Limit circle problem for a Fuchsian differential operator on a torus

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Abstract

In this short note, we study spectral properties of the simple operator $P = -\partial_x(\sin x\partial_x)$ on the one-dimensional torus. We prove that P is not essential self-adjoint and give its four proofs. Moreover, we prove discreteness of the spectrum of its self-adjoint extension.

1 Introduction

In this short note, we consider the following second order differential operator:

$$P = -\partial_x(\sin x\partial_x) \quad \text{on} \quad \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}.$$

We denote the symbol of P by p:

$$p(x,\xi) = (\sin x)\xi^2, \quad (x,\xi) \in T^*\mathbb{T} = \mathbb{T}\times\mathbb{R}.$$

The main theorem of this note is the following:

Theorem 1.1. *The symmetric operator P is not essential self-adjoint on $C^\infty(\mathbb{T})$.*

Remark 1.2. This theorem holds if we replace P by $P+V$, where V is a first order symmetric differential operator. In fact, the method in Section 3 can be applied with $P+V$.

The purpose of this note is to collect various proofs of Theorem 1.1. To prove Theorem 1.1, we construct the distributional eigenfunctions for P associated with the complex eigenvalues in various ways. In Section 3, we use the method developed in [5], which is an analog of the standard construction of generalized eigenfunctions in scattering theory. In section 4, we use the a priori estimate (so-called the radial source/sink estimates) and determine the regularity of eigenfunctions of P by using microlocal analysis. In Section 5, we only use the Fourier analysis and directly compute the regularity of the eigenfunctions from the recurrence formula which is equivalent to the eigenvalue equation for P. On the other hand, in Section 6, we prove Theorem 1.1 just by using the integration by parts.

As an analogy of [5, Corollary 1.5], we obtain the following theorem.

Theorem 1.3. *Each self-adjoint extension of $P|_{C^\infty(\mathbb{T})}$ has a discrete spectrum.*

The proof of this theorem is given in the end of Section 4. Its proof is essentially due to the radial sink estimate and the fact that the radial sink for P is isolated in the characteristic set of p. Although we can prove Theorem 1.3 by an alternative proof which is similar to [5, Corollary 1.5], we omit its proof. While the proofs in Section 5 and Section 6 are very short, the proofs in Section 3 and Section 4 make the connection between the classical trajectories and the quantum dynamics clear.

It is believed that the completeness of classical trajectories and essential self-adjointness of the corresponding differential operators are closely related. This is because essential self-adjointness of a differential operator P is equivalent to existence and uniqueness of solutions to a time-dependent Schrödinger equation

$$i\partial_t u + Pu = 0, \quad u|_{t=0} \in L^2.$$

Hence it seems important to prove essential self-adjointness or not essential self-adjointness of differential operators from microlocal point of view.

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2 Pseudodifferential operators

Let (M, g) be a closed Riemannian manifold with dimension n and let us denote

$$\langle \xi \rangle := (1 + |\xi|_g^2)^{\frac{1}{2}}, \quad |\xi|_g^2 := \sum_{j,k=1}^{n} g^{jk}(x)\xi_j\xi_k,$$

where the left hand side is independent of the choice of the trivialization of T^*M. We denote by the Kohn-Nirenberg symbol classes by S^k:

$$S^k := \{a \in C^\infty(T^*M) \mid |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta}\langle \xi \rangle^{k-|\beta|}\}$$

for $k \in \mathbb{R}$. We also denote the sets of all pseudodifferential operators of order k by $\mathrm{Op}S^k$. Moreover, we fix the quantization $\mathrm{Op}(a)$ of $a \in S^k$ (see [1, Proposition E.15]). For $A \in \mathrm{Op}S^k$, we denote its principal symbol by $\sigma(A) \in S^k$ (see [1, Proposition E.14]).

Lemma 2.1. *[1, Proposition E.23] Let* $A \in \mathrm{Op}S^{2k+1}$ *with* $k \in \mathbb{R}$ *and* $\mathrm{Re}\,\sigma(A) \geq 0$. *Then there exists* $C > 0$ *such that*

$$\mathrm{Re}\,(u, Au)_{L^2(M)} \geq -C\|u\|_{H^k(M)}^2, \quad u \in C^\infty(M).$$

In our case ($M = \mathbb{T}$), we can take Op such that $\mathrm{Op}(a)$ is formally self-adjoint for real-valued symbol a.

We recall the definition of the radial source/sink from [1]. Let $p \in S^k$ be a real-valued symbol with $k > 0$. We denote the projection map by

$$\kappa : T^*M \setminus 0 \to S^*M = \partial T^*M, \quad \kappa(x,\xi) = (x, \frac{\xi}{|\xi|_g})$$

We write the radial compactification $\overline{T}^*M = T^*M \cup \partial T^*M$. Then it follows that $\overline{T}^*M$ becomes a manifold with boundary and that the vector field $\langle\xi\rangle^{1-k}H_p$ on $\overline{T}^*M$ generates the complete flow

$$\varphi_t = e^{t\langle\xi\rangle^{1-k}H_p} : \overline{T}^*M \to \overline{T}^*M.$$

Lemma 2.2. *[1, Definition E.50] We say that a non-empty invariant set $L \subset \{\langle\xi\rangle^{-k}p = 0\} \cap \partial\overline{T}^*M$ is a radial source for p if there exists a neighborhood $U \subset \overline{T}^*M$ of L such that*

$$\kappa(\varphi_t(x,\xi)) \to L, \quad t \to -\infty,$$
$$|\varphi_t(x,\xi)|_g \geq Ce^{\theta|t|}|\xi|_g, \quad t \leq 0$$

uniformly in $(x,\xi) \in U \cap T^*M$ with constants $C, \theta > 0$. We say that L is a radial sink if L is a radial source for $-p$.

Consider a formally self-adjoint operator $P \in \mathrm{Op}S^k$ with $k > 0$ and a real-valued principal symbol p. We assume

$$\langle\xi\rangle^{1-k}H_p \text{ vanishes at } L. \tag{2.1}$$

Theorem 2.3. *[1, Theorem E.52 and exercise 37] Assume that L is a radial source for p and (2.1) is satisfied. Let $s \in \mathbb{R}$ satisfy*

$$(s + \frac{1-k}{2})\frac{H_p\langle\xi\rangle}{\langle\xi\rangle} < 0 \quad \textit{on} \quad L. \tag{2.2}$$

Then there exists $a \in C^\infty(T^*M; [0,1])$ with $a = 1$ near the conic neighborhood of L such that

$$\|Au\|_{H^s(M)} \leq C\|Pu\|_{H^{s-k+1}(M)} + C\|u\|_{H^{-N}(M)}$$

for $A = \mathrm{Op}(a)$, $N > 0$ and $u \in H^{s_0+0}(M)$, where s_0 satisfies (2.3) and $s > s_0$.

