ON GAN-GROSS-PRASAD CONJECTURE FOR $(\mathrm{U}(2n), \mathrm{U}(1))$ AND $(\mathrm{SO}(5), \mathrm{SO}(2))$

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Abstract. In this paper, we announce the results given in a joint work with Masaaki Furusawa (Osaka City University) on Gan-Gross-Prasad conjecture for $(\mathrm{U}(2n), \mathrm{U}(1))$ and $(\mathrm{SO}(5), \mathrm{SO}(2))$.

1. $(\mathrm{U}(2n), \mathrm{U}(1))$-case

Let F be a number field. We denote its ring of adeles by $\mathbb{A} = \mathbb{A}_F$. Let ψ be a non-trivial character of $\mathbb{A}/F$. For a place v of F, we denote by F_v the completion of F at v. For an algebraic group K defined over F, we write its F_v-rational points $K(F_v)$ by K_v. Let E be a quadratic extension field of F and $\mathbb{A}_E$ be its ring of adeles. We write $E_v = E \otimes F_v$. Let σ denote the non-trivial element of $\mathrm{Gal}(E/F)$. Let us denote by $N_{E/F}$ the norm map from E to F. Let χ_E denote the quadratic character of $\mathbb{A}^\times/F^\times$ corresponding to E/F. We define a character ψ_E of $\mathbb{A}_E/E$ by $\psi_E(x) = \psi\left(\frac{x+\sigma(x)}{2}\right)$. *Throughout this paper, we fix a character Λ of $\mathbb{A}_E^\times/E^\times$ such that the restriction of Λ to $\mathbb{A}^\times$ is trivial.*

Let $(V, (\ ,\ )_V)$ be a $2n$-dimensional hermitian spaces over E with a non-degenerate hermitian pairing $(\ ,\ )_V$. Suppose that the Witt index of V is at least $n-1$. Then we have Witt decomposition $V = \mathbb{H}^{n-1} \oplus L$ where $\mathbb{H}$ is a hyperbolic plane over E and L is a 2-dimensional hermitian space over E. Then $\mathcal{G}_n$ is defined as the set of F-isomorphism classes of the unitary group $\mathrm{U}(V)$ for such V. By abuse of notation, we shall often identify $\mathrm{U}(V)$ with its isomorphism classes in $\mathcal{G}_n$. We may decompose V as a direct sum

$$V = X^+ \oplus L \oplus X^-$$

where $X^\pm$ are totally isotropic $(n-1)$-dimensional subspaces of V which are dual to each other and orthogonal to L. We take a basis $\{e_1, \dots, e_{n-1}\}$ of X^+ and a basis $\{e_{-1}, \dots, e_{-n+1}\}$ of X^-, respectively so that

$$(e_i, e_{-j})_V = \delta_{i,j} \tag{1.0.1}$$

for $1 \le i, j \le n-1$, where $\delta_{i,j}$ denotes Kronecker's delta.

Suppose $G = \mathrm{U}(V) \in \mathcal{G}_n$ and let us fix an anisotropic vector $e \in L$. Let P' be the maximal parabolic subgroup of G preserving the isotropic subspace X^-. Let M' and S' denote the Levi part and the unipotent part of P' respectively. We define a character χ_e of $S'(\mathbb{A})$ by

$$\chi_e \begin{pmatrix} 1_{n-1} & A & B \\ & 1_2 & A' \\ & & 1_{n-1} \end{pmatrix} = \psi_E((Ae, e_{n-1})).$$

Here, w_r denotes the $r\times r$ matrix with ones on the sinister diagonal, zero elsewhere.

Let U_{n-1} denote the group of upper unipotent matrices in $\mathrm{Res}_{E/F}\mathrm{GL}_{n-1}$. For $u\in U_{n-1}$, we define $\check{u}\in P'$ by

$$\check{u}=\begin{pmatrix} u & & \\ & 1_2 & \\ & & u^* \end{pmatrix}.$$

Then we define an unipotent subgroup S of P' by

$$S=S'S'' \quad \text{where} \quad S''=\{\check{u}: u\in U_{n-1}\}$$

and we extend χ_e to a character of $S(\mathbb{A})$ by putting

$$\chi_e(\check{u})=\psi(u_{1,2}+\cdots+u_{n-2,n-1}) \quad \text{for} \quad u\in U_{n-1}(\mathbb{A}).$$

