ASYMPTOTIC EXPANSIONS FOR THE MULTIPLE LAPLACE-MELLIN TRANSFORM OF LERCH ZETA-FUNCTIONS AND APPLICATIONS

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Abstract. Let $s \in \mathbb{C}$ be variable, $a, \lambda \in \mathbb{R}$ parameters with $a > 0$, $\phi(s, a, \lambda)$ the Lerch zeta-function defined below, and $\phi^*(s, a, \lambda)$ its slight modification obtained by extracting the only singularity at $s = 1$ (if $\lambda \in \mathbb{Z}$) of the Hurwitz zeta-function $\zeta(s, a)$. We denote by $(\phi^*)^{(m)}(s, a, \lambda)$ for any $m \in \mathbb{Z}$ the m-th derivative (with respect to s) if $m \geq 0$, while the $|m|$-th primitive (with its initial point at $s + \infty$) if $m \leq 0$. It is shown in the present article that complete asymptotic expansions exist for the multiple Laplace-Mellin transform (with respect to s) of $(\phi^*)^{(m)}(s, a, \lambda)$ for any $m \in \mathbb{Z}$ if $a > 1$, when the multivariate pivotal parameter $\boldsymbol{z} = (z_1, \ldots, z_d) \in \mathbb{C}^d$ (of the transforms) becomes both small and large through an appropriate poly-sector (Theorems 5 and 6), which extends our previous results on one dimensional case (Theorems 1 and 2). Further consideration on the excluded singular part of $\zeta(s, a)$ at $s = 1$ is supplemented to establish complete asymptotic expansions for the Lapalce-Mellin transform of $\zeta^{(m)}(s, a)$ (Theorems 7 and 8). A topic on complete asymptotic expansions for certain mean values of multiple zeta-functions, which is positioned on a sligntly different direction of research, is discussed in the final section (Theorems 9–13). Several open problems, relevant to the present study, are to be posed along with the presentation of our results (Problems 1–5).

1. Introduction

Let $s = \sigma + it$ be a complex variable, $z = x + iy$ complex parameter, a and λ real parameters with $a > 0$, and write $e(s) = e^{2\pi is}$ throughout the article. The Lerch zeta-function $\phi(s, a, \lambda)$ is defined by the Dirichlet series

$$\phi(s, a, \lambda) = \sum_{l=0}^{\infty} e(\lambda l)(a + l)^{-s} \qquad (\sigma = \operatorname{Re} s > 1) \tag{1.1}$$

and its meromorphic continuation over the whole s-plane (cf. [14][15]); this reduces to the exponential zeta-function $\zeta_\lambda(s) = e(\lambda)\phi(s, 1, \lambda)$ if $a = 1$, to the Hurwitz zeta-function $\zeta(s, a)$ if $\lambda \in \mathbb{Z}$, and hence to the Riemann zeta-function $\zeta(s) = \zeta_\lambda(s) = \zeta(s, 1)$ for $\lambda \in \mathbb{Z}$. Let $\delta_{\mathbb{Z}}(x)$ be the symbol which equals 1 or 0 according to $x \in \mathbb{Z}$ or otherwise, and set

$$\psi(s, a) = \frac{a^{1-s}}{s - 1}. \tag{1.2}$$

2020 *Mathematics Subject Classification.* Primary 11M35; Secondary 11M06, 11M32.

Key words and phrases. Riemann zeta-function, Dirichlet L-function, Lerch zeta-function, Hurwitz zeta-function, Laplace-Mellin transform, Riemann-Liouville transform, Mellin-Barnes integral, asymptotic expansion, weighted mean value.

The present research was supported in part by Grant-in-Aid for Scientific Research (No. 26400021) from JSPS.

The main object of study is a slight modification $\phi^*(s,a,\lambda)$ of $\phi(s,a,\lambda)$, defined by

$$(1.3)\qquad \phi^*(s,a,\lambda)=\phi(s,a,\lambda)-\delta_{\mathbb{Z}}(\lambda)\psi(s,a)=\begin{cases}\zeta(s,a)-\dfrac{a^{1-s}}{s-1} & \text{if } \lambda\in\mathbb{Z},\\ \phi(s,a,\lambda) & \text{if otherwise,}\end{cases}$$

which removes the only (possible) singularity at $s=1$. Let $(\phi^*)^{(m)}(s,a,\lambda)$ for any $m\in\mathbb{Z}$ denote the m-th derivative (with respect to s) if $m\geq 0$, while the $|m|$-th primitive (with its initial point at $s+\infty$; see (2.3) below) if $m\leq 0$. We have shown in our previous study [7][8] that complete asymptotic expansions exist for the Laplace-Mellin and Riemann-Liouville transforms (with respect to the variable s) of $(\phi^*)^{(m)}(s,a,\lambda)$ for any $m\in\mathbb{Z}$ if $a>1$, when the pivotal parameter $z\in\mathbb{C}$ (of the transforms) tends both to 0 and ∞ through appropriate sectors (see Theorems 1–4). The principal aim of the present article is to show that similar expansions still exist for the multiple Laplace-Mellin transform of $(\phi^*)^{(m)}(s,a,\lambda)$ for any $m\in\mathbb{Z}$ if $a>1$, when the (multivariate) pivotal parameter $\boldsymbol{z}=(z_1,\ldots,z_d)\in\mathbb{C}^d$ becomes both small and large through an appropriate poly-sector (Theorems 5 and 6).

It is seen from (1.3) that the original Hurwitz zeta-function $\zeta(s,a)=\zeta^*(s,a)+\psi(s,a)$ (including the singularity at $s=1$) is excluded from our initial consideration. It is in fact possible to modify our method to study asympțics for the Laplace-Mellin transform of the (excluded) singular term $\psi(s,a)$; this makes up for establishing complete asymptotic expansions for (the restored) $\zeta^{(m)}(s,a)$ $(a>1)$ and for $\{\zeta(s)-1\}^{(m)}$ with any $m\in\mathbb{Z}$ (Theorem 7 and 8).

The article is organized as follows. After preparing several necessary notations, we review our previous results in the next section. Section 3 is devoted to presenting our results on the asymptotics for the multiple Laplace-Mellin transform of $(\phi^*)^{(m)}(s,a,\lambda)$, while those for the $\zeta^{(m)}(s,a)$ and $\{\zeta(s)-1\}^{(m)}$ are given in Section 4. In the final section, we discuss some results on complete asymptotic expansions for certain mean values of multiple zeta-functions. Several open problems, which are relevant to the present study, are to be posed along with the statement of our results (Problems 1–5).

2. Notation and previous results

Let $\Gamma(s)$ denote the gamma function, α and β complex numbers with positive real parts, $f(z)$ a function holomorphic in the sector $|\arg z|<\pi$, and write $X_+=\max(0,X)$ for any $X\in\mathbb{R}$. We introduce the Laplace-Mellin and Riemann-Liouville (or Erdélyi-Köber) transforms of $f(z)$, in the forms

$$(2.1)\qquad \mathcal{LM}^{\alpha}_{z;\tau}f(\tau)=\frac{1}{\Gamma(\alpha)}\int_0^\infty f(z\tau)\tau^{\alpha-1}e^{-\tau}d\tau,$$

$$(2.2)\qquad \mathcal{RL}^{\alpha,\beta}_{z;\tau}f(\tau)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^\infty f(z\tau)\tau^{\alpha-1}(1-\tau)_+^{\beta-1}d\tau$$

with the normalization gamma multiples, provided that the integrals converge; the factor $\tau^{\alpha-1}$ secures the convergence of the integrals as $\tau\to 0^+$, while $e^{-\tau}$ and $(1-\tau)_+^{\beta-1}$ have effects to extract the portions of $f(z)$ corresponding to $\tau=O(z)$. It is to be remarked here that an overview of asymptotic results on the integral transforms of various zeta-functions is given in [7, Sect. 1][8, Sect. 1]. Next let $f^{(m)}(s)$ $(m\in\mathbb{Z})$ for any entire function $f(s)$ denote its m-th derivative if $m\geq 0$, while its $|m|$-th primitive if $m\leq 0$, defined inductively by

$$f^{(m)}(s)=\int_{s+\infty}^{s}f^{(m+1)}(w)dw=-\int_{0}^{+\infty}f^{(m+1)}(s+u)du, \tag{2.3}$$

subject to convergence, where the path of integration is the horizontal half-line.

