Bohr-Sommerfeld quantization conditions and eigenvalue splitting for a double-well model of matrix Schrödinger operators

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1 Introduction

The spectral analysis of scalar Schrödinger operators has witnessed a lot of progress in the last decades thanks to the impulse of various techniques from perturbation theory and semiclassical and microlocal analysis. We primarily refer to the monographs [DS, DZ, Ma, HS, Zw] and references therein. On the contrary, in the case of systems of Schrödinger operators the literature is much less rich and only few results are available. These systems play an important role in many problems in quantum physics and quantum chemistry where they either represent the original Hamiltonian which describes the physical system or a convenient model to simplify the study. A typical example arise in the framework of the Born-Oppenheimer approximation of molecular dynamics which allows for a drastic reduction of problem size when dealing with molecular systems. Roughly speaking, it states that the study of the molecular Hamiltonian which describes the dynamics of systems of nuclei and electrons is reduced to that of a $N \times N$ system of pseudodifferential operators of the form

$$\mathcal{P}(h) = \begin{pmatrix} P_1(h) & 0 & \cdots & 0 \\ 0 & P_2(h) & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & P_N(h) \end{pmatrix} + h\mathcal{R}(x; hD_x),$$

where each $P_j(h) := -h^2\Delta + V_j(x)$, $j = 1, ..., N$, is a scalar Schrödinger operator with potential V_j corresponding to an electronic energy-level and $\mathcal{R}(x; hD_x)$ is a $N \times N$ matrix pseudodifferential operator of order less than one. In this context the semiclassical parameter $h > 0$ represents the square root of the quotient between the electronic and nuclear masses.

In the literature there has been some works devoted to the study of the spectral properties of systems of coupled Schrödinger operators most of them concern the study of quantum resonances. Martinez [Ma1] obtained exponential bound on the widths (imaginary parts) of resonances for a two-level system without crossing at the classical level, i.e., in the phase space. This happens when the two potentials do not cross or the energy considered is lower than that of the crossing. This result has been improved in the one-dimensional case in [Ba, Na]. See also [GM1, GM2] for recent results in this direction. In [FLN] the authors studied the resonances of a two-level matrix Schrödinger operator in dimension 2 with linear conical intersection. They established a generalized Bohr-Sommerfeld quantization condition and an asymptotic description of the set of resonances using a decomposition of this model into a direct sum of first order systems on the real half line and the exact WKB method. In a serie of recent works, Fujiié, Martinez and Watanabe [FMW1, FMW2, FMW3] and Ashida [As] studied the asymptotic distribution of resonances near a given energy-level for a model of one-dimensional two-by-two system of coupled

Schrödinger operators with energy-level crossing of the form (1.1). They established precise asymptotics on both the real and imaginary parts of the resonances in the semiclassical limit $h \to 0^+$. We also refer to the recent work of Higuchi [Hi] where an absence of resonances result was proved for a two-by-two system with crossings.

In this paper we review the recent results of [AF] where the asymptotic distribution of the eigenvalues in the semiclassical limit of a one-dimensional 2×2 matrix Schrödinger operator was studied. The considered Hamiltonian is of the form

$$\mathcal{P}(h) = \begin{pmatrix} P_1(h) & h\mathcal{R} \\ h\mathcal{R}^* & P_2(h) \end{pmatrix} \tag{1.1}$$

acting on the Hilbert space $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$, where the diagonal elements

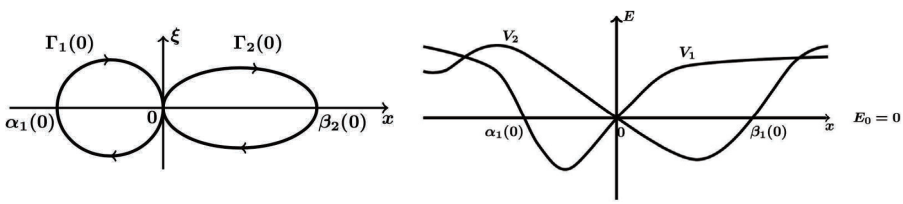
$$P_j(h) := -h^2 \frac{d^2}{dx^2} + V_j(x) \quad (j = 1, 2),$$

are semiclassical Schrödinger operators on the real line with smooth real-valued potentials, and the anti-diagonal elements, $\mathcal{R}$ and its formal adjoint $\mathcal{R}^*$, are first-order semiclassical differential operators which play the role of the interaction (see (2.2)). We fix an energy-level $E \in \mathbb{R}$, say $E = 0$, and we assume that each potential V_j, $j = 1, 2$, admits a simple well at this level, denoted $]\alpha_j(0), \beta_j(0)[$ with $\alpha_j(0) \leq \beta_j(0)$ (we refer to the next section for the precise assumption). Under this assumption, in the phase space $\mathbb{R}^2_{x,\xi} = T^*\mathbb{R}$, the characteristic set

$$\Gamma_j(E) = \{(x, \xi) \in \mathbb{R}^2; \xi^2 + V_j(x) = E\} \quad (j = 1, 2) \tag{1.2}$$

for E near 0 is a simple smooth closed curve. In this case, the spectrum of each scalar operator P_j, $j = 1, 2$, is discrete near 0, consists on h-dependent eigenvalues subject to the Bohr-Sommerfeld quantization rule (2.4). If the interaction is absent, i.e., $\mathcal{R} \equiv 0$, then the spectrum of $\mathcal{P}(h)$ near 0 is just the union of the spectra of P_1 and P_2. Under the interaction, one naturally expects that the eigenvalues of the system are approximated in the semiclassical limit by the union of those of P_1 and P_2. We studied the accuracy of this approximation in the case where the two characteristic sets $\Gamma_1(0)$ and $\Gamma_2(0)$ cross to each other, which in particular implies an energy-level crossing at the level of the potentials. Assuming that the energy-level crossing occurs at $x = 0$ and setting $E_0 := V_1(0) = V_2(0)$, we studied the following two cases:

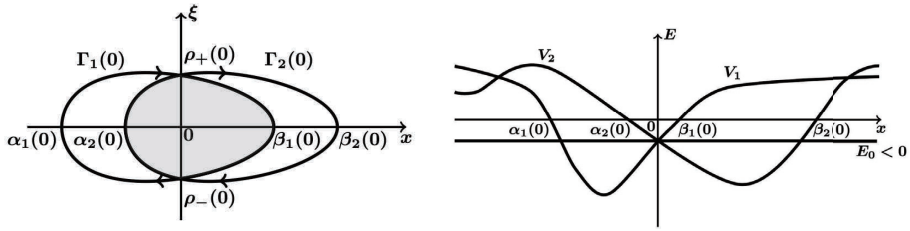
- **Tangential case:** $\Gamma_1(0)$ and $\Gamma_2(0)$ intersect tangentially at one point (see Figure 1a).



(a) Characteristic sets (b) Potentials

Figure 1: Tangential case

- **Transversal case:** $\Gamma_1(0)$ and $\Gamma_2(0)$ intersect transversally at two points (see Figure 2a).