Theorem 2.4. *[1, Theorem E.54 and exercise 36] Assume that L is a radial sink for p and (2.1) is satisfied. Fix a conic neighborhood V of L. Let $s \in \mathbb{R}$ satisfy*

$$(s + \frac{1-k}{2})\frac{H_p\langle\xi\rangle}{\langle\xi\rangle} > 0 \quad \textit{on} \quad L. \tag{2.3}$$

Then there exists $a \in C^\infty(T^*M; [0,1])$ with $a = 1$ near the conic neighborhood of L and $b \in C^\infty(T^*M; [0,1])$ supported away from a conic neighborhood of L and $\mathrm{supp}\, b \subset V$ *such that*

$$\|Au\|_{H^s(M)} \leq C\|Pu\|_{H^{s-k+1}(M)} + C\|Bu\|_{H^s(M)} + C\|u\|_{H^{-N}(M)}$$

for $A = \mathrm{Op}(a)$, $B = \mathrm{Op}(b)$ $N > 0$ and $u \in \mathcal{D}'(M)$.

3 First proof, the method in [5]

In this section, we apply the method developed in [5] with our operator P and prove Theorem 1.1. We recall

$$p(x,\xi) = (\sin x)\xi^2, \quad H_p = 2(\sin x)\xi\partial_x - (\cos x)\xi^2\partial_\xi.$$

3.1 Construction of an escape function

In this subsection, we construct an escape function which is needed for the definition of the anisotropic Sobolev space. Let $\rho \in C^\infty(\mathbb{R}; [0,1])$ and $\chi \in C^\infty(\mathbb{R}; [0,1])$ satisfying

$$\rho(t) = \begin{cases} 1 \text{ if } t \geq \frac{1}{2}, \\ -1 \text{ if } t \leq -\frac{1}{2}, \end{cases} \qquad t\rho(t) \geq 0,\ \rho'(t) \geq 0,\ \inf_{|t|\geq 1/4} |\rho(t)| > 0,\ \inf_{|t|\leq 1/4} \rho'(t) > 0,$$

and

$$\chi(t) = \begin{cases} 1 \text{ if } t \geq 2, \\ 0 \text{ if } t \leq 1, \end{cases} \qquad \chi'(t) \geq 0.$$

We define

$$\eta(x) = \cos x, \quad \chi_1(\xi) = \chi(\xi) - \chi(-\xi), \quad m(x,\xi) = \rho(\eta(x))\chi_1(\xi) \in S^0.$$

Lemma 3.1. *There exists $C > 0$ such that*

$$H_p(m(x,\xi)\log\langle\xi\rangle) \leq -C\langle\xi\rangle \quad \textit{for} \quad |\xi| \geq 2.$$

Proof. We note $\xi\chi_1(\xi) \geq 0$ and

$$(H_p\eta)(x,\xi) = -2(\sin^2 x)\xi^2,\ H_p(\rho(\eta)) \leq 0,\ mH_p\log\langle\xi\rangle = -\eta(x)\rho(\eta(x))\chi_1(\xi)\xi^3\langle\xi\rangle^{-2} \leq 0.$$

In particular, we have

$$(H_pm)(x,\xi)\log\langle\xi\rangle \leq 0,\ m(x,\xi)(H_p\log\langle\xi\rangle) \leq 0$$

for $|\xi| \geq 2$. For $|\eta| \geq 1/4$ and $|\xi| \geq 2$, we have

$$\begin{aligned} H_p(m(x,\xi)\log\langle\xi\rangle) =&(H_pm)(x,\xi)\log\langle\xi\rangle + m(x,\xi)(H_p\log\langle\xi\rangle) \\ \leq& m(x,\xi)(H_p\log\langle\xi\rangle) \\ \leq& -C\langle\xi\rangle. \end{aligned}$$

For $|\eta| \leq 1/4$ and $|\xi| \geq 2$, we obtain

$$\begin{aligned} H_p(m(x,\xi)\log\langle\xi\rangle) =&(H_pm)(x,\xi)\log\langle\xi\rangle + m(x,\xi)(H_p\log\langle\xi\rangle) \\ \leq&(H_pm)(x,\xi)\log\langle\xi\rangle \\ \leq& -C\langle\xi\rangle. \end{aligned}$$

This completes the proof. □

3.2 Fredholm estimate

Let m be a symbol constructed in the above subsection and $t > 0$. Take an invertible operator $A_m \in \mathrm{Op}S^{\frac{1}{2}+m(x,\xi)}$ satisfying $(A.1)$:

$$a_{tm}t(x,\xi) := \langle\xi\rangle^{\frac{1}{2}+tm(x,\xi)}, \quad A_{tm} - \mathrm{Op}(a_{tm}) \in \mathrm{Op}S^{-\infty}.$$

Define

$$P_{tm} := A_{tm} P A_{tm}^{-1}. \tag{3.1}$$

By the asymptotic expansion (see [3, Lemma 3.2] for the Anosov vector field), we have

$$P_{tm} = P + it\mathrm{Op}(H_p(m \log\langle\xi\rangle)) + \mathrm{Op}S^{+0}. \tag{3.2}$$

The main result of this subsection is the following Fredholm estimates.

Proposition 3.2. *Let $t > 0$. Then for any $N > 0$, there exists $C > 0$ such that*

$$\|u\|_{H^{\frac{1}{2}}(\mathbb{T})} \leq C\|(P_{tm} - z)u\|_{H^{-\frac{1}{2}}(\mathbb{T})} + C\|u\|_{H^{-N}(\mathbb{T})}, \tag{3.3}$$

$$\|u\|_{H^{\frac{1}{2}}(\mathbb{T})} \leq C\|(P_{tm} - z)^* u\|_{H^{-\frac{1}{2}}(\mathbb{T})} + C\|u\|_{H^{-N}(\mathbb{T})}, \tag{3.4}$$

for $z \in \mathbb{C}$ and $u \in \mathcal{D}'(\mathbb{T})$. Here P_{tm}^* is the formal adjoint operator of P_{tm}. Moreover, if $\mathrm{Im}\, z >> 1$*, then the term $\|u\|_{H^{-N}(\mathbb{T})}$ in* (3.3) *and* (3.4) *can be removed.*

Remark 3.3. We obtain the Fredholm estimates uniformly in $\mathrm{Re}\, z$, which is different from [5, Proposition 3.4]. This seems reflect the property of H_p: The trapped set of H_p lies only int the zero section of $T^*\mathbb{T}$.

