Weyl-type lower bound for non-scattering energies for acoustic-type equations

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1 Introduction

1.1 Non-scattering energy

In this lecture note, we review some results for *non-scattering energies* (NSEs) of acoustic equations and some related topics. In particular, the main purpose is the proof of a Weyl-type lower bound for the number of NSEs. The original paper of our argument is Morioka-Shoji [14].

Let us consider the scattering theory for time-independent acoustic-type equations on a Riemannian manifold (M, g) with a flat end. The assumption is as follows. Let M be a connected and non-compact C^∞-Riemannian manifold of dimension $d \geq 2$. We assume that M is split into two parts $M = \mathcal{K} \cup \Omega^e$ where $\mathcal{K}$ is a connected and compact subset, and Ω^e which is called *end* of M is diffeomorphic to a connected exterior domain in $\mathbf{R}^d$ with smooth boundary. Then we naturally identify Ω^e with a connected exterior domain $\mathbf{R}^d \setminus \overline{\Omega^i_0}$ where Ω^i_0 is a connected and bounded domain in $\mathbf{R}^d$ with the Euclidean metric. In the following, Ω^i and Γ denote the interior and the boundary of $\mathcal{K}$ respectively. The metric $g = (g_{kl})^d_{k,l=1}$ is positive-definite on M and satisfies $g_{kl} = \delta_{kl}$ on $\overline{\Omega^e}$. We denote by Δ_g the Laplacian on M which is given by

$$\Delta_g = \frac{1}{\sqrt{g}} \sum_{k,l=1}^{d} \frac{\partial}{\partial x_k} \left(\sqrt{g} g^{kl} \frac{\partial}{\partial x_l} \right),$$

in local coordinates x. By the assumption, Δ_g coincides the usual Euclidean Laplacian Δ on Ω^e.

We consider the following time-independent acoustic-type equation

$$-\Delta_g u = \lambda n u \quad \text{on} \quad M, \quad \lambda > 0, \tag{1.1}$$

where the coefficient $n \in C(M)$ satisfies the following conditions :

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- $n\big|_{\mathcal{K}} \in C^\infty(\mathcal{K})$,
- $\mathrm{supp}(n(p) - 1) = \mathcal{K}$,
- $n(p) > 0$ for all $p \in M$,
- $\partial_\nu n(p) \neq 0$ for all $p \in \Gamma$ where ∂_ν is the outward normal derivative on Γ i.e. $\partial_\nu n$ does not change its sign on Γ.

Remark. The forth assumption can be replaced by "$n(p) \neq 1$ for all $p \in \Gamma$" and so on. These kinds of singularities of n across Γ have a crucial role in our arguments.

In the time-independent scattering theory, we study generalized eigenfunctions of (1.1). Roughly speaking, the generalized eigenfunction can be written as $u = u^i + u^s$ at infinity where u^i is the incident wave satisfying $-\Delta u^i = \lambda u^i$ on $\mathbf{R}^d$, and u^s is the scattered wave associated with u^i. If we take $u^i(x) = e^{i\sqrt{\lambda}x\cdot\omega}$ for $\omega \in S^{d-1}$, the scattered wave u^s satisfies the asymptotic behavior

$$u^s(x) = C(\lambda)|x|^{-(d-1)/2} e^{i\sqrt{\lambda}|x|} A(\lambda;\omega,\theta) + O(|x|^{-(d+1)/2}), \quad |x| \to \infty,$$

for $\theta := x/|x| \in S^{d-1}$. Here the function $A(\lambda;\omega,\theta)$ is the *scattering amplitude*. We replace u^i by the Herglotz wave

$$u^i(x) = (2\pi)^{-d/2} \int_{S^{d-1}} e^{i\sqrt{\lambda}x\cdot\omega} \phi(\omega) d\Sigma, \quad \phi \in L^2(S^{d-1}),$$

where $d\Sigma$ is the measure on S^{d-1} induced by the Euclidean measure. Then we obtain

$$u^s(x) = C(\lambda)|x|^{-(d-1)/2} e^{i\sqrt{\lambda}|x|} (A(\lambda)\phi)(\theta) + O(|x|^{-(d+1)/2}), \quad |x| \to \infty,$$

where the operator $A(\lambda)$ is a compact on $L^2(S^{d-1})$ and its integral kernel is $A(\lambda;\omega,\theta)$. Thus the operator $A(\lambda)$ determine the far-field pattern of u^s.

Now we consider the case where $A(\lambda)$ has the eigenvalue 0 in $L^2(S^{d-1})$. Then there exists a non-trivial solution $\phi_0 \in L^2(S^{d-1})$ to the equation $A(\lambda)\phi_0 = 0$. In this case, the scattered wave u^s associated with the Herglotz wave

$$u^i(x) = (2\pi)^{-d/2} \int_{S^{d-1}} e^{i\sqrt{\lambda}x\cdot\omega} \phi_0(\omega) d\Sigma$$

satisfies $u^s(x) = O(|x|^{-(d+1)/2})$ as $|x| \to \infty$. The Rellich type uniqueness theorem ([17], [20]) and the unique continuation property for Helmholtz equations imply that u^s vanishes on $\overline{\Omega^e}$. Thus the scattered wave cannot be observed at infinity even though the generalized eigenfunction u of the equation (1.1) is perturbed by the coefficient n and the metric g. Therefore, we define the notion of NSE as follows.

Definition 1.1 *If there exists a non-trivial solution $\phi \in L^2(S^{d-1})$ to the equation $A(\lambda)\phi = 0$, we call the corresponding $\lambda > 0$ a non-scattering energy (NSE).*

Here we would like to make some comments on history. As far as the authors know, there is only one paper by Colton-Monk [6] for the existence of NSE of acoustic equations. Note that they considered the case $M = \mathbf{R}^3$ with a spherically symmetric inhomogeneity n. There are some examples of inhomogeneities (for acoustic equations) or potentials (for Schrödinger equations) such that they do not have NSEs (see [9], [3], [7], [15]). NSEs naturally appear in inverse scattering problems. For some numerical methods, NSEs cause some difficulties.