(a) Characteristic sets

(b) Potentials

Figure 2: Transversal case

At the level of the potentials, the tangential case corresponds to the degenerate situation where the crossing level coincides with 0, i.e, $E_0 = 0$ (see Figure 1b), while the transversal case corresponds to the situation where the crossing level is below 0, i.e., $E_0 < 0$ (see Figure 2b). In these cases, the interaction between the two wells is stronger than in the case without crossing. This interaction is observed as eigenvalue splitting when the two action integrals along $\Gamma_1(E)$ and $\Gamma_2(E)$ coincide. We proved that the splitting is of polynomial order in h in both cases and gave precise estimates for it. In particular we computed explicitly the leading coefficients which reflect the geometry of the crossing.

For the background of this study and the detailed proofs, we send the readers to [AF]. Here we only state the main results and we give a very brief sketch of the method. One can also finds an account on the existing literature in relation with the eigenvalue splitting phenomena in the introduction of the above paper.

2 Main results

2.1 Precise assumptions

We suppose the following conditions on the potentials V_1, V_2 and the interaction operator $\mathcal{R}$.

Assumption 2.1. *For each $j = 1, 2$, we assume that*

(i) V_j is smooth and real-valued on $\mathbb{R}$, and it admits limits as $x \to \pm\infty$ such that

$$\lim_{x\to\pm\infty} V_j(x) > 0.$$

Moreover, there exists $\delta_0 > 1$ such that

$$V_j'(x)^2 + |V_j''(x)| = \mathcal{O}(|x|^{-\delta_0}) \quad (j = 1, 2), \quad \text{as } |x| \to +\infty.$$

(ii) There exist two real points $\alpha_j < \beta_j$ such that

$$\frac{V_j(x)}{(x-\alpha_j)(x-\beta_j)} > 0, \quad \forall x \in \mathbb{R}. \tag{2.1}$$

Assumption 2.1 (ii) is the so called simple-well condition. It means that V_j admits a simple well $]\alpha_j, \beta_j[$ at the energy-level $E = 0$. In the following, when E varies near 0, we denote by $\alpha_j(E)$ and $\beta_j(E)$ the zeros of $V_j(x) - E$ near α_j and β_j respectively. These are the so-called turning points.

Assumption 2.2. *$\mathcal{R}$ is a first-order semiclassical differential operator of the form*

$$\mathcal{R} = \mathcal{R}(x; hD_x) = r_0(x) + ir_1(x)hD_x, \quad D_x = \frac{1}{i}\frac{d}{dx}, \tag{2.2}$$

with r_0, r_1 smooth real-valued functions bounded together with all their derivatives on $\mathbb{R}$.

For $\varepsilon > 0$ possibly depending on h, we set $I(\varepsilon) := [-\varepsilon, \varepsilon]$. Under the above assumptions, the operator $\mathcal{P}(h)$ is self-adjoint in $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ and its spectrum is discrete in $I(\varepsilon)$ for sufficiently small $\varepsilon > 0$.

For $E \in I(\varepsilon)$, we define the action integrals along the characteristic sets $\Gamma_j(E)$ defined by (1.2)

$$\mathcal{A}_j(E) := \frac{1}{2}\int_{\Gamma_j(E)} \xi dx = \int_{\alpha_j(E)}^{\beta_j(E)} \sqrt{E - V_j(t)}dt \quad (j = 1, 2). \tag{2.3}$$

The functions $\mathcal{A}_1$ and $\mathcal{A}_2$ are smooth and strictly increasing near 0. It is well known (see e.g. [Ol, Ya]) that the eigenvalues of the scalar operator P_j in $I(\varepsilon)$, for $\varepsilon > 0$ small enough, are approximated by the roots of the Bohr-Sommerfeld quantization rule

$$\cos\left(\frac{\mathcal{A}_j(E)}{h}\right) = 0 \quad (j = 1, 2). \tag{2.4}$$

Set

$$\mathcal{U}_h(\varepsilon) := \mathcal{U}_h^{(1)}(\varepsilon) \cup \mathcal{U}_h^{(2)}(\varepsilon), \quad \mathcal{U}_h^{(j)}(\varepsilon) := \left\{E = E(h) \in I(\varepsilon),\ E \text{ satisfies } (2.4)\right\} \quad (j = 1, 2).$$

The elements of $\mathcal{U}_h(\varepsilon)$ will play the role of reference points near which the eigenvalues of the operator $\mathcal{P}(h)$ are localized. In the following, we assume without loss of generality that the energy-level crossing occurs at $x = 0$ and $\alpha_1(0) \leq \alpha_2(0) \leq 0$. We set

$$E_0 := V_1(0) = V_2(0), \quad v_j := V_j'(0),\ j = 1, 2.$$

2.2 Tangential case

Throughout this paragraph, we assume the following condition on the characteristic sets $\Gamma_1(0)$ and $\Gamma_2(0)$.

Assumption 2.3 (Tangential case)**.** *$\Gamma_1(0)$ and $\Gamma_2(0)$ intersect tangentially at one point $(x, \xi) = (0, 0)$, and their interior domains are disjoint (see Figure 1).*

Remark 2.1. *Under the assumption 2.1 (ii), this condition is equivalent to that the two potentials V_1 and V_2 cross at the origin with value 0 and with derivatives of different signs (see Figure 1b), namely,*

$$\beta_1(0) = \alpha_2(0) = 0, \quad E_0 = 0, \quad v_1 > 0,\ v_2 < 0.$$

The first result is the following Bohr-Sommerfeld type quantization condition which determines the eigenvalues of $\mathcal{P}(h)$ on any interval centered at 0 of size $\mathcal{O}(h^{\frac{2}{3}})$.

Theorem 2.1. *Let Assumptions 2.1, 2.2 and 2.3 hold and fix $C_0 > 0$.*

(i) There exists a smooth function $m(E; h)$ of E defined in $I(C_0h^{\frac{2}{3}})$ for sufficiently small h and satisfying the estimate

$$m(E; h) = \mathcal{O}(h^{\frac{2}{3}}),$$

such that

$$E = E(h) \in \sigma\left(\mathcal{P}(h)\right) \cap I(C_0h^{\frac{2}{3}}) \quad \textit{iff} \quad \cos\left(\frac{\mathcal{A}_1(E)}{h}\right)\cos\left(\frac{\mathcal{A}_2(E)}{h}\right) = m(E; h). \tag{2.5}$$

(ii) Assume moreover that $\mathcal{A}_1(E) = \mathcal{A}_2(E)$. *Then we have*

$$m(E;h) = \mathcal{D}_{tang}(E)^2 h^{\frac{2}{3}} + \mathcal{O}(h), \quad \mathcal{D}_{tang}(E) := 2|r_0(0)\mu_A(h^{-\frac{2}{3}}E)|, \tag{2.6}$$

uniformly for $E \in I(C_0 h^{\frac{2}{3}})$, *where*

$$\mu_A(t) := \int_0^{+\infty} \mathrm{Ai}\left(v_1^{-\frac{2}{3}}(v_1\eta - t)\right) \mathrm{Ai}\left(-|v_2|^{-\frac{2}{3}}(|v_2|\eta + t)\right) d\eta.$$

Here $\mathrm{Ai}(t)$ *is the Airy function solution to the Airy equation* $u''(t) = tu(t)$ *defined by*

$$\mathrm{Ai}(t) = \frac{1}{\pi}\int_0^{+\infty} \cos\left(\frac{\eta^3}{3} + t\eta\right) d\eta.$$

This result entails the following one about the location of the eigenvalues of the operator $\mathcal{P}(h)$ and the eigenvalue splitting in the case $\mathcal{A}_1(E) = \mathcal{A}_2(E)$.

