Uniform convexity on a complete geodesic space
完備測地距離空間における一様凸性

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1 Introduction

A Banach space is a generalization of Hilbert spaces and we often assume additional conditions for the space when we obtain results for nonlinear analysis. Uniform convexity is one of such conditions and, for instance, the following convex minimization theorem is obtained for uniformly convex Banach spaces.

Theorem 1 ([3]). *Let E be a uniformly convex real Banach space and let $f : E \to]-\infty, +\infty]$ be a proper lower semicontinuous convex function such that $f(z_n) \to \infty$ for $\{z_n\} \subset E$ satisfying $\|z_n\| \to \infty$. Then, there exists a point $x_0 \in E$ such that*

$$f(x_0) = \inf_{x \in E} f(x).$$

On the other hand, we know that a Hadamard space is another generalization of Hilbert spaces. It is defined as a complete metric space having a particular convexity structure and it also has various useful properties that Hilbert spaces have. We can also obtain the following convex minimization theorem.

Theorem 2 ([1]). *Let X be a Hadamard space and let $f : X \to]-\infty, +\infty]$ be a proper lower semicontinuous convex function such that $f(z_n) \to \infty$ for $\{z_n\} \subset X$ satisfying $d(z_n, w) \to \infty$ for some $w \in X$. Then, there exists a point $x_0 \in X$ such that*

$$f(x_0) = \inf_{x \in X} f(x).$$

In this work, we obtain a similar result as above under the assumptions that are satisfied for both Banach spaces and Hadamard spaces.

2 Preliminaries

Let E be a real Banach space. Then we know that the following propositions are equivalent:

- E is uniformly convex;
- If $r > 0$, $z \in E$ and $\{x_n\}, \{y_n\} \subset E$ satisfy

$$\lim_{n\to\infty} \|x_n - z\| = \lim_{n\to\infty} \|y_n - z\| = \lim_{n\to\infty} \|(\frac{1}{2}x_n + \frac{1}{2}y_n) - z\| = r,$$

then

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Moreover, we can prove the following lemma on uniformly convex real Banach spaces.

Proposition 1. *Let E be a uniformly convex real Banach space. Then,*

$$\|(tx + (1-t)y) - z\|^2 \leq t\|x - z\|^2 + (1-t)\|y - z\|^2$$

for $x, y, z \in X$ and $t \in [0, 1]$.

Let E be a real Banach space. Then the following propositions are equivalent:

- E is reflexive;
- $\bigcap_{n=1}^{\infty} C_n$ is a nonempty set for any sequence $\{C_n\} \subset 2^E$ of nonempty bounded closed convex subsets which is decreasing with respect to inclusion.

We know that if E is uniformly convex real Banach space, then E is reflexive and strictly convex. For more details about the properties of uniformly convex real Banach spaces, see [2, 3, 4].

Let (X, d) be a metric space and $x, y \in X$. A *geodesic path* from x to y is an isometry $c : [0, d(x, y)] \to X$ such that $c(0) = x$, $c(d(x, y)) = y$ and $d(c(s), c(t)) = |s - t|$ for every $s, t \in [0, d(x, y)]$. If a geodesic exists for every $x, y \in X$, then we call X a *geodesic space*. The image of a geodesic path from x to y is called a *geodesic segment* joining x and y. A geodesic segment joining x and y is not necessarily unique in general. When it is unique, this geodesic segment with endpoints x and y is denoted by $[x, y]$. For $x, y \in X$ and $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that $d(x, z) = (1-t)d(x, y)$ and $d(y, z) = td(x, y)$. We denote it by $z = tx \oplus (1-t)y$. A geodesic triangle with vertices $x, y, z \in X$ is the union of geodesic segments $[x, y]$, $[y, z]$ and $[z, x]$. We denote it by $\triangle(x, y, z)$.

For $\triangle(x,y,z)$ in a uniquely geodesic space X, there exist points $\bar{x},\bar{y},\bar{z}\in\mathbb{R}^2$ such that $d(x,y)=\|\bar{x}-\bar{y}\|_{\mathbb{R}^2}$, $d(y,z)=\|\bar{y}-\bar{z}\|_{\mathbb{R}^2}$, $d(z,x)=\|\bar{z}-\bar{x}\|_{\mathbb{R}^2}$, where $\|\cdot\|_{\mathbb{R}^2}$ is the Euclidean norm on $\mathbb{R}^2$. The triangle having such vertices $\bar{x}$, $\bar{y}$ and $\bar{z}$ in $\mathbb{R}^2$ is called a comparison triangle of $\triangle(x,y,z)$. Notice that it is unique up to an isometry of $\mathbb{R}^2$. For a specific choice of comparison triangles, we denote it by $\bar{\triangle}(\bar{x},\bar{y},\bar{z})$. A point $\bar{p}\in[\bar{x},\bar{y}]$ is called a comparison point for $p\in[x,y]$ if $d(x,p)=\|\bar{x}-\bar{p}\|_{\mathbb{R}^2}$.