We define a subgroup D_e of G by

$$D_e=\left\{\begin{pmatrix} 1_{n-1} & & \\ & h & \\ & & 1_{n-1}\end{pmatrix}: h\in \mathrm{U}(L),\, he=e\right\}$$

and let $R_e=D_eS$. Then the elements of $D_e(\mathbb{A})$ stabilize a character χ_e of $R_e(\mathbb{A})$ by conjugation. We note that

$$D_e(F)\simeq \mathrm{U}_1(F):=\{a\in E^\times : \bar{a}a=1\}.$$

We may regard Λ as a character of $D_e(\mathbb{A})$ by $d\mapsto \Lambda(\det d)$. Then we define a character $\chi_{e,\Lambda}$ of $R_e(\mathbb{A})$ by

$$\chi_{e,\Lambda}(ts)=\Lambda(t)\chi_e(s) \quad \text{for} \quad t\in D_e(\mathbb{A}), s\in S(\mathbb{A}).$$

For a cusp form φ on $G(\mathbb{A}_F)$, we define the (e,ψ,Λ)-Bessel period of φ by

$$B_{e,\psi,\Lambda}(\varphi)=\int_{D_e(F)\backslash D_e(\mathbb{A}_F)}\int_{S(F)\backslash S(\mathbb{A}_F)} \chi_{e,\Lambda}^{-1}(ts)\varphi(ts)\,ds\,dt.$$

We say that an irreducible cuspidal automorphic representation (π, V_π) of $\mathrm{U}(V)$ has (e,ψ,Λ)-Bessel period if $B_{e,\psi,\Lambda}(-)\not\equiv 0$ on V_π.

When φ is a cusp form on the similitude unitary group $\mathrm{GU}(V)$, we define (e,ψ,Λ)-Bessel period of φ by $B_{e,\psi,\Lambda}(\varphi):=B_{e,\psi,\Lambda}(\varphi|_{\mathrm{U}(V,\mathbb{A})})$. Our first result is a proof of Gan-Gross-Prasad conjecture in the case of $(\mathrm{U}(2n),\mathrm{U}(1))$ (see [4, Conjecture 24.1]).

Theorem 1.1. *Let π be an irreducible cuspidal tempered automorphic representation of $G(\mathbb{A})$ for $G\in\mathcal{G}_n$. Then the following two conditions are equivalent :*

(1) $L\left(\frac{1}{2},\pi\times\Lambda\right)\neq 0$.
(2) *There exists $G'\in\mathcal{G}_n$ and an irreducible cuspidal tempered automorphic representation π' of $G'(\mathbb{A})$ such that π' is nearly equivalent to π and π' has (e,ψ,Λ)-Bessel period.*

More precisely, the following two conditions are equivalent :

(1) $L\left(\frac{1}{2},\pi\times\Lambda\right)\neq 0$ *and* $\mathrm{Hom}_{R_{e,v}}(\pi_v,\chi_{e,\Lambda,v})\neq 0$
(2) π *has (e,ψ,Λ)-Bessel period.*

Remark 1.1. *The first equivalence was proved in Jiang-Zhang* [10, Theorem 6.10] *under the assumption that* [10, Conjecture 6.8] *holds for π.*

Let us explain our idea of proof of Theorem 1.1. Let (π, V_π) be as in Theorem 1.1. First, we note that the second equivalene follows from the first one and the uniqueness of the element in a L-packet which has (e, ψ_v, Λ_v)-Bessel period by Beuzart-Plessis [2, 3]. Hence, it suffices to show the first claim.

Let us define a quasi-split unitary group $\mathbb{G}_n^-$ by

$$\mathbb{G}_n^-(F) = \left\{ g \in \mathrm{GL}_{2n}(E) : {}^t\bar{g} J_n g = J_n \right\} \quad \text{where} \quad J_n = \begin{pmatrix} & 1_n \\ -1_n & \end{pmatrix}.$$

