RESTRICTION OF EISENSTEIN SERIES AND STARK-HEEGNER POINTS

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This short note is written based on my talk on a joint work with Shunsuke Yamana [HY20] in the RIMS conference "Analytic, geometric and p-adic aspects of automorphic forms and L-functions" during January 20-24, 2020. [1] The author thanks the organizers for their hospitality during the conference.

1. The work of Darmon, Pozzi and Vonk

Let F be a real quadratic field and let $\mathfrak{d}$ be the different of $F/\mathbf{Q}$. Let $x \mapsto \overline{x}$ denote the non-trivial automorphism of F and let $\mathrm{N} : F \to \mathbf{Q}$, $\mathrm{N}(x) = x\overline{x}$ be the norm map. Let Δ_F be the discriminant of $F/\mathbf{Q}$. Let $\mathrm{Cl}^+(\mathcal{O}_F)$ be the narrow ideal class group of F. Let $\phi : \mathrm{Cl}^+(\mathcal{O}_F) \to \overline{\mathbf{Q}}^\times$ be an odd narrow ideal class character, i.e. $\phi((\delta)) = -1$ for any $\delta \in \mathcal{O}_F$ with $\overline{\delta} = -\delta$. Let $L(s, \phi)$ be the Hecke L-function attached to ϕ. Fix an odd rational prime p unramified in F. Fix an embedding $\iota_p : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$ throughout.

To each odd character ϕ of the narrow ideal class group of a real quadratic field F, we associate a one-variable p-adic family $E_k^{(p)}(1, \phi)$ of Hilbert Eisenstein series on $\Gamma_0(p)$ over a real quadratic field F defined by the q-expansion:

$$E_k^{(p)}(1, \phi)(z_1, z_2) = L^{(p)}(1 - k, \phi) + \sum_{\beta \in \mathfrak{d}_F^{-1}, \beta \gg 0} \sigma_{k-1,\phi}^{(p)}(\beta \mathfrak{d}_F) q^\beta,$$

$$q^\beta = \exp(2\pi\sqrt{-1}(\beta z_1 + \overline{\beta} z_2)),$$

where $L^{(p)}(1 - k, \phi) = (1 - \phi(p)p^{k-1})L(1 - k, \phi)$ and

$$\sigma_{k-1,\phi}^{(p)}(\mathfrak{a}) = \sum_{\mathfrak{b}|\mathfrak{a},\,(\mathfrak{b},p)=1} \mathrm{N}\mathfrak{b}^{k-1}\phi(\mathfrak{b}).$$

Consider the elliptic modular form $G_{2k}(\phi)$ of weight $2k$ defined by

$$G_{2k}(\phi) := e_{\mathrm{ord}} \left(E_k^{(p)}(1, \phi)(z, z) \right),$$

where e_{ord} is Hida's p-ordinary projector. Then one can interpolate the function $k \in \mathbf{Z}_{\geq 2} \mapsto G_{2k}(\phi)$ into a locally analytic functions on $\mathbf{Z}_p$ valued in the space of p-adic elliptic modular forms. Suppose that

$$p \text{ is inert in } F.$$

[1] The paper [HY20] was finished later after the conference. The text of this note has some overlapping with the content in [HY20].

It can be verified that

$$e_{\mathrm{ord}} G_k(\phi)|_{k=1} = 0.$$

In [DPV19], the authors investigate the spectral decomposition of the derivative

$$\frac{d}{dk}\left(e_{\mathrm{ord}} G_k(\phi)\right)|_{k=1} = \sum_f \lambda'_f \cdot f. \in M_2(\Gamma_0(p)),$$

where f runs over all the normalized Hecke eigenforms in $M_2(\Gamma_0(p))$. The main results of [DPV19] show that if f is an Eisenstein series, then λ'_f is essentially the p-adic logarithm of elliptic units over F in [DD06], while if f is a cusp form, then λ'_f can be expressed in terms of the product of special values of the L-function for f and the p -adic logarithms of Stark-Heegner points or introduced in [Dar01].

Stark-Heegner points are local points defined by theory of p-adic double integration on the product of p-adic upper half planes and conjectured to be rational over Hilbert class fields of F. The work [DPV19] may shed some light on the global nature of Stark-Heegner points in the future provided one has some K-theoretic construction of Hilbert Eisenstein series.