It has been shown in our previous study [7][8] that complete asymptotic expansions exist for

$$\begin{aligned}\mathcal{LM}^{\alpha}_{z;\tau}(\phi^*)^{(m)}(s+\tau,a,\lambda)&=\frac{1}{\Gamma(\alpha)}\int_0^{\infty}(\phi^*)^{(m)}(s+z\tau,a,\lambda)\\&\quad\times\tau^{\alpha-1}e^{-\tau}d\tau,\end{aligned} \tag{2.4}$$

$$\begin{aligned}\mathcal{RL}^{\alpha,\beta}_{z;\tau}(\phi^*)^{(m)}(s+\tau,a,\lambda)&=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^{\infty}(\phi^*)^{(m)}(s+z\tau,a,\lambda)\\&\quad\times\tau^{\alpha-1}(1-\tau)_+^{\beta-1}d\tau\end{aligned} \tag{2.5}$$

with any $m\in\mathbb{Z}$ if $a>1$, together with those for their iterations, when both $z\to 0$ and $z\to\infty$ through appropriate sectors. We introduce here the Hadamard type operator with the initial point at $s+\infty$, defined for any $(r,s)\in\mathbb{C}^2$ by

$$\mathcal{I}^{r}_{\infty,s}f(s)=\frac{1}{\Gamma(r)\{e(r)-1\}}\int_{\infty}^{(0+)}f(s+z)z^{r-1}dz, \tag{2.6}$$

if $f(s+x)$ belongs to the class $x^{1-\operatorname{Re}r}L^1_x[0,+\infty[$ (as a function of x). Here the path of integration is a contour which starts from ∞, proceeds along the real axis to a sufficiently small $\delta>0$, encircles the origin counter-clockwise, and returns to ∞; $\arg z$ varies from 0 to 2π along the contour. The auxiliary zeta-function $\phi^*_r(s,a,\lambda)$ is defined for any $(r,s)\in\mathbb{C}^2$ and for any $a,\lambda\in\mathbb{R}$ with $a>1$ by

$$\phi^*_r(s,a,\lambda)=\mathcal{I}^{r}_{\infty,s}\phi^*(s,a,\lambda), \tag{2.7}$$

which is crucial in describing our results, and also of some interests in itself, since it interpolates the generalized Euler-Stieltjes constants $\gamma_m(a,\lambda)$ (associated with the Lerch zeta-function), defined by

$$\phi(s,a,\lambda)=(s-1)^{-1}+\sum_{m=0}^{\infty}\gamma_m(a,\lambda)(s-1)^m\qquad(0<|s-1|<1)$$

(cf. [3, p.41, 1.8(1.123)]), when $a>1$ as

$$\gamma_m(a,\lambda)=\frac{(-1)^m}{m!}\bigl\{\phi^*_{-m}(1,a,\lambda)+\log^m a\bigr\}\qquad(m=0,1,\ldots).$$

Theorem 1 ([8, Theorem 1]). *Let α be any complex number with $\operatorname{Re}\alpha>0$, a and λ real parameters with $a>1$, and m any integer. Then for any complex s and any integer $N\geq 0$, in the sector $|\arg z|<\pi$ we have*

$$\begin{aligned}\mathcal{LM}^{\alpha}_{z;\tau}(\phi^*)^{(m)}(s+\tau,a,\lambda)&=(-1)^m\sum_{n=0}^{N-1}\frac{(-1)^n(\alpha)_n}{n!}\phi^*_{-n-m}(s,a,\lambda)z^n\\&\quad+R^{1,+}_{m,N}(s,a,\lambda;z).\end{aligned} \tag{2.8}$$

Here the reminder $R^{1,+}_{m,N}(s,a,\lambda;z)$ satisfies the estimate

$$R^{1,+}_{m,N}(s,a,\lambda;z)=O\bigl\{(|t|+1)^{\max(0,\lfloor 2-\sigma\rfloor)}|z|^N\bigr\} \tag{2.9}$$

as $z \to 0$ through $|\arg z| \le \pi - \eta$ with any small $\eta > 0$, where the implied O-constant depend at most on α, a, σ, m, N and η.

Theorem 2 ([8, Theorem 2]). *Let a, λ and m be as in Theorem 1, and β any complex number with $\operatorname{Re}\beta > 0$. Then for any complex s and any integer $N \ge 0$, in the sector $|\arg z| < \pi$ we have*

$$\mathcal{LM}^{\alpha}_{z;\tau}(\phi^*)^{(m)}(s+\tau,a,\lambda) = (-1)^m \sum_{n=0}^{N-1} \frac{(-1)^n(\alpha)_n}{n!} \phi^*_{\alpha+n-m}(s,a,\lambda) z^{-\alpha-n} + R^{1,-}_{m,N}(s,a,\lambda;z). \tag{2.10}$$

Here the reminder $R^{1,-}_{m,N}(s,a,\lambda;z)$ satisfies the estimate

$$R^{1,-}_{m,N}(s,a,\lambda;z) = O\bigl\{(|t|+1)^{\max(0,\lfloor 2-\sigma\rfloor)}|z|^{-\operatorname{Re}\alpha-N}\bigr\} \tag{2.11}$$

as $z \to \infty$ through the sector $|\arg z| \le \pi - \eta$ with any small $\eta > 0$, where the implied O-constant depends at most on σ, α, a, m, N and η.

The case $(s,z) = (\sigma, it) \in \mathbb{R} \times i\mathbb{R}$ of Theorem 2 yields the following result.

Corollary 2.1 ([8, Corollary 2.1]). *Let α, a, λ and m be as in Theorem 1. Then for any real σ and for any $N \ge 0$ we have the asymptotic expansion, as $t \to \pm\infty$,*

$$\begin{aligned}\mathcal{LM}^{\alpha}_{t;\tau}&(\phi^*)^{(m)}(\sigma+i\tau,a,\lambda)\\ &= (-1)^m \sum_{n=0}^{N-1} \frac{(-1)^n(\alpha)_n}{n!} \phi^*_{\alpha+n-m}(\sigma,a,\lambda)(e^{(\operatorname{sgn} t)\pi i/2}|t|)^{-\alpha-n} + O\bigl(|t|^{-\operatorname{Re}\alpha-N}\bigr),\end{aligned} \tag{2.12}$$

where the implied O-constant depends at most on σ, α, λ, a, m and N.

The following Theorems 3 and 4 give the complete asymptotic expansions for the Riemann-Liouville transform $\mathcal{RL}^{\alpha,\beta}_{z;\tau}(\phi^*)^{(m)}(s+\tau,a,\lambda)$ as $z \to 0$ and $z \to \infty$ respectively.

Theorem 3 ([8, Theorem 3]). *Let a, λ and m be as in Theorem 1, and β complex number with $\operatorname{Re}\beta > 0$. Then for any complex s and any integer $N \ge 0$, in the sector $|\arg z| < \pi$ we have*

$$\mathcal{RL}^{\alpha,\beta}_{z;\tau}(\phi^*)^{(m)}(s+\tau,a,\lambda) = (-1)^m \sum_{n=0}^{N-1} \frac{(-1)^n(\alpha)_n}{(\alpha+\beta)_n n!} \phi^*_{-n-m}(s,a,\lambda) z^{n} + R^{2,+}_{m,N}(s,a,\lambda;z). \tag{2.13}$$

Here the reminder $R^{2,+}_{m,N}(s,a,\lambda;z)$ satisfies the estimate

$$R^{2,+}_{m,N}(s,a,\lambda;z) = O\bigl\{(|t|+1)^{\max(0,\lfloor 2-\sigma\rfloor)}|z|^{N}\bigr\} \tag{2.14}$$

as $z \to 0$ through the sector $|\arg z| \le \pi - \eta$ with any small $\eta > 0$, where the implied O-constant depends at most on α, a, σ, m, N and η.

We write, for $\alpha_h, \beta_k \in \mathbb{C}$ $(k = 1,\dots,m;\ k = 1,\dots,n)$,

$$\Gamma\binom{\alpha_1,\dots,\alpha_m}{\beta_1,\dots,\beta_n} = \frac{\prod_{h=1}^{m}\Gamma(\alpha_h)}{\prod_{k=1}^{n}\Gamma(\beta_k)},$$

and set $\varepsilon(z) = \operatorname{sgn}(\arg z)$ for any $z \in \mathbb{C}$ in the sectors $|\arg z| > 0$.