Lemma 3.4. *We consider the Banach space*

$$\tilde{D}_{tm} := \{u \in H^{\frac{1}{2}}(\mathbb{T}) \mid P_{tm} u \in H^{-\frac{1}{2}}(\mathbb{T})\}$$

equipped with the graph norm of P_{tm}. Then it follows that $C^\infty(\mathbb{T})$ is dense in $\tilde{D}_m$.

Remark 3.5. This lemma holds if we replace P_{tm} by the general pseudodifferential operator $Q \in \mathrm{Op}S^2$. See [1, Lemma E.45].

Proof. Let $\chi \in C_c^\infty(\mathbb{R}; [0,1])$ satisfying $\chi(t) = 1$ on $|t| \leq 1$ and $\chi(t) = 0$ on $|t| \geq 2$. Set $A_R := \mathrm{Op}(\chi(\frac{|\xi|}{R}))$ for $R \geq 1$. We note that $[P_m, A_R]$ is uniformly bounded in $\mathrm{Op}\mathbb{T}$ and converges to 0 in $\mathrm{Op}S^{1+0}$.

Now let $u \in \tilde{D}_{tm}$ and set $u_R := A_R u \in C^\infty(\mathbb{T})$. Clearly, we have $u_R \to u$ in $H^{\frac{1}{2}}(\mathbb{T})$. Moreover, we have

$$Pu_R = A_R P u + [P, A_R]u \to Pu \quad \text{in} \quad H^{-\frac{1}{2}}(\mathbb{T}).$$

This completes the proof.

□

Lemma 3.6. *There exits $C > 0$ such that for $u \in C^\infty(\mathbb{T})$,*

$$-(u, \mathrm{Op}(H_p(m \log\langle\xi\rangle)u)_{L^2(\mathbb{T})} \geq C\|u\|_{H^{\frac{1}{2}}}^2 - C\|u\|_{H^{+0}}^2.$$

Proof. This lemma follows from Lemmas 2.1, 3.1 and the formula (3.2).

□

Proof of Proposition 3.2. We only deal with (3.3). The inequality (3.4) is similarly proved. By virtue of Lemma 3.4, it suffices to prove (3.3) for $u \in C^\infty(\mathbb{T})$. Lemma 3.6 implies

$$\begin{aligned}\mathrm{Im}\,(u,(P_{tm}-z)u)_{L^2(\mathbb{T})} =& t(u,\mathrm{Op}(H_p(m\log\langle\xi\rangle))u) - \mathrm{Im}\,z\|u\|^2_{L^2(\mathbb{T})} + O(\|u\|^2_{H^{+0}(\mathbb{T})})\\ \leq& -Ct\|u\|^2_{H^{\frac{1}{2}}} - \mathrm{Im}\,z\|u\|^2_{L^2} + O(\|u\|^2_{H^{+0}(\mathbb{T})})\end{aligned}$$

for $u \in C^\infty(\mathbb{T})$. The Cauchy-Schwarz inequality and the interpolation inequality

$$\|u\|^2_{H^{+0}(\mathbb{T})} \leq \varepsilon\|u\|^2_{H^{\frac{1}{2}}(\mathbb{T})} + C\|u\|^2_{H^{-N}(\mathbb{T})}, \quad \forall\varepsilon > 0, \quad N > 0,$$

we obtain

$$Ct\|u\|^2_{H^{\frac{1}{2}}} + \mathrm{Im}\,z\|u\|^2_{L^2} \leq C_1\|(P_{tm}-z)u\|^2_{H^{\frac{1}{2}}(\mathbb{T})} + C_1\|u\|^2_{H^{-N}(\mathbb{T})}.$$

This implies (3.3). Moreover, if $\mathrm{Im}\,z >> 1$, the term $\|u\|^2_{H^{-N}(\mathbb{T})}$ in the left hand side can be removed. □

From Proposition 3.2 and the proof in [5, Corollary 3.6], we obtain the following corollary.

Corollary 3.7. *Consider a family of bounded operators*

$$P_{tm} - z : \tilde{D}_{tm} \to H^{-\frac{1}{2}}(\mathbb{T}). \tag{3.5}$$

Then it follows that the (3.5) *is an analytic family of Fredholm operators with index* 0. *Moreover, there exists a discrete subset* $S_t \subset \mathbb{C}$ *such that the map* 3.5 *is invertible for* $z \in \mathbb{C} \setminus S_t$.

3.3 WKB solutions

In this subsection, we construct an approximate eigenfunction of P which wavefront set lies in the incoming region (the radial sink) for p.

First, we consider the WKB state

$$u_-(x) := \chi(x)\int_{\mathbb{R}} a_0(\xi)e^{ix\cdot\xi}d\xi, \tag{3.6}$$

where $\chi \in C_c^\infty(\mathbb{R};[0,1])$ and $a_0 \in C^\infty(\mathbb{R})$

$$\chi(x) = \begin{cases}1 & \text{for} \quad |x| \leq \frac{\pi}{4},\\ 0 & \text{for} \quad |x| \geq \frac{\pi}{2},\end{cases} \qquad a_0(\xi) = \begin{cases}\frac{1}{\xi} & \text{for} \quad -\xi \geq 2,\\ 0 & \text{for} \quad -\xi \leq 1.\end{cases} \tag{3.7}$$

We shall see that u_- is the approximate eigenfunction for P. By virtue of its support condition, we can regard u_- as a smooth function on $\mathbb{T}$. We note that u_- is a Lagrangian distribution ([4, Definition 25.1.1]) associated with the conic Lagrangian submanifold

$$L_{-,0} := \{(0,\xi) \in \mathbb{T}\times\mathbb{R} \mid \xi < 0\}.$$

Moreover, we have

$$u_- \in C^\infty(\mathbb{T}\setminus\{0\}) \cap H^{\frac{1}{2}-0}(\mathbb{T}), \quad u_- \notin H^{\frac{1}{2}}(\mathbb{T}), \quad \mathrm{WF}(u_-) \subset L_{-,0}. \tag{3.8}$$

Lemma 3.8. *We have* $Pu_- \in H^{\frac{3}{2}-0}(\mathbb{T})$.