2. Statement of the main result

Now we introduce our recent work on partial generalizations of [DPV19] to the two-variable setting by introducing the cyclotomic variable. For $x \in \mathbf{Z}_p^\times$, let $\boldsymbol{\omega}(x)$ be the Teichmüller lift of $x \pmod p$ and let $\langle x \rangle := x\boldsymbol{\omega}^{-1}(x) \in 1+p\mathbf{Z}_p$. Let $\mathscr{X} := \{x \in \mathbf{C}_p \mid |x|_p \le 1\}$ be the p-adic closed unit disk and let $A(\mathscr{X})$ be the ring of rigid analytic functions on $\mathscr{X}$. For each ideal $\mathfrak{m} \lhd \mathcal{O}_F$ corpime to p, define $\boldsymbol{\sigma}_\phi(\mathfrak{m}) \in A(\mathscr{X} \times \mathscr{X})$ by

$$\boldsymbol{\sigma}_\phi(\mathfrak{m})(k,s) = \sum_{\mathfrak{a} \lhd \mathcal{O}_F,\, \mathfrak{a}|\mathfrak{m}} \phi(\mathfrak{a}) \left\langle \mathrm{N}(\mathfrak{a}) \right\rangle^{\frac{k-s}{2}} \left\langle \mathrm{N}(\mathfrak{m}\mathfrak{a}^{-1}) \right\rangle^{\frac{s-2}{2}}.$$

Let $\mathscr{X}^{\mathrm{cl}} := \left\{k \in \mathbf{Z}^{\ge 2} \mid k \equiv 2 \pmod{2(p-1)}\right\}$ be the set of classical points in $\mathscr{X}$. Let $h = \#\mathrm{Cl}^+(\mathcal{O}_F)$. Fix a set $\{\mathfrak{t}_\lambda\}_{\lambda=1,\dots,h}$ of representatives of the narrow ideal class group $\mathrm{Cl}^+(\mathcal{O}_F)$ with $(\mathfrak{t}_\lambda, p\mathcal{O}_F) = 1$. For each classical point $k \in \mathscr{X}^{\mathrm{cl}}$, the classical Hilbert-Eisenstein series $E_{\frac{k}{2}}(1,\phi)$ on $\mathrm{SL}_2(\mathcal{O}_F)$ of parallel weight $\frac{k}{2}$ is determined by the normalized Fourier coefficients

$$c(\mathfrak{m}, E_{\frac{k}{2}}(1,\phi)) = \boldsymbol{\sigma}_\phi(\mathfrak{m})(k,s), \quad c_\lambda(0, E_{\frac{k}{2}}(1,\phi)) = 4^{-1}L(1-k/2,\phi).$$

Let I_F be the set of integral ideals of F. Let $\mathfrak{n} \in I_F$ and p be coprime. Let $\mathcal{M}^{(2)}(\mathfrak{n})$ be the space of two-variable p-adic families of Hilbert *semi-cusp* forms[2] of tame level $\mathfrak{n}$, which consists of functions

$$f\colon I_F \to \mathscr{A}(\mathscr{X} \times \mathscr{X}), \quad \mathfrak{m} \mapsto c(\mathfrak{m}, f)$$

[2]Recall that a Hilbert semi-cusp form is a Hilbert modular form having no constant in the Fourier expansion around the cusps at the infinity.

such that the specialization $f(k,s) = \{c(\mathfrak{m}, f)(k,s)\}$ is the set of normalized Fourier coefficients of a p-adic Hilbert semi-cusp forms of parallel weight k on $\Gamma_0(p\mathfrak{n})$ for (k,s) in a p-adically dense subset $U \subset \mathbf{Z}_p \times \mathbf{Z}_p$. Define $\widehat{E}_\phi^{\{p\}} : I_F \to A(\mathscr{X} \times \mathscr{X})$ by the data

$$c(\mathfrak{m}, \widehat{E}_\phi^{\{p\}}) = \boldsymbol{\sigma}_\phi(\mathfrak{m}) \text{ if } (\mathfrak{m}, p\mathcal{O}_F) = 1,$$
$$c(\mathfrak{m}, \widehat{E}_\phi^{\{p\}}) = 0 \text{ otherwise.}$$