Theorem 4 ([8, Theorem 4])**.** *Let α, β, a, λ and m be as in Theorem 3. Then for any complex s, and any integers N_j $(j=1,2)$ with $N_1 \geq \lfloor \operatorname{Re}\beta \rfloor$ and $N_2 \geq \lfloor \operatorname{Re}\alpha \rfloor$, in the sectors $0 < |\arg z| < \pi$ we have*

$$
\begin{aligned}
(2.15)\qquad & \mathcal{RL}^{\alpha,\beta}_{z;\tau}(\phi^*)^{(m)}(s+\tau,a,\lambda) \\
&= (-1)^m \Gamma\binom{\alpha+\beta}{\beta} e^{-\varepsilon(z)\pi i\alpha}\left\{\sum_{n=0}^{N_1-1}\frac{(-1)^n(\alpha)_n(1-\beta)_n}{n!}\right. \\
&\qquad \left.\times\, \phi^*_{\alpha+n-m}(s,a,\lambda)(e^{-\varepsilon(z)\pi i}z)^{-\alpha-n} + R^{2,-}_{1,m,N_1}(s,a,\lambda;z)\right\} \\
&\quad + (-1)^m \Gamma\binom{\alpha+\beta}{\alpha} e^{\varepsilon(z)\pi i\beta}\left\{\sum_{n=0}^{N_2-1}\frac{(-1)^n(\beta)_n(1-\alpha)_n}{n!}\right. \\
&\qquad \left.\times\, \phi^*_{\beta+n-m}(s+z,a,\lambda)z^{-\beta-n} + R^{2,-}_{2,m,N_2}(s,a,\lambda;z)\right\}.
\end{aligned}
$$

Here the reminder $R^{2,-}_{j,m,N_j}(s,a,\lambda;z)$ $(j=1,2)$ satisfy the estimates

$$
(2.16)\qquad
\begin{aligned}
R^{2,-}_{1,m,N_1}(s,a,\lambda;z) &= O\bigl\{(|t|+1)^{\max(0,\lfloor 2-\sigma\rfloor)}|z|^{-\operatorname{Re}\alpha-N_1}\bigr\}, \\
R^{2,-}_{2,m,N_2}(s,a,\lambda;z) &= O\bigl\{(|t+y|+1)^{\max(0,\lfloor 2-\sigma-x\rfloor)}|z|^{-\operatorname{Re}\beta-N_2}\bigr\}
\end{aligned}
$$

as $z\to\infty$ through the sector $\eta \leq |\arg z| \leq \pi-\eta$ with any small $\eta>0$, where the constant implied in the first O-symbol depends at most on σ, α, β, a, m, N_1 and η, while that in the second at most on σ, x, α, β, a, m, N_2 and η.

Remark. Let ${}_1F_1\left({\kappa \atop \nu};Z\right)$ and $U(\kappa;\nu;Z)$ denote Kummer's confluent hypergeometric functions of the first and second kind defined by (4.1) and (4.2) below respectively. Then Stokes' phenomenon for confluent hypergeometric functions, which is revealed in the connection formula

$$
\begin{aligned}
(2.17)\qquad {}_1F_1\left({\kappa \atop \nu};Z\right) &= \Gamma\binom{\nu}{\nu-\kappa}e^{\varepsilon(Z)\pi i\kappa}U(\kappa;\nu;Z) + \Gamma\binom{\nu}{\kappa}e^{\varepsilon(Z)\pi i(\kappa-\nu)}e^Z \\
&\qquad \times U(\nu-\kappa;\nu;e^{-\varepsilon(Z)\pi i}Z)
\end{aligned}
$$

for $|\arg Z|>0$ (cf. [2, p.259, 6.7(7)][13, p.265, (10.5)]), in fact effects splitting the shape of the asymptotic expansions into the two sectors $0<|\arg z|<\pi$.

The case $(s,z)=(\sigma,it)\in\mathbb{R}\times i\mathbb{R}$ of Theorem 4 yields the following result.

Corollary 4.1 ([8, Corollary 4.1])**.** *Let a, λ, α, β, m be as in Theorem 4. Then for any real σ, and for any integers N_j $(j=1,2)$ with $N_1 \geq \lfloor \operatorname{Re}\beta \rfloor$ and $N_2 \geq \lfloor \operatorname{Re}\alpha \rfloor$, we have* *the asymptotic expansion, as $t \to \pm\infty$,*

$$\begin{aligned}(2.18)\quad &\mathcal{RL}^{\alpha,\beta}_{t;\tau}(\phi^*)^{(m)}(\sigma+i\tau,a,\lambda)\\ &= (-1)^m \Gamma\binom{\alpha+\beta}{\beta} e^{-(\operatorname{sgn} t)\pi i\alpha}\biggl\{\sum_{n=0}^{N_1-1}\frac{(-1)^n(\alpha)_n(1-\beta)_n}{n!}\phi^*_{\alpha+n-m}(\sigma,a,\lambda)\\ &\quad\times\bigl(e^{-(\operatorname{sgn} t)\pi i/2}|t|\bigr)^{-\alpha-n}+O\bigl(|t|^{-\operatorname{Re}\alpha-N_1}\bigr)\biggr\}\\ &\quad+(-1)^m \Gamma\binom{\alpha+\beta}{\alpha} e^{(\operatorname{sgn} t)\pi i\beta}\biggl\{\sum_{n=0}^{N_2-1}\frac{(-1)^n(\beta)_n(1-\alpha)_n}{n!}\phi^*_{\beta+n-m}(\sigma+it,a,\lambda)\\ &\quad\times\bigl(e^{(\operatorname{sgn} t)\pi i/2}|t|\bigr)^{-\beta-n}+O\bigl(|t|^{\max(0,\lfloor 2-\sigma\rfloor)-\operatorname{Re}\beta-N_2}\bigr)\biggr\},\end{aligned}$$

where the constant implied in the first O-symbol depends at most on σ, α, β, a, m and N_1, while that in the second at most on σ, α, β, a, m and N_2.

We pose here the first problem on the Laplace-Mellin and the Riemann-Liouville transforms of Lerch zeta-functions.

Problem 1. *Deduce the asymptotics for $|\phi^{(m)}(\sigma+i\tau,a,\lambda)|^2$ $(m=0,1,\ldots)$, under application of the operators $\mathcal{LM}^{\alpha}_{t;\tau}$ and $\mathcal{RL}^{\alpha,\beta}_{t;\tau}$, as $t\to\pm\infty$, and further for the product $\phi^{(m_1)}(s_1+\tau,a,\lambda)\phi^{(m_2)}(s_2-\tau,a,-\lambda)$ $(m_1,m_2=0,1,\ldots)$ apart from the poles, under application of $\mathcal{LM}^{\alpha}_{z;\tau}$ and $\mathcal{RL}^{\alpha,\beta}_{z;\tau}$, as both $z\to 0$ and $z\to\infty$ through appropriate sectors.*

3. Asymptotics for the multiple Laplace-Mellin transform

To describe our results, several notations are prepared in what follows.