Proof. By the Taylor theorem, we have

$$P = -\partial_x(x\partial_x) + \partial_x(h(x)x^3\partial_x) \quad \text{on} \quad \operatorname{supp}\chi, \tag{3.9}$$

where h is smooth near $\operatorname{supp}\chi$. A direct calculation gives $\partial_x(x\partial_x)(u_-) \in C^\infty(\mathbb{T})$. Moreover, since $\partial_x(h(x)x^3\partial_x)$ is the second order differential operator which vanishes at 0 of third order and since u_- is the Lagrangian distribution, then we have $\partial_x(h(x)x^3\partial_x)u_- \in H^{\frac{3}{2}-0}(\mathbb{T})$. This completes the proof.

□

Next, we shall construct the approximate eigenfunction for $P - z$ for $z \in \mathbb{C}$. Let χ be as in (3.7) and consider

$$u_{-,z}(x) := \chi(x)\int_{\mathbb{R}} a_z(\xi)e^{ix\cdot\xi}d\xi,$$

where $a_z \in C^\infty(\mathbb{R})$ is supported in $-\xi \geq 1$ and is determined later. Moreover, we impose

$$a_z \in S_{-1}, \quad S_k := \{a \in C^\infty(\mathbb{R}) \mid |\partial_\xi^\alpha a(\xi)| \leq C_\alpha\langle\xi\rangle^{k-|\alpha|}\}. \tag{3.10}$$

Then it follows that $u_{-,z}$ is a Lagrangian distribution associated with $L_{-,0}$ and satisfies (3.8). A direct calculation (as in Lemma 3.8, use (3.9)) gives

$$(P-z)u_{-,z}(x) = i\int_{\mathbb{R}} \left(\xi^2\partial_\xi a_z(\xi) + \xi a_z(\xi) + iza_z(\xi)\right)e^{ix\cdot\xi}d\xi + H^{\frac{3}{2}-0}(\mathbb{T}) \text{ for } |x| \leq \pi/4$$

and $(P-z)u_{-,z} \in C^\infty$ on $\mathbb{T}\setminus\{|x| \leq \pi/4\}$. If we take $a_z \in C^\infty(\mathbb{R})$ as

$$a_z(\xi) = \begin{cases} \frac{1}{\xi} + \frac{iz}{\xi^2} & \text{for} \quad -\xi \geq 2 \\ 0 \quad \text{for} & -\xi \leq 1, \end{cases}$$

then we have $a_z \in S_{-1}$ and $\xi^2\partial_\xi a_z(\xi) + \xi a_z(\xi) + iza_z(\xi) \in S_{-2}$. Consequently, we obtain the following lemma.

Lemma 3.9. *For* $z \in \mathbb{C}$, *we have* $(P-z)u_{-,z} \in H^{\frac{3}{2}-0}(\mathbb{T})$ *and* $u_{-,z}$ *satisfies* (3.8). *In particular,* $u_{-,z} \neq 0$.

Remark 3.10. A finer construction gives existence of an approximate eigenfunction $u_{-,z}$ satisfying (3.8) and $(P-z)u_{-,z} \in C^\infty(\mathbb{T})$.

3.4 Existence of generalized eigenfunctions

In this subsection, we construct a generalized eigenfunction of P. In order to show Theorem 1.1, it suffices to prove the following proposition.

Proposition 3.11. *Let* $t > 0$ *small enough satisfying* $H^{\frac{3}{2}-0}(\mathbb{T}) \subset H^{\frac{1}{2}+tm(x,\xi)}(\mathbb{T}) \subset L^2(\mathbb{T})$ *and let* $z \in \mathbb{C}\setminus S_t$. *Then there exists* $u \in L^2(\mathbb{T})\setminus\{0\}$ *such that* $(P-z)u = 0$ *in the distributional sense.*

Proof. By (3.1) and Corollary 3.7, it follows that the map

$$P - z : D_{tm} := \{u \in H^{\frac{1}{2}+tm(x,\xi)}(\mathbb{T}) \mid Pu \in H^{-\frac{1}{2}+tm(x,\xi)}(\mathbb{T})\} \to H^{-\frac{1}{2}+tm(x,\xi)}(\mathbb{T}) \quad (3.11)$$

is a Fredholm operator with index and that $z \to P - z$ is analytic. Moreover, (3.11) is invertible for $z \in \mathbb{C} \setminus S_t$, where S_t is same as in Corollary 3.7. We denote the inverse of (3.11) by $R_+(z)$ for $z \in \mathbb{C} \setminus S_t$.

Take $u_{-,z} \neq 0$ satisfying (3.8) and $(P - z)u_{-,z} \in H^{\frac{3}{2}-0}(\mathbb{T})$. Set

$$u_{+,z} := -R_+(z)(P - z)u_{-,z}, \quad u_z := u_{+,z} + u_{-,z}.$$

Then we have $u_z \in L^2(\mathbb{T})$ and $(P-z)u_z = 0$. Moreover, we have $u_z \neq 0$. In fact,$u_{-,z} \neq 0$, its wavefront condition (3.8) and the construction of the escape function m imply $u_z = u_{+,z} + u_{-,z} \neq 0$.

□

4 Second proof, via radial point estimates

In this section, we give another Fredholm estimate which is different from the last section and is similar to the estimate in [6].

4.1 Hamilton dynamics

We recall $p(x, \xi) = (\sin x)\xi^2$ for $(x, \xi) \in T^*\mathbb{T} = \mathbb{T} \times \mathbb{R}$. Set

$$L_{\mp,0} = \{(x, \xi) \in \partial T^*\mathbb{T} \mid x = 0,\ \xi = \pm\infty\}, \quad L_{\pm,\pi} = \{(x, \xi) \in \partial T^*\mathbb{T} \mid x = \pi,\ \xi = \pm\infty\}.$$

Proposition 4.1. *It follows that*

- *$L_{-,0}$ and $L_{+,\pi}$ are radial sources for p, $L_{+,0}$ and $L_{-,\pi}$ are radial sinks for p,*

in the sense of Definition 2.2.