Let X be a uniquely geodesic space. If for any $p,q\in\triangle(x,y,z)$, and for their comparison points $\bar{p},\bar{q}\in\bar{\triangle}(\bar{x},\bar{y},\bar{z})$, the CAT(0) inequality

$$d(p,q)\le\|\bar{p}-\bar{q}\|_{\mathbb{R}^2}$$

holds, then we call X a CAT(0) space. If X is complete, then X is said to be a Hadamard space. A subset C of X is said to be *convex* if $tx\oplus(1-t)y\in C$ for every $x,y\in C$ and $t\in[0,1]$. For a subset S of X, a *closed convex hull of* S is defined as the intersection of all closed convex sets including S, and we denote it by $\operatorname{clco} S$.

Let X be a CAT(0) space. From the CAT(0) inequality, it is easy to see that

$$d(tx\oplus(1-t)y,z)^2\le td(x,z)^2+(1-t)d(y,z)^2-t(1-t)d(x,y)^2$$

for every $x,y,z\in X$ and $t\in[0,1]$.

The following proposition shows that a CAT(0) space has a similar property to the uniform convexity.

Proposition 2. *Let X be a* CAT(0) *space, let $\{x_n\},\{y_n\}\subset X$, let $z\in X$ and let $r\in\,]0,\infty[$. If*

$$\lim_{n\to\infty}d(x_n,z)=\lim_{n\to\infty}d(y_n,z)=\lim_{n\to\infty}d(\frac{1}{2}x_n\oplus\frac{1}{2}y_n,z)=r,$$

then

$$\lim_{n\to\infty}d(x_n,y_n)=0.$$

Proof. For $\{x_n\},\{y_n\}\subset X$, $z\in X$ and $r\in\,]0,\infty[$, we suppose that

$$\lim_{n\to\infty}d(x_n,z)=\lim_{n\to\infty}d(y_n,z)=\lim_{n\to\infty}d(\frac{1}{2}x_n\oplus\frac{1}{2}y_n,z)=r.$$

Then,

$$d(\frac{1}{2}x_n\oplus\frac{1}{2}y_n,z)^2\le\frac{1}{2}d(x_n,z)^2+\frac{1}{2}d(y_n,z)^2-\frac{1}{4}d(x_n,y_n)^2,$$
$$d(x_n,y_n)^2\le 2d(x_n,z)^2+2d(y_n,z)^2-4d(\frac{1}{2}x_n\oplus\frac{1}{2}y_n,z)^2.$$

Letting $n\to\infty$, we have

$$\lim_{n\to\infty}d(x_n,y_n)=0.$$

This is the desired result. □

For more details about the properties of Hadamard spaces, see [1].

3 Uniform convexity of a complete geodesic space

Let X be a uniquely geodesic space. If

$$d(tx \oplus (1-t)y, z)^2 \leq td(x,z)^2 + (1-t)d(y,z)^2$$

for $x, y, z \in X$ and $t \in [0,1]$, then X is said to satisfy the condition (D).

Let X be a uniquely geodesic space. X is said to be *sequentially uniformly convex* if X satisfies the condition (D) and, for $r > 0$, $z \in X$ and $\{x_n\}, \{y_n\} \subset X$, it holds that

$$\lim_{n\to\infty} d(x_n, y_n) = 0$$

whenever

$$\lim_{n\to\infty} d(x_n, z) = \lim_{n\to\infty} d(y_n, z) = \lim_{n\to\infty} d(\frac{1}{2}x_n \oplus \frac{1}{2}y_n, z) = r.$$

Uniformly convex real Banach spaces and Hadamard spaces are sequentially uniformly convex complete uniquely geodesic spaces.