We shall consider the theta lift of π to $\mathbb{G}_n^-$. Let us recall that we can define the Weil representation $\omega_{\psi,\Lambda}$ of $\mathbb{G}_n \times \mathrm{U}(V)$ associated to $\chi_\Lambda = (\Lambda, \Lambda)$ and ψ. Then, we may define the theta function by

$$\theta^\phi_{\psi,\Lambda}(g,h) = \sum_{x \in V^n(F)} \omega_{\psi,\Lambda}(g,h)\phi(x)$$

for $\phi \in \mathcal{S}(V^n(\mathbb{A}))$. Then for $\phi \in \mathcal{S}(V^n(\mathbb{A}))$ and a cusp form f on $\mathrm{U}(V)(\mathbb{A})$, we can define the theta lift of f to $\mathbb{G}_n^-(\mathbb{A})$ by

$$\theta_{\psi,\Lambda}(f;\phi)(g) = \int_{U(V)(F)\backslash U(V)(\mathbb{A})} \theta^\phi_{\psi,\Lambda}(g,h)\overline{f(h)}\, dh.$$

For an irreducible cuspidal automorphic representation (π, V_π) of $G(\mathbb{A})$ with $G = \mathrm{U}(V) \in \mathcal{G}_m$, we define

$$\theta(\pi,\psi,\Lambda) = \langle \theta_{\psi,\Lambda}(f;\phi) : f \in V_\pi,\ \phi \in \mathcal{S}(V^n(\mathbb{A}))\rangle.$$

Then we note that when $\theta(\pi,\psi,\Lambda)$ is cuspidal, we have

$$\theta(\pi,\psi,\Lambda) \simeq \otimes_v \theta(\pi_v,\psi_v,\Lambda_v)$$

because of the Howe duality by Gan-Takeda [6] at finite places and Howe [9] at archimedean places. Let N_n be the group of upper unipotent matrices in $\mathbb{G}_n^-$. Then it is a unipotent radical of Borel subgroup of $\mathbb{G}_n^-$. For $\lambda \in F^\times$, let $\psi_{N_n,\lambda}$ be a non-degenerate character of $N_n(\mathbb{A})$ defined by

$$\psi_{N_n,\lambda}(u) = \psi_E\left(-\sum_{i=1}^{n-1} u_{i,i+1} + \frac{\lambda}{2} u_{n,2n} \right).$$

For an automorphic form f on $\mathbb{G}_n^-(\mathbb{A})$, we define $\psi_{N_n,\lambda}$-Whittaker period of f by

$$W_{\psi,\lambda}(f;g) = \int_{N_n(F)\backslash N_n(\mathbb{A})} f(u)\psi^{-1}_{N_n,\lambda}(u)\, du.$$

The following proposition is a key ingredient of our proof.

Proposition 1.1. *Let (π, V_π) be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$ with $G = \mathrm{U}(V) \in \mathcal{G}_m$.*

(1) *Suppose $m < n$. Then $W_{\psi,\lambda} \equiv 0$ on $\theta(\pi,\psi,\Lambda)$*
(2) *Suppose that $m = n$. Then for $\phi \in \mathcal{S}(V^n(\mathbb{A}))$ and $\varphi \in \pi$, we have*

$$W_{\psi,-(e,e)}(\theta^\phi_{\psi,\Lambda}(\varphi);1) = \int_{R'_e(\mathbb{A})\backslash G(\mathbb{A})} \omega_{\psi,\Lambda}(g,1)\phi(e_{-1},\ldots,e_{-n+1},e)\overline{B_{e,\psi,\Lambda}(\pi(g)\varphi)}\, dg$$

where

$$R'_e = \{ g \in G : ge_{-1} = e_{-1}, \ldots, ge_{-n+1} = e_{-n+1}, ge = e\}.$$

In this case, π has (e,ψ,Λ)-Bessel period if and if $\Theta_{V,\mathbb{W}_n}(\pi,\psi,\Lambda)$ is $\psi_{N_n,-(e,e)}$-generic.

Let us explain our idea of proof of of the first claim in Theorem 1.1. Suppose that π has (e,ψ,Λ)-Bessel period. By Proposition 1.1 (2), the theta lift $\theta_{\psi,\Lambda}(\pi)$ is not zero. Further, by Proposition 1.1 (1), we see that it should be cuspidal, and thus it is irreducible because of the Howe duality at each place. Then Yamana [16] shows that

$$L\left(\frac{1}{2},\pi\times\Lambda\right)\neq 0.$$

On the other hand, suppose that we have $L\left(\frac{1}{2},\pi\times\Lambda\right)\neq 0$. Let (π_g,V_{π_g}) be the irreducible cuspidal $\psi_{(e,e)}$-globally generic automorphic representation of $\mathbb{G}_n^-(\mathbb{A})$ which is nearly equivalent to π. The existence of π_g follows from the endoscopic classification [11, 13] and the descent method [7].