We set $\mathbf{1}=(1,\ldots,1)$, for any d-dimensional complex vectors $\boldsymbol{x}=(x_1,\ldots,x_d)$, $\boldsymbol{y}=(y_1,\ldots,y_d)$,

$$\langle\boldsymbol{x},\boldsymbol{y}\rangle = x_1\overline{y_1}+\cdots+x_d\overline{y_d}\qquad\text{and}\qquad\langle\boldsymbol{x}\rangle=\langle\boldsymbol{x},\mathbf{1}\rangle=x_1+\cdots+x_d,$$

and write $\boldsymbol{x}_{d-1}=(x_1,\ldots,x_{d-1})\in\mathbb{C}^{d-1}$ and further if $x_d\neq 0$,

$$\frac{\boldsymbol{x}_{d-1}}{x_d}=\Bigl(\frac{x_1}{x_d},\ldots,\frac{x_{d-1}}{x_d}\Bigr).$$

Let α, $\boldsymbol{\beta}=(\beta_1,\ldots,\beta_d)$ and γ be complex parameters. The (fourth) Lauricella hypergeometric function of d-variables x_j $(j=1,\ldots,d)$ is defined by the multiple series

$$F_D^{(d)}\binom{\alpha,\beta_1,\ldots,\beta_d}{\gamma};x_1,\ldots,x_d\Bigr)=\sum_{k_1,\ldots,k_d=0}^{\infty}\frac{(\alpha)_{k_1+\cdots+k_d}(\beta_1)_{k_1}\cdots(\beta_d)_{k_d}}{(\gamma)_{k_1+\cdots+k_d}k_1!\cdots k_d!}x_1^{k_1}\cdots x_d^{k_d}$$

for all α, $\beta_j\in\mathbb{C}$ $(j=1,\ldots,d)$ and $\gamma\in\mathbb{C}\setminus\mathbb{Z}_{\le 0}$, where the series converges absolutely in the poly-disk $|x_j|<1$ $(j=1,\ldots,d)$ (cf. [16, p.228, 8.6(8.6.4)]); this is continued to a one-valued holomorphic function of $(\alpha,\boldsymbol{\beta},\gamma,\boldsymbol{x})$ for all $(\alpha,\boldsymbol{\beta},\gamma)\in\mathbb{C}^{d+1}\times(\mathbb{C}\setminus\mathbb{Z}_{\le 0})$ and in the sector $|\arg(1-x_j)-\varphi_0|<\pi/2$ $(j=1,\ldots,d)$ with any fixed $\varphi_0\in[-\pi/2,\pi/2]$. The notations

$$(\boldsymbol{\beta})_{\boldsymbol{k}}=(\beta_1)_{k_1}\cdots(\beta_k)_{k_d},\qquad \boldsymbol{k}!=k_1!\cdots k_d!\qquad\text{and}\qquad \boldsymbol{x}^{\boldsymbol{k}}=x_1^{k_1}\cdots x_d^{k_d}$$

for $\boldsymbol{k}=(k_1,\ldots,k_d)\in(\mathbb{Z}_{\geq 0})^d$ and $\boldsymbol{x}=(x_1,\ldots,x_d)\in\mathbb{C}^d$ allows us to rewrite $F_D^{(d)}$ in a more concise form

$$F_D^{(d)}\binom{\alpha,\boldsymbol{\beta}}{\gamma};\boldsymbol{x}\Big)=\sum_{\boldsymbol{k}\geq\boldsymbol{0}}\frac{(\alpha)_{\langle\boldsymbol{k}\rangle}(\boldsymbol{\beta})_{\boldsymbol{k}}}{(\gamma)_{\langle\boldsymbol{k}\rangle}\boldsymbol{k}!}\boldsymbol{x}^{\boldsymbol{k}}.$$

Our chief concern in this section is a multiple extension of the Laplace-Mellin transform of $(\phi^*)^{(m)}(s+\langle\boldsymbol{\tau}\rangle,a,\lambda)$, in the form

$$\begin{aligned}(3.1)\quad \mathcal{LM}^{\boldsymbol{\alpha}}_{\boldsymbol{z};\boldsymbol{\tau}}(\phi^*)^{(m)}(s+\langle\boldsymbol{\tau}\rangle,a,\lambda)=&\frac{1}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_d)}\int_0^\infty\cdots\int_0^\infty(\phi^*)^{(m)}(s+\langle\boldsymbol{z},\boldsymbol{\tau}\rangle,a,\lambda)\\&\times\tau_1^{\alpha_1-1}\cdots\tau_d^{\alpha_d-1}e^{-\tau_1-\cdots-\tau_d}d\tau_1\cdots d\tau_d,\end{aligned}$$

where $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_d)\in\mathbb{C}^d$ with $\operatorname{Re}\alpha_j>0$ $(j=1,\ldots,d)$ and $\boldsymbol{z}=(z_1,\ldots,z_d)\in\mathbb{C}^d$.

Let (u) for any $u\in\mathbb{R}$ denote the vertical straight path from $u-i\infty$ to $u+i\infty$. The following Theorem 5 gives a complete asymptotic expansion in the ascending order of z_d as $z_d\to 0$ through the sector $|\arg z_d-\theta_0|<\pi/2$ with any fixed $\theta_0\in[-\pi/2,\pi/2]$, while the remaining parameter $\boldsymbol{z}_{d-1}$ moves within the poly-sector $|\arg z_j-\theta_0|<\pi/2$ upon satisfying $z_j\asymp z_d$ $(j=1,\ldots,d-1)$.

Theorem 5. *Let a, λ and m be as in Theorem 1, $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_d)$ any complex vector with $\operatorname{Re}\alpha_j>0$ $(j=1,\ldots,d)$, and θ_0 any angle fixed with $\theta_0\in[-\pi/2,\pi/2]$. Then for any complex s and any integer $N\geq 0$, in the poly-sector $|\arg z_j-\theta_0|<\pi/2$ $(j=1,\ldots,d)$ we have*

$$\begin{aligned}(3.2)\qquad &\mathcal{LM}^{\boldsymbol{\alpha}}_{\boldsymbol{z};\boldsymbol{\tau}}(\phi^*)^{(m)}(s+\langle\boldsymbol{\tau}\rangle,a,\lambda)\\&=(-1)^m\sum_{n=0}^{N-1}\frac{(-1)^n(\langle\boldsymbol{\alpha}\rangle)_n}{n!}F_D^{(d-1)}\binom{-n,\boldsymbol{\alpha}_{d-1}}{\langle\boldsymbol{\alpha}\rangle};\boldsymbol{1}-\frac{\boldsymbol{z}_{d-1}}{z_d}\Big)\phi^*_{-n-m}(s,a,\lambda)z_d^n\\&\quad+R^+_{m,N}(s,a,\lambda;\boldsymbol{z}).\end{aligned}$$

Here the reminder $R^+_{m,N}(s,a,\lambda;z)$ is expressed as

$$\begin{aligned}(3.3)\qquad R^+_{m,N}(s,a,\lambda;\boldsymbol{z})=&\frac{(-1)^m}{2\pi i}\int_{(u_N^+)}\Gamma\binom{\langle\boldsymbol{\alpha}\rangle+w,-w}{\langle\boldsymbol{\alpha}\rangle}F_D^{(d-1)}\binom{-w,\boldsymbol{\alpha}_{d-1}}{\langle\boldsymbol{\alpha}\rangle};\boldsymbol{1}-\frac{\boldsymbol{z}_{d-1}}{z_d}\Big)\\&\times\phi^*_{-w-m}(s,a,\lambda)z_d^w dw\end{aligned}$$

with a constant u_N^+ satisfying $\max(-\operatorname{Re}\langle\boldsymbol{\alpha}\rangle,N-1)<u_N^+<N$. Further if $\boldsymbol{z}$ is in the poly-sector $|\arg z_j-\theta_0|\leq\pi/2-\eta$ $(j=1,\ldots,d)$ with any small $\eta>0$, and satisfies

$$c_1|z_d|\leq|z_j|\leq c_2|z_d|\quad(j=1,\ldots,d-1)$$

for some constants $c_k>0$ $(k=1,2)$, then the estimates

$$\begin{aligned}(3.4)\qquad &F_D^{(d-1)}\binom{-n,\boldsymbol{\alpha}_{d-1}}{\langle\boldsymbol{\alpha}\rangle};\boldsymbol{1}-\frac{\boldsymbol{z}_{d-1}}{z_d}\Big)=O(1),\\&R^+_{m,N}(s,a,\lambda;\boldsymbol{z})=O\bigl\{(|t|+1)^{\max(0,\lfloor 2-\sigma\rfloor)}|z_d|^N\bigr\}\end{aligned}$$

follow for all $N>n\geq 0$ as $z_d\to 0$ through the sector $|\arg z_d-\theta_0|\leq\pi/2-\eta$ with any small $\eta>0$, where the implied O-constants depend at most on $\boldsymbol{\alpha}$, σ, a, m, N, c_1, c_2 and η; this shows that (3.2) gives a complete asymptotic expansion in the ascending order of z_d as $z_d\to 0$ through $|\arg z_d-\theta_0|<\pi/2$.