Corollary 2.1. *Let Assumptions 2.1, 2.2 and 2.3 hold and fix* $C_0 > 0$. *Then, we have*

(1) The eigenvalues $E \in I(C_0 h^{\frac{2}{3}})$ *of* $\mathcal{P}(h)$ *satisfy*

$$\mathrm{dist}\left(E, \mathcal{U}_h(C_0 h^{\frac{2}{3}})\right) = \mathcal{O}(h^{\frac{4}{3}}).$$

(2) Assume moreover that $\mathcal{A}_1(E) = \mathcal{A}_2(E) =: \mathcal{A}(E)$. *Then, the operator* $\mathcal{P}(h)$ *has exactly two eigenvalues* $E_+(h)$ *and* $E_-(h)$ *in a neighborhood of size* $h^{\frac{4}{3}}$ *of each element* $E \in \mathcal{U}_h(C_0 h^{\frac{2}{3}})$, *and they satisfy*

$$|E_+(h) - E_-(h)| = 2\frac{\mathcal{D}_{tang}(E)}{\mathcal{A}'(E)} h^{\frac{4}{3}} + \mathcal{O}(h^{\frac{5}{3}}) \quad \text{as } h \to 0^+.$$

2.3 Transversal case

Now, we consider the transversal crossing case, namely, we assume that

Assumption 2.4 (Transversal case)**.** $\Gamma_1(0)$ *and* $\Gamma_2(0)$ *intersect transversally at two points* $\rho_\pm(0) := (0, \pm\sqrt{|E_0|})$ *(see Figure 2).*

Remark 2.2. *Under the assumption 2.1 (ii), this condition is equivalent to that the two potentials* V_1 *and* V_2 *cross transversally at the origin with negative value* E_0 *(see Figure 2b), namely,*

$$\alpha_1(0) < \alpha_2(0) < 0 < \beta_1(0) < \beta_2(0), \quad E_0 < 0, \quad v_1 > v_2.$$

Remark 2.3. *Notice that under the assumption 2.4, for* E *close enough to* 0, $\Gamma_1(E)$ *and* $\Gamma_2(E)$ *still cross transversally at two points* $\rho_\pm(E) := (0, \pm\sqrt{E - E_0})$. *Notice also that in this case, the derivatives of* V_1 *and* V_2 *at* $x = 0$ *dont have to be of different signs.*

We shall also need the following microlocal ellipticity condition on the interaction operator $\mathcal{R}$ at the crossing points $\rho_\pm(0)$. Let $r(x,\xi) := r_0(x) + ir_1(x)\xi$, $(x,\xi) \in \mathbb{R}^2$, be the symbol of the operator $\mathcal{R}$.

Assumption 2.5. $r(\rho_\pm(0)) \neq 0$.

The following Bohr-Sommerfeld type quantization condition determines the eigenvalues of $\mathcal{P}(h)$ in any interval centered at $E = 0$ of size $\mathcal{O}(\varepsilon)$, for $\varepsilon > 0$ sufficiently small.

Theorem 2.2. *Let assumptions 2.1, 2.2, 2.4 and 2.5 hold.*

(i) There exists a smooth function $m(E;h)$ of E defined in $I(\varepsilon)$ for sufficiently small h with

$$m(E;h) = \mathcal{O}(h^{\frac{1}{6}}), \tag{2.7}$$

such that

$$E = E(h) \in \sigma\left(\mathcal{P}(h)\right) \cap I(\varepsilon) \quad \textit{iff} \quad \cos\left(\frac{\mathcal{A}_1(E)}{h}\right)\cos\left(\frac{\mathcal{A}_2(E)}{h}\right) = m(E;h). \tag{2.8}$$

(ii) Assume moreover that $\mathcal{A}_1(E) = \mathcal{A}_2(E)$. Then, we have $m(E;h) = \mathcal{O}(h)$, more precisely,

$$m(E;h) = \mathcal{D}_{trans}(E)^2 h + \mathcal{O}(h^{\frac{3}{2}}), \tag{2.9}$$

uniformly for $E \in I(\varepsilon)$, where

$$\mathcal{D}_{trans}(E) := \sqrt{\frac{\pi}{v_1 - v_2}}(E - E_0)^{-\frac{1}{4}}|W(\rho_+(E))|\left|\cos\left(\frac{\mathcal{B}(E)}{h} + \frac{\pi}{4} + arg\left(W(\rho_+(E))\right)\right)\right|, \tag{2.10}$$

where $\mathcal{B}(E)$ is the action defined by

$$\mathcal{B}(E) := \int_{\alpha_2(E)}^{0} \sqrt{E - V_2(t)}dt + \int_{0}^{\beta_1(E)} \sqrt{E - V_1(t)}dt. \tag{2.11}$$

As a consequence, we get the following result.

Corollary 2.2. *Let assumptions 2.1, 2.2, 2.4 and 2.5 hold. Then, we have*

(i) The eigenvalues $E \in I(\varepsilon)$ of $\mathcal{P}(h)$ satisfy

$$\operatorname{dist}(E, \mathcal{U}_h(\varepsilon)) = \mathcal{O}(h^{\frac{13}{12}}).$$

(ii) Assume moreover that $\mathcal{A}_1(E) = \mathcal{A}_2(E) =: \mathcal{A}(E)$. Then the above estimate holds with $h^{\frac{3}{2}}$ instead of $h^{\frac{13}{12}}$. More precisely, the operator $\mathcal{P}(h)$ has exactly two eigenvalues $E_+(h)$ and $E_-(h)$ in a neighborhood of size $h^{\frac{3}{2}}$ of each element $E \in \mathcal{U}_h(\varepsilon)$, and they satisfy

$$|E_+(h) - E_-(h)| = 2\frac{\mathcal{D}_{trans}(E)}{\mathcal{A}'(E)}h^{\frac{3}{2}} + \mathcal{O}\big(h^{\frac{7}{4}}\big), \quad \textit{as } h \to 0^+.$$

Remark 2.4. *The condition $\mathcal{A}_1(E) = \mathcal{A}_2(E)$ near $E = 0$ holds for example when $V_1(x) = V_2(-x)$ and when $V_1(x) = V_2(x + a)$, for some $a \in \mathbb{R}$.*

3 Outline of the proofs

In this section we give the main ideas of the proofs of the previous results. The core of the proofs relies on two steps. We consider the eigenvalue problem

$$(\mathcal{P}(h) - E)w = 0, \quad E = E(h) \text{ near } 0. \tag{3.1}$$

In the first step we construct two L^2 solutions to the system (3.1) on the half-line $(-\infty, 0]$ and two other L^2 solutions on the half-line $[0, +\infty)$. We denote these solutions by

$$w_{1,L}, w_{2,L} \in L^2(\mathbb{R}_-) \oplus L^2(\mathbb{R}_-), \quad w_{1,R}, w_{2,R} \in L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_+). \tag{3.2}$$