Let V_v' be a hermitian space over E_v such that

$$\varepsilon\left(\frac{1}{2},\pi_v^\circ\times\Lambda_v,\psi_v^{-1}\right)=\omega_{\pi_v^\circ}(-1)\varepsilon(V_v')\varepsilon(\mathbb{W}_v) \tag{1.0.2}$$

where $\varepsilon(V_v')$ and $\varepsilon(\mathbb{W}_v)$ denote the Hasse invariant of V_v' and $\mathbb{W}$, respectively. Then by the epsilon dichotomy, we see that the local theta lift $\theta(\pi_v^\circ,\psi_v^{-1},\chi_{\Lambda_v}^\square)$ of π_v° to $\mathrm{U}(V_v')$ is not zero. Here, we note that the epsilon dichotomy is proved by Harris-Kudla-Sweet [8, Theorem 6.1] and Gan-Ichino [5, Theorem 11.1] when v is finite and inert, and we can show it using Paul [15] when v is real and inert and using the functional equation of local L-factors when v is split. From our assumption $L\left(\frac{1}{2},\pi\times\Lambda\right)\neq 0$, we see that

$$\prod_v\varepsilon\left(\frac{1}{2},\pi_v^\circ\times\Lambda_v,\psi_v^{-1}\right)=1,$$

and thus $\prod_v\varepsilon(V_v')=1$. This means that there is a hermitian space V' over E such that $V'(F_v)\simeq V_v'$. On the other hand, by Yamana [16], the theta lift $\theta(\pi_g,\psi^{-1},\Lambda^{-1})$ of π_g to $\mathrm{U}(V')$ is non-zero. Then by Proposition 1.1, $\theta(\pi_g,\psi^{-1},\Lambda^{-1})$ has (e,ψ,Λ)-Bessel period, and it is nearly equivalent to π because of local computation of theta lifts by Kudla [12]. Hence, we may take $\pi'=\theta(\pi_g,\psi^{-1},\Lambda^{-1})$.

2. $(\mathrm{SO}(5),\mathrm{SO}(2))$-case

Let D be a quaternion algebra over F containing E, which is possibly split. Let us denote its reduced norm by N_D and reduced trace by Tr_D. We denote the canonical involution of D by $x\mapsto\bar{x}$ for $x\in D$. Then we define a similitude quaternionic unitary group H_D over F by

$$H_D(F):=\left\{g\in\mathrm{GL}_2(D):{}^t\overline{g}\begin{pmatrix}&1\\1&\end{pmatrix}g=\lambda_D(g)\begin{pmatrix}&1\\1&\end{pmatrix},\ \lambda_D(g)\in F^\times\right\}.$$

Here, $\overline{g}=\left(\begin{smallmatrix}\bar{x}&\bar{y}\\\bar{z}&\bar{w}\end{smallmatrix}\right)$ for $g=\left(\begin{smallmatrix}x&y\\z&w\end{smallmatrix}\right)\in\mathrm{GL}_2(D)$. When $D(F)\simeq\mathrm{Mat}_{2\times 2}(F)$, we know that H_D is isomorphic to a similitude symplectic group H of degree 2 defined by

$$H(F)=\left\{g\in\mathrm{GL}_4(F):{}^tgJg=\lambda(g)J,\ \lambda(g)\in F^\times\right\}$$

where

$$J=\begin{pmatrix}&1_2\\-1_2&\end{pmatrix}.$$

Also, we define their isometry groups by

$$H_D^1 = \{h \in H_D : \lambda_D(h) = 1\} \quad \text{and} \quad H^1 = \{h \in H : \lambda(h) = 1\}.$$