Theorem 3 (Strict convexity)**.** *Let X be a sequentially uniformly convex uniquely geodesic space. For $r > 0$ and $x, y, z \in X$ with $x \neq y$, if*

$$d(x,z) = d(y,z) = r,$$

then

$$d(\frac{1}{2}x \oplus \frac{1}{2}y, z) < r.$$

Proof. For $r > 0$ and $x, y, z \in X$ with $x \neq y$, we suppose that $d(x,z) = d(y,z) = r$. If $r \leq d(\frac{1}{2}x \oplus \frac{1}{2}y, z)$, then, since

$$r^2 \leq d(\frac{1}{2}x \oplus \frac{1}{2}y, z)^2 \leq \frac{1}{2}d(x,z)^2 + \frac{1}{2}d(y,z)^2 = r^2,$$

we have $d(\frac{1}{2}x \oplus \frac{1}{2}y, z) = r$. From sequential uniform convexity, we have $d(x,y) = 0$. This is a contradiction. Therefore, we have that

$$d(\frac{1}{2}x \oplus \frac{1}{2}y, z) < r.$$

This is the desired result. □

Let X be a metric space and let $T : X \to X$ be a mapping. We denote *the set of fixed points of* T by $F(T)$, that is, $F(T) = \{x \in X : Tx = x\}$. If $F(T) \neq \varnothing$ and

$$d(Tx, u) \leq d(x, u)$$

for $x \in X$ and $u \in F(T)$, then we say that T is *quasi-nonexpansive*.

Theorem 4. *Let X be a sequentially uniformly convex uniquely geodesic space and let $T : X \to X$ be a quasi-nonexpansive mapping. Then $F(T)$ is closed and convex.*

Proof. First, we show that $F(T)$ is closed. Let $\{x_n\} \subset F(T)$ be a sequence such that $x_n \to x_0$. Since T is quasi-nonexpansive, we have

$$\begin{aligned} d(x_0, Tx_0) &\leq d(x_0, x_n) + d(x_n, Tx_0) \\ &\leq 2d(x_n, x_0). \end{aligned}$$

Therefore, since $x_n \to x_0$, we have $d(x_0, Tx_0) = 0$ and thus $x_0 \in F(T)$. Hence, $F(T)$ is closed.

Next, we show that $F(T)$ is convex. Let $x, y \in F(T)$, $\alpha \in [0, 1]$ and $z = \alpha x \oplus (1-\alpha) y$. Since T is quasi-nonexpansive, we have

$$\begin{aligned} d(x, Tz) &\leq d(x, z), \\ d(y, Tz) &\leq d(y, z). \end{aligned}$$

Moreover, since

$$d(x, y) \leq d(x, Tz) + d(y, Tz) \leq d(x, z) + d(y, z) = d(x, y),$$

we have $d(x, Tz) = d(x, z)$ and $d(y, Tz) = d(y, z)$. we suppose that $z \neq Tz$. Then,

$$\begin{aligned} d(x, Tz) &= d(x, z) = s, \\ d(y, Tz) &= d(y, z) = t. \end{aligned}$$

From Theorem 3, we have

$$\begin{aligned} d(\frac{1}{2} z \oplus \frac{1}{2} Tz, x) &< s, \\ d(\frac{1}{2} z \oplus \frac{1}{2} Tz, y) &< t \end{aligned}$$

and thus

$$d(x, y) \leq d(\frac{1}{2} z \oplus \frac{1}{2} Tz, x) + d(\frac{1}{2} z \oplus \frac{1}{2} Tz, y) < s + t = d(x, z) + d(y, z) = d(x, y).$$

This is a contradiction. Therefore, $z = Tz$ and $z \in F(T)$. Hence, we have $F(T)$ is convex. □

Let X be a metric space. For a point $x \in X$ and a nonempty subset $C \subset X$, the distance between them is defined by $d(x, C) = \inf_{y \in C} d(x, y)$.

Theorem 5 (The nearest point theorem)**.** *Let X be a sequentially uniformly convex complete uniquely geodesic space and let C be a nonempty closed convex subset of X. Then, for $x \in X$, there exists a unique point $y_0 \in C$ such that $d(x, y_0) = d(x, C)$.*