This construction is made by means of the method of successive construction of serie solutions established in [FMW1, FMW3] starting from suitable solutions to the underlying scalar equations

$$(P_j(h) - E)u = 0 \quad (j = 1, 2). \tag{3.3}$$

The eigenvalues of $\mathcal{P}(h)$ are characterized as the energies such that the four solutions (3.2) are linearly dependent. More precisely, the quantization condition is given by

$$\mathcal{W}_h(E) = 0,$$

where $\mathcal{W}_h(E)$ stands for the Wronskian of $w_{1,L}, w_{2,L}, w_{1,R}$ and $w_{2,R}$. The asymptotic behavior of $\mathcal{W}_h(E)$ with respect to h depends on the behaviors of the solutions to the scalar equations (3.3) which in theirs turn depend on the case, that is, the tangential case (near the crossing-level) or the transversal case (above the crossing-level):

- **Tangential case:** In this case, estimates on the first terms of the constructed series solutions are sufficient to get the precise quantization condition (2.5). The result in this case is achieved only by means of this tool.
- **Transversal case:** In this case, due to the existence of a region where both potentials V_1 and V_2 are below $E = 0$ (see Figure 2b), the solutions to the scalar equations (3.3) are both oscillating in this region which makes the convergence of our series solutions slower and then requires many terms computation for a satisfactory quantization condition for the eigenvalues of the system. At this step we content ourselves with a quantization condition with a rough error estimate. This in particular ensures the existence of the eigenvalues but only give a rough estimate on their location.

To obtain a precise quantization condition in the transversal case, we use a microlocal approach that relies on the study of the behavior of the corresponding eigenfunctions microlocally near the characteristic set

$$\mathrm{Char}(\mathcal{P}(h) - E) = \Gamma_1(E) \cup \Gamma_2(E), \quad E \in I(\varepsilon). \tag{3.4}$$

The key point in this method consists in the computation of the microlocal transfer matrix that link the microlocal data at the crossing points of $\mathrm{Char}(\mathcal{P}(h) - E)$. In [AF], we derived this microlocal transfer matrix in the framework of a general non necessarliy self-adjoint pseudodifferential system. We present this result in details in the next section. We send the readers to the end of section 5 in the above paper for the details of the derivation of the quantization condition from the microlocal connection formulae.

4 Microlocal transfer matrix at a crossing point for a general pseudodifferential system

We send the reader to the textbooks [DS, Ma, Zw] for the details of the different notions of semiclassical and microlocal analysis used in our study. We introduce the class of symbols

$$S^0 := \left\{ q = q(\cdot, \cdot; h) \in C^\infty(T^*\mathbb{R}; \mathbb{C});\ \left|\partial_x^\alpha \partial_\xi^\beta q(x, \xi; h)\right| = \mathcal{O}_{\alpha,\beta}(1),\ \forall \alpha, \beta \in \mathbb{N} \right\}.$$

For a symbol $q \in S^0$, the corresponding h-pseudodifferential operator denoted $\mathcal{Q}(h) = \mathrm{Op}_h^w(q)$ can be defined using the h-Weyl quantization by

$$\mathcal{Q}(h)u(x) := \frac{1}{2\pi h} \int_{T^*\mathbb{R}} e^{i(x-y)\xi/h} q\left(\frac{x+y}{2}, \xi; h\right) u(y) dy d\xi, \quad u \in C_0^\infty(\mathbb{R}).$$

Let $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{R}_1$ and $\mathcal{R}_2$ be four h-pseudodifferential operators with symbols $q_1(x, \xi)$, $q_2(x, \xi)$, $r_1(x, \xi)$ and $r_2(x, \xi)$ respectively. Consider, microlocally near $\rho_0 = (0, 0) \in T^*\mathbb{R}$, the two-by-two h-pseudodifferential system

$$\mathcal{Q}(h) := \begin{pmatrix} \mathcal{Q}_1 & h\mathcal{R}_1 \\ h\mathcal{R}_2 & \mathcal{Q}_2 \end{pmatrix}. \tag{4.1}$$

We study the microlocal solutions near ρ_0 to the system

$$\mathcal{Q}(h)u = 0. \tag{4.2}$$

We make the following assumptions on the symbols q_1, q_2, r_1 and r_2.

Assumption 4.1. *The symbols $q_1, q_2 \in S^0$ are real-valued, and satisfy the following conditions:*

$$q_1(\rho_0) = q_2(\rho_0) = 0, \tag{4.3}$$

$$\partial_\xi q_1(\rho_0)\partial_\xi q_2(\rho_0) \neq 0 \quad \text{and} \quad \{q_1, q_2\}(\rho_0) \neq 0, \tag{4.4}$$

where $\{q_1, q_2\}(x,\xi) := (\partial_\xi q_1 \partial_x q_2 - \partial_x q_1 \partial_\xi q_2)(x,\xi)$ denotes the Poisson bracket of q_1, q_2.

Assumption 4.2. *The symbols $r_1, r_2 \in S^0$ satisfy the ellipticity condition at ρ_0:*

$$r_j(\rho_0) \neq 0 \quad (j = 1, 2). \tag{4.5}$$

Let $\Gamma_{\mathcal{Q}}$ be the characteristic set of $\mathcal{Q}(h)$ given by

$$\Gamma_{\mathcal{Q}} = \Gamma_{q_1} \cup \Gamma_{q_2} \quad \text{with} \quad \Gamma_{q_j} := \{(x,\xi) \in T^*\mathbb{R}; q_j(x,\xi) = 0\} \quad (j = 1, 2).$$

The condition (4.4) means that Γ_{q_1} and Γ_{q_2} intersect transversally at ρ_0. Since the operator $\mathcal{Q}$ is microlocally elliptic outside $\Gamma_{\mathcal{Q}}$, it follows from standard arguments of microlocal analysis that the solutions to the system (4.2) are microlocally supported in a neighborhood of $\Gamma_{\mathcal{Q}}$. First, we study these microlocal solutions away from the crossing point ρ_0, that is, near the four curves

$$\Gamma^+_{q_1} := \{(x,\xi) \in \Gamma_{q_1};\, q_2(x,\xi) > 0\}, \quad \Gamma^-_{q_1} := \{(x,\xi) \in \Gamma_{q_1};\, q_2(x,\xi) < 0\},$$

$$\Gamma^+_{q_2} := \{(x,\xi) \in \Gamma_{q_2};\, q_1(x,\xi) > 0\}, \quad \Gamma^-_{q_2} := \{(x,\xi) \in \Gamma_{q_2};\, q_1(x,\xi) < 0\}.$$

In the following proposition we give a basis of WKB microlocal solutions to the system (4.2) on each of the four curves $(\Gamma^\pm_{q_j})_{j=1,2}$.