In the following, we prove that $L_{-,0}$ is a radial sink only. The other part of Proposition of 4.1 is similarly proved. Set

$$U = \{(x, \xi) \in T^*\mathbb{T} \mid x \in (-\pi/4, \pi/4),\ \xi > 1\} \quad \text{and} \quad \tilde{H}_p = \langle \xi \rangle^{-1} H_p.$$

We denote the integral curve of $\tilde{H}_p$ with a initial data (x_0, ξ_0) by $(z(t), \zeta(t))$:

$$\begin{cases} \frac{d}{dt}z(t) = 2(\sin z(t))\langle \zeta(t) \rangle^{-1}\zeta(t), \\ \frac{d}{dt}\zeta(t) = -(\cos z(t))\langle \zeta(t) \rangle^{-1}\zeta(t)^2, \end{cases} \quad \begin{cases} z(0) = x, \\ \zeta(0) = \xi. \end{cases} \quad (4.1)$$

First, we prove that the any trajectory through $\{\xi = 0\}$ must be constant.

Lemma 4.2. *Let $x \in \mathbb{T}$ and $\xi = 0$. Then $z(t) = x$ and $\zeta(t) = 0$ for any $t \in \mathbb{R}$.*

Proof. A pair $(z(t), \zeta(t)) = (x, 0)$ is a solution to (4.1). By the uniqueness of solutions to ODE, we obtain our conclusion. □

Now the proof of Proposition 4.1 reduces to Lemma 4.3 and 4.4 below.

Lemma 4.3. *There exists $\theta > 0$ such that for each initial value $(x, \xi) \in U$,*

$$\zeta(t) \geq e^{\theta|t|}\xi, \quad \textit{for} \quad t \leq 0.$$

Proof. Let $(x, \xi) \in U$. First, we show $|z(t)| < \pi/4$ for all $t \leq 0$. We set $S = \{t \leq 0 \mid z(t) \in (-\pi/4, \pi/4)\}$. We note that 0 belongs to S. Suppose $(-\infty, 0] \setminus S \neq \emptyset$ holds. Setting $s_0 = \sup(-\infty, 0] \setminus S$, we have $|z(s_0)| = \pi/4$ and $|z(s)| < \pi/4$ for all $s_0 < s \leq 0$. This contradicts to

$$\frac{d}{dt}|z(t)|^2|_{t=s_0} = 2z(s_0)(\sin z(s_0))\frac{\zeta(s_0)}{\langle\zeta(s_0)\rangle} = \frac{\sqrt{2}\pi}{2}\frac{\zeta(s_0)}{\langle\zeta(s_0)\rangle} \geq 0,$$

which follows from the equation (4.1) and Lemma 4.2. Thus we have $S = (-\infty, 0]$.

Then there exists $\theta > 0$ such that

$$\zeta'(t) \leq -\theta\zeta(t)$$

for $t \leq 0$. From a simple calculation, we obtain

$$\zeta(t) \geq e^{\theta|t|}\xi \quad \text{for} \quad t \leq 0.$$

□

Lemma 4.4. *For each initial value $(x, \xi) \in U$, we have $z(t) \to 0$ as $t \to -\infty$.*

Proof. Let $(x, \xi) \in U$. As is shown in the proof of Lemma 4.3, we have $|z(t, x, \xi)| < \pi/4$ for $t \leq 0$. Thus we have

$$\frac{d}{dt}|z(t)|^2 = 2z(t)\sin z(t)\frac{\zeta(t)}{\langle\zeta(t)\rangle} \geq c|z(t)|^2$$

for $t \leq 0$, where we use $x \sin x \geq cx^2$ for $x \in (-\pi/4, \pi/4)$ with $c > 0$. Thus we have

$$|z(t)|^2 \leq |z(0)|^2 e^{-c|t|} \to 0$$

as $t \to -\infty$. □

4.2 Fredholm estimates in Sobolev spaces

The main theorem of this subsection is the following:

Theorem 4.5. *For $z \in \mathbb{C}$, we define*

$$d(z) := \dim \mathrm{Ker}_{\,\mathcal{D}'(\mathbb{T})}(P - z), \quad d^*(z) = \dim \mathrm{Ker}_{\,H^{\frac{1}{2}-0}(\mathbb{T})}(P - z),$$

Then we have $d^*(z) = 2$ for $z \in \mathbb{C}$ and

$$d(z) = 2 \ \textit{for} \ z \neq 0, \quad d(0) = 3.$$

Moreover, if $(P - z)u = 0$ with $u \in \mathcal{D}'(\mathbb{T}) \setminus \{0\}$ and $z \in \mathbb{C}$, then we have

$$u \ \textit{is not a constant function} \ \Rightarrow u \in H^{\frac{1}{2}-0}(\mathbb{T}) \setminus H^{\frac{1}{2}+0}(\mathbb{T}).$$

Theorem 1.1 directly follows from Theorem 4.5. In the following of this subsection, we shall prove Theorem 4.5. Let us define

$$\mathcal{X}^s = \{u \in H^{s+1}(\mathbb{T}) \mid Pu \in H^s(\mathbb{T})\}, \quad \mathcal{Y}^s = H^s(\mathbb{T})$$

for $s > -\frac{1}{2}$. From the results of the last subsection, Theorems 2.3 and 2.4, we obtain the following proposition.

Proposition 4.6. *For $s > -\frac{1}{2}$, $N > 0$ and $\varepsilon > 0$, we have*

$$\|u\|_{\mathcal{X}^s} \leq C\|(P-z)u\|_{\mathcal{Y}^s} + C\|u\|_{H^{-N}(\mathbb{T})} \quad \text{for} \quad u \in H^{\frac{1}{2}+\varepsilon}(\mathbb{T}) \tag{4.2}$$

$$\|u\|_{H^{-s}(\mathbb{T})} \leq C\|(P-z)u\|_{H^{-s-1}(\mathbb{T})} + C\|u\|_{H^{-N}(\mathbb{T})} \quad \text{for} \quad u \in \mathcal{D}'(\mathbb{T}). \tag{4.3}$$

Remark 4.7. When applying the radial sink estimate (Theorem 2.4), we use the fact that we can take the control region (supp b in Theorem 2.4) as supported in the elliptic set for p.

Now we study the Fredholm property of P from $\mathcal{X}^s$ to $\mathcal{Y}^s$. As a warm up, we prove P is a bounded operator.

Lemma 4.8. *The operator $P : \mathcal{X}^s \to \mathcal{Y}^s$ is bounded.*

Proof. It suffices to prove that P is a closed operator. Take a sequence $u_n \in \mathcal{X}^s$ such that $u_n \to u$ in $\mathcal{X}^s$ and $Pu_n \to w$ in $\mathcal{Y}^s$ for some $u \in \mathcal{X}^s$ and $w \in \mathcal{Y}^s$. By the definition of $\mathcal{X}^s$, we have $Pu_n \to Pu$ in $H^s(\mathbb{T}) = \mathcal{Y}^s$. This implies $w = Pu$ and $u \in \mathcal{X}^s$. This completes the proof. □

Next proposition assures P is a Fredholm operator.