Moreover, we define

$$H_D(\mathbb{A})^+ = \{h \in H(\mathbb{A}) : \nu(h) \in N_{E/F}(\mathbb{A}_E^\times)\}$$

and

$$H_D(F)^+ = \{h \in H(F) : \nu(h) \in N_{E/F}(E)\}.$$

We note that for any orthogonal space V over F such that $\dim V = 5$ and its Witt index is at least 1, there is a quaternion algebra D over F such that

$$PH_D \simeq \mathrm{SO}(V).$$

Let P_D be Siegel parabolic subgroup of H_D with the Levi decomposition $P_D = M_{H_D} N_{H_D}$ where

$$M_{H_D}(F) = \left\{ \begin{pmatrix} g & \\ & \lambda \cdot \bar{g}^{-1} \end{pmatrix} : g \in D^\times, \lambda \in F^\times \right\}$$

and

$$N_{H_D}(F) = \left\{ \begin{pmatrix} 1 & a \\ & 1 \end{pmatrix} : a \in D^1(F), \right\}.$$

Here, we define $D^1 = \{a \in D : \mathrm{Tr}_D(a) = 0\}$. For $S_D \in D^1(F)$, let us define a character $\psi_{S_D,D}$ of $N_{H_D}(\mathbb{A})$ by

$$\psi_{S_D,D}\left(\begin{pmatrix} 1 & a \\ & 1 \end{pmatrix} \right) = \psi(\mathrm{Tr}_D(S_D a)).$$

Then the identity component of the stabilizer T_{D,S_D} of $\psi_{S_D,D}$ in M_{H_D} is

$$\left\{ \begin{pmatrix} h & \\ & N_D(h) \cdot \bar{h}^{-1} \end{pmatrix} : h \in T_{S_D} \right\}.$$

where

$$T_{S_D} = \left\{ t \in D^\times : \bar{t} S_D t = N_D(t) S_D \right\}.$$

We write this torus of H_D by the same symbol T_{S_D}. Let us take $S_D \in D^1(F)$ such that $T_{S_D}(F) \simeq E^\times$. Then we may regard Λ as a character of $T_{S_D}(\mathbb{A})/T_{S_D}(F)$. Let π_D be an irreducible cuspidal automorphic representation of H_D with trivial central character and we denote its space by V_{π_D}. Then we define (S_D, ψ, Λ)-Bessel period of $f \in V_{\pi_D}$ by

$$B_{S_D,\psi,\Lambda}(f) = \int_{\mathbb{A}^\times T_{D,S_D}(F) \backslash T_{D,S_D}(\mathbb{A})} \int_{N_{H_D}(F) \backslash N_{H_D}(\mathbb{A})} f(tu) \Lambda^{-1}(t) \psi_{S_D,D}^{-1}(u) \, dt \, du.$$

We say that π_D has (S_D, ψ, Λ)-Bessel period when $B_{S_D,\psi,\Lambda}(-) \not\equiv 0$ on V_{π_D}. Our second result studies Gan-Gross-Prasad conjecture in the case of $(\mathrm{SO}(5), \mathrm{SO}(2))$.

Theorem 2.1. *Let Λ be as above and we denote by $\theta(\Lambda)$ the automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ associated to Λ. Let (π, V_π) be an irreducible cuspidal tempered automorphic representation of $H_D(\mathbb{A}_F)$ with trivial central character.*

(1) *If π has (S_D, ψ, Λ)-Bessel period, then we have*

$$L\left(\frac{1}{2}, \pi \times \theta(\Lambda)\right) \neq 0.$$

(2) *Assume that* [1, Conjecture 9.4.2, Conjecture 9.5.4] *holds for* PH_D*. Then there exist a quaternion algebra* D' *over* F *and an irreducible cuspidal automorphic representation* π' *of* $H_{D'}(\mathbb{A})$ *such that it is nearly equivalent to* π *and* π' *has* (S_D, ψ, Λ)*-Bessel period if and only if*

$$L\left(\frac{1}{2}, \pi \times \theta(\Lambda)\right) \neq 0. \tag{2.0.1}$$

In this case, D' *and* π' *are unique.*

Remark 2.1. *This theorem is proved in* [10] *using different method except for the uniqueness.*

As in the unitary group case, we consider pull-back of periods for theta lifts. Let us take $\eta \in E^\times$ such that $\eta^\sigma = -\eta$. Then we define a similitude quaternionic unitary group $\mathrm{GU}_{3,D}^\eta = \mathrm{GU}_{3,D}$ of degree 3 over F by

$$\mathrm{GU}_{3,D}(F) = \left\{g \in \mathrm{GL}_3(D) : {}^t\bar{g}J_\eta g = \mu_D(g)J_\eta,\ \mu_D(g) \in F^\times\right\}$$

where

$$J_\eta = \begin{pmatrix} & & \eta \\ & \eta & \\ \eta & & \end{pmatrix}.$$

We denote the identity component of $\mathrm{GU}_{3,D}$ by $\mathrm{GSU}_{3,D}$.