Proposition 4.1 (WKB basis of solutions)**.** *Let $j \in \{1, 2\}$. On each of the curves $\Gamma^\pm_{q_j}$, the space of microlocal solutions to the system (4.2) is one-dimensional and there exist $f^\pm_{q_j}$ such that*

$$\mathcal{Q} f^\pm_{q_j} \sim 0 \ \text{ microlocally on } \Gamma^\pm_{q_j},$$

and $f^\pm_{q_j}$ have the following WKB form

$$f^\pm_{q_j}(x;h) \sim \begin{pmatrix} a_{q_j}(x;h) \\ \\ b_{q_j}(x;h) \end{pmatrix} e^{i\phi_{q_j}(x)/h} \ \text{ microlocally on } \Gamma^\pm_{q_j}, \tag{4.6}$$

where the phase function ϕ_{q_j} is defined as the unique solution of the eikonal equation

$$\begin{cases} q_j(x, \phi'_{q_j}(x)) = 0, \\ \phi_{q_j}(0) = 0, \end{cases} \tag{4.7}$$

and a_{q_j}, b_{q_j} are symbols of the form $a_{q_j}(x;h) \sim \sum_{k\geq 0} h^k a_{q_j,k}(x)$, $b_{q_j}(x;h) \sim \sum_{k\geq 0} h^k b_{q_j,k}(x)$ with leading terms given by

$$a_{q_1,0}(x) = \exp\left(-\int_0^x \frac{\partial_x\partial_\xi q_1(t, \phi'_{q_1}(t)) + \phi''_{q_1}(t)\partial^2_\xi q_1(t, \phi'_{q_1}(t))}{2\partial_\xi q_1(t, \phi'_{q_1}(t))} dt \right),$$

$$b_{q_1,0}(x) = 0 \quad ; \quad b_{q_1,1}(x) = -\frac{r_2(x,\phi'_{q_1}(x))}{q_2(x,\phi'_{q_1}(x))} a_{q_1,0}(x),$$

and

$$b_{q_2,0}(x) = \exp\left(-\int_0^x \frac{\partial_x \partial_\xi q_2(t,\phi'_{q_2}(t)) + \phi''_{q_2}(t)\partial_\xi^2 q_2(t,\phi'_{q_2}(t))}{2\partial_\xi q_2(t,\phi'_{q_2}(t))} dt \right),$$

$$a_{q_2,0}(x) = 0 \quad ; \quad a_{q_2,1}(x) = -\frac{r_1(x,\phi'_{q_2}(x))}{q_1(x,\phi'_{q_2}(x))} b_{q_2,0}(x).$$

On $\Gamma^\pm_{q_1}$, the operator $\mathcal{Q}_1$ is of real principal type while $\mathcal{Q}_2$ is elliptic, and the same is true on $\Gamma^\pm_{q_2}$ by interchanging $\mathcal{Q}_1$ and $\mathcal{Q}_2$. Hence microlocally on each of the four curves $(\Gamma^\pm_{q_j})_{j=1,2}$, the system (4.2) is reduced to a scalar one-dimensional equation. Thus, the space of microlocal solutions on each of these curves is one-dimensional. The construction of the $f^\pm_{q_j}$ is based on formal computations using standard pseudodifferential calculus. We refer to [AF] for the details of this construction. The main result of this section is the following:

Theorem 4.1 (Microlocal transfer matrix)**.** *Let Assumptions 4.1 and 4.2 hold, and let $u(x;h) \in L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ be a solution to the system $\mathcal{Q}u = 0$ microlocally in a small neighborhood of ρ_0 such that*

$$u \sim t^\pm_j f^\pm_{q_j} \text{ microlocally on } \Gamma^\pm_{q_j},$$

for some scalar complex numbers $t^\pm_j = t^\pm_j(h)$, $j = 1, 2$. Then, there exist classical symbols of order 0, $\mu = \mu(h) \sim \sum_{k\geq 0} h^k \mu_k$ and $\widehat{\mu} = \widehat{\mu}(h) \sim \sum_{k\geq 0} h^k \widehat{\mu}_k$ such that

$$\begin{pmatrix} t^+_1 \\ t^+_2 \end{pmatrix} = \mathcal{T}(h) \begin{pmatrix} t^-_1 \\ t^-_2 \end{pmatrix} \quad \text{with} \quad \mathcal{T}(h) := \begin{pmatrix} \kappa_{1,1}(h) & h^{\frac{1}{2} - ih\mu}\kappa_{1,2}(h) \\ h^{\frac{1}{2} - ih\widehat{\mu}}\kappa_{2,1}(h) & \kappa_{2,2}(h) \end{pmatrix}, \tag{4.8}$$

where $\kappa_{j,k}(h) \sim \sum_{n\geq 0} h^n \kappa^n_{j,k}$ are symbols with leading terms given by

$$\kappa^0_{1,1} = \kappa^0_{2,2} = 1,$$

$$\kappa^0_{1,2} = -e^{i\frac{\pi}{4}} \left(r_1 \sqrt{\left| \frac{2\pi\partial_\xi q_2}{\partial_\xi q_1 \{q_1, q_2\}} \right|} \right)_{|(x,\xi)=\rho_0}, \quad \kappa^0_{2,1} = \sigma e^{-i\frac{\pi}{4}} \left(r_2 \sqrt{\left| \frac{2\pi\partial_\xi q_1}{\partial_\xi q_2 \{q_1, q_2\}} \right|} \right)_{|(x,\xi)=\rho_0},$$

where $\sigma := -sgn(\partial_\xi q_1 \partial_\xi q_{2|(x,\xi)=\rho_0})$.

The rest of this paper devoted to the proof of Theorem 4.1 which relies on many steps. The first step consists in the reduction of the system (4.2) to a scalar equation using the ellipticity condition (4.5) and then to solve this equation by means of a normal form in the spirit of [Sj, CdvPa].

Step 1: Reduction to a scalar equation and normal form

Setting $u = {}^t(u_1, u_2)$ and using the ellipticity of $\mathcal{R}_1$ at ρ_0 according to assumption (4.5), the system (4.2) is reduced microlocally near the origin to a scalar equation of u_1. More precisely, there exists a small neighborhood $\mathcal{V} \subset \mathbb{R}^2$ of ρ_0 such that microlocally in $\mathcal{V}$, the system $\mathcal{Q}u \sim 0$ is reduced to

$$\begin{cases} \mathcal{L}u_1 \sim 0, \\ u_2 \sim -h^{-1}\mathcal{R}_1^{-1}\mathcal{Q}_1 u_1, \end{cases} \tag{4.9}$$

where $\mathcal{R}_1^{-1}$ denotes a parametrix of $\mathcal{R}_1$ in $\mathcal{V}$ and $\mathcal{L}$ is the h-pseudodifferential operator defined by

$$\mathcal{L} := \mathcal{R}_1\mathcal{Q}_2\mathcal{R}_1^{-1}\mathcal{Q}_1 - h^2\mathcal{R}_1\mathcal{R}_2,$$

with semiclassical Weyl symbol $\ell(x,\xi;h) = \sum_{j\geq 0} h^j \ell_j(x,\xi)$. In particular, by the pseudodifferential symbolic calculus, we have

$$\ell_0 = q_1 q_2 \ ; \ \ell_1(\rho_0) = \frac{i}{2}\{q_1, q_2\}(\rho_0).$$

The crossing point ρ_0 is a hyperbolic fixed point of the Hamiltonian vector field of ℓ. We set

$$\alpha := \partial_x q_1(\rho_0),\ \beta := \partial_\xi q_1(\rho_0),\ \gamma := \partial_x q_2(\rho_0),\ \delta := \partial_\xi q_2(\rho_0),\ D := \{q_1, q_2\}(\rho_0). \tag{4.10}$$

Without loss of generality, we assume that $\beta\delta > 0$ and $D > 0$.