1.2 Remark for resonant tunneling effect

Here we mention related topics in quantum mechanics. Typical example is the 1D Schrödinger equation

$$-\psi'' + V\psi = \lambda\psi \quad \text{on} \quad \mathbf{R},$$

with a double-barrier potential

$$V(x) = \begin{cases} V_0, & |x| \in (r, r+\epsilon), \\ 0, & \text{otherwise}, \end{cases}$$

for fixed $r > 0$ and small $\epsilon > 0$. When we take the incident wave $e^{i\sqrt{\lambda}x}$ with energy $\lambda > 0$ coming from $x = -\infty$, the generalized eigenfunction $\psi_+ \in L^\infty(\mathbf{R})$ is given by

$$\psi_+(x) = \begin{cases} e^{i\sqrt{\lambda}x} + r_+(\lambda)e^{-i\sqrt{\lambda}x}, & x \leq -r-\epsilon, \\ t_+(\lambda)e^{i\sqrt{\lambda}x}, & x \geq r+\epsilon. \end{cases}$$

Similarly, we can have the generalized eigenfunction $\psi_- \in L^\infty(\mathbf{R})$ satisfying

$$\psi_-(x) = \begin{cases} t_-(\lambda)e^{-i\sqrt{\lambda}x}, & x \leq -r-\epsilon, \\ e^{-i\sqrt{\lambda}x} + r_-(\lambda)e^{i\sqrt{\lambda}x}, & x \geq r+\epsilon. \end{cases}$$

The S-matrix is defined by

$$S(\lambda) = \begin{bmatrix} t_+(\lambda) & r_-(\lambda) \\ r_+(\lambda) & t_-(\lambda) \end{bmatrix} \in \mathrm{U}(2).$$

It is well-known that there may exist infinite number of $\lambda > 0$ such that $|t_+(\lambda)| = 1$ (i.e. $|r_+(\lambda)| = 0$). We call this phenomenon the *resonant-tunneling effect* (See [5], [19]). The similar phenomenon has been derived for quantum walks (see [13]).

We can consider the similar problem for multi-dimensional Schrödinger equations : $(-\Delta + V)\psi = \lambda\psi$ on $\mathbf{R}^d$. As has been mentioned above, the authors do not know results for existence of resonant-tunneling effect of multi-dimensional cases. Note that our argument in this article does not work for the Schrödinger operator $-\Delta + V$ on $\mathbf{R}^d$, since we use a singularity of the coefficient of Δ for acoustic equations.

2 Results

Since the scattered wave u^s vanishes if $\lambda > 0$ is a NSE, we can reduce the problem to the following boundary value problem. If $u = u^i + u^s$ is the generalized eigenfunction to (1.1) such that u^s vanishes on $\overline{\Omega^e}$, we have

$$(-\Delta_g - \lambda n)v = 0 \quad \text{in} \quad \Omega^i, \tag{2.1}$$

$$(-\Delta - \lambda)w = 0 \quad \text{in} \quad \Omega_0^i, \tag{2.2}$$

$$v = w, \quad \partial_\nu v = \partial_\nu w \quad \text{on} \quad \Gamma, \tag{2.3}$$

where $v = u\big|_{\overline{\Omega^i}}$ and $w = u^i\big|_{\overline{\Omega_0^i}}$. We call the system of boundary value problems (2.1)-(2.3) the *interior transmission eigenvalue problem (ITEP)*. If the system (2.1)-(2.3) has a non-trivial solution in $H^2(\Omega^i) \times H^2(\Omega_0^i)$, we call the corresponding $\lambda \in \mathbf{C}$ an *interior transmission eigenvalue (ITE)*.

Remark. The system (2.1)-(2.3) is non self-adjoint. Then there exist complex ITEs. In general, the discreteness of ITEs is not trivial. Fortunately, we can prove the discreteness of ITEs under our assumptions.

Obviously, we obtain the following inclusion relation.

Lemma 2.1 *We have* $\{\mathrm{NSE}\} \subset \{\mathrm{ITE}\}$. *If the set* $\{\mathrm{ITE}\}$ *is a discrete subset in* $\mathbf{C}$*, the set* $\{\mathrm{NSE}\}$ *is a discrete subset in* $(0, \infty)$.

Remark. The converse relation $\{\mathrm{NSE}\} \supset \{\mathrm{ITE}\}$ does not hold. The relation $\{\mathrm{ITE}\} \cap (0, \infty) \subset \{\mathrm{NSE}\}$ is also nontrivial. We have to remove *singular ITEs* for this assertion. The notion of singular ITEs will be defined later.

Now we state our results. For the proof of them, we basically adopt the argument of Lakshtanov-Vainberg [12]. They studied the Dirichlet-to-Neumann map (D-N map) on the boundary. Then we do not have to impose topological assumptions for Ω^i and Ω_0^i.

First one is the discreteness of NSEs. This is a direct consequence of the discreteness of $\{\mathrm{ITE}\}$.

Theorem 2.2 *The set* $\{\mathrm{NSE}\}$ *is a discrete subset in* $(0, \infty)$ *with only possible accumulation points at* 0 *and infinity.*

Moreover, we can show the existence of infinitely many NSEs by proving a Weyl-type lower bound for the number of NSEs. Here we put $\gamma = \mathrm{sign}(\partial_\nu n\big|_\Gamma) = 1$ or -1 and

$$V_n = (2\pi)^{-d} \int_{\Omega^i} \int_{\{\xi \in \mathbf{R}^d \, ; \, \langle g^{-1}(x)\xi, \xi \rangle \leq n(x)\}} d\xi dx,$$

$$V_0 = (2\pi)^{-d} \mathrm{vol}(\Omega_0^i) \mathrm{vol}(B_d),$$

where B_d is the unit ball in $\mathbf{R}^d$.

Theorem 2.3 *Let $\alpha > 0$ be sufficiently small. Suppose that $C_\gamma := V_n - 2V_0 > 0$ for $\gamma = 1$ or $C_\gamma := V_0 - 2V_n > 0$ for $\gamma = -1$. Taking account multiplicities of each NSE, $N_{NSE}(\lambda) := \#\{\text{NSE} \in (\alpha, \lambda]\}$ satisfies*

$$N_{NSE}(\lambda) \geq C_\gamma \lambda^{d/2} + O(\lambda^{(d-1)/2}),$$

as $\lambda \to \infty$.

Remark. The condition $V_n - 2V_0 > 0$ for $\gamma = 1$ holds for n such that $n(x) < 1$ near the boundary Γ and $n(x)$ is sufficiently large inside of Ω^i. Similarly, the condition $V_0 - 2V_n > 0$ for $\gamma = -1$ holds for n such that $n(x) > 1$ near the boundary Γ and $n(x)$ is sufficiently small inside of Ω^i.

Theorem 2.3 follows from the same kind of estimates for ITEs. In order to derive a key lemma, we introduce the notion of singular ITEs here.