Let Q_{G_D} be a maximal parabolic subgroup with the Levi decomposition $Q_{G_D} = M_{G_D}U_{G_D}$ where

$$M_{G_D} = \left\{\begin{pmatrix} g & & \\ & h & \\ & & \bar{g}^{-1} \end{pmatrix} : h \in T_\eta\right\}, \quad U_{G_D} = \left\{\begin{pmatrix} 1 & A' & B \\ & 1 & A \\ & & 1 \end{pmatrix} \in \mathrm{GSU}_{3,D}\right\}.$$

Here, we write $T_\eta = \{h \in D^\times : \bar{h}\eta h = N_D(h)\eta\}$. We note that $T_\eta(F) \simeq E^\times$. For $X \in D^\times$, we define a character $\psi_{X,D}$ of $U_{G,D}(\mathbb{A})$ by

$$\psi_{X,D}\left(\begin{pmatrix} 1 & A' & B \\ & 1 & A \\ & & 1 \end{pmatrix}\right) = \psi\left(\mathrm{Tr}_D(XA)\right).$$

Then the identity component of the stabilizer of $\psi_{X,D}$ in M_{G_D} is

$$M_X = \left\{\begin{pmatrix} N_D(h) \cdot \overline{(h^X)}^{-1} & & \\ & h & \\ & & h^X \end{pmatrix} : h \in T_\eta\right\}$$

where $h^X = XhX^{-1}$. For a character χ of $\mathbb{A}_E^\times$, we regard χ as a character of $T_\eta(\mathbb{A})$, and we also define a character χ_D of $M_X(\mathbb{A})$ by

$$\chi_D\left(\begin{pmatrix} N_D(h) \cdot (h^X)^* & & \\ & h & \\ & & h^X \end{pmatrix}\right) = \chi(h).$$

Then we define (X, χ)-Bessel period of a cusp form φ on $\mathrm{PGSU}_{3,D}(\mathbb{A})$ by

$$\mathcal{B}_{X,\chi,D}(\varphi) = \int_{U_{G_D}(F)\backslash U_{G_D}(\mathbb{A})} \int_{\mathbb{A}^\times M_X(F)\backslash M_X(\mathbb{A})} \varphi(hu)\chi_D^{-1}(h)\psi_{X,D}(u)^{-1}\, dh\, du.$$

We note that

$$\mathrm{PGSU}_{3,D} \simeq \mathrm{PGU}(2,2) \simeq \mathrm{PGSO}_{4,2} \quad \text{or} \quad \mathrm{PGSU}_{3,D} \simeq \mathrm{PGU}(3,1)$$

according to D is split or not. Then for a cusp form φ on $\mathrm{PGSU}_{3,D}(\mathbb{A})$, we can define (e,ψ,Λ)-Bessel period of φ.

Let us denote the standard basis of $Y=D^3$ by

$$y_-={}^t(1,0,0),\quad e={}^t(0,1,0),\quad y_+={}^t(0,0,1).$$

Then we have $y_+J_\eta{}^ty_-=\eta$ and $eJ_\eta{}^te=\eta$. We define $Y_{D,\pm}=\langle y_\pm\rangle$ and $Y_{D,0}=\langle e\rangle$. Let us denote the standard basis of $X_D=D^2$ by

$$x_+=(1,0),\quad x_-=(0,1)$$

We denote $X_{D,\pm}=\langle x_\pm\rangle$. Now, we regard $Z_D=X_D\otimes Y_D$ as a symplectic space over F with the symplectic form defiend by

$$(x_1\otimes y_1,x_2\otimes y_2)_D=\mathrm{Tr}_D\left(x_1\begin{pmatrix}0&1\\1&0\end{pmatrix}{}^t\overline{x_2}\cdot\overline{{}^ty_1J_\eta\overline{y_2}}\right).$$