The following Theorem 6 gives a complete asymptotic expansion in the descending order of z_d as $z_d \to \infty$ through the sector $|\arg z_d - \theta_0| < \pi/2$ with any fixed $\theta_0 \in [-\pi/2, \pi/2]$, while the remainig parameter $\boldsymbol{z}_{d-1}$ moves within the poly-sector $|\arg z_j - \theta_0| < \pi/2$ upon satisfying $z_j \asymp z_d$ $(j = 1, \ldots, d-1)$.

Theorem 6. *Let a, λ, m, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$ and θ_0 be as in Theorem 5. Then for any complex s and any integer $N \geq 0$, in the poly-sector $|\arg z_j - \theta_0| < \pi/2$ $(j = 1, \ldots, d)$ we have*

$$\begin{aligned}(3.5)\qquad &\mathcal{LM}^{\boldsymbol{\alpha}}_{\boldsymbol{z};\boldsymbol{\tau}}(\phi^*)^{(m)}(s + \langle\boldsymbol{\tau}\rangle, a, \lambda)\\ &= \sum_{n=0}^{N-1} \frac{(-1)^{n+m}(\langle\boldsymbol{\alpha}\rangle)_n}{n!} F_D^{(d-1)}\left(\begin{matrix}\langle\boldsymbol{\alpha}\rangle + n, \boldsymbol{\alpha}_{d-1}\\ \langle\boldsymbol{\alpha}\rangle\end{matrix}; \mathbf{1} - \frac{\boldsymbol{z}_{d-1}}{z_d}\right)\\ &\quad \times \phi^*_{\langle\boldsymbol{\alpha}\rangle+n-m}(s, a, \lambda) z_d^{-\langle\boldsymbol{\alpha}\rangle - n} + R^-_{m,N}(s, a, \lambda; \boldsymbol{z}).\end{aligned}$$

Here the reminder $R^-_{m,N}(s, a, \lambda; z)$ is expressed as

$$\begin{aligned}(3.6)\qquad R^-_{m,N}(s, a, \lambda; \boldsymbol{z}) &= \frac{(-1)^m}{2\pi i}\int_{(u_N^-)} \Gamma\left(\begin{matrix}\langle\boldsymbol{\alpha}\rangle + w, -w\\ \langle\boldsymbol{\alpha}\rangle\end{matrix}\right) F_D^{(d-1)}\left(\begin{matrix}-w, \boldsymbol{\alpha}_{d-1}\\ \langle\boldsymbol{\alpha}\rangle\end{matrix}; \mathbf{1} - \frac{\boldsymbol{z}_{d-1}}{z_d}\right)\\ &\quad \times \phi^*_{-w-m}(s, a, \lambda) z_d^w dw\end{aligned}$$

with a constant u_N^- satisfying $-\operatorname{Re}\langle\boldsymbol{\alpha}\rangle - N < u_N^- < \min(-\operatorname{Re}\langle\boldsymbol{\alpha}\rangle - N + 1, 0)$. Further if $\boldsymbol{z}$ is in the poly-sector $|\arg z_j - \theta_0| \leq \pi/2 - \eta$ $(j = 1, \ldots, d)$ with any small $\eta > 0$, and satisfies

$$c_1|z_d| \leq |z_j| \leq c_2|z_d| \quad (j = 1, \ldots, d-1)$$

for some constants $c_k > 0$ $(k = 1, 2)$, then the estimates

$$\begin{aligned}(3.7)\qquad &F_D^{(d-1)}\left(\begin{matrix}\langle\boldsymbol{\alpha}\rangle + n, \boldsymbol{\alpha}_{d-1}\\ \langle\boldsymbol{\alpha}\rangle\end{matrix}; \mathbf{1} - \frac{\boldsymbol{z}_{d-1}}{z_d}\right) = O(1),\\ &R^-_{m,N}(s, a, \lambda; \boldsymbol{z}) = O\left\{(|t| + 1)^{\max(0, \lfloor 2-\sigma \rfloor)}|z_d|^{-\operatorname{Re}\langle\boldsymbol{\alpha}\rangle - N}\right\}\end{aligned}$$

follow for all $N > n \geq 0$ as $z_d \to \infty$ through the sector $|\arg z_d - \theta_0| \leq \pi/2 - \eta$ with any small $\eta > 0$, where the implied O-constants depend at most on $\boldsymbol{\alpha}$, σ, a, m, N, c_1, c_2 and η; this shows that (3.5) gives a complete asymptotic expansion in the descending order of z_d as $z_d \to \infty$ through $|\arg z_d - \theta_0| < \pi/2$.

The case $(s, \boldsymbol{z}) = (\sigma, i\boldsymbol{t}) \in \mathbb{R} \times (i\mathbb{R})^d$ with $\boldsymbol{t} = (t_1, \ldots, t_d)$ and $\theta_0 = (\operatorname{sgn} t_d)\pi/2$ of Theorem 6 implies the complete asymptotic expansion in the descending order of t_d as $t_d \to \pm\infty$.

Corollary 6.1. *Let a, λ, m, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$ and θ_0 be as in Theorem 6, and $\boldsymbol{t} = (t_1, \ldots, t_d) \in \mathbb{R}^d$. Then for any integer $N \geq 0$ we have the asymptotic expansion, as $t_d \to \pm\infty$,*

$$\begin{aligned}(3.8)\qquad &\mathcal{LM}^{\boldsymbol{\alpha}}_{\boldsymbol{t};\boldsymbol{\tau}}(\phi^*)^{(m)}(\sigma + i\langle\boldsymbol{\tau}\rangle, a, \lambda)\\ &= \sum_{n=0}^{N-1} \frac{(-1)^{n+m}(\langle\boldsymbol{\alpha}\rangle)_n}{n!} F_D^{(d-1)}\left(\begin{matrix}\langle\boldsymbol{a}\rangle + n, \boldsymbol{\alpha}_{d-1}\\ \langle\boldsymbol{\alpha}\rangle\end{matrix}; \mathbf{1} - \frac{\boldsymbol{t}_{d-1}}{t_d}\right)\\ &\quad \times \phi^*_{\langle\boldsymbol{\alpha}\rangle+n-m}(\sigma, a, \lambda)(e^{(\operatorname{sgn} t)\pi i/2}|t_d|)^{-\langle\boldsymbol{\alpha}\rangle - n} + O\left(|t_d|^{-\operatorname{Re}\langle\boldsymbol{\alpha}\rangle - N}\right),\end{aligned}$$

while the other parameter $\boldsymbol{t}_{d-1}$ *moves as* $t_j \to \pm\infty$ *upon satisfying*

$$c_1 t_d \le t_j \le c_2 t_d \quad (j = 1, \ldots, d-1)$$

for some constants $c_k > 0$ $(k = 1, 2)$*, where the implied* O*-constant depends at most on* $\boldsymbol{\alpha}$*,* a*,* σ*,* m*,* N*,* c_1 *and* c_2*.*

Problem 2. *Find the complete asymptotic expansions for the multiple Riemann-Liouville transform* $\mathcal{RL}^{\boldsymbol{\alpha};\boldsymbol{\beta}}_{\boldsymbol{z};\boldsymbol{\tau}}(\phi^*)^{(m)}(s+\langle \boldsymbol{z}, \boldsymbol{\tau}\rangle, a, \lambda)$ *when* $\boldsymbol{z} \in \mathbb{C}^d$ *becomes both small and large through appropriate poly-sectors.*

Problem 3. *Find a good class of multiple zeta-functions which appropriates to apply the operators* $\mathcal{LM}^{\boldsymbol{\alpha}}_{\boldsymbol{z};\boldsymbol{\tau}}$ *and* $\mathcal{RL}^{\boldsymbol{\alpha};\boldsymbol{\beta}}_{\boldsymbol{z};\boldsymbol{\tau}}$*, and further deduce the asymptotic expansions with respect to* $\boldsymbol{z} \in \mathbb{C}^d$ *under application of these operators.*

4. Supplemented asymptotics for (restored) $\mathcal{LM}^{\alpha}_{z;\tau}\zeta(s+\tau, a)$

We recall the extraction of the singular part $\psi(s,a)$ from $\zeta(s,a)$ as $\zeta^*(s,a) = \zeta(s,a) - \psi(s,a)$, see (1.3). Note further that our asymptotic results are obtained under the restriction $a > 1$, and hence the case of $\zeta(s) = \zeta(s,1)$ is excluded from our initial consideration. We supplement in this section the corresponding results on the singular part $\psi(s,a)$ to make up for establishing the asymptotic expansions for (restored) $\mathcal{LM}^{\alpha}_{z;\tau}\zeta(s+\tau, a)$, although the presentation of the resulting formulae becomes rather involved.