In our one-dimensional case, we have the following normal form for the quantization $\mathcal{L}$:

Lemma 4.1. *There exist a small neighborhood $\Omega \subset \mathbb{R}^2$ of $(0,0)$, a Fourier integral operator U with associated canonical transformation κ sending $\mathcal{V}$ to Ω and $\kappa(\rho_0) = (0,0)$, and a classical symbol $F(t;h) \sim \sum_{k\geq 0} h^k F_k(t) \in C^\infty$ defined near $t = 0$, with*

$$F(0;h) = -\frac{i}{2}h + \mu h^2, \tag{4.11}$$

where $\mu = \mu(h) \sim \sum_{k\geq 0} h^k \mu_k$ is a classical symbol of order 0, such that

$$UF(\mathcal{L};h)U^{-1} \sim \mathcal{G} := \frac{1}{2}(yhD_y + hD_y \cdot y) \quad \text{microlocally in } \ \Omega. \tag{4.12}$$

Proof. The normal form (4.12) is due to [Sj]. Notice that in this work, this result was proved for self-adjoint operators, but it still holds for our non-self-adjoint operator $\mathcal{L}$ which is self-adjoint at the principal level $\mathrm{Op}_h^w(\ell_0)$. In the following, we prove (4.11).

The FIO U is associated with the canonical transform $\kappa : (x,\xi) \mapsto (y,\eta)$ satisfying

$$F(\ell(x,\xi;h)) = y\eta.$$

In particular, we can choose

$$\kappa(x,\xi) = \kappa_0(x,\xi) + \mathcal{O}((x,\xi)^2), \quad \kappa_0(x,\xi) = \frac{1}{\sqrt{D}}\left(\gamma x + \delta\xi, \alpha x + \beta\xi\right).$$

After a normalization, we can write U^{-1} in the form

$$U^{-1}v(x;h) = \int_{\mathbb{R}} e^{i\psi(x,y)/h} c(x,y;h) v(y) dy,$$

where $c(x,y;h) \sim \sum_{k\geq 0} c_k(x,y)$ is a symbol with $c_0(0,0) = 1$ and the phase function $\psi(x,y)$ is a generating function of κ^{-1}, in the sense that $\kappa^{-1} : (y, -\nabla_y\psi) \mapsto (x, \nabla_x\psi)$. In particular, near $(x,y) = \rho_0$, we have

$$\psi(x,y) = \frac{1}{2\delta}(-\gamma x^2 + 2\sqrt{D}xy - \beta y^2) + \mathcal{O}((x,y)^3). \tag{4.13}$$

At the levels of principal and sub-principal symbols, the relation (4.12) implies that

$$F_0(\ell_0(\kappa^{-1}(y,\eta))) = y\eta,$$

$$F_1(\ell_0(\kappa^{-1}(y,\eta))) + \ell_1(\kappa^{-1}(y,\eta))F_0'(\ell_0(\kappa^{-1}(y,\eta))) = 0.$$

In particular, the first equation at $(y,\eta) = (0,0)$ implies that $F_0(0) = 0$ and $F_0'(0) = \frac{1}{D}$, and the second one gives

$$F_1(0) = -\ell_1(\rho_0)F_0'(0) = -\frac{i}{2},$$

since $\ell_1(\rho_0) = \frac{iD}{2}$. Thus the symbol $F(0;h)$ has the form (4.11). □

Step 2: Microlocal solutions near the crossing point

Setting $\tilde{u}_1 := Uu_1$, the equation $\mathcal{L}u_1 \sim 0$ microlocally in $\mathcal{V}$ is equivalent to

$$\mathcal{G}\tilde{u}_1 \sim F(0;h)\tilde{u}_1 \quad \text{microlocally in } \Omega, \tag{4.14}$$

which can be rewritten as

$$y\tilde{u}_1' \sim i\mu h\tilde{u}_1 \quad \text{microlocally in } \Omega.$$

The space of microlocal solutions of this equation is two dimensional and a basis is given by the two functions

$$g_\mu^+(y) := H(y)y^{i\mu h}, \quad g_\mu^-(y) := H(-y)|y|^{i\mu h}, \tag{4.15}$$

where H denotes the Heaviside function, i.e., $H(y) = 1$ for $y \geq 0$ and $H(y) = 0$ for $y < 0$. In particular, we have

$$\mathrm{FS}(g_\mu^\pm) = \{\pm y > 0, \eta = 0\} \cup \{y = 0\}.$$

Thus, $u_1^\pm := U^{-1}g_\mu^\pm$ are solutions to the equation $\mathcal{L}u_1 \sim 0$ microlocally in $\mathcal{V}$, and we have

$$\mathrm{FS}(u_1^\pm) \cap \mathcal{V} \subset \left(\Gamma_{q_1}^\pm \cup \Gamma_{q_2}\right) \cap \mathcal{V}.$$

More precisely, we have the following asymptotic formulae for $u_1^\pm$.

Proposition 4.2. *There exist symbols $\sigma^\pm(x;h) \sim \sum_{k\geq 0} h^k\sigma_k^\pm(x)$, $\eta^\pm(x;h) \sim \sum_{k\geq 0} h^k\eta_k^\pm(x)$, with leading terms given by*

$$\sigma_0^\pm(x) = \sqrt{\frac{\delta}{\beta}}e^{-i\frac{\pi}{4}} + \mathcal{O}(x), \quad \eta_0^\pm(x) = \pm\frac{i\delta}{\sqrt{D}x}(1 + \mathcal{O}(x)),$$

such that, modulo $\mathcal{O}(h^\infty)$ as $h \to 0^+$, we have

$$u_1^+(x;h) = \begin{cases} \sqrt{2\pi h}\,\sigma^+(x;h)e^{i\phi_{q_1}(x)/h} + h^{1+i\mu h}\eta_+(x;h)e^{i\phi_{q_2}(x)/h} & (x > 0) \\ h^{1+i\mu h}\,\eta_+(x;h)e^{i\phi_{q_2}(x)/h} & (x < 0) \end{cases}$$

and

$$u_1^-(x;h) = \begin{cases} h^{1+i\mu h}\,\eta_-(x;h)e^{i\phi_{q_2}(x)/h} & (x > 0) \\ \sqrt{2\pi h}\,\sigma^-(x;h)e^{i\phi_{q_1}(x)/h} + h^{1+i\mu h}\eta_-(x;h)e^{i\phi_{q_2}(x)/h} & (x < 0). \end{cases}$$

Proof. We only prove the formula for u_1^+. By definition, we have

$$u_1^+(x;h) = U^{-1}g_\mu^+(x;h) = \int_0^{+\infty} e^{i\psi(x,y)/h}c(x,y;h)y^{i\mu h}dy, \tag{4.16}$$

where the phase function ψ satisfies (4.13). The right hand side of (4.16) is an oscillatory integral and up to $\mathcal{O}(h^\infty)$, its asymptotic behavior as $h \to 0$ is governed by the contributions of the critical points of the phase function $y \mapsto \psi(x,y)$ and the end point $y = 0$ of $y \mapsto y^{i\mu h}$.

