< r + \varepsilon, \\ d(w_n, x) &> r - \varepsilon. \end{aligned}$$

for all $n \ge n_0$. Therefore, we have

$$\begin{aligned} \cos(r-\varepsilon) &> \cos d(w_n, x) \\ &> \cos(r+\varepsilon) \frac{\sin t_n d(x_n, y_n) + \sin(1-t_n) d(x_n, y_n)}{\sin d(x_n, y_n)}. \end{aligned}$$

Thus we get

$$\frac{\sin t_n d(x_n, y_n) + \sin(1-t_n) d(x_n, y_n)}{\sin d(x_n, y_n)} < \frac{\cos(r-\varepsilon)}{\cos(r+\varepsilon)}.$$

It follows that

$$\begin{aligned} \frac{\cos(r-\epsilon)}{\cos(r+\epsilon)} &> \frac{\sin t_n d(x_n, y_n) + \sin(1-t_n) d(x_n, y_n)}{\sin d(x_n, y_n)} \\ &= \frac{2 \sin \dfrac{d(x_n, y_n)}{2} \cos \dfrac{(1-2t_n) d(x_n, y_n)}{2}}{2 \sin \dfrac{d(x_n, y_n)}{2} \cos \dfrac{d(x_n, y_n)}{2}} \end{aligned}$$

$$= \frac{\cos \dfrac{(1-2t_n)d(x_n,y_n)}{2}}{\cos \dfrac{d(x_n,y_n)}{2}}.$$

Since $0 < b \leq t_n \leq 1-b < 1$, we have

$$1 > 1-2b \geq 1-2t_n \geq -1+2b > -1,$$

that is, we have

$$|1-2t_n| \leq 1-2b < 1.$$

Thus, we obtain

$$\cos \frac{d(x_n,y_n)}{2} < \cos \frac{(1-2b)d(x_n,y_n)}{2} \leq \cos \frac{|1-2t_n|d(x_n,y_n)}{2}$$

Therefore, we get

$$1 < \frac{\cos \dfrac{(1-2b)d(x_n,y_n)}{2}}{\cos \dfrac{d(x_n,y_n)}{2}} \leq \frac{\cos \dfrac{|1-2t_n|d(x_n,y_n)}{2}}{\cos \dfrac{d(x_n,y_n)}{2}} < \frac{\cos(r-\varepsilon)}{\cos(r+\varepsilon)}.$$

Letting $\varepsilon \to 0$, we have

$$\frac{\cos(r-\varepsilon)}{\cos(r+\varepsilon)} \to 1.$$

This implies

$$\frac{\cos \dfrac{(1-2b)d(x_n,y_n)}{2}}{\cos \dfrac{d(x_n,y_n)}{2}} = 1. \tag{1}$$

For any subsequence $\{d(x_{n_i},y_{n_i})\}$ of $\{d(x_n,y_n)\}$, there exists $\{d(x_{n_{i_j}},y_{n_{i_j}})\} \subset \{d(x_{n_i},y_{n_i})\}$ such that $\{d(x_{n_{i_j}},y_{n_{i_j}})\}$ converges to $d \in \mathbb{R}$. Replacing x_n with $x_{n_{i_j}}$ in (1), and letting $j \to \infty$ we have $\cos \frac{(1-2b)d}{2} / \cos \frac{d}{2} = 1$. Therefore, we have

$$\begin{aligned} 0 &= \cos \frac{(1-2b)d}{2} - \cos \frac{d}{2} \\ &= \cos \frac{(1-2b)d}{2} - \cos \frac{d}{2} \\ &= -2\sin \frac{2(1-b)d}{2} \sin \frac{-2bd}{2}. \end{aligned}$$

This shows $\sin \frac{2(1-b)d}{2} = 0$ or $\sin \frac{-2bd}{2} = 0$. Since $b \neq 0, 1$, we have $d = 0$, and hence $\lim_{n\to\infty} d(x_n,y_n) = 0$. □

The next lemma is regarding Δ-demiclosedness of generalized nonexpanisive mappings.

Lemma 1. *Let X be an admissible complete* CAT(1) *space, T a generalized nonexpansive mapping of X into itself. Suppose $\{x_{n_k}\} \subset \{x_n\}, x_0 \in \mathrm{AC}(\{x_{n_k}\})$ and $d(x_n, Tx_n) \to 0$. Then $x_0 \in F(T)$, that is, T is Δ-demiclosed.*

Proof. Let $x_0 \in X$. Then we have

$$\begin{aligned}\limsup_{k\to\infty} d(x_0, x_{n_k}) &\le \limsup_{k\to\infty} d(Tx_0, x_{n_k}) \\ &\le \limsup_{k\to\infty}(\mu d(Tx_{n_k}, x_{n_k}) + d(x_{n_k}, x_0)) \\ &\le \mu \limsup_{k\to\infty} d(Tx_{n_k}, x_{n_k}) + \limsup_{n\to\infty} d(x_0, x_{n_k}) \\ &= \limsup_{k\to\infty} d(x_0, x_{n_k}).\end{aligned}$$

So, we have $\limsup_{k\to\infty} d(x_0, x_{n_k}) = \limsup_{k\to\infty} d(Tx_0, x_{n_k})$. From the uniqueness of the asymptotic center of $\{x_{n_k}\}$, we get $Tx_0 \in \mathrm{AC}(\{x_{n_k}\})$. This implies $x_0 \in F(T)$. □

The next is about the existence of fixed point of generalized nonexpansive T.