Let ${}_1F_1\binom{\kappa}{\nu}; Z)$ and $U(\kappa;\nu;Z)$ denote Kummer's confluent hypergeometric function of the first and second kind, defined respectively by

$$(4.1)\qquad {}_1F_1\binom{\kappa}{\nu}; Z\Big) = \sum_{k=0}^{\infty} \frac{(\kappa)_k}{(\nu)_k k!} Z^k \qquad (|Z| < +\infty)$$

for $(\kappa,\nu) \in \mathbb{C} \times (\mathbb{C} \setminus \mathbb{Z}_{\le 0})$ (cf. [2, p.248, 6.1(1)]), and

$$(4.2)\qquad U(\kappa;\nu;Z) = \frac{1}{\Gamma(\kappa)\{e(\kappa) - 1\}} \int_{\infty}^{(0+)} e^{-Zw} w^{\kappa-1}(1+w)^{\nu-\kappa-1} dw$$

for $|\arg Z| < \pi/2$ and for all $(\kappa,\nu) \in \mathbb{C}^2$ (cf. [2, p.255, 6.5(2)]), where the domain of Z in the latter expression can be extended to $|\arg Z| < 3\pi/2$ by rotating suitably the path of integration (cf. [2, p.273, 6.11.2(9)]). We write $s = \sigma + it$ and $r = \rho + i\tau$ with real coordinates throughout the following, and introduce for any $(r,s) \in \mathbb{C} \times (\mathbb{C} \setminus \{1\})$ the auxiliary function

$$(4.3)\qquad \mathcal{I}^r_{\infty,s}\psi(s,a) = \psi_r(s,a).$$

Then for any $a > 1$ the evaluation

$$\begin{aligned}\psi_r(s,a) &= a^{1-s}\log^{1-r} a \cdot U(1; 2-r; (s-1)\log a)\\ &= \frac{a^{1-s}\log^{1-r} a}{r-1} {}_1F_1\binom{1}{2-r}; (s-1)\log a\Big) + \Gamma(1-r)(s-1)^{r-1},\end{aligned}$$

is in fact valid for all $(r,s) \in \mathbb{C}^2$ with $|\arg(s-1)| < \pi$, where the second equality follows from the connection formula

$$(4.4)\qquad U(\kappa;\nu;Z) = \Gamma\binom{1-\nu}{\kappa-\nu+1}{}_1F_1\binom{\kappa}{\nu}; Z\Big) + \Gamma\binom{\nu-1}{\kappa} Z^{1-\nu} {}_1F_1\binom{\kappa-\nu+1}{2-\nu}; Z\Big)$$

for any $(\kappa,\nu)\in\mathbb{C}\times(\mathbb{C}\setminus\mathbb{Z})$ (cf. [2, p.257, 6.5(7)]); this further implies the estimate

$$\begin{aligned}\psi_r(s,a)&\ll(|\tau|+1)^{-\max(0,\lfloor\rho\rfloor)}(|\tau|+|t|+1)^{\max(0,\lfloor\rho\rfloor)}\\&\quad+e^{-\pi|\tau|/2-\tau\arg(s-1)}(|\tau|+1)^{1/2-\rho}(|t|+1)^{\rho-1}\end{aligned}$$

in the same region of (r,s) above.

We can show along with the lines above the following Theorems 7 and 8, which give complete asymptotic expansions for $\mathcal{LM}^{\alpha}_{z;\tau}\zeta^{(m)}(s+\tau,a)$ if $a>1$ as both $z\to 0$ and $z\to\infty$ through a sector narrower than $|\arg z|<\pi$.

Theorem 7. *Let α, a, λ and m be as in Theorem 1, the complex variable s located in the sector $|\arg(s-1)|<\pi$, and set $\theta(s)=\arg(s-1)$ and $\delta(s)=\pi-|\theta(s)|(>0)$. Then for any integer $N\geq 0$, in the sector*

$$\max\Bigl(-\pi,-\frac{3\pi}{2}+\theta(s)\Bigr)<\arg z<\min\Bigl(\pi,\frac{3\pi}{2}+\theta(s)\Bigr)$$

we have

$$\begin{aligned}\mathcal{LM}^{\alpha}_{z;\tau}\zeta^{(m)}(s+\tau,a)&=(-1)^m\sum_{n=0}^{N-1}\frac{(-1)^n(\alpha)_n}{n!}\bigl\{\zeta^*_{-n-m}(s,a)+\psi_{-n-m}(s,a)\bigr\}z^n\\&\quad+R^+_{m,N}(s,a;z).\end{aligned}\tag{4.5}$$

Here the reminder $R^+_{m,N}(s,a;z)$ satisfies the estimate

$$R^+_{m,N}(s,a;z)=O\bigl\{(|t|+1)^{\max(0,\lfloor 2-\sigma\rfloor,N-m)}|z|^N\bigr\}\tag{4.6}$$

as $z\to 0$ through the sector

$$\max\Bigl(-\pi,-\frac{3\pi}{2}+\theta(s)\Bigr)+\eta\leq\arg z\leq\min\Bigl(\pi,\frac{3\pi}{2}+\theta(s)\Bigr)-\eta$$

with any small $0<\eta<\delta(s)$, where the implied O-constant depends at most on α, σ, a, m, N and η.

Theorem 8. *Let α, a, λ, m and $\theta(s)$ be as in Theorem 7. Then for any complex variable s with $|\theta(s)|<\pi$, and any integer $N\geq 0$, in the sector*

$$\max\Bigl(-\pi,-\frac{3\pi}{2}+\theta(s)\Bigr)<\arg z<\min\Bigl(\pi,\frac{3\pi}{2}+\theta(s)\Bigr)$$

we have

$$\begin{aligned}\mathcal{LM}^{\alpha}_{z;\tau}\zeta^{(m)}(s+\tau,a)&=(-1)^m\sum_{n=0}^{N-1}\frac{(-1)^n(\alpha)_n}{n!}\bigl\{\zeta^*_{\alpha+n-m}(s,a)+\psi_{\alpha+n-m}(s,a)\bigr\}z^{-\alpha-n}\\&\quad+R^-_{m,N}(s,a;z).\end{aligned}\tag{4.7}$$

Here the reminder $R^-_{m,N}(s,a;z)$ satisfies the estimate

$$R^-_{m,N}(s,a;z)=O\bigl\{(|t|+1)^{\max(0,\lfloor 2-\sigma\rfloor,\operatorname{Re}\alpha+N-m)}|z|^{-\operatorname{Re}\alpha-N}\bigr\}\tag{4.8}$$

as $z\to\infty$ through the sector

$$\max\Bigl(-\pi,-\frac{3\pi}{2}+\theta(s)\Bigr)+\eta\leq\arg z\leq\min\Bigl(\pi,\frac{3\pi}{2}+\theta(s)\Bigr)-\eta$$

with any small $0<\eta<\delta(s)$, where the implied O-constant depends at most on α, σ, a, m, N and η.

One can observe for any $s_0 \in \mathbb{C}$ fixed in the sector $|\arg(s_0-1)| < \pi$ that the vertical lines $z = e^{(\operatorname{sgn} t)\pi i/2}|t|$ with all $t \in \mathbb{R} \setminus \{0\}$ are included in the sector

$$\max\Bigl(-\pi, -\frac{3\pi}{2} + \theta(s_0)\Bigr) < \arg z < \min\Bigl(\pi, \frac{3\pi}{2} + \theta(s_0)\Bigr),$$

which allows us to deduce from Theorem 8 the following corollary.

Corollary 8.1. *Let α, a, λ, m and $\theta(s)$ be as in Theorem 8. Then for any complex s_0 fixed with $|\theta(s_0)| < \pi$, and for any integer $N \geq 0$ we have the asymptotic expansion, as $t \to \pm\infty$,*

$$\begin{aligned}(4.9)\qquad \mathcal{LM}^{\alpha}_{t;\tau}\zeta^{(m)}(s_0+i\tau,a) &= (-1)^m \sum_{n=0}^{N-1} \frac{(-1)^n(\alpha)_n}{n!}\bigl\{\zeta^{*}_{\alpha+n-m}(s_0,a) + \psi_{\alpha+n-m}(s_0,a)\bigr\}\\ &\quad\times \bigl(e^{(\operatorname{sgn} t)\pi i/2}|t|\bigr)^{-\alpha-n} + O(|t|^{-\operatorname{Re}\alpha - N}),\end{aligned}$$

where the implied O-constant depends at most on α, a, $\operatorname{Re} s_0$, m and N.