For $x > 0$, the function $y \mapsto \psi(x,y)$ has a positive non degenerate critical point $y_c(x)$ which behaves like $y_c(x) = \frac{\sqrt{D}}{\beta}x + \mathcal{O}(x^2)$ as $x \to 0$. The critical value $\psi(x, y_c(x))$ coincides with the generating function $\phi_{q_1}(x)$ of Γ_{q_1}, and $\psi(x, y_c(x)) = -\frac{\alpha}{2\beta}x^2 + \mathcal{O}(x^3)$ as $x \to 0$. Moreover, we have $\partial_y^2\psi(x, y_c(x)) = -\frac{\beta}{\delta} + \mathcal{O}(x) < 0$. Then, for a cutoff function $\chi(y) \in C_0^\infty(\mathbb{R})$ identically 1 near 0 and supported in a small neighborhood of $y = 0$ so that $y_c(x) \notin \operatorname{supp}\chi$, we have, by the stationary phase theorem (see e.g. [Ma] Corollary 2.6.3),

$$\int_0^{+\infty} e^{i\psi(x,y)/h}c(x,y;h)y^{i\mu h}(1 - \chi(y))dy = \sqrt{2\pi h}\,\sigma^+(x;h)e^{i\phi_{q_1}(x)/h}, \tag{4.17}$$

where $\sigma^+(x;h) \sim \sum_{k\geq 0} h^k \sigma_k^+(x)$ is a symbol with leading term

$$\sigma_0^+(x) = e^{-i\frac{\pi}{4}} |\partial_y^2 \psi(x, y_c(x))|^{-\frac{1}{2}} c_0(x, y_c(x)) = \sqrt{\frac{\delta}{\beta}} e^{-i\frac{\pi}{4}} + \mathcal{O}(x). \tag{4.18}$$

On the other hand, we have

$$\int_0^{+\infty} e^{i\psi(x,y)/h} c(x,y;h) y^{i\mu h} \chi(y) dy = h^{1+i\mu h} \eta_+(x;h) e^{i\phi_{q_2}(x)/h},$$

where $\eta_+(x;h) \sim \sum_{k\geq 0} h^k \eta_k^+(x)$ with

$$\eta_0^+(x) = \frac{i\delta}{\sqrt{D}x} c_0(x,0) = \frac{i\delta}{\sqrt{D}x}(1 + \mathcal{O}(x)).$$

For the study of this contribution from the endpoint, we develop $\psi(x,y)$ in Taylor expansion at $y=0$: $\psi(x,y) = \phi_{q_2}(x) + \left(\frac{\sqrt{D}}{\delta} + \mathcal{O}(x)\right) y + \mathcal{O}(y^2)$. Then using the fact that $D \neq 0$, we reduce the integral to a Laplace integral after a change of variable which eliminates the term $\mathcal{O}(y^2)$. The above asymptotic formula results from the term by term integration which is known as Watson's lemma.

For $x < 0$, there is no positive critical points of $y \mapsto \psi(x,y)$, and hence on this side, the asymptotic expansion of u_1^+ comes only from the endpoint of the integral, which can be computed similarly as above. □

Now, we construct another pair of solutions $v^\pm = {}^t(v_1^\pm, v_2^\pm)$ to the system (4.2) that are microlocally zero on one of $\Gamma_{q_2}^+$ and $\Gamma_{q_2}^-$. To do this, we proceed in a similar way as above but now by reducing the system (4.2) to a scalar equation of v_2 instead of v_1. Setting $v = {}^t(v_1, v_2)$ and using the ellipticity of $\mathcal{R}_2$ at ρ_0, the system $\mathcal{Q}v \sim 0$ is reduced microlocally near ρ_0 to

$$\begin{cases} \widehat{\mathcal{L}} v_2 \sim 0, \\ v_1 \sim -h^{-1} \mathcal{R}_2^{-1} \mathcal{Q}_2 v_2, \end{cases} \tag{4.19}$$

where $\mathcal{R}_2^{-1}$ denotes a parametrix of $\mathcal{R}_2$ in a neighborhood of ρ_0 and $\widehat{\mathcal{L}}$ is the h-pseudodifferential operator defined by

$$\widehat{\mathcal{L}} := \mathcal{R}_2 \mathcal{Q}_1 \mathcal{R}_2^{-1} \mathcal{Q}_2 - h^2 \mathcal{R}_2 \mathcal{R}_1.$$

As before, we can construct two microlocal solutions $v_2^\pm$using a normal form reduction.

Proposition 4.3. *There exist microlocal solutions $v_2^\pm$ to $\widehat{\mathcal{L}} v_2 \sim 0$ in a neighborhood of ρ_0 such that, modulo $\mathcal{O}(h^\infty)$ as $h \to 0^+$, we have*

$$v_2^+(x;h) = \begin{cases} h^{1+i\widehat{\mu}h}\, \widehat{\eta}_+(x;h) e^{i\phi_{q_1}(x)/h} & (x > 0) \\ \sqrt{2\pi h}\, \widehat{\sigma}^+(x;h) e^{i\phi_{q_2}(x)/h} + h^{1+i\widehat{\mu}h} \widehat{\eta}_+(x;h) e^{i\phi_{q_1}(x)/h} & (x < 0) \end{cases}$$

and

$$v_2^-(x;h) = \begin{cases} \sqrt{2\pi h}\, \widehat{\sigma}^-(x;h) e^{i\phi_{q_2}(x)/h} + h^{1+i\widehat{\mu}h} \widehat{\eta}_-(x;h) e^{i\phi_{q_1}(x)/h} & (x > 0) \\ h^{1+i\widehat{\mu}h}\, \widehat{\eta}_-(x;h) e^{i\phi_{q_1}(x)/h} & (x < 0). \end{cases}$$

where $\widehat{\sigma}^\pm(x;h) \sim \sum_{k\geq 0} h^k \widehat{\sigma}_k^\pm(x)$, $\widehat{\eta}^\pm(x;h) \sim \sum_{k\geq 0} h^k \widehat{\eta}_k^\pm(x)$, $\widehat{\mu}(h) \sim \sum_{k\geq 0} h^k \widehat{\mu}_{-,k}$ with

$$\widehat{\sigma}_0^\pm(x) = \sqrt{\frac{\beta}{\delta}} e^{i\frac{\pi}{4}} + \mathcal{O}(x), \quad \widehat{\eta}_0^\pm(x) = \pm \frac{i\beta}{\sqrt{D}x}(1 + \mathcal{O}(x)).$$