Definition 2.4 *We call $\lambda \in (0, \infty)$ a singular ITE if there exist $v \in H^2(\Omega^i)$ and $w \in H^2(\Omega_0^i)$ such that*

$$\begin{aligned} (-\Delta_g - \lambda n)v = 0 \quad &in \quad \Omega^i, \\ (-\Delta - \lambda)w = 0 \quad &in \quad \Omega_0^i, \\ v = w = 0, \quad \partial_\nu v = \partial_\nu w \quad &on \quad \Gamma. \end{aligned}$$

The multiplicity of λ is defined by the dimension of the subspace of $L^2(\Gamma)$ spanned by $\partial_\nu v (= \partial_\nu w)$ above.

Let $\sigma_D(-n^{-1}\Delta_g)$ and $\sigma_D(-\Delta)$ be the set of Dirichlet eigenvalues of $-n^{-1}\Delta_g$ on Ω^i and $-\Delta$ on Ω_0^i, respectively. Definition 2.4 implies $\lambda \in \sigma_D(-n^{-1}\Delta_g) \cap \sigma_D(-\Delta)$ if λ is a singular ITE.

Then we can state key lemmas as follows.

Lemma 2.5 *If $\lambda \in (0, \infty)$ is a non-singular ITE, λ is also a NSE.*

Lemma 2.6 *Let*

$$N_T^{reg}(\lambda) = \#\{\text{non-singular ITEs} \in (\alpha, \lambda]\}, \quad N_T^{sng}(\lambda) = \#\{\text{singular ITEs} \in (\alpha, \lambda]\}.$$

Then we have

$$N_T^{reg}(\lambda) + N_T^{sng}(\lambda) \geq \gamma(V_n - V_0)\lambda^{d/2} + O(\lambda^{(d-1)/2}),$$

and

$$N_T^{sng}(\lambda) \leq V_n \lambda^{d/2} + O(\lambda^{(d-1)/2}), \quad \text{and} \quad V_0 \lambda^{d/2} + O(\lambda^{(d-1)/2}),$$

as $\lambda \to \infty$.

It follows from Lemma 2.6 that

$$N_T^{reg}(\lambda) \geq C_\gamma \lambda^{d/2} + O(\lambda^{(d-1)/2}), \quad \lambda \to \infty.$$

Then Lemma 2.5 implies Theorem 2.3.

In order to prove these key lemmas, we use the one-to-one relation between the operator $A(\lambda)$ and the D-N map on Γ. This means that the far-field pattern of the scattered wave and the boundary measurement on Γ are equivalent. This fact is well-known in inverse scattering problems (see [10] and so on). For inverse scattering problems, one can avoid Dirichlet eigenvalues in Ω^i or Ω_0^i. However, we have to deal with them for our argument. Thus we derive this equivalence on a subspace of $L^2(\Gamma)$ by using the Laurent expansion of the D-N map at each Dirichlet eigenvalue.

We mention some related works on the Weyl-type estimate for ITEs. As has been mentioned above, our argument is based on Lakshtanov-Vainberg [12]. Lemma 2.6 has been derived in [12] for the case where Ω^i is a domain in the Euclidean space. Recently Petkov-Vodev [16] gives a sharp estimate for the number of ITEs lying on the complex plane not only real ITEs.

3 Summary of scattering theory

Let $H = -n^{-1}\Delta_g$ and $H_0 = -\Delta$. H is self-adjoint on $L_n^2(M) := L^2(M, ndV_g)$. We can prove the limiting absorption for $R(z) = (H - z)^{-1}$ and $R_0(z) = (H_0 - z)^{-1}$, $z \in \mathbf{C} \setminus [0, \infty)$. In fact, we have $R(\lambda \pm i0) \in \mathbf{B}(\mathcal{B}(M); \mathcal{B}^*(M))$ and $R_0(\lambda \pm i0) \in \mathbf{B}(\mathcal{B}(\mathbf{R}^d); \mathcal{B}^*(\mathbf{R}^d))$ for $\lambda > 0$. Here $\mathcal{B}$-$\mathcal{B}^*$ are Agmon-Hörmander's spaces (see [1]).

Let $\mathbf{h}_\lambda$ be the Hilbert space on the sphere S^{d-1} with its inner product

$$(\psi, \phi)_{\mathbf{h}_\lambda} = \frac{\lambda^{(d-2)/2}}{2} \int_{S^{d-1}} \phi(\theta)\overline{\psi(\theta)} d\Sigma, \quad \lambda > 0.$$

We define the restriction of the Fourier transform by

$$\mathcal{F}_0(\lambda) f(\theta) = (2\pi)^{-d/2} \int_{\mathbf{R}^d} e^{-i\sqrt{\lambda}x\cdot\theta} f(x)dx, \quad \theta \in S^{d-1}.$$

Note that $\mathcal{F}_0(\lambda) \in \mathbf{B}(\mathcal{B}(\mathbf{R}^d); \mathbf{h}_\lambda)$. The adjoint operator $\mathcal{F}_0(\lambda)^* \in \mathbf{B}(\mathbf{h}_\lambda; \mathcal{B}^*(\mathbf{R}^d))$ gives the Herglotz wave with its pattern $\phi \in \mathbf{h}_\lambda$.

We also define the distorted Fourier transform $\mathcal{F}_\pm(\lambda)$ by

$$\mathcal{F}_\pm(\lambda) = \mathcal{F}_0(\lambda) \left(\chi_e - V^* R(\lambda \pm i0)\right) \in \mathbf{B}(\mathcal{B}(M); \mathbf{h}_\lambda),$$

where χ_e is a smooth function on M such that χ_e vanishes in a neighborhood of $\mathcal{K}$ and $\chi_e = 1$ at infinity, and $V = H\chi_e - \chi_e H_0$. Then $\mathcal{F}_\pm(\lambda)^* \in \mathbf{B}(\mathbf{h}_\lambda; \mathcal{B}^*(M))$ is the eigenoperator of H i.e. $\mathcal{F}_\pm(\lambda)^*\phi$ for $\phi \in \mathbf{h}_\lambda$ is a generalized eigenfunction of H. Moreover, $\mathcal{F}_-(\lambda)^*\phi - \chi_e\mathcal{F}_0(\lambda)^*\phi$ for $\phi \in \mathbf{h}_\lambda$ gives the outgoing scattered wave of the form

$$\mathcal{F}_-(\lambda)^*\phi = \chi_e\mathcal{F}_0(\lambda)^*\phi - R(\lambda + i0)V\mathcal{F}_0(\lambda)^*\phi.$$

Then we have the asymptotic behavior

$$\mathcal{F}_-(\lambda)^*\phi \simeq \mathcal{F}_0(\lambda)^*\phi - C_+(\lambda)|x|^{-(d-1)/2}e^{i\sqrt{\lambda}|x|}(A(\lambda)\phi)(\theta),$$

at infinity for $A(\lambda) = \mathcal{F}_+(\lambda)V\mathcal{F}_0(\lambda)^*$. Here $f \simeq g$ for $f, g \in L^2_{loc}(M)$ means

$$f \simeq g \Leftrightarrow \lim_{R\to\infty} \frac{1}{R}\int_{\{x\in\Omega^e;|x|<R\}} |f(x) - g(x)|^2 dx = 0.$$

Thus NSEs can be defined as has been introduced in Section 1.