The result above on $\zeta(s,a)$ can in fact be transferred to that below on $\zeta(s)$ through the relation $\zeta(s) = 1 + \zeta(s,2)$. Note here that the primitives $\{\zeta(s)-1\}^{(m)}$ exist for all non-positive integers m; however $\zeta^{(m)}(s)$ does not for such m, since $\lim_{\sigma\to+\infty}\zeta(\sigma+it) = 1$ with any real t. We can therefore show the following formula for $\{\zeta(s)-1\}^{(m)}$ insted for $\zeta^{(m)}(s)$ $(m \in \mathbb{Z})$ itself.

Corollary 8.2. *Let s_0, α, a, λ and m be as in Theorem 8. Then for any integer $N \geq 0$ we have the asymptotic expansion, as $t \to \pm\infty$,*

$$\begin{aligned}(4.10)\qquad &\mathcal{LM}^{\alpha}_{t;\tau}\{\zeta(s_0+i\tau)-1\}^{(m)}\\ &= (-1)^m \sum_{n=0}^{N-1} \frac{(-1)^n(\alpha)_n}{n!}\bigl\{\zeta^{*}_{\alpha+n-m}(s_0,2) + \psi_{\alpha+n-m}(s_0,2)\bigr\}\bigl(e^{(\operatorname{sgn} t)\pi i/2}|t|\bigr)^{-\alpha-n}\\ &\quad + O(|t|^{-\operatorname{Re}\alpha - N}),\end{aligned}$$

where the implied O-constant depends at most on α, a, $\operatorname{Re} s_0$, m and N.

Problem 4. *Find the complete asymptotic expansions for the Riemann-Liouville transform $\mathcal{RL}^{\alpha,\beta}_{z;\tau}\zeta^{(m)}(s+\tau,a)$ with any $m \in \mathbb{Z}$ as both $z \to 0$ and $z \to \infty$ through appropriate sectors.*

5. Some mean values of double Hurwitz zeta-functions

Let $\boldsymbol{s} = (s_1, s_2)$ be complex variables, $\boldsymbol{a} = (a_1, a_2)$ real parameters with $a_j > 0$ $(j = 1,2)$, write $s_j = \sigma_j + it_j$ $(j = 1,2)$, and set $\boldsymbol{e}_1 = (1,0)$ and $\boldsymbol{e}_2 = (0,1)$ throughout the following. We here introduce the double Hurwitz zeta-function $\widetilde{\zeta}_2(\boldsymbol{s};\boldsymbol{a})$ and the double Plejeri-Minak zeta-function $\zeta_2(\boldsymbol{s};\boldsymbol{a})$, defined respectively by

$$(5.1)\qquad \widetilde{\zeta}_2(\boldsymbol{s};\boldsymbol{a}) = \sum_{l_1,l_2=0}^{\infty} (a_1+a_2+l_1+l_2)^{-s_1}(a_2+l_2)^{-s_2}$$

for $\sigma_1 > 1$ and $\sigma_2 > 1$, and

$$(5.2)\qquad \zeta_2(\boldsymbol{s};\boldsymbol{a}) = \sum_{l=0}^{\infty}(a_1+a_2+l)^{-s_1}(a_2+l)^{-s_2}$$

for $\sigma_1+\sigma_2>1$, both with their meromorphic continuations to the whole $\boldsymbol{s}$-space $\mathbb{C}^2$. The following theorems can in fact be shown for the mean values with respect to the parameters $\boldsymbol{x}=(x_1,x_2)\in(\mathbb{R}_{\geq 0})^2$ in $\widetilde{\zeta}_2(\boldsymbol{s};\boldsymbol{a}+\boldsymbol{x})$.

Theorem 9. *For any integer $N\geq 0$, and for any complex $\boldsymbol{s}=(s_1,s_2)$ in the region $\sigma_1+\sigma_2>1-N$, except the points on*

$$\widetilde{E_1}=\{\boldsymbol{s}\in\mathbb{C}^2\mid\langle\boldsymbol{s}\rangle=2-n\ \textit{or}\ s_1=1+n\quad(n=0,1,\ldots)\},\tag{5.3}$$

we have the formula

$$\int_0^1\widetilde{\zeta}_2(\boldsymbol{s};\boldsymbol{a}+x_1\boldsymbol{e}_1)dx_1=\frac{\zeta(\langle\boldsymbol{s}\rangle-1)}{s_1-1}+S_{1,N}(\boldsymbol{s};\boldsymbol{a})+R_{1,N}(\boldsymbol{s};\boldsymbol{a}),\tag{5.4}$$

where

$$S_{1,N}(\boldsymbol{s};\boldsymbol{a})=\sum_{n=0}^{N-1}\frac{(s_1)_n}{(n+1)!}a_1^{n+1}\zeta_2(\boldsymbol{s}+n\boldsymbol{e}_1;\boldsymbol{a}),\tag{5.5}$$

$$R_{1,N}(\boldsymbol{s};\boldsymbol{a})=\frac{(s_1)_N}{(N+1)!}a_1^{N+1}\sum_{l=0}^{\infty}(a_2+l)^{1-\langle\boldsymbol{s}\rangle}\int_{a_2+l}^{\infty}\frac{x^{s_1-2}}{(a_1+x)^{s_1+N}}dx.\tag{5.6}$$

Here the estimates

$$\begin{aligned}\zeta_2(\boldsymbol{s}+n\boldsymbol{e}_1;\boldsymbol{a})&=O(a_2^{1-\sigma_1-\sigma_2-n}),\\ R_{1,N}(\boldsymbol{s};\boldsymbol{a})&=O(a_2^{1-\sigma_1-\sigma_2-N})\end{aligned}\tag{5.7}$$

follow for all $N>n\geq 0$ in the same region of $\boldsymbol{s}$ above; this shows that the formula (5.4) gives a complete asymptotic expansion in the descending order of a_2 as $a_2\to+\infty$, while $a_1>0$ is fixed.

Theorem 10. *For any integer $N\geq 0$ and for any complex $\boldsymbol{s}=(s_1,s_2)$ in the region $\sigma_1<1+N$ and $\sigma_2>1-N$, except the points on*

$$\widetilde{E_2}=\{\boldsymbol{s}\in\mathbb{C}^2\mid\langle\boldsymbol{s}\rangle=2-n\ \textit{or}\ s_2=1+n\ (n=0,1,\ldots)\},\tag{5.8}$$

we have the formula

$$\begin{aligned}\int_0^1\widetilde{\zeta}_2(\boldsymbol{s};\boldsymbol{a}+x_2\boldsymbol{e}_2)dx_2&=\Gamma\Big(\begin{matrix}1-s_2,\langle\boldsymbol{s}\rangle-1\\ s_1\end{matrix}\Big)\zeta(\langle\boldsymbol{s}\rangle-1,a_1)\\&\quad-S_{2,N}(\boldsymbol{s}\ \boldsymbol{a})-R_{2,N}(\boldsymbol{s};\boldsymbol{a}),\end{aligned}\tag{5.9}$$

where

$$S_{2,N}(\boldsymbol{s};\boldsymbol{a})=\sum_{n=0}^{N-1}\frac{(s_1)_n}{(1-s_2)_{n+1}}a_2^{s_1+n}\zeta_2(\boldsymbol{s}+n\boldsymbol{e}_1,\langle\boldsymbol{a}\rangle),\tag{5.10}$$

$$R_{2,N}(\boldsymbol{s};\boldsymbol{a})=\frac{(s_1)_N}{(1-s_2)_N}a_2^{s_1+N}\sum_{l=0}^{\infty}(a_1+l)^{1-\langle\boldsymbol{s}\rangle}\int_{a_1+l}^{\infty}\frac{x^{\langle\boldsymbol{s}\rangle-2}}{(a_2+x)^{s_1+N}}dx.\tag{5.11}$$