By definition, for $(k,s) \in \mathscr{X}^{\mathrm{cl}} \times \mathscr{X}^{\mathrm{cl}}$ with $k \geq 2s$, we have

$$\widehat{E}_\phi^{\{p\}}(k,s) = \langle \Delta_F \rangle^{\frac{s-2}{2}} \cdot \theta^{\frac{s-2}{2}} E^{\{p\}}_{\frac{k+4-2s}{2}}(1,\phi),$$

where $E_k^{\{p\}}(1,\phi)$ is the *p-depletion* of $E_k(1,\phi)$ and θ is Serre's differential operator defined by $c(\mathfrak{m}, \theta f) = \Delta_F^{-1} \mathrm{N}_{F/\mathbf{Q}}(\mathfrak{m}) c(\mathfrak{m}, f)$. Therefore, $\widehat{E}_\phi^{\{p\}}(k,s)$ is a p-adic Hilbert modular form of parallel weight k for all $(k,s) \in \mathbf{Z}_p^2$, and $\widehat{E}_\phi^{(p)} \in \mathcal{M}^{(2)}(\mathcal{O}_F)$. For each prime ideal $\mathfrak{q}$, define $\mathbf{U}_\mathfrak{q} \colon \mathcal{M}^{(2)}(\mathfrak{n}) \to \mathcal{M}^{(2)}(\mathfrak{n}\mathfrak{q})$ by $c(\mathfrak{m}, \mathbf{U}_\mathfrak{q} f) := c(\mathfrak{m}\mathfrak{q}, f)$. Let N be a positive integer such that $p \nmid N$ and

$$\text{(Splt)} \qquad N\mathcal{O}_F = \mathfrak{N}\overline{\mathfrak{N}}, \quad (\mathfrak{N}, \overline{\mathfrak{N}}) = 1.$$

Define $\boldsymbol{E}_\phi \in \mathcal{M}^{(2)}(\mathfrak{N})$ by

$$\boldsymbol{E}_\phi := \prod_{\mathfrak{q} \mid \mathfrak{N}} (1 - \phi(\mathfrak{q})^{-1} \langle \mathrm{N}(\mathfrak{q}) \rangle^{\frac{2s-2-k}{2}} \mathbf{U}_\mathfrak{q}) \cdot \widehat{E}_\phi^{\{p\}}$$

and the *diagonal restriction* $\boldsymbol{G}_\phi \in A(\mathscr{X} \times \mathscr{X})[\![q]\!]$ of $\boldsymbol{E}_\phi$ by

$$\boldsymbol{G}_\phi := \sum_{n>0} \Big(\sum_{\beta \in \mathfrak{d}_+^{-1}, \mathrm{Tr}(\beta) = n} c(\beta\mathfrak{d}, \boldsymbol{E}_\phi) \Big) q^n,$$

where $\mathfrak{d}_+^{-1}$ is the additive semigroup of totally positive elements in $\mathfrak{d}^{-1}$.

By definition $\boldsymbol{G}_\phi(k,s)$ is the q-expansion of a p-adic elliptic modular on $\Gamma_0(pN)$ of weight k obtained from the diagonal restriction of $\widehat{E}_\phi^{\{p\}}(k,s)$ for $(k,s) \in \mathscr{X}^{\mathrm{cl}} \times \mathscr{X}^{\mathrm{cl}}$ with $k \geq 2s$. Let $\mathscr{U}$ be an appropriate neighborhood around $2 \in \mathscr{X}$. Let $\mathbf{S}^{\mathrm{ord}}(N)$ be the space of ordinary $A(\mathscr{U})$-adic elliptic cusp forms on $\Gamma_0(Np)$, consisting of q-expansion $\boldsymbol{f} = \sum_{n>0} c(n, \boldsymbol{f}) q^n \in A(\mathscr{U})[\![q]\!]$ such that the weight k specialization $\boldsymbol{f}_k$ is a p-ordinary cusp forms of weight k on $\Gamma_0(pN)$ for $k \in \mathscr{X}^{\mathrm{cl}}$. By Hida theory, we know $\mathbf{S}^{\mathrm{ord}}(N)$ is a free $A(\mathscr{U})$-module of finite rank. It can be shown that the image $e\boldsymbol{G}_\phi$ under Hida's ordinary projector actually belongs to $\mathbf{S}^{\mathrm{ord}}(N) \widehat{\otimes}_{A(\mathscr{U})} A(\mathscr{U} \times \mathscr{X})$, where $A(\mathscr{U})$ is regarded as a subring of $A(\mathscr{U} \times \mathscr{X})$ via the pull-back of the first projection $\mathscr{U} \times \mathscr{X} \to \mathscr{U}$. We can thus decompose