4 From boundary data to scattering data

4.1 Interior D-N map

The D-N map is defined by

$$\Lambda_n(\lambda)f = \partial_\nu v \quad \text{on} \quad \Gamma, \tag{4.1}$$

where v is a solution to the Dirichlet problem

$$(-n^{-1}\Delta_g - \lambda)v = 0 \quad \text{in} \quad \Omega^i, \quad v = f \quad \text{on} \quad \Gamma. \tag{4.2}$$

Note that the argument in this subsection is similar if we replace (4.1) and (4.2) by

$$\Lambda_0(\lambda)f = \partial_\nu w \quad \text{on} \quad \Gamma, \tag{4.3}$$

$$(-\Delta - \lambda)w = 0 \quad \text{in} \quad \Omega^i_0, \quad w = f \quad \text{on} \quad \Gamma. \tag{4.4}$$

The one-to-one relation between $A(\lambda)$ and $\Lambda_n(\lambda)$ for $\lambda \notin \sigma_D(-n^{-1}\Delta_g)$ can be seen in Isakov-Nachman [10], Eskin [8], Isozaki-Kurylev [11] and so on.

Lemma 4.1 *For $\lambda \notin \sigma_D(-n^{-1}\Delta_g)$, $A(\lambda)$ and $\Lambda_n(\lambda)$ determine each other.*

Thus we have the relations

$$A(\lambda) \Leftrightarrow \Lambda_n(\lambda), \quad 0 \Leftrightarrow \Lambda_0(\lambda),$$

for $\lambda \notin \sigma_D(-n^{-1}\Delta_g) \cup \sigma_D(-\Delta)$.

It is well-known that $\Lambda_n(\lambda)$ has a simple pole at $\lambda = \mu \in \sigma_D(-n^{-1}\Delta_g)$ as follows. Let $\{\phi_j\}_{j=1}^{m_\mu}$ be the orthonormal eigenfunctions associated with $\mu \in \sigma_D(-n^{-1}\Delta_g)$. Here we choose the Hilbert space $L^2_n(\Omega^i)$ with the inner product

$$(f, g)_{L^2_n(\Omega^i)} = \int_{\Omega^i} f(x)\overline{g(x)}n(x)dV_g.$$

Then we obtain the Laurent expansion of $\Lambda_n(\lambda)$.

Lemma 4.2 *The D-N map $\Lambda_n(\lambda)$ is meromorphic with respect to $\lambda \in \mathbf{C}$ and the first order poles at every $\lambda = \mu \in \sigma_D(-n^{-1}\Delta_g)$. Moreover, $\Lambda_n(\lambda)$ satisfies the following representations. In a small neighborhood of $\mu \in \sigma_D(-n^{-1}\Delta_g)$, we have*

$$\Lambda_n(\lambda) = \frac{Q_\mu}{\mu - \lambda} + T_\mu(\lambda),$$

where Q_μ is the residue of $\Lambda_n(\lambda)$ at $\lambda = \mu$ given by

$$Q_\mu f = -\sum_{j=1}^{m_\mu} \int_\Gamma \partial_\nu \phi_j \cdot f dS \, \partial_\nu \phi_j, \quad f \in H^{3/2}(\Gamma),$$

and $T_\mu(\lambda) \in \mathbf{B}(H^{3/2}(\Gamma); H^{1/2}(\Gamma))$ is analytic in a small neighborhood of μ.

For $\Lambda_0(\lambda)$, the similar expansion holds for each eigenvalue $\mu \in \sigma_D(-\Delta)$. The range of Q_μ is a finite dimensional subspace

$$B_n(\mu) := \mathrm{Span}\{\partial_\nu \phi_1, \ldots, \partial_\nu \phi_{m_\mu}\}.$$

Similarly, $B_0(\mu)$ for $\mu \in \sigma_D(-\Delta)$ is defined by the finite dimensional subspace as above for $n = 1$ on Ω_0^i. In the following, we use the auxiliary operators $D_n(\lambda)$ and $D_0(\lambda)$ defined by

$$D_n(\lambda) = \begin{cases} \Lambda_n(\lambda), & \lambda \notin \sigma_D(-n^{-1}\Delta_g), \\ T_\lambda(\lambda), & \lambda \in \sigma_D(-n^{-1}\Delta_g), \end{cases}$$

$$D_0(\lambda) = \begin{cases} \Lambda_0(\lambda), & \lambda \notin \sigma_D(-\Delta), \\ T_{0,\lambda}(\lambda), & \lambda \in \sigma_D(-\Delta), \end{cases}$$

where $T_{0,\mu}(\lambda)$ is the regular part of the Laurent expansion of $\Lambda_0(\lambda)$ at $\mu \in \sigma_D(-\Delta)$. Note that the operator $D_n(\lambda)$ is defined on the subspace $B_n(\lambda)^\perp \cap H^{3/2}(\Gamma)$ for $\lambda \in \sigma_D(-n^{-1}\Delta_g)$.

4.2 From $D_n(\lambda)$ to $A(\lambda)$

As has been considered for the proof of Lemma 4.1, we consider the exterior Dirichlet problem. We have only to study the case $\lambda \in \sigma_D(-n^{-1}\Delta_g)$.