Here the estimates

$$\begin{aligned}\zeta_2(\boldsymbol{s}+n\boldsymbol{e}_1,\langle\boldsymbol{a}\rangle)&=O(a_1^{1-\sigma_1-n}),\\ R_{2,N}(\boldsymbol{s};\boldsymbol{a})&=O(a_1^{1-\sigma_1-N})\end{aligned}\tag{5.12}$$

follow for all $N>n\geq 0$ in the same region of $\boldsymbol{s}$ above; this shows that the formula (5.9) gives a complete asymptotic expansion in the descending order of a_1 as $a_1\to+\infty$, while

a_2 is fixed. Furthermore, in the same region of $\boldsymbol{s}$ above, for any integer $K \geq 0$, we have the expression

$$R_{2,N}(\boldsymbol{s};\boldsymbol{a}) = \sum_{k=1}^{K} \frac{(-1)^{k-1}(2-\langle\boldsymbol{s}\rangle)_{k-1}(s_1)_{N-k}}{(1-s_2)_N} a_2^{s_1+N}\zeta_2(k, s_1+N-k;\boldsymbol{a}) + \frac{(-1)^{K}(2-\langle\boldsymbol{s}\rangle)_{K}(s_1)_{N-K}}{(1-s_2)_N} a_2^{s_1+N}\sum_{l=0}^{\infty}(a_1+l)^{1-\langle\boldsymbol{s}\rangle} \times \int_{a_1+l}^{\infty} \frac{x^{\langle\boldsymbol{s}\rangle-K-2}}{(a_2+x)^{s_1+N-K}}dx, \tag{5.13}$$

which gives a complete asymptotic expansion when $s_j \to \infty$ $(j = 1, 2)$, so as that $\boldsymbol{s} \in \mathbb{C}^2$ is on any hyper-plane $\langle\boldsymbol{s}\rangle = c \in \mathbb{C}$ except the points on $\widetilde{E_2}$.

Theorem 11. *A complete asymptotic expansion similar to that in Theorem 10 exists for the double mean value*

$$\int_0^1\int_0^1 \widetilde{\zeta}_2(\boldsymbol{s};\boldsymbol{a}+x_1\boldsymbol{e}_1+x_2\boldsymbol{e}_2)dx_1dx_2. \tag{5.14}$$

Theorem 12. *Complete asymptotic expansions in the descending order of q as $q \to +\infty$ exist for the discrete mean values*

$$\sum_{r_j=0}^{q-1} \widetilde{\zeta}_2\Bigl(\boldsymbol{s};\frac{\boldsymbol{a}+r_j\boldsymbol{e}_j}{q}\Bigr) \quad (j = 1, 2). \tag{5.15}$$

Theorem 13. *Complete asymptotic expansions in the descending order of q as $q \to +\infty$ exist for the hybrid mean values*

$$\sum_{r_j=0}^{q-1} \int_0^1 \widetilde{\zeta}_2\Bigl(\boldsymbol{s};\frac{\boldsymbol{a}+x_i\boldsymbol{e}_i+r_j\boldsymbol{e}_j}{q}\Bigr)dx_i, \tag{5.16}$$

where $(i, j) = (1, 2)$ or $(2, 1)$.

Let $\boldsymbol{s} = (s_1, \ldots, s_d)$ be complex variables, and $\boldsymbol{a} = (a_1, \ldots, a_d)$ real parameters with $a_j > 0$ $(j = 1, \ldots, d)$. Then the multiple Hurwitz zeta-function $\widetilde{\zeta_d}(\boldsymbol{s};\boldsymbol{a})$ is defined by

$$\widetilde{\zeta}_d(\boldsymbol{s};\boldsymbol{a}) = \sum_{l_1,\ldots,l_d=0}^{\infty} \prod_{j=1}^{d}\Bigl\{\sum_{i=j}^{d}(a_i+l_i)\Bigr\}^{-s_j} = \sum_{l_1,\ldots,l_d=0}^{\infty} \prod_{j=1}^{d}(a_j+a_{j+1}+\cdots+a_d+l_j+l_{j+1}+\cdots+l_d)^{-s_j} \tag{5.17}$$

for $\sum_{j=1}^{d}\sigma_j > d$, and its meromorphic continuation to the whole $\boldsymbol{s}$-space $\mathbb{C}^d$. It is reasonable from the observation of the theorems above to pose the following Problem 5.

Problem 5. *Find all the 'asymptotic phenomena' as above for mixed mean values of $\widetilde{\zeta_d}(\boldsymbol{s};\boldsymbol{a}+\boldsymbol{x})$ appropriately averaged with respect to the (discrete or continuous) parameter $\boldsymbol{x} \in [0, 1]^d$.*

It is to be remarked that there are

$$\sum_{j=1}^{d}\binom{d}{j}3^j=4^d-1$$

possibilities of formulation for continuous and discrete mean values, together with their hybridization, for $\widetilde{\zeta}_d(\boldsymbol{s};\boldsymbol{a}+\boldsymbol{x})$ with respect to the parameter $\boldsymbol{x}\in[0,1]^d$.

Let $L(s,\chi)$ denote the Dirichlet L-functions attached to a Dirichlet character χ modulo $q(\geq 1)$. We note that the direction of research above proceeds from our previous study of the discrete and continuous mean squares $\sum_{\chi(\bmod q)}|L(s,\chi)|^2$, $\sum_{r=1}^{q}|\zeta(s,r/q)|^2$ and $\int_0^1|\zeta(s,1+x)|^2dx$, given in [9], [10], [11] and [12], also of the continuous and multiple mean squares $\int_0^1|\phi(s,1+x,\lambda)|^2dx$ and $\int_0^1\cdots\int_0^1|\phi(s,a+x_1+\cdots+x_m,\lambda)|^2dx_1\cdots dx_m$ $(m=1,2,\ldots)$ in [4] and [6], and further of the higher power moments $\sum_{r=1}^{q}|\zeta(s,r/q)|^{2k}$ $(k=2,3,\ldots)$ in [1].

6. Outline of the proofs of Theorems 5 and 6

We write $w=u+iv$ with real coordinates u and v. A key to prove Theorems 5 and 6 is the Mellin-Barnes type integral expression

$$\begin{aligned}\mathcal{LM}^{\boldsymbol{\alpha}}_{\boldsymbol{z};\boldsymbol{\tau}}(\phi^*)^{(m)}&(s+\langle\boldsymbol{\tau}\rangle,a,\lambda)\\&=\int_{(u)}\Gamma\binom{\langle\boldsymbol{\alpha}\rangle+w,-w}{\langle\boldsymbol{\alpha}\rangle}F_D^{(d-1)}\binom{-w,\boldsymbol{\alpha}_{d-1};\mathbf{1}-\dfrac{\boldsymbol{z}_{d-1}}{z_d}}{\gamma}\phi^*_{-w-m}(s,a,\lambda)z_d^w\,dw\end{aligned}\tag{6.1}$$

for any $m\in\mathbb{Z}$, and a constant u with $-\operatorname{Re}\langle\boldsymbol{\alpha}\rangle<u<0$. The vertical estimate

$$\phi^*_{-w-m}(s,a,\lambda)\ll(|v|+|t|+1)^{\max(0,\lfloor 2-\sigma\rfloor)}\tag{6.2}$$

can in fact be shown for any $(w,s)\in\mathbb{C}^2$ (cf. [8, Lemma 3]). It follows from (6.2) that the integral in (6.1) converges absolutely for all $s\in\mathbb{C}$ and $\boldsymbol{z}\in\mathbb{C}^d$ in the sector $|\arg z_j-\theta_0|<\pi/2$ $(j=1,\ldots,d)$ with any fixed $\theta_0\in[-\pi/2,\pi/2]$; this provides the analytic continuation of $\mathcal{LM}^{\boldsymbol{\alpha}}_{\boldsymbol{z};\boldsymbol{\tau}}(\phi^*)^{(m)}(s+\langle\boldsymbol{\tau}\rangle,a,\lambda)$ to the same region of $(s,\boldsymbol{z})$ above.

Theorems 5 and 6 are established respectively by moving the path (u) to the right upon yielding the complete asymptotic expansion as $z_d\to 0$ (Theorem 5), while to the left upon that as $z_d\to\infty$ (Theorem 6).

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