$$e\boldsymbol{G}_\phi = \sum_{\boldsymbol{f}} \mathcal{L}_{\boldsymbol{E}_\phi, \boldsymbol{f}} \cdot \boldsymbol{f} + (\text{old forms}), \quad \mathcal{L}_{\boldsymbol{E}_\phi, \boldsymbol{f}} \in A(\mathscr{U} \times \mathscr{X}),$$

where $\boldsymbol{f}$ runs over the set of primitive Hida families of tame conductor N. We shall call $\mathcal{L}_{\boldsymbol{E}_\phi,\boldsymbol{f}} \in A(\mathscr{U} \times \mathscr{X})$ the twisted triple product p-adic L-function attached to the p-adic Hilbert Eisenstein series $\boldsymbol{E}_\phi$ and a primitive Hida family $\boldsymbol{f}$. We provide the following derivative formula for $\mathcal{L}_{\boldsymbol{E}_\phi,\boldsymbol{f}}$, which partially generalizes [DPV19, Theorem C(2)] to elliptic newforms of split tame conductor.

Theorem (H.- and S. Yamana)**.** *Let E be an elliptic curve over $\mathbf{Q}$ of conductor pN with N satisfying* (Splt)*. Let $\boldsymbol{f} \in A(\mathscr{U})[\![q]\!]$ be a primitive Hida family of tame level N such that the weight two specialization $\boldsymbol{f}_2$ is the elliptic newform associated with E. Suppose that p is inert in F. Then $\mathcal{L}_{\boldsymbol{E}_\phi,\boldsymbol{f}}(2, s) = 0$ and*

$$\frac{d}{dk}\big(\mathcal{L}_{\boldsymbol{E}_\phi,\boldsymbol{f}}(k, s+1)\big)|_{k=2} = \frac{1}{2}(1+\phi(\mathfrak{N})^{-1}w_N)\cdot \log_E P_\phi \cdot L_p(E,s) \times \frac{c_f}{m_E^2 2^{\alpha(E)}} \langle \Delta_F \rangle^{\frac{s-1}{2}},$$

where

- $\log_E P_\phi$ *is the p-adic loagrithm of the twisted Stark-Heegner point $P_\phi \in E(F_p) \otimes \mathbf{Q}(\phi)$ introduced in* [Dar01, (182)],
- $L_p(E,s)$ *is the Mazur-Tate-Teitelbaum cyclotomic p-adic L-function for E,*
- $c_f \in \mathbf{Z}^{>0}$ *is the congruence number for f, $m_E \in \mathbf{Q}^\times$ is the Mainn constant for E and* $2^{\alpha(E)} = [\mathrm{H}_1(E(\mathbf{C}),\mathbf{Z}) : \mathrm{H}_1(E(\mathbf{C}),\mathbf{Z})^+ \oplus \mathrm{H}_1(E(\mathbf{C}),\mathbf{Z})]$.

Remark 2.1.

- The definition of Stark-Heegner points P_ϕ for odd ϕ in [Dar01] depends on a choice of the purely imaginary period Ω_E^-. In the above theorem, we require $(\sqrt{-1})^{-1}\Omega_E^-$ to be positive.
- Our main motivation for this two-variable generalization is that we have the non-vanishing of the p-adic L-function $L_p(E,s)$ thanks to Rohrlich's theorem [Roh84], so $\log_E P_\phi$ can be computed from the twisted triple product p-adic L-function even when the central value $L(E,1)$ vanishes.
- The Eisenstein contribution in the spectral decomposition in Part (2) of [DPV19, Theorem C] is connected with the p-adic logatithms of elliptic units over F, while in our two-variable setting, $e\boldsymbol{G}_\phi$ is a p-adic family of cusp forms, so we do not get any information for elliptic units.