Now let us define the exterior D-N map by

$$\Lambda^e_\pm(\lambda) f = \partial^e_\nu u^e_\pm \quad \text{on} \quad \Gamma, \tag{4.5}$$

for $f \in H^{3/2}(\Gamma)$. Here we define

$$\partial^e_\nu v(x) := \lim_{y \to x, y \in \Omega^e} \nu(x) \cdot \nabla v(y), \quad x \in \Gamma,$$

and $u^e_\pm$ is the outgoing (for $+$) or incoming (for $-$) solution with Sommerfeld's radiation condition to the equation

$$(-\Delta - \lambda)u^e_\pm = 0 \quad \text{in} \quad \Omega^e, \quad u_\pm = f \quad \text{on} \quad \Gamma. \tag{4.6}$$

We can derive the equation to (4.6) by using the layer potential method. Let us define the operator $\delta \in \mathbf{B}(L^2(\Gamma); H^{-1/2}(M))$ and $\delta_0 \in \mathbf{B}(L^2(\Gamma); H^{-1/2}(\mathbf{R}^d))$ by

$$\int_M \delta f \cdot \overline{v}\, n dV_g = \int_\Gamma f \cdot \overline{\delta^* v}\, dS, \quad f \in L^2(\Gamma), \quad v \in H^{1/2}(M),$$
$$\int_{\mathbf{R}^d} \delta_0 f \cdot \overline{v}\, n dV_g = \int_\Gamma f \cdot \overline{\delta_0^* v}\, dS, \quad f \in L^2(\Gamma), \quad v \in H^{1/2}(\mathbf{R}^d),$$

where δ^* and δ_0^* are trace operators to Γ.

Due to $R(\lambda \pm i0) \in \mathbf{B}(H^{-1/2}_{loc}(M); H^{3/2}_{loc}(M))$ and $R_0(\lambda \pm i0) \in \mathbf{B}(H^{-1/2}(\mathbf{R}^d); H^{3/2}(\mathbf{R}^d))$, we have

$$R(\lambda \pm i0)\delta f \in H^{3/2}_{loc}(M), \quad R_0(\lambda \pm i0)\delta_0 f \in H^{3/2}_{loc}(\mathbf{R}^d),$$

for $f \in L^2(\Gamma)$.

Letting χ^i and χ^e be the characteristic function of Ω^i and Ω^e, respectively. Then we put

$$u_\pm = \chi^i u^i + \chi^e u^e_\pm, \tag{4.7}$$

where u^i is the solution to (4.2), and $u^e_\pm$ is the solution to (4.6) with the radiation condition. Note that we assume that $u^i \in H^2(\Omega^i) \cap E_n(\lambda)^\perp$ if $\lambda \in \sigma_D(-n^{-1}\Delta_g)$. The next lemma follows from the integration by parts.

Lemma 4.3 *(1) We have*

$$u_\pm = R(\lambda \pm i0)\delta(D_n(\lambda) - \Lambda^e_\pm(\lambda))f,$$

for $f \in H^{3/2}(\Gamma)$ when $\lambda \notin \sigma_D(-n^{-1}\Delta_g)$ or $f \in H^{3/2}(\Gamma) \cap B_n(\lambda)^\perp$ when $\lambda \in \sigma_D(-n^{-1}\Delta_g)$.
(2) We have

$$(D_n(\lambda)f, g)_{L^2(\Gamma)} = (f, D_n(\lambda)g)_{L^2(\Gamma)},$$
$$(\Lambda^e_\pm(\lambda)f, g)_{L^2(\Gamma)} = (f, \Lambda^e_\mp(\lambda)g)_{L^2(\Gamma)},$$

for $f, g \in H^{3/2}(\Gamma)$ when $\lambda \notin \sigma_D(-n^{-1}\Delta_g)$ or $f \in H^{3/2}(\Gamma) \cap B_n(\lambda)^\perp$ when $\lambda \in \sigma_D(-n^{-1}\Delta_g)$.

Let us introduce the operator $M_\pm(\lambda)$ which is equivalent to $D_n(\lambda)$ by

$$M_\pm(\lambda)f = \delta^* R(\lambda \pm i0)\delta f, \quad f \in H^{1/2}(\Gamma).$$

Thus the asymptotic behavior and the jump relation on Γ of the layer potential of Lemma 4.3 implies the following properties.

Lemma 4.4 *(1) $M_{\pm}(\lambda)$ is one to one on $H^{1/2}(\Gamma)$ for $\lambda \notin \sigma_D(-n^{-1}\Delta_g)$. If $\lambda \in \sigma_D(-n^{-1}\Delta_g)$, we have $\mathrm{Ker}M_{\pm}(\lambda) \subset H^{1/2}(\Gamma) \cap B_n(\lambda)$.*
(2) Let $\lambda \notin \sigma_D(-n^{-1}\Delta_g)$. Then $D_n(\lambda) - \Lambda^e_{\pm}(\lambda)$ is an isomorphism from $H^{3/2}(\Gamma)$ to $H^{1/2}(\Gamma)$ and we have $M_{\pm}(\lambda) = (D_n(\lambda) - \Lambda^e_{\pm}(\lambda))^{-1}$.
(3) Let $\lambda \in \sigma_D(-n^{-1}\Delta_g)$. We put $\widetilde{D}_n(\lambda) = D_n(\lambda) - \Lambda^e_{\pm}(\lambda)$ on $H^{3/2}(\Gamma) \cap B_n(\lambda)^{\perp}$. Then $\widetilde{D}_n(\lambda)$ is an isomorphism from $H^{3/2}(\Gamma) \cap B_n(\lambda)^{\perp}$ to $\mathrm{Ran}\widetilde{D}_n(\lambda)$, and $M_{\pm}(\lambda)\big|_{\mathrm{Ran}\widetilde{D}_n(\lambda)} = \widetilde{D}_n(\lambda)^{-1}$.

We need another operator associated with the exterior Dirichlet problem. Let $\mathcal{G}_{\pm}(\lambda) \in \mathbf{B}(H^{3/2}(\Gamma); \mathbf{h}_{\lambda})$ be defined by

$$\mathcal{G}_{\pm}(\lambda)f = \mathcal{F}_0(\lambda)\left((-\Delta - \lambda)(\chi_e u^e_{\pm})\right).$$

By the definition, $\mathcal{G}_{\pm}(\lambda)$ depends on the shape of Ω^e but is independent of n.

Lemma 4.5 *For any $f \in H^{3/2}(\Gamma)$, we have*

$$u^e_{\pm} \simeq C_{\pm}(\lambda)|x|^{-(d-1)/2} e^{\pm i\sqrt{\lambda}|x|}(\mathcal{G}_{\pm}(\lambda)f)(\pm\theta).$$

Moreover, we have

$$\mathcal{G}_{\pm}(\lambda)f = \mathcal{F}_{\pm}(\lambda)\delta(D_n(\lambda) - \Lambda^e_{\pm}(\lambda))f,$$

for $\lambda \notin \sigma_D(-n^{-1}\Delta_g)$ and $f \in H^{3/2}(\Gamma)$ or $\lambda \in \sigma_D(-n^{-1}\Delta_g)$ and $f \in H^{3/2}(\Gamma) \cap B_n(\lambda)^{\perp}$.

