EPSILON DICHOTOMY FOR LINEAR MODELS

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This is an expanded version of my talk at the RIMS conference "Analytic, geometric and p-adic aspects of automorphic forms and L-functions". It surveys some recent work towards a conjecture of Prasad and Takloo-Bighash on epsilon dichotomy for linear models. I thank the organizers of the conference for their hospitality. I am also grateful to Miyu Suzuki for her help during the preparation of this note. This work is partially supported by the NSF grant DMS #1901862.

1. The theorem

Let E/F be a quadratic extension of local nonarchimedean fields of characteristic zero and $\eta : F^\times/NE^\times \to \{\pm 1\}$ the quadratic character associated to this extension. We fix a nontrivial additive character $\psi : F \to \mathbb{C}^\times$. Let A be a central simple algebra (CSA) over F of dimension $4n^2$ with a fixed embedding $E \to A$ and let B be the centralizer of E in A. Then B is a CSA over E of dimension n^2. Let $G = A^\times$ and $H = B^\times$, both viewed as algebraic groups over F. Let π be an irreducible admissible representation of G. We say that π is H-distinguished if

$$\mathrm{Hom}_H(\pi, \mathbb{C}) \neq 0.$$

This space is at most one dimensional [BM]. Let $G' = \mathrm{GL}_{2n}(F)$ and π' be the Jacquet–Langlands transfer of π to G'. Let $\epsilon(\pi') = \pm 1$ be the local root number. The following theorem is the combination of [Séc, SX, Xue].

Theorem 1.1. *Let the notation be as above. If π is H-distinguished then the following two conditions hold:*

(1) *the Langlands parameter of π' takes values in the $\mathrm{Sp}_{2n}(\mathbb{C})$;*
(2) $\epsilon(\pi')\epsilon(\pi' \otimes \eta)\eta(-1)^n = (-1)^r$*, where r is the split rank of G.*

Conversely, if π' is a square integrable representation that satisfies conditions (1) and (2) above, and assume that either (a) $G = \mathrm{GL}_n(D)$ where D is a quaternion algebra over F (possibly split) or (b) the residue characteristic of F is odd, then π is H-distinguished.

2. The proof

The proof consists of two steps. The first is to handle the case π being supercuspidal. This is achieved independently in [Séc] and [Xue]. The second is to reduce the square integrable case to

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the supercuspidal case. This is achieved in [SX]. The method used in [Xue] has the potential of handling square integrable representations directly. We will propose a conjecture in this direction.

There are two approaches in the first step. The approach of [Xue] is via the relative trace formula proposed by Guo [Guo96] and the approach of [Séc] is via the theory of types. They each have their strength and weakness. The relative formula approach so far handles only the case where π' is supercuspidal or $G = \mathrm{GL}_n(D)$ where D is a quaternion algebra. The type theory approach does not work well if the residue characteristic of F is even. Let me briefly comment on the relative trace formula approach. In essence, the relative trace formula compares spectral and geometric information on $H\backslash G/H$ and $H'\backslash G'/H'$. There is a notion of matching of elliptic regular semisimple elements in $H\backslash G/H$ and $H'\backslash G'/H'$ and also test functions on G and on G'. The key observation of [Xue] is that there is an involution on $C_c^\infty(G')$, given by conjugation by the longest Weyl group element of G'. If we apply this involution to the spectral data, we obtain the local root numbers from the local functional equation. If we apply this involution to the geometric data, we obtain the factor $(-1)^r$. Thus local root numbers are connected to $(-1)^r$ via the relative trace formula.

At some point in this approach, we need to consider the matching of test functions on G' which are matrix coefficients of π' after integration along the center. This is the reason why we need to handle the supercuspidal case first because only matrix coefficients of supercuspidal representations are compactly supported modulo the center. However, I expect the following conjecture.

Conjecture 2.1. *If π' is an irreducible square integrable representation of G' satisfying conditions (1) and (2) in the Theorem. Assume that either r is odd or π is supercuspidal. Then there exists a (relative) pseudo-coefficient for π'.*

It is also believed that Howe's finiteness conjecture holds for symmetric spaces. The analogous result for Lie algebras was established in [RR96]. Once we have Conjecture 2.1 and Howe's finiteness conjecture, we should be able to prove Theorem 1.1 directly in full generality (without assuming (a) or (b)).

The reason to believe this conjecture is that from the view point of spectral data, a matrix coefficient of π' should match a matrix coefficient of π. If π is supercuspidal, then its matrix coefficient should be compactly supported modulo center and its transfer back to G' should be the pseudo-coefficient that we are looking for. A key step in the proof of this conjecture should be to show that the set of elliptic regular semisimple orbits on which the matrix coefficient has nonzero orbital integral is compact.

The second step is of a completely different nature. It is well-known that irreducible square integrable representations are given by segments. More precisely every irreducible square integrable representation of G can be written as the unique irreducible quotient of

$$\mathrm{Ind}_P^G\left(\rho\nu^{(1-t)l_\rho/2} \boxtimes \rho\nu^{(3-k)l_\rho/2} \boxtimes \cdots \boxtimes \rho\nu^{(t-1)l_\rho/2}\right),$$

where

- P is the upper triangular parabolic subgroup of G whose Levi subgroup is isomorphic to t copies of $\mathrm{GL}_k(D)$;
- ρ is an irreducible supercuspidal representation of $\mathrm{GL}_k(D)$;
- ν is the reduced norm of $\mathrm{GL}_k(D)$ and l_ρ is a certain integer.

Given information on π it is relatively straightforward to deduce information for ρ. The converse implication is the nontrivial part of the second step. Essentially we need to prove that if the full induced representation as above is H-distinguished, then so is π. It is known from [BM] that the space of H-invariant linear forms on the full induced representation is one dimensional. Therefore we need to prove that any H-invariant linear form on the full induced representation factors through π. By some Jacquet module computation this is reduced to the case $t = 2$. Assume that $t = 2$. We observe the following.

- There is an explicit H-invariant linear form $J(\cdot, s)$ on $\mathrm{Ind}_P^G(\rho\nu^s \boxtimes \rho\nu^{-s})$, called the open intertwining period. It is nonzero at $s = -l_\rho/2$.
- The kernel of the projection $\mathrm{Ind}_P^G(\rho\nu^s \boxtimes \rho\nu^{-s}) \to \pi$ is the image of the standard intertwining operator $M(s)$ at $s = l_\rho/2$.

It follows that we just need to prove that $J(M(s)\cdot, -s)$ has a zero at $s = l_\rho/2$. The technical heart of [SX] is to prove a functional equation

$$J(M(s)\cdot, -s) = \alpha(s)J(\cdot, s) \tag{2.1}$$

where the zero and poles of $\alpha(s)$ are the same as

$$\gamma(-2s, \mathrm{JL}(\rho)^\vee, \wedge^2, \psi)^{-1}\gamma(2s, \mathrm{JL}(\rho), \mathrm{Sym}^2, \psi)^{-1}.$$

Here JL stands for the Jacquet–Langlands transfer. The desired fact that $J(M(s)\cdot, -s)$ has a zero at $s = l_\rho/2$ follows immediately.

The computation of $\alpha(s)$ is neverthelss quite technical. Our proof is inspired by [Mat] and uses global-to-local arguments. We are not sure if one can prove this proposition using purely local methods because of the appearance of Jacquet–Langlands transfer. The proof uses the global counterpart of intertwining periods and the global functional equation analogous to (2.1). Global intertwining periods appear as regularized periods of Eisenstein series and the functional equation of intertwining periods is naturally a consequence of that of Eisenstein series.

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