Proposition 4.9. *$P - z : \mathcal{X}^s \to \mathcal{Y}^s$ is an analytic family of Fredholm operators for $z \in \mathbb{C}$.*

Proof. We take $\varepsilon > 0$ such that $s + 1 > \frac{1}{2} + \varepsilon$. First, we prove $\mathrm{Ker}_{\mathcal{X}^s}(P - z)$ is finite dimensional. Take a sequence $u_n \in \mathrm{Ker}_{\mathcal{X}^s}(P - z)$ with $\|u_n\|_{\mathcal{X}^s} = 1$. It suffices to show that u_n has a convergent subsequence in $\mathcal{X}^s$. Since the natural injection $\mathcal{X}^s \hookrightarrow H^{\frac{1}{2}+\varepsilon}(\mathbb{T})$ is compact, u_n has a convergent subsequence in $H^{\frac{1}{2}+\varepsilon}(\mathbb{T})$. We also denotes the subsequence by u_n. Using (4.2), we have

$$\|u_n - u_m\|_{\mathcal{X}^s} \leq C\|u_n - u_m\|_{H^{\frac{1}{2}+\varepsilon}} \to 0$$

as $n, m \to \infty$. Thus, u_n is Cauchy in $\mathcal{X}^s$ and hence converges in $\mathcal{X}^s$.

Next, we show that $P - z$ has a closed range. By virtue of [4, Proposition 19..1.3], it suffices to prove that any sequence $u_n \in \mathcal{X}^s$ such that u_n is bounded and $(P - z)u_n$ is convergent has a convergent subsequence. Take a sequence $u_n \in \mathcal{X}^s$ such that u_n is bounded and $(P - z)u_n$ is convergent. By using the compactness of the natural injection $\mathcal{X}^s \hookrightarrow H^{\frac{1}{2}+\varepsilon}(\mathbb{T})$, it follows that $u_{n_k} \to u$ in $H^{\frac{1}{2}+\varepsilon}(\mathbb{T})$ for some $u \in H^{\frac{1}{2}+\varepsilon}(\mathbb{T})$ for a subsequence u_{n_k}. Due to (4.2), u_{n_k} is convergent in $\mathcal{X}^s$. It easily follows that $u \in \mathcal{X}^s$ and $u_{n_k} \to u$ in $\mathcal{X}^s$.

Finally, we show that the kernel of $(P - z)^* : (\mathcal{Y}^s)^* = H^{-s}(\mathbb{T}) \to (\mathcal{X}^s)^*$ is finite dimensional. Note that if $(P - z)^*u = 0$ for $u \in H^{-s}(\mathbb{T})$, then $(P - \bar{\lambda})u = 0$ in the distribution sense since $C^\infty(\mathbb{T}) \subset \mathcal{X}^s$. Take a sequence $u_n \in H^{-s}(\mathbb{T})$ such that $(P -$

$z)^*u_n = 0$ and $\|u_n\|_{H^{-s}(\mathbb{T})} = 1$. If we take $N > 0$ large, the natural injection $H^{-s}(\mathbb{T}) \hookrightarrow H^{-N}(\mathbb{T})$ is compact. Then u_n has a convergent subsequence u_{j_k} in $H^{-N}(\mathbb{T})$. By using (4.3), it follows that u_{j_k} is convergent in $H^{-s}(\mathbb{T})$.

□

The next lemma specifies the regularity of eigenfunctions of P. From the next lemma, it turns out that if $(P - z)u = 0$ with $u \in \mathcal{D}'(\mathbb{T})$, then we have $u \in C^\infty(\mathbb{T})$ or $u \in H^{\frac{1}{2}-0}(\mathbb{T}) \setminus H^{\frac{1}{2}+0}(\mathbb{T})$.

Lemma 4.10. *For $\lambda \in \mathbb{C}$ and $s > -\frac{1}{2}$,*

$$\begin{aligned} &\mathrm{Ker}\,_{\mathcal{X}^s}(P - z) = \{u \in C^\infty(\mathbb{T}) \mid (P - z)u = 0\}, \\ &\mathrm{Ker}\,_{H^{-s}(\mathbb{T})}((P - z)^*) = \{u \in H^{\frac{1}{2}-0}(\mathbb{T}) \mid (P - \bar{z})u = 0\}. \end{aligned}$$

Proof. The first equality directly follows from (4.2). We show that the second equality. If $u \in \mathrm{Ker}\,_{H^{-s}(\mathbb{T})}((P - z)^*)$, then $(P - \bar{z})u = 0$ in a distribution sense since $C^\infty(\mathbb{T}) \subset \mathcal{X}^s$. By using (4.3), we have $u \in H^{\frac{1}{2}-0}(\mathbb{T})$. Conversely, suppose $u \in H^{\frac{1}{2}-0}(\mathbb{T})$ and $(P - \bar{\lambda})u = 0$. Note that $u \in H^{-s}(\mathbb{T})$. Then, for $w \in C^\infty(\mathbb{T})$,

$$0 = (u, (P - z)w) = ((P - z)^*u, w).$$

Using Lemma 4.11 below, we obtain $(P - z)^*u = 0$. □

Lemma 4.11. *$C^\infty(\mathbb{T})$ is dense in $\mathcal{X}^s$.*

Proof. Let $\chi \in C_c^\infty(\mathbb{R})$ such that $\chi(t) = 1$ on $t \leq 1$. Let $u \in \mathcal{X}^s$. Set $u_R(x) = \mathrm{Op}(\chi(\frac{|\xi|}{R}))u(x) \in C^\infty(\mathbb{T})$ for $R \geq 1$. Then, $u_R \to u$ in $H^{s+1}(\mathbb{T})$ and

$$Pu_R = [P, \mathrm{Op}(\chi(\frac{|\xi|}{R}))]u + \mathrm{Op}(\chi(\frac{|\xi|}{R}))Pu.$$

Since $Pu \in H^s(\mathbb{T})$, then $\mathrm{Op}(\chi(\frac{|\xi|}{R}))Pu \to Pu$ in $H^s(\mathbb{T})$. Moreover, since $u \in H^s(\mathbb{T})$ and $[P, \mathrm{Op}(\chi(\frac{|\xi|}{R}))]$ is uniformly bounded in $\mathrm{Op}S^1$ and converges to 0 in $\mathrm{Op}S^{1+0}$, we obtain $[P, \mathrm{Op}(\chi(\frac{|\xi|}{R}))]u \to 0$. Thus, $u_R \to u$ in $\mathcal{X}^s$. □

We can calculate the eigenfunctions of P with 0-eigenvalue.