If we replace Ω^i and n by Ω^i_0 and 1 respectively, Lemmas 4.3-4.5 also hold.
The operator $\mathcal{G}_{\pm}(\lambda)$ has the following properties.

Lemma 4.6 *(1) $\mathcal{G}_{\pm}(\lambda)$ is one to one on $H^{3/2}(\Gamma)$.*
(2) The range of $\mathcal{G}_{\pm}(\lambda)^*$ is dense in $L^2(\Gamma)$.

Now we can state the equivalence between the scattering data $A(\lambda)$ and the boundary measurement $D_n(\lambda)$ from Lemmas 4.4-4.6.

Lemma 4.7 *$A(\lambda)$ and $M_+(\lambda)$ determine each other in the sense of*

$$\mathcal{G}_+(\lambda)M_+(\lambda)\mathcal{G}_-(\lambda)^* = A^e(\lambda) - A(\lambda),$$

where $A^e(\lambda)$ is an operator of the scattering data associated with the exterior Dirichlet problem.

This lemma implies the relation

$$A(\lambda) \Leftrightarrow D_n(\lambda), \quad 0 \Leftrightarrow D_0(\lambda),$$

for any $\lambda > 0$. As a consequence, we obtain the following fact.

Corollary 4.8 *If $\lambda \in (0, \infty)$ is a non-singular ITE, λ is a NSE.*

Overall, the arguments in this section are technical. For details of the proofs, see the section 4 in our original paper [14].

5 Discreteness of ITEs

For the proof of discreteness of ITEs, we can apply the analytic Fredholm theory and the theory of parameter dependent elliptic operators. In fact, non-singular ITEs are characterized by $\lambda \in \mathbf{C}$ such that the kernel of the operator $\Lambda_n(\lambda) - \Lambda_0(\lambda)$ is non-trivial. Thus if the inverse $(\Lambda_n(\lambda) - \Lambda_0(\lambda))^{-1}$ exists, the corresponding λ is not an ITE.

5.1 Parametrix of Dirichlet problems

Non-singular ITEs are characterized by the kernel of $\Lambda_n(\lambda) - \Lambda_0(\lambda)$. Namely, $\mathrm{Ker}(\Lambda_n(\lambda) - \Lambda_0(\lambda))$ is defined by

$$\mathrm{Ker}(\Lambda_n(\lambda) - \Lambda_0(\lambda)) = \begin{cases} \{f \in H^{3/2}(\Gamma) \ ; \ (\Lambda_n(\lambda) - \Lambda_0(\lambda))f = 0\}, & \text{if } \lambda \text{ is not a pole,} \\ \{f \in H^{3/2}(\Gamma) \ ; \ Q_{\lambda_0} f = T_{\lambda_0}(\lambda_0) f = 0\}, & \text{if } \lambda \text{ is a pole,} \end{cases}$$

where Q_{λ_0} is the residue and $T_{\lambda_0}(\lambda)$ is the regular part at a pole λ_0. Then λ is a non-singular ITE if and only if $\mathrm{Ker}(\Lambda_n(\lambda) - \Lambda_0(\lambda))$ is non-trivial. The multiplicity of non-singular ITEs are given by $\mathrm{dimKer}(\Lambda_n(\lambda) - \Lambda_0(\lambda))$.

In order to apply the analytic Fredholm theory, we compute the parametrix of Dirichlet problems. We consider

$$(-\Delta_g - \lambda n)u = 0 \quad \text{in} \quad \Omega^i, \quad u = f \quad \text{on} \quad \Gamma, \tag{5.1}$$

for $f \in H^{3/2}(\Gamma)$. By using the parametrix of this problem, we can derive the symbol of $\Lambda_n(\lambda)$.

Let $\{\chi_j\}$ be a partition of unity on Γ such that the support of each χ_j is sufficiently small. We take a coordinate patch $\{V_j\}$ on Γ such that $\chi_j \in C_0^\infty(V_j)$. Let U_j be a small open subset in Ω^i such that $\overline{U_j} \cap \Gamma = \overline{V_j}$. We can take an open set $\widetilde{U}_j \subset \mathbf{R}^d$ which is diffeomorphic to U_j. Without loss of generality, we can assume that there exists a constant $\epsilon_0 > 0$ such that $\widetilde{U}_j = \{y \in \mathbf{R}^d \ ; \ |y| < \epsilon_0, y_d > 0\}$, the boundary V_j is identified with the set $\widetilde{V}_j = \{y \in \mathbf{R}^d \ ; \ |y| < \epsilon_0, y_d = 0\}$, and $g^{kl}(y)$ satisfies $g^{kd}(y', 0) = g^{dk}(y', 0) = 0$ and $g^{dd}(y', 0) = 1$ for any $(y', 0) \in \widetilde{V}_j$, $k = 1, \ldots, d-1$, by using a suitable change of variables. In particular, we have $T^* U_j = \widetilde{U}_j \times \mathbf{R}^d$, and $y \in \widetilde{U}_j$ gives a local coordinate of U_j.

Under this setting, we can construct a parametrix for (5.1) near the boundary Γ. In fact, we identify $-\Delta_g - \lambda n$ with

$$\begin{aligned} A = &- \frac{\partial^2}{\partial y_d^2} - \sum_{k,l=1}^{d-1} a_{kl}(y) \frac{\partial^2}{\partial y_k \partial y_l} - 2 \sum_{k=1}^{d-1} a_{kd}(y) \frac{\partial^2}{\partial y_k \partial y_d} \\ &- \sum_{k=1}^{d} b_k(y) \frac{\partial}{\partial y_k} - \lambda c(y), \end{aligned}$$

for smooth coefficients a_{kl}, b_k, and n. The symbol of the operator A is given by

$$a(y,\xi,\lambda)=\xi_d^2+\sum_{k,l=1}^{d-1}a_{kl}(y)\xi_k\xi_l+2\sum_{k=1}^{d-1}a_{kd}(y)\xi_k\xi_d-i\sum_{k=1}^{d}b_k(y)\xi_k-\lambda n(y).$$