Proof. For $x \in X$, let $d = d(x, C)$. Then, for $n \in \mathbb{N}$, we can take a sequence $\{y_n\} \subset C$ such that

$$d \leq d(x, y_n) \leq d + \frac{1}{n}.$$

Then, we have $d(x, y_n) \to d$. We suppose that $\{y_n\}$ is not a Cauchy sequence. That is, we suppose that there exists $\epsilon > 0$ such that for any $i \in \mathbb{N}$, there exists $m_i, n_i \geq i$ such that $d(y_{m_i}, y_{n_i}) \geq \epsilon$. In this way, we take two sequences $\{y_{m_i}\}, \{y_{n_i}\} \subset \{y_n\}$. Then,

$$\lim_{i\to\infty} d(x, y_{m_i}) = \lim_{i\to\infty} d(x, y_{n_i}) = d$$

and we have

$$\begin{aligned} d^2 &\leq d(x, \frac{1}{2} y_{m_i} \oplus \frac{1}{2} y_{n_i})^2 \\ &\leq \frac{1}{2} d(x, y_{m_i})^2 + \frac{1}{2} d(x, y_{n_i})^2 \to d^2. \end{aligned}$$

Hence, from sequential uniform convexity of X, we have

$$\lim_{i\to\infty} d(y_{m_i}, y_{n_i}) = 0.$$

This is a contradiction and thus $\{y_n\}$ is a Cauchy sequence. Since X is complete and C is closed, there exists $y_0 \in C$ such that $y_n \to y_0$. Therefore, we have

$$d(x, y_0) = \lim_{n\to\infty} d(x, y_n) = d = d(x, C).$$

Next, we show the uniqueness of y_0. We suppose that $y_0, z_0 \in C$ satisfying $y_0 \neq z_0$ and $d(x, y_0) = d(x, z_0) = d(x, C)$. Then, from Theorem 3, we have

$$d(x, \frac{1}{2} y_0 \oplus \frac{1}{2} z_0) < d(x, C).$$

This is a contradiction. Therefore, for $x \in X$, there exists a unique point $y_0 \in C$ such that $d(x, y_0) = d(x, C)$. □

Let X be a sequentially uniformly convex complete uniquely geodesic space and let C be a nonempty closed convex subset of X. Then for $x \in X$, there exists a unique point $y_x \in C$ such that

$$d(x, y_x) = d(x, C).$$

We call such a mapping defined by $P_C x = y_x$, the *metric projection* of X onto C.

Theorem 6 (Reflexivity)**.** *Let X be a sequentially uniformly convex complete uniquely geodesic space and let $\{C_n\} \subset 2^X$ be a sequence of nonempty bounded closed convex subsets which is decreasing with respect to inclusion. That is, $C_1 \supset C_2 \supset \cdots \supset C_n \supset \cdots$. Then, $\bigcap_{n=1}^{\infty} C_n$ is nonempty.*

Proof. Since C_n is nonempty bounded closed convex subset for $n \in \mathbb{N}$, for $x \in X$, we can take a sequence $\{x_n\} \subset X$ by $x_n = P_{C_n}x$. Then $\{d(x, x_n)\}$ is bounded and increasing real sequence and hence $\{d(x, x_n)\}$ has a limit $c \in [0, \infty[$. That is, we have

$$\lim_{n\to\infty} d(x, x_n) = c.$$

First, we show that $\{x_n\}$ converges to some point $x_0 \in X$. If $c = 0$, since $d(x, P_{C_n}x) \to 0$, we have $x_n \to x_0$ as $x_0 = x$. Hence, we suppose that $c > 0$. Suppose that $\{x_n\}$ is not a Cauchy sequence. That is, there exists $\epsilon > 0$ such that for any $i \in \mathbb{N}$, there exists $m_i, n_i \geq i$ such that $d(x_{m_i}, x_{n_i}) \geq \epsilon$. Without loss of generality, we can suppose that $m_i \geq n_i$. In this way, we take two sequences $\{y_{m_i}\}, \{y_{n_i}\} \subset \{y_n\}$. Then,

$$\lim_{i\to\infty} d(x, x_{m_i}) = \lim_{i\to\infty} d(x, x_{n_i}) = c$$

and we have

$$d(x, \frac{1}{2}x_{m_i} \oplus \frac{1}{2}x_{n_i})^2 \leq \frac{1}{2}d(x, x_{m_i})^2 + \frac{1}{2}d(x, x_{n_i})^2$$

and thus

$$\limsup_{i\to\infty} d(x, \frac{1}{2}x_{m_i} \oplus \frac{1}{2}x_{n_i}) \leq c.$$

Since $x_{m_i}, x_{n_i} \in C_{n_i}$, we have

$$d(x, x_{n_i}) = d(x, P_{C_{n_i}}x) \leq d(x, \frac{1}{2}x_{m_i} \oplus \frac{1}{2}x_{n_i})$$

and hence we have

$$c \leq \liminf_{i\to\infty} d(x, \frac{1}{2}x_{m_i} \oplus \frac{1}{2}x_{n_i}).$$

Therefore, we have

$$\lim_{i\to\infty} d(x, \frac{1}{2}x_{m_i} \oplus \frac{1}{2}x_{n_i}) = c.$$

From sequential uniform convexity of X, we have $\lim_{i\to\infty} d(x_{m_i}, x_{n_i}) = 0$. This is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence and thus there exists $x_0 \in X$ such that $x_n \to x_0$.

