Maass forms on $\mathrm{GL}(2)$ over division quaternion algebras of discriminant p

Hiroaki Narita, Ameya Pitale, Siddhesh Wagh

1 Introduction

In this talk, we will present a construction of Maass forms, that violate the Ramanujan conjecture, on 5-dimensional hyperbolic spaces. To provide some context, let us remind the reader of another famous example of modular forms that violate the Ramanujan conjecture – the Saito-Kurokawa lifts.

Saito-Kurokawa lifts: Let $f \in S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$, with k even, and let $h \in S^+_{k-1/2}(\Gamma_0(4))$ be the corresponding cusp form in the Kohnen plus space. Let $\{c(n)\}$ be the Fourier coefficients of h. For T half integral, positive definite, symmetric 2×2 matrix, define

$$A(T) := \sum_{d|\gcd(T)} c\Big(\frac{\det(2T)}{d^2}\Big) d^{k-1}.$$

1.1 Theorem. *With $A(T)$ as above, the function $F_f(Z) = \sum_T A(T)\exp(2\pi i \mathrm{Tr}(TZ))$ is a Siegel cusp form of weight k with respect to $\mathrm{Sp}_4(\mathbb{Z})$.*

Let us list some of the properties of the Saito-Kurokawa lifts (see [2] for details).

1. Explicit formula for Fourier coefficients.
2. The map $f \mapsto F_f$ is linear and injective.
3. Relation between L-functions
$$L(s, F_f, \mathrm{spin}) = \zeta(s-k+1)\zeta(s-k+2)L(s, f).$$
4. The map $f \mapsto F_f$ preserves Hecke eigenforms.
5. If F_f is a Hecke eigenform, then let $\pi_F = \otimes_p \pi_p$ be the irreducible cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$ generated by F_f. Then, for every $p < \infty$, the local representation π_p is not tempered, i.e. F_f violates the generalized Ramanujan conjecture.
6. Characterization of lifts as the Maass space: For $T = \begin{bmatrix} m & r/2 \\ r/2 & n \end{bmatrix}$, write $A(T) = A(m, r, n)$. Then a Siegel cusp form F with Fourier coefficients $A(T)$ is a Saito-Kurokawa lift if and only if we have
$$A(m, r, n) = \sum_{d|\gcd(m,r,n)} d^{k-1} A(\frac{mn}{d^2}, \frac{r}{d}, 1).$$

2 Maass forms on 5-dimensional hyperbolic space

Let B be a definite division quaternion algebra over $\mathbb{Q}$. Let us make the assumption that the discriminant of B is a prime number p.

Let G be the algebraic group such that $G(\mathbb{Q}) = \mathrm{GL}_2(B)$. Then $G(\mathbb{R}) = \mathrm{GL}_2(\mathbb{H})$, where $\mathbb{H}$ is the Hamiltonian quaternions. We have the Iwasawa decomposition: $\mathrm{GL}_2(\mathbb{H}) = ZNAK$, where Z is the center, and K is the maximal compact, and

$$N = \{n(x) = \left[\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right] : x \in \mathbb{H}\}, A = \{a_y : \begin{bmatrix} \sqrt{y} & \\ & \sqrt{y}^{-1} \end{bmatrix} : y \in \mathbb{R}^+\}.$$

We have

$$G/ZK \simeq \{\left[\begin{smallmatrix} y & x \\ & 1 \end{smallmatrix}\right] : x \in \mathbb{H}, y \in \mathbb{R}^+\},$$

a realization of the 5-dimensional hyperbolic space $\mathbb{H}_5$. For a discrete subgroup $\Gamma \subset \mathrm{GL}_2(\mathbb{H})$ and $r \in \mathbb{C}$ we denote by $\mathcal{M}(\Gamma, r)$ the space of smooth functions F on $\mathrm{GL}_2(\mathbb{H})$ satisfying the following conditions:

1. $\Omega \cdot F = -\frac{1}{2}(\frac{r^2}{4} + 1)F$, where Ω is the Casimir operator,
2. for any $(z, \gamma, g, k) \in Z \times \Gamma \times G \times K$