Taylor's theorem implies that $a(y,\xi,\lambda)$ can be expanded by homogeneous functions as

$$\begin{aligned}a(y,\xi,\lambda)&=a_0(z;\xi',\xi_d)+a_1(z;y'-z',y_d,\xi',\xi_d)\\&\quad+\sum_{m=2}^{N}a_m(z;y'-z',y_d,\xi',\xi_d,\lambda)+a'_N(z;y'-z',y_d,\xi',\xi_d,\lambda),\end{aligned}$$

where $z=(z',0)\in\widetilde{V}_j$,

$$a_0(z;\xi',\xi_d)=\xi_d^2+\sum_{k,l=1}^{d-1}g^{kl}(z)\xi_k\xi_l,$$

and a'_N is the remainder term. Letting

$$D_{y_d}=-i\frac{\partial}{\partial y_d},\quad \widehat{D}_{\xi'}=\prod_{j=1}^{d-1}i\frac{\partial}{\partial\xi_j},\quad \rho(z;\xi')=\left(\sum_{k,l=1}^{d-1}g^{kl}(z)\xi_k\xi_l\right)^{1/2},$$

we define the differential operator $\widehat{A}$ by $\widehat{A}=\sum_{m=0}^{N}\widehat{A}_m+\widehat{A}'_N$ where

$$\begin{aligned}\widehat{A}_0&=a_0(z;\xi',D_{y_d})=-\frac{\partial^2}{\partial y_d^2}+\rho(z;\xi')^2,\\\widehat{A}_1&=a_1(z;\widehat{D}_{\xi'},y_d,\xi',D_{y_d}),\\\widehat{A}_m&=a_m(z;\widehat{D}_{\xi'},y_d,\xi',D_{y_d},\lambda),\\\widehat{A}'_N&=a'_N(z;\widehat{D}_{\xi'},y_d,\xi',D_{y_d},\lambda),\end{aligned}$$

for $2\le m\le N$.

We consider the function E of the form $E(z;y_d,\xi')=\sum_{m=0}^{N}E_m(z;y_d,\xi')$. Then we have

$$\widehat{A}E=\sum_{j=0}^{2N}\sum_{m,k\le N,m+k=j}\widehat{A}_mE_k+\widehat{A}'_NE.$$

If E is a solution to the system of differential equations

$$\widehat{A}_0E_0=0, \tag{5.2}$$

$$\widehat{A}_0E_1+\widehat{A}_1E_0=0, \tag{5.3}$$

$$\vdots$$

$$\sum_{l=0}^{m}\widehat{A}_{m-l}E_l=0, \tag{5.4}$$

for $2 \le m \le N$, with boundary conditions $E_0(z;0,\xi') = 1$, $E_m(z;0,\xi') = 0$ for $m \neq 0$, and $\lim_{y_d \to \infty} E_m(z;y_d,\xi') = 0$ for all m, then $\widehat{A}E$ satisfies

$$\widehat{A}E = \sum_{j=N+1}^{2N} \sum_{m,k \le N, m+k=j} \widehat{A}_m E_k + \widehat{A}'_N E, \quad E(z;0,\xi') = 1. \tag{5.5}$$

When $\rho(z;\xi') \neq 0$, there exists a unique solution E to (5.2)-(5.4) with the conditions $E_0(z;0,\xi') = 1$, $E_m(z;0,\xi') = 0$ for $m \ge 1$, and $\lim_{y_d \to \infty} E_m(z;y_d,\xi') = 0$ for all m.

The parametrix for (5.1) near the boundary Γ is given as follows. Let $\beta(\xi') \in C^\infty(\mathbf{R}^{d-1})$ such that $\beta(\xi') = 0$ in a small neighborhood of 0 and $\beta(\xi') = 1$ for large $|\xi'|$. For $f \in H^{3/2}(\widetilde{V}_j)$ with a small support, we define

$$Q_m f(y) = (2\pi)^{-d+1} \int_{\mathbf{R}^{d-1}} e^{iy'\cdot\xi'} \beta(\xi') \int_{\mathbf{R}^{d-1}} e^{-iz'\cdot\xi'} E_m(z;y_d,\xi') f(z') dz' d\xi',$$

and put

$$R_N = \sum_{m=0}^{N} Q_m.$$

Thus we have

$$R_N f(y) = \int_{\mathbf{R}^{d-1}} r_N(z;y'-z',y_d) f(z') dz',$$

where

$$r_N(z;y',y_d) = (2\pi)^{-d+1} \sum_{m=0}^{N} \int_{\mathbf{R}^{d-1}} e^{iy'\cdot\xi'} \beta(\xi') E_m(z;y_d,\xi') d\xi'.$$

It follows that R_N is a parametrix near the boundary.

Lemma 5.1 *For $f \in H^{3/2}(\widetilde{V}_j)$ with a small support and sufficiently large $N > 0$, we have $a(y,D_y,\lambda) R_N f \in H^s(\widetilde{U}_j)$ with $s < N - d/2 + 5/2$ and $R_N f\big|_{y_d=0} - f \in C^\infty(\widetilde{V}_j)$.*

As a consequence, we can compute the symbol of the D-N map.

Lemma 5.2 *(1) The full symbol of $\Lambda_n(\lambda)$ is formally given by*

$$\Lambda_n(z';\xi',\lambda) = -\beta(\xi') \sum_{k=0}^{\infty} \frac{\partial E_k}{\partial y_d}(z;0,\xi'), \quad (z',\xi') \in T^*\widetilde{V}_j.$$

If λ is a pole of $\Lambda_n(\lambda)$, this formula gives the full symbol of the regular part of $\Lambda_n(\lambda)$ in view of the Laurent expansion.
(2) The principal symbol of $\Lambda_n(\lambda) - \Lambda_0(\lambda)$ is given by

$$\frac{\lambda \beta(\xi') \partial_\nu n(z)}{4\rho(z';\xi')^2}, \quad (z,\xi') \in T^*\Gamma.$$

By the similar argument, we also have an expansion of the D-N map in view of the theory of parameter-dependent elliptic operators (see [2]). Then we have

Lemma 5.3 *Let* $L(\tau) = \tau^{-2}e^{-2i\theta}(\Lambda_n(\tau^2 e^{2i\theta}) - \Lambda_0(\tau^2 e^{2i\theta}))$ *for* $\tau > 0$ *and* $\theta \in \mathbf{R}$ *with* $\theta \neq 0$ *modulo* π. *Then* $L(\tau)$ *is uniformly parameter elliptic of order* -2 *and regularity* ∞. *Its principal symbol is given by*

$$\frac{\partial_\nu n(z)}{4(\rho(z;\xi')^2 - \tau^2 e^{2i\theta})}, \quad (z,\xi') \in T^*\Gamma.$$

In view of Lemma 5.2, $\Lambda_n(\lambda) - \Lambda_0(\lambda)$ is Fredholm for $\lambda \in \mathbf{C} \setminus \{0\}$. Theorem 2.2 is follows from the following analytic Fredholm theory (see [4]) and the invertibility of $L(\tau)$ for large $\tau > 0$.