Lemma 4.12. *For $C_0, C_1, C_2 > 0$, set*

$$u_0(x) = C_0,\ u_1(x) = C_1 H(x),\ u_2(x) = C_2 \log|\tan\frac{x}{2}|,$$

where $H(x) = 1$ on $[0, \pi]$ and $H(x) = 0$ on $(\pi, 2\pi)$. Then $Pu_j(x) = 0$ in the distributional sense. Moreover, $u_0 \in \mathrm{Ker}\,_{\mathcal{X}^s}(P)$ and $u_1, u_2 \in \mathrm{Ker}\,_{H^{-s}(\mathbb{T})}(P^*)$.

Proof. This lemma follows from a direct calculation. □

Proposition 4.13. *We have $\mathrm{Ind}(P - z) = -2$, $d(0) = 3$ and $d(z) = 2$ for $z \neq 0$. Moreover, if $z \in \mathbb{C} \setminus \{0\}$, then*

$$\dim \mathrm{Ker}\,_{\mathcal{X}^s}(P - z) = 0,\ \dim \mathrm{Ker}\,_{H^{-s}(\mathbb{T})}((P - z)^*) = 2.$$

Proof. First, we prove $\mathrm{Ind}(P-z)=-2$. If we write $u=\sum_{k\in\mathbb{Z}} e^{ikx}a_k \in \mathcal{D}'(\mathbb{T})$, $Pu=zu$ is equivalent to

$$k(k-1)a_{k-1}-k(k+1)a_{k+1}=2iza_k. \tag{4.4}$$

In fact, we have

$$\begin{aligned}
Pu=zu \Leftrightarrow & -i\partial_x\Big(\frac{(e^{ix}-e^{-ix})}{2i}\sum_{k\in\mathbb{Z}} ka_k e^{ikx}\Big)=zu\\
\Leftrightarrow & \partial_x\Big(\sum_{k\in\mathbb{Z}} ka_k e^{i(k+1)x}-ka_k e^{i(k-1)x}\Big)=-2zu\\
\Leftrightarrow & i\sum_{k\in\mathbb{Z}}(k(k-1)a_{k-1}-k(k+1)a_{k+1})e^{ikx}=-2zu\\
\Leftrightarrow & k(k-1)a_{k-1}-k(k+1)a_{k+1}=2iza_k.
\end{aligned}$$

This implies $d(z)=2$ for $z\neq 0$ and $d(0)=3$. Let $z=0$. Note that (4.4) is uniquely solved if a_0, a_1, a_{-1} are determined. Moreover, any solutions to $Pu=0$ with $u\in\mathcal{D}'(\mathbb{T})$ can be written as

$$u(x)=a_0+a_1\sum_{k=0}^{\infty}\frac{1}{2k+1}e^{i(2k+1)x}+a_{-1}\sum_{k=0}^{\infty}\frac{1}{2k-1}e^{-i(2k-1)x}.$$

In particular, $Pu=0$ has just three linearly independent solutions in $\mathcal{D}'(\mathbb{T})$. By Lemma 4.12, we conclude that $\mathrm{Ind}P=1-3=-2$. The stability of Fredhollm index under the continuous perturbation implies $\mathrm{Ind}(P-z)=-2$ for $z\in\mathbb{C}$.

Suppose $z\in\mathbb{C}\setminus\mathbb{R}$. Let $u\in\dim\mathrm{Ker}_{\mathcal{X}^s}(P-z)$. Since $u\in C^\infty(\mathbb{T})$ and $P=P^*$ formally, then an integration by parts gives $u=0$.

Next, suppose $z\in\mathbb{R}\setminus\{0\}$. Then (4.4) gives $a_0=0$. Moreover, $\{a_k\}_{k>0}$ is uniquely determined by (4.4) and $a_1\in\mathbb{C}$. Similarly, $\{a_k\}_{k<0}$ is uniquely determined by (4.4) and $a_{-1}\in\mathbb{C}$. Consequently, (4.4) has just two solutions. $\mathrm{Ind}(P-z)=-2$ implies that $Pu=zu$ also has just two distributional solutions. □

Proof of Theorem 4.5. Theorem 4.5 follows from Lemma 4.10 and Proposition 4.13. □

4.3 Discreteness of the spectrum

Proof of Theorem 1.3. By (4.3), we have

$$\|u\|_{H^{\frac{1}{2}-0}}\leq C\|Pu\|_{L^2}+C\|u\|_{L^2} \tag{4.5}$$

for $u\in L^2(\mathbb{T})$ if the right hand side is bounded. This implies that there is a continuous inclusion $D_{max}=D((P|_{C^\infty(\mathbb{T})})^*)\subset H^{1/2-0}(\mathbb{T})$. Now fix a self-adjoint extension of P and let D be its domain. By virtue of [5, Proposition 5.1], it suffices to prove that the inclusion $D\subset L^2(\mathbb{T})$ is compact. Then $D\subset D_{max}$ is a continuous inclusion. Since the inclusion $H^{1/2-0}(\mathbb{T})\subset L^2(\mathbb{T})$ is compact, the inclusion $D\subset L^2(\mathbb{T})$ is also compact. □

5 Third proof, via Fourier analysis

In this section, we give a shorter proof of Theorem 1.1 via Fourier analysis. We construct $u \in H^{\frac{1}{2}-0}(\mathbb{T}) \setminus \{0\}$ satisfying

$$(P-i)u = 0. \tag{5.1}$$

If we write $u(x) = \sum_{k=-\infty}^{\infty} a_k e^{ikx}$, then the above equation is equivalent to

$$k(k-1)a_{k-1} - k(k+1)a_{k+1} = -2a_k. \tag{5.2}$$

For a proof, see after (4.4).