Lemma 2. *Let X be an admissible complete* CAT(1) *space, T a generalized nonexpansive mapping of X into itself. Let $x_1 \in X$. Let $\{x_n\}$ be a sequence defined by the Picard-Mann hybrid iteration as follows:*

$$\begin{aligned} x_{n+1} &= Ty_n; \\ y_n &= (1-\alpha_n)x_n \oplus \alpha_n Tx_n, \end{aligned}$$

where $\alpha_n \in]b, c[\subset]0, 1[$. If $\{x_n\}$ is spherically bounded and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then $F(T) \neq \varnothing$.

Proof. Since $\{x_n\}$ is spherically bounded, $\mathrm{AC}(\{x_n\})$ is singleton. Let $\{p\} = \mathrm{AC}(\{x_n\})$. Since T is generalized nonexpansive, we have

$$d(x_n, Tp) \le \mu d(x_n, Tx_n) + d(x_n, p).$$

From this inequality and the assumpution, we get

$$\begin{aligned}\limsup_{n\to\infty} d(x_n, Tp) &\le \limsup_{n\to\infty} \mu d(x_n, Tx_n) + \limsup_{n\to\infty} d(x_n, p) \\ &\le \limsup_{n\to\infty} d(x_n, p).\end{aligned}$$

Then we have $r(Tp, \{x_n\}) \le r(p, \{x_n\})$. Since $\mathrm{AC}(\{x_n\})$ is singleton, we have $Tp = p$. It implies $F(T) \neq \varnothing$. □

Lemma 3. *Let X be an admissible complete* CAT(1) *space, T a generalized nonexpansive mapping of X into itself. Let $\{\alpha_n\} \subset [0,1]$ and $x_1 \in X$. Suppose $\{x_n\}$ is a sequence defined by the same as Lemma 8. If $F(T) \neq \varnothing$, then $\{x_n\}$ is spherically bounded and there exists $\lim_{n\to\infty} d(x_n, p) < \frac{\pi}{2}$ for all $p \in F(T)$.*

Proof. Let $p \in F(T)$. Then we have

$$\begin{aligned}
\cos d(x_{n+1}, p) &= \cos d(Ty_n, p) \\
&\geq \cos(\mu d(Tp, p) + d(y_n, p)) \\
&= \cos d(y_n, p) \\
&= \cos d((1-\alpha_n)x_n \oplus \alpha_n Tx_n, p) \\
&\geq (1-\alpha_n)\cos d(x_n, p) + \alpha_n \cos d(Tx_n, p) \\
&\geq (1-\alpha_n)\cos d(x_n, p) + \alpha_n \cos d(x_n, p) \\
&= \cos d(x_n, p).
\end{aligned}$$

Since $\{d(x_n, p)\}$ is nonincreasing and bounded below, there exists $\lim\limits_{n\to\infty} d(x_n, p) < d(x_1, p) < \dfrac{\pi}{2}$. It also shows that $\{x_n\}$ is spherically bounded. □

Theorem 5. *Let X be an admissible complete* CAT(1) *space, and T a generalized nonexpansive mapping of X into itself. Let $x_1 \in X$. Suppose $\{x_n\} \subset X$ is a sequence defined by Picard-Mann hybrid iteration as follows:*

$$\begin{cases} x_{n+1} &= Ty_n; \\ y_n &= (1-\alpha_n)x_n \oplus \alpha_n Tx_n, \end{cases}$$

where $\alpha_n \in]b, c[\subset]0,1[$ for $n \in \mathbb{N}$. Then, the following are equivalent:

(i) *$\{x_n\}$ is spherically bounded and* $\lim_{n\to\infty} d(x_n, Tx_n) = 0$,
(ii) *$F(T) \neq \varnothing$.*

Proof. By Lemma 2, (i) implies (ii). We will show (ii) implies (i). Let $p \in F(T)$. From Lemma 3, we know there exists $\lim_{n\to\infty} d(x_n, p) < \frac{\pi}{2}$ and $\{x_n\}$ is spherically bounded. Thus we put $\lim_{n\to\infty} d(x_n, p) = c_p$. First, we show that $\lim_{n\to\infty} d(y_n, p) = c_p$. From the proof of Lemma 3, we have

$$d(x_{n+1}, p) \leq d(y_n, p) \leq d(x_n, p).$$

Since $\lim_{n\to\infty} d(x_n, p) = \lim_{n\to\infty} d(x_{n+1}, p) = c_p$, we get

$$\lim_{n\to\infty} d(y_n, p) = c_p. \tag{2}$$

Furthermore, since a generalized nonexpansive mapping T is quasi-nonexpansive, we have

$$d(Tx_n, p) \leq d(x_n, p).$$

Thus, we get

$$\limsup_{n\to\infty} d(Tx_n, p) \le c_p. \tag{3}$$

From the assumption, (2), (3) and Theorem 4, we have $\lim\limits_{n\to\infty} d(x_n, Tx_n) = 0$. □

The following two theorems are the main results in this paper on Δ-convergence and strong convergence.