Theorem 5.4 *Let* $D \subset \mathbf{C}$ *be a connected open domain, and* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *are Hilbert spaces. Suppose that a* $\mathbf{B}(\mathcal{H}_1;\mathcal{H}_2)$*-valued function* $A(z)$ *for* $z \in D$ *is finitely meromorphic and Fredholm in* D. *If there exists its bounded inverse* $A(z_0)^{-1}$ *at a point* $z_0 \in D$, *then* $A(z)^{-1}$ *is finitely meromorphic and Fredholm in* D.

6 Weyl-type lower bound for the number of NSEs

Theorem 2.3 follows from Weyl's law for Dirichlet eigenvalues of $-n^{-1}\Delta_g$ in Ω^i and $-\Delta$ in Ω_0^i. See Theorem 1.2.1 in [18].

Theorem 6.1 *Let*

$$N_n(\lambda) = \#\{\mu \in \sigma_D(-n^{-1}\Delta_g) \ ; \ \mu \leq \lambda\}, \quad N_0(\lambda) = \#\{\mu \in \sigma_D(-\Delta) \ ; \ \mu \leq \lambda\},$$

for $\lambda > 0$. *We have*

$$N_n(\lambda) = V_n \lambda^{d/2} + O(\lambda^{(d-1)/2}), \quad N_0(\lambda) = V_0 \lambda^{d/2} + O(\lambda^{(d-1)/2}),$$

as $\lambda \to \infty$.

By using this estimate, we evaluate the number of $\mu \in (0,\lambda)$ for large $\lambda > 0$ such that $\mathrm{Ker}(\Lambda_n(\mu) - \Lambda_0(\mu)) \neq \{0\}$. In order to avoid the compactness of the operator $\Lambda_n(\lambda) - \Lambda_0(\lambda)$, we define the auxiliary operator

$$\widetilde{\Lambda}(\lambda) = \gamma(-\Delta_\Gamma + 1)^{3/4}(\Lambda_n(\lambda) - \Lambda_0(\lambda))(-\Delta_\Gamma + 1)^{3/4},$$

where $-\Delta_\Gamma$ is the positive Laplacian on Γ. Thus λ is a non-singular ITE if and only if $\mathrm{Ker}\widetilde{\Lambda}(\lambda)$ is non-trivial.

Let $\{\lambda_j^T\}$ be the set of ITEs lying in (α, ∞) for sufficiently small $\alpha > 0$. We put

$$N_T(\lambda) = \#\{j \ ; \ \alpha < \lambda_j^T \leq \lambda\},$$

taking into account the multiplicities of ITEs where $\lambda_1^T \leq \lambda_2^T \leq \cdots$. Letting $\mu_k(\lambda) \in \sigma_D(\widetilde{\Lambda}(\lambda))$, we evaluate $N_T(\lambda)$ by the number of $\lambda \in (\alpha, \infty)$ such that $\mu_k(\lambda) = 0$ for some k. We define

$$N_-(\lambda) = \#\{k \ ; \ \mu_k(\lambda) < 0\},$$

for $\lambda \notin \sigma_D(-n^{-1}\Delta_g) \cup \sigma_D(-\Delta)$. Suppose that $\tau \in \mathbf{R}$ moves from α to ∞. Since $\mu_k(\tau)$ is meromorphic with respect to τ, $N_-(\tau)$ changes only when some $\mu_k(\tau)$ pass through 0 or τ passes through a pole of $\widetilde{\Lambda}(\tau)$. When τ moves from α to $\lambda > \alpha$, $\mathcal{N}_0(\lambda)$ denotes the change of $N_-(\lambda) - N_-(\alpha)$ due to the first case and $\mathcal{N}_{-\infty}(\lambda)$ denotes the change of $N_-(\lambda) - N_-(\alpha)$ due to the second case. By the definition, we have

$$N_-(\lambda) - N_-(\alpha) = \mathcal{N}_0(\lambda) + \mathcal{N}_{-\infty}(\lambda).$$

We put

$$\delta\mathcal{N}_{-\infty}(\lambda) = N_-(\lambda + \epsilon) - N_-(\lambda - \epsilon),$$

at a pole λ of $\widetilde{\Lambda}(\lambda)$ for sufficiently small $\epsilon > 0$. We can show

$$\delta\mathcal{N}_{-\infty}(\lambda) = \#\{j \ ; \ \mathrm{res}_{\tau=\lambda}\mu_j(\tau) > 0\} - \#\{j \ ; \ \mathrm{res}_{\tau=\lambda}\mu_j(\tau) < 0\}.$$

Moreover, we also have

$$|\delta\mathcal{N}_{-\infty}(\lambda) + \gamma(m_n(\lambda) - m_0(\lambda))| \leq m(\lambda),$$

at a pole of $\widetilde{\Lambda}(\lambda)$ where

$$\begin{aligned} m_n(\lambda) &= \mathrm{dimRan}Q_\lambda, \quad m_0(\lambda) = \mathrm{dimRan}Q_{0,\lambda}, \\ m(\lambda) &= \dim(\mathrm{Ran}Q_\lambda \cap \mathrm{Ran}Q_{0,\lambda}), \end{aligned}$$

for the residues Q_λ and $Q_{0,\lambda}$ of $\Lambda_n(\lambda)$ and $\Lambda_0(\lambda)$, respectively. Taking the summation of this inequality on poles in $(\alpha, \lambda]$, we can see

$$\left| \mathcal{N}_{-\infty}(\lambda) + \gamma \sum_{\alpha < \lambda' \leq \lambda} (m_n(\lambda') - m_0(\lambda')) \right| \leq N_T^{sng}(\lambda).$$

Plugging this inequality and $N_-(\lambda) - N_-(\alpha) = \mathcal{N}_0(\lambda) + \mathcal{N}_{-\infty}(\lambda)$, we have

$$\begin{aligned} N_T(\lambda) &\geq \mathcal{N}_0(\lambda) + N_T^{sng}(\lambda) \\ &\geq \gamma(N_n(\lambda) - N_0(\lambda)) - N_-(\alpha) \\ &\geq \gamma(V_n - V_0)\lambda^{d/2} + O(\lambda^{(d-1)/2}), \end{aligned}$$

as $\lambda \to \infty$ in view of Theorem 6.1.

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