3. The idea of the proof

We briefly explain the idea of the proof. Let $\mathcal{L}_p(\boldsymbol{f}/F, \phi, k)$ be the (odd) square-root p-adic L-function associated with the primitive Hida family $\boldsymbol{f}$ and the character ϕ constructed in [BD09, Definition 3.4] with $w_\infty = -1$ and let $L_p(\boldsymbol{f}, k, s)$ be the Mazur-Kitagawa two-variable p-adic L-function so that $L_p(\boldsymbol{f},2,s)$ is the cyclotomic p-adic L-function for $\boldsymbol{f}_2$. We prove the following factorization formula of $\mathcal{L}_{\boldsymbol{E}_\phi,\boldsymbol{f}}$:

$$C^*(k)\cdot\mathcal{L}_{\boldsymbol{E}_\phi,\boldsymbol{f}}(k,s+1)=\mathcal{L}_p(\boldsymbol{f}/F,\phi,k)\cdot L_p(\boldsymbol{f},k,s), \tag{3.1}$$

where $C^*(k)$ is a meromorphic function on $\mathscr{X}$ holomorphic at all classical points $k\in\mathscr{X}^{\mathrm{cl}}$ with $C^*(2)=1$. By the very construction, the square root p-adic L-function $\mathcal{L}_p(\boldsymbol{f}/F,\phi,k)$ interpolates the toric period integrals $B^{\phi}_{\boldsymbol{f}_k}$. Thus we get $\mathcal{L}_{\boldsymbol{E}_\phi,\boldsymbol{f}}(2,s)=\mathcal{L}_p(\boldsymbol{f}/F,\phi,2)=0$ by a classical theorem of Saito and Tunnell. Moreover, from the formula [BD09, Corollary 2.6], it is not difficult to deduce that the first derivative of $\mathcal{L}_p(\boldsymbol{f}/F,\phi,k)$ at $k=2$ is $2^{-1}(1+w_N\phi(\mathfrak{N})^{-1})\log_E P_\phi$, and hence we obtain Theorem from (3.1). The factorization formula (3.1) is established by the explicit interpolation formulae on both sides. In particular, the interpolation formula for $\mathcal{L}_{\boldsymbol{E}_\phi,\boldsymbol{f}}(k,s)$ is the most technical part of this paper. Roughly speaking, for $(k,s)\in\mathscr{X}^{\mathrm{cl}}\times\mathscr{X}^{\mathrm{cl}}$ with $k\geq 2s$, Hida's p-adic Rankin-Selberg method shows that $\mathcal{L}_{\boldsymbol{E}_\phi,\boldsymbol{f}}(k,s)$ is interpolated by the inner product between the diagonal restriction of a nearly holomorphic Hilbert Eisenstein series $\boldsymbol{E}_\phi(k,s)$ and $\boldsymbol{f}_k$. Therefore, a result of Keaton and Pitale [KP19, Proposition 2.3] tells us that $\mathcal{L}_{\boldsymbol{E}_\phi,\boldsymbol{f}}(k,s)$ is a product of (i) the Waldspurger toric period integral $B^{\phi}_{\boldsymbol{f}_k}$ of $\boldsymbol{f}_k$ over F twisted by ϕ, (ii) the special value $L(\boldsymbol{f}_k,s)$ of the L-function for $\boldsymbol{f}_k$ and (iii) local zeta integrals $Z_{\mathcal{D}}(s,B_{W_v})$ for every place of $\mathbf{Q}$. Now items (i) and (ii) are basically interpolated by $\mathcal{L}(\boldsymbol{f}/K,\phi,k)$ and $L_p(\boldsymbol{f},k,s)$, so our task is to evaluate explicitly these local zeta integrals, which occupy the main body of Section 4. By the explicit interpolation formulae of these p-adic L-functions, we find immediately that the ratio C^* between $\mathcal{L}_p(\boldsymbol{f}/F,\phi,k)\cdot L_p(\boldsymbol{f},k,s)$ and $\mathcal{L}_{\boldsymbol{E}_\phi,\boldsymbol{f}}(k,s+1)$ is independent of s, and hence C^* is a meromorphic function in k only. Finally, by a standard argument using Rohrlich's result on the non-vanishing of the cyclotomic p-adic L-functions for elliptic modular forms, we can conclude that $C^*(k)$ is holomorphic at all $k\in\mathscr{X}^{\mathrm{cl}}$ and $C^*(2)$ is essentially the congruence number.

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