Summing up, we have then constructed 4 microlocal solutions to the system (4.2) microlocally in a small neighborhood $\mathcal{V}$ of ρ_0

$$u^\pm = {}^t(u_1^\pm, u_2^\pm), \quad v^\pm = {}^t(v_1^\pm, v_2^\pm),$$

with

$$\mathrm{FS}(u^\pm) \cap \mathcal{V} \subset \left(\Gamma_{q_1}^\pm \cup \Gamma_{q_2}\right) \cap \mathcal{V}, \quad \mathrm{FS}(v^\pm) \cap \mathcal{V} \subset \left(\Gamma_{q_2}^\pm \cup \Gamma_{q_1}\right) \cap \mathcal{V},$$

where $u_1^\pm$ and $v_2^\pm$ are defined above and

$$u_2^\pm \sim -h^{-1}\mathcal{R}_1^{-1}\mathcal{Q}_1 u_1^\pm, \;\; v_1^\pm \sim -h^{-1}(\mathcal{R}_2)^{-1}\mathcal{Q}_2 v_2^\pm.$$

Step 3: Connecting the solutions to the basis elements and conclusion

Now we connect our microlocal solutions $u^\pm$ and $v^\pm$ to the WKB solutions $f_{q_j}^\pm$, $j = 1, 2$, given by Proposition 4.1 and we deduce the transfer matrix at the crossing point. The following result is an immediate consequence of Propositions 4.2 and 4.3.

Proposition 4.4. *There exist symbols*

$$A_{q_1}^\pm(h) \sim \sum_{k\geq 0} h^k A_{q_1,k}^\pm, \; A_{q_2}^{\pm,\pm}(h) \sim \sum_{k\geq 0} h^k A_{q_2,k}^{\pm,\pm}, \; B_{q_1}^{\pm,\pm}(h) \sim \sum_{k\geq 0} h^k B_{q_1,k}^{\pm,\pm}, \; B_{q_2}^\pm(h) \sim \sum_{k\geq 0} h^k B_{q_2,k}^\pm,$$

with leading terms

$$A_{q_1,0}^+ = A_{q_1,0}^- = \sqrt{\frac{2\pi\delta}{\beta}} e^{-i\frac{\pi}{4}}, \;\; A_{q_2,0}^{+,+} = A_{q_2,0}^{+,-} = -A_{q_2,0}^{-,+} = -A_{q_2,0}^{-,-} = \frac{i\sqrt{D}}{r_1(\rho_0)}, \tag{4.20}$$

$$B_{q_2,0}^+ = B_{q_2,0}^- = \sqrt{\frac{2\pi\beta}{\delta}} e^{i\frac{\pi}{4}}, \;\; -B_{q_1,0}^{+,+} = -B_{q_1,0}^{+,-} = B_{q_1,0}^{-,+} = B_{q_1,0}^{-,-} = \frac{i\sqrt{D}}{r_2(\rho_0)}, \tag{4.21}$$

such that

$$u^+ \sim \begin{cases} A_{q_1}^+ h^{\frac{1}{2}} f_{q_1}^+ & \text{on } \Gamma_{q_1}^+ \cap \mathcal{V} \\ 0 & \text{on } \Gamma_{q_1}^- \cap \mathcal{V} \\ A_{q_2}^{+,+} h^{i\mu h} f_{q_2}^+ & \text{on } \Gamma_{q_2}^+ \cap \mathcal{V} \\ A_{q_2}^{+,-} h^{i\mu h} f_{q_2}^- & \text{on } \Gamma_{q_2}^- \cap \mathcal{V} \end{cases}, \quad u^- \sim \begin{cases} 0 & \text{on } \Gamma_{q_1}^+ \cap \mathcal{V} \\ A_{q_1}^- h^{\frac{1}{2}} f_{q_1}^+ & \text{on } \Gamma_{q_1}^- \cap \mathcal{V} \\ A_{q_2}^{-,+} h^{i\mu h} f_{q_2}^+ & \text{on } \Gamma_{q_2}^+ \cap \mathcal{V} \\ A_{q_2}^{-,-} h^{i\mu h} f_{q_2}^- & \text{on } \Gamma_{q_2}^- \cap \mathcal{V} \end{cases}$$

$$v^+ \sim \begin{cases} B_{q_2}^+ h^{\frac{1}{2}} f_{q_2}^+ & \text{on } \Gamma_{q_2}^+ \cap \mathcal{V} \\ 0 & \text{on } \Gamma_{q_2}^- \cap \mathcal{V} \\ B_{q_1}^{+,+} h^{i\widehat{\mu} h} f_{q_1}^+ & \text{on } \Gamma_{q_1}^+ \cap \mathcal{V} \\ B_{q_1}^{+,-} h^{i\widehat{\mu} h} f_{q_1}^- & \text{on } \Gamma_{q_1}^- \cap \mathcal{V} \end{cases}, \quad v^- \sim \begin{cases} 0 & \text{on } \Gamma_{q_2}^+ \cap \mathcal{V} \\ B_{q_2}^- h^{\frac{1}{2}} f_{q_2}^+ & \text{on } \Gamma_{q_2}^- \cap \mathcal{V} \\ B_{q_1}^{-,+} h^{i\widehat{\mu} h} f_{q_1}^+ & \text{on } \Gamma_{q_1}^+ \cap \mathcal{V} \\ B_{q_1}^{-,-} h^{i\widehat{\mu} h} f_{q_1}^- & \text{on } \Gamma_{q_1}^- \cap \mathcal{V}. \end{cases}$$

We set

$$\begin{pmatrix} t_1^+(h) \\ \\ t_2^+(h) \end{pmatrix} = \begin{pmatrix} s_{1,1}(h) & s_{1,2}(h) \\ \\ s_{2,1}(h) & s_{2,2}(h) \end{pmatrix} \begin{pmatrix} t_1^-(h) \\ \\ t_2^-(h) \end{pmatrix}.$$

Observe that if $t_1^-(h) = 1$ and $t_2^-(h) = 0$ then u should be equal to $(B_{q_1}^{+,-} h^{i\widehat{\mu} h})^{-1} v^+$ microlocally near ρ_0, and therefore we have

$$s_{2,1}(h) = t_2^+(h) = h^{\frac{1}{2} - i\widehat{\mu} h} \kappa_{2,1}(h) \quad \text{with} \;\; \kappa_{2,1}(h) := \frac{B_{q_2}^+(h)}{B_{q_1}^{+,-}(h)},$$

$$s_{1,1}(h) = t_1^+(h) = \frac{B_{q_1}^{+,+}(h)}{B_{q_1}^{+,-}(h)}.$$

Analogously, if $t_1^-(h) = 0$ and $t_2^-(h) = 1$ then u should be equal to $(A_{q_2}^{+,-} h^{i\mu h})^{-1} u^+$ microlocally near ρ_0, and therefore we have

$$s_{1,2}(h) = t_1^+(h) = h^{\frac{1}{2} - i\mu h} \kappa_{1,2}(h) \quad \text{with} \quad \kappa_{1,2}(h) := \frac{A_{q_1}^+(h)}{A_{q_2}^{+,-}(h)},$$

$$s_{2,2}(h) = t_2^+(h) = \frac{A_{q_2}^{+,+}(h)}{A_{q_2}^{+,-}(h)}.$$

This ends the proof of Theorem 4.1.

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