Lemma 5.1. *Let $\{a_k\}_{k=-\infty}^{\infty}$ be a sequence satisfying* (5.2)*. Then we have $|a_k| \leq C\langle k\rangle^{-1}$ for $k \in \mathbb{Z}$. In particular, if $u \in \mathcal{D}'(\mathbb{T})$ satisfying* (5.1)*, then we have $u \in H^{\frac{1}{2}-0}(\mathbb{T})$.*

Proof. We only deal with the case of $k \geq 0$. Set $b_k = ka_k$. Then the equation (5.2) is equivalent to

$$b_{k+2} = b_k + \frac{2}{(k+1)^2} b_{k+1}.$$

First, we prove that for each integer $n \geq 1$, we have

$$|b_k| = O(k^{\frac{1}{n}}) \quad \text{as} \quad k \to \infty. \tag{5.3}$$

To see this, it suffices to prove that $|b_k| \leq C_1 k^{1/n}$ and $|b_{k+1}| \leq C_1(k+1)^{1/n}$ imply $|b_{k+2}| \leq C_1(k+2)^{1/n}$ for large k. We observe

$$\begin{aligned} |b_{k+2}|^n = |b_k + \frac{2}{k+1} b_{k+1}|^n \leq & C_1^n (k^{\frac{1}{n}} + 2(k+1)^{\frac{1}{n}-2})^n \\ \leq & C_1^n (k+1)(1 + \frac{1}{(k+1)^2})^n. \end{aligned}$$

Thus, to prove $|b_{k+2}| \leq C_1(k+2)^{1/n}$, we only need to prove

$$(1 + \frac{1}{(k+1)^2})^n \leq \frac{k+2}{k+1} = 1 + \frac{1}{k+1}. \tag{5.4}$$

Since the left hand side is $1 + O(\frac{1}{(k+1)^2})$ as $k \to \infty$, the inequality (5.4) holds for large k. Thus we have $|b_{k+2}| \leq C_1(k+2)^{1/n}$ for large k.

Next, we prove $|b_k| = O(1)$ as $k \to \infty$. Set $r_k = 2b_{k+1}/(k+1)^2$. Using (5.3) with $n = 1/2$, we have $\sum_{k=1}^{\infty} |r_k| < \infty$. This implies

$$|b_{2m+1} - b_1| = |\sum_{k=1}^{2m-1} (b_{k+2} - b_k)| \leq \sum_{k=1}^{\infty} |r_k|, \quad |b_{2m} - b_2| = |\sum_{k=2}^{2m-2} (b_{k+2} - b_k)| \leq \sum_{k=1}^{\infty} |r_k|$$

for $m \geq 2$. Consequently, we obtain $|b_k| = O(1)$ and $|a_k| = O(1/k)$ as $k \to \infty$. $\square$

Now we take a sequence $\{a_k\}_{k=-\infty}^{\infty}$ satisfying $a_1 = a_2 = 1$. Then the function $u(x) = \sum_{k=-\infty}^{\infty} a_k e^{ikx}$ satisfies $u \in H^{\frac{1}{2}-0}(\mathbb{T}) \setminus \{0\}$ and $(P-i)u = 0$. Moreover, we have $(P+i)\bar{u} = 0$ with $\bar{u} \neq 0$. This completes the proof of Theorem 1.1.

6 Fourth proof, via integration by parts

In this section, we prove 1.1 just by using integration by parts.

Proof of Theorem 1.1. Since the maximal domain of P is $\{u \in L^2(\mathbb{T}) \mid Pu \in L^2(\mathbb{T})\}$, it suffices to find $u, v \in L^2(\mathbb{T})$ with $Pu, Pv \in L^2(\mathbb{T})$ satisfying

$$(Pu, v)_{L^2(\mathbb{T})} \neq (u, Pv)_{L^2(\mathbb{T})}.$$

Let $u(x) = \log|x|\chi(x)$ and $v(x) = (H(x)+2H(-x))\chi(x)$ where $\chi \in C_c^\infty((-\frac{\pi}{2}, \frac{\pi}{2}))$ is a real valued function which is $\chi(x) = 1$ on $|x| \leq \frac{\pi}{4}$. Note that $u, v \in L^2(\mathbb{T})$ and $Pu, Pv \in L^2(\mathbb{T})$. Then, a direct calculation gives $(Pu, v)_{L^2(\mathbb{T})} - (u, Pv)_{L^2(\mathbb{T})} = -v(+0) + v(-0) = -3 \neq 0$. □

Remark 6.1. Using the technique in the the proof above, we can easily prove that $P = -\partial_x(x\partial_x)$ is not essential self-adjoint on $C_c^\infty(\mathbb{R})$.

A Anisotropic Sobolev space

In this appendix, we recall the definition and some standard properties of variable order Sobolev spaces, which are described in [2, Appendix].

Let (M, g) be a closed Riemannian manifold. For a real valued symbol $m \in S^0$ and $\rho \in (\frac{1}{2}, 1)$, we set

$$a_m(x,\xi) = \langle\xi\rangle^{m(x,\xi)}, \quad S_\rho^{m(x,\xi)} := \{a \in C^\infty(T^*M) \mid |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C\langle\xi\rangle^{m(x,\xi)+|\alpha|\rho-(1-\rho)|\beta|}\}.$$

Then it follows that $a_m \in S_\rho^{m(x,\xi)}$ is elliptic in the sense of [2, Definition 8]. From [2, Corollary 4], we deduce that there exists an operator $A_m \in \mathrm{Op}S_\rho^{m(x,\xi)}$ satisfying

$$A_m - \mathrm{Op}(a_m) \in \mathrm{Op}S_\rho^{m(x,\xi)-(2\rho-1)}.$$

Moreover, the operator A_m is formally self-adjoint and invertible in $C^\infty(M) \to C^\infty(M)$ (hence, also in $\mathcal{D}'(M) \to \mathcal{D}'(M)$). If M admits the quantization Op such that $\mathrm{Op}(a)$ is formally self-adjoint for any real-valued sybbol a (for example, $M = \mathbb{T}$, see [7, §5.3]), then [2, Lemma 12] implies that we can take A_m as

$$A_m - \mathrm{Op}(a_m) \in \mathrm{Op}S^{-\infty}. \tag{A.1}$$

Now we define the anisotropic Sobolev space.

Definition 1. For a real-valued symbol $m \in S^0$, we define

$$H^{m(x,\xi)} = \{u \in \mathcal{D}'(M) \mid A_m u \in L^2(M)\}, \quad (u, w)_{H^m} = (A_m u, A_m w).$$

The Hilbert space $H^{m(x,\xi)}$ with the inner metric $(\cdot,\cdot)_{H^m}$ is called the anisotropic Sobolev space of order m.

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