Theorem 6. *Let X be an admissible complete* CAT(1) *space. Let T be a generalized nonexpansive mapping of X into itself and $F(T) \neq \varnothing$. Let $x_1 \in X$. Suppose that $\{x_n\} \subset X$ is a sequence defined by Picard-Mann hybrid iteration as follows:*

$$\begin{aligned} x_{n+1} &= Ty_n; \\ y_n &= (1-\alpha_n)x_n \oplus \alpha_n Tx_n, \end{aligned}$$

where $\alpha_n \in]b,c[\subset]0,1[$ for $n \in \mathbb{N}$. Then, $\{x_n\}$ is Δ-convergent to an element of $F(T)$.

Proof. Since $F(T) \neq \varnothing$ and from Theorem 5, we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $\{x_n\}$ is spherically bounded. Thus $\mathrm{AC}(\{x_n\})$ is singleton, let $\mathrm{AC}(\{x_n\}) = \{x_0\}$. For an arbitrary subsequence $\{x_{n_k}\}$ of $\{x_n\}$, let $q \in \mathrm{AC}(\{x_{n_k}\})$. Since $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ by Lemma 1, we get $q \in F(T)$. Furthermore, from Lemma 3, there exists $\lim_{n\to\infty} d(q, x_n)$. Then we have

$$\begin{aligned} r(q, \{x_{n_k}\}) &= \limsup_{k\to\infty} d(q, x_{n_k}) \\ &\le \limsup_{k\to\infty} d(x_0, x_{n_k}) \\ &\le \limsup_{n\to\infty} d(x_0, x_n) \\ &\le \limsup_{n\to\infty} d(q, x_n) \\ &= \lim_{n\to\infty} d(q, x_n) \\ &= \limsup_{k\to\infty} d(q, x_{n_k}) = r(q, \{x_{n_k}\}). \end{aligned}$$

Therefore, we have $\limsup\limits_{k\to\infty} d(q, x_{n_k}) = \limsup\limits_{k\to\infty} d(x_0, x_{n_k})$. From the uniqueness of the asymptotic center of $\{x_{n_k}\}$, we get $x_0 = q \in F(T)$. This implies $\{x_n\}$ is Δ-convergent to an element of $F(T)$. □

Theorem 7. *Let X be an admissible complete* CAT(1) *space, T a generalized nonexpansive mapping of X into itself and $F(T) \neq \varnothing$. Let $x_1 \in X$. Suppose that $\{x_n\} \subset X$ is defined by Picard-Mann hybrid iteration as follows:*

$$x_{n+1} = Ty_n;$$

$$y_n = (1 - \alpha_n)x_n \oplus \alpha_n T x_n,$$

where $\alpha_n \in]b, c[\subset]0, 1[$ *for* $n \in \mathbb{N}$ *and suppose* $\{x_n\}$ *is spherically bounded. Then,* $\{x_n\}$ *is convergent to an element of* $F(T)$ *if and only if* $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, *where* $d(x, F(T)) = \inf_{p\in F(T)} d(x, p)$.

Proof. Necessity is obvious. To prove sufficiency, we assume $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. From Lemma 3, we have

$$d(x_{n+1}, F(T)) \leq d(x_n, F(T)).$$

Thus, there exists $\lim_{n\to\infty} d(x_n, F(T))$ and we have $\lim_{n\to\infty} d(x_n, F(T)) = 0$ from the assumption. Next we show $\{x_n\}$ is a Cauchy sequence. Put $\varepsilon > 0$. From $\lim_{n\to\infty} d(x_n, F(T)) = 0$, there exists $n_0 \in \mathbb{N}$ such that

$$d(x_n, F(T)) < \frac{\varepsilon}{4},$$

for all $n \geq n_0$. Thus, we have

$$\inf_{p\in F(T)} d(x_{n_0}, p) < \frac{\varepsilon}{4}.$$

Therefore, there exists $p^* \in F(T)$ such that $d(x_{n_0}, p^*) < \frac{\varepsilon}{2}$. For all $n, m \geq n_0$, we get

$$\begin{aligned} d(x_m, x_n) &< d(x_m, p^*) + d(p^*, x_n) \\ &\leq 2d(x_{n_0}, p^*) < \varepsilon. \end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $q \in X$ such that $x_n \to q$. Fathermore, since $\lim_{n\to\infty} d(x_n, F(T)) = 0$, we get $d(q, F(T)) = 0$. Thus we have $q \in F(T)$. Therefore, $\{x_n\}$ is convergent to an element of $F(T)$. □

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