As a polarization of $Z_D=Z_{D,+}+Z_{D,-}$ of symplectic space Z_D, we may take

$$Z_{D,\pm}=X_D\otimes Y_{D,\pm}+X_{D,\pm}\otimes Y_{D,0}.$$

As in the unitary group case, we may define the theta function $\theta_\psi^\phi(g,h)$ on $\mathrm{GSU}_{3,D}(\mathbb{A})\times H_D(\mathbb{A})^+$ for $\phi\in\mathcal{S}(Z_{D,+}(\mathbb{A}))$. Then we define a theta lift of a cusp form f on $H_D(\mathbb{A})^+$ to $\mathrm{GSU}_{3,D}(\mathbb{A})$ by

$$\theta_\psi(f;\phi)=\int_{H_D^1(F)\backslash H_D^1(\mathbb{A})}\theta_\psi(g,h)\overline{f(h)}\,dh.$$

Also, for an irreducible cuspidal automorphic representation (π_0,V_{π_0}) of $H_D(\mathbb{A})^+$, we define

$$\theta(\pi_0,\psi)=\langle\theta_\psi(f;\phi):f\in V_{\pi_0},\phi\in\mathcal{S}(Z_{D,+}(\mathbb{A}))\rangle.$$

Then we note that

$$\theta(\pi,\psi)\simeq\otimes\theta(\pi_v,\psi_v)$$

with the local theta lift $\theta(\pi_v,\psi_v)$ of π_v to $\mathrm{GSU}_{3,D}(F_v)$. The following observation is a key ingredient to connect $(\mathrm{SO}(5),\mathrm{SO}(2))$-case and $(\mathrm{U}(4),\mathrm{U}(1))$-case.

Proposition 2.1. *For a cusp form f on $H_D(\mathbb{A})^+$ and $\phi\in\mathcal{S}(Z_{D,+}(\mathbb{A}))$, we have*

$$\mathcal{B}_{X,\chi,D}(\theta(f:\phi))=\int_{N_{H_D}(\mathbb{A})\backslash H_D^1(\mathbb{A})}B_{D,S_{D,X},\chi^{-1}}(\pi(g)f)\omega(g,1)\phi(v_{D,X})\,dg$$

where we define $v_{D,X}=(x_-,-\eta^{-1}Xx_+)$ and $S_{D,X}=X\eta\overline{X}$.

Let us explain our idea of proof of Theorem 2.1. Suppose that π has (S_D,ψ,Λ)-Bessel period. Then by the uniqueness of Bessel model, there is unique irreducible constituent π^B of $\pi|_{H_D(\mathbb{A})^+}$ which has (S_D,ψ,Λ)-Bessel period. Let us take X such that $S_{D,X}=S_D$. Then by Proposition 2.1, the theta lift of π to $\mathrm{GSU}_{3,D}(\mathbb{A})$ is non-zero, and it has (X,Λ)-Bessel period. Then we see that

$$L\left(1/2,\theta_\psi(\pi^B)\times\Lambda\right)\neq 0\tag{2.0.2}$$

by Theorem 1.1. Now, we note that

$$L(s,\pi_v\times\theta(\Lambda)_v)=L\left(s,\theta_\psi(\pi^B)_v\times\Lambda_v\right).$$

at almost all finite places v, and thus the claim (1) follows from (2.0.2).

Suppose that $L\left(1/2,\pi\times\theta(\Lambda)\right)\neq 0$. Let (π_g,V_{π_g}) be an irreducible cuspidal globally generic automorphic representation of $H(\mathbb{A})$ which is nearly equivalent to π. Then by [14], the theta lift of π_g to $\mathrm{PGSO}_{4,2}\simeq\mathrm{PGU}(2,2)$ is non-zero. We denote this automorphic representation of GU(2, 2) by (Σ, V_Σ), and applying Theorem 1.1 to Σ, we get an irreducible cuspidal automorphic representation Σ' of $\mathrm{PGU}(V)$ such that $\mathrm{U}(V) \in \mathcal{G}_4$, Σ' is nearly equivalent to Σ and Σ' has (e, ψ, Λ)-Bessel period. Let us take a quaternion algebra D' such that $\mathrm{PGU}(V) \simeq \mathrm{PGSU}_{3,D'}$. Then by Proposition 2.1, we see that the theta lift of Σ' to $H_{D'}$ satisfies the required condition.

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