Next, we show that $x_0 \in \bigcap_{n=1}^{\infty} C_n$. For $n_0 \in \mathbb{N}$, if $n \geq n_0$, since $\{x_n\} \subset C_{n_0}$, we have that $x_0 \in C_{n_0}$. Therefore, $x_0 \in \bigcap_{n=1}^{\infty} C_n$ and it completes the proof. □

4 Minimization theorem for a convex function

In this section, we prove a minimization theorem for a convex function defined on a uniformly convex geodesic space. we obtain the following lemmas.

Lemma 1. *Let X be a sequentially uniformly convex complete uniquely geodesic space and $f : X \to]-\infty, +\infty]$ be a proper lower semicontinuous convex function. Then, f is bounded below on a bounded set.*

Proof. Let $S \subset X$ be a bounded set and let $C = \operatorname{clco} S$. If $\inf_{x \in C} f(x) = -\infty$, then we can define a sequence of subsets $\{C_k\} \subset 2^X$ by

$$C_k = \{z \in C : f(z) \le -k\}$$

and C_k is bounded closed convex set for every $k \in \mathbb{N}$. Moreover, we have $C_1 \supset C_2 \supset \cdots \supset C_k \supset \cdots$. From Theorem 6, $\bigcap_{k=1}^{\infty} C_k$ is nonempty. This is a contradiction. Therefore,

$$-\infty < \inf_{x \in C} f(x) \le \inf_{x \in S} f(x)$$

and it completes the proof. □

Lemma 2. *Let X be a sequentially uniformly convex complete uniquely geodesic space and let $f : X \to]-\infty, +\infty]$ be a proper lower semicontinuous convex function satisfying $f(z_n) \to \infty$ for $\{z_n\} \subset X$ such that $d(z_n, w) \to \infty$ for some $w \in X$. Then, f is bounded below on X.*

Proof. Let $M = \inf_{x \in X} f(x)$. Since f is proper, we have $M \in [-\infty, +\infty[$. Then, there exists a sequence $\{z_n\} \subset X$ such that $f(z_n) \to M$. If $\{z_n\}$ is not a bounded sequence, then there exists a subsequence $\{z_{n_i}\} \subset \{z_n\}$ such that $d(w, z_{n_i}) \to \infty$ for $w \in X$. From assumption of f, we have $f(z_{n_i}) \to \infty$. This is a contradiction. Therefore, $\{z_n\}$ is a bounded sequence. From Lemma 1, we have $\{f(z_n)\}$ is bounded below. Hence, we have $M > -\infty$ and it completes the proof. □

Theorem 7 (Minimization theorem)**.** *Let X be a sequentially uniformly convex complete uniquely geodesic space and $f : X \to]-\infty, +\infty]$ be a proper lower semicontinuous convex function and $f(z_n) \to \infty$ for $\{z_n\} \subset X$ such that $d(z_n, w) \to \infty$ for some $w \in X$. Then, there exists a point $x_0 \in X$ such that*

$$f(x_0) = \inf_{x \in X} f(x).$$

Proof. Let $M = \inf_{x \in X} f(x)$. From Lemma 2, we have $M \in \mathbb{R}$. Then, we can define a sequence of subsets $\{C_n\} \subset 2^X$ by

$$C_n = \{z \in X : M \le f(z) \le M + \frac{1}{n}\}$$

and C_n is a nonempty bounded closed convex set for any $n \in \mathbb{N}$. Moreover, $\{C_n\}$ satisfies that $C_1 \supset C_2 \supset \cdots \supset C_n \supset \cdots$. Then, from Theorem 6, we have $\bigcap_{n=1}^{\infty} C_n \neq \varnothing$. Therefore, $f(x_0) = M$ for $x_0 \in \bigcap_{n=1}^{\infty} C_n$ and hence

$$f(x_0) = \inf_{x \in X} f(x).$$

This is the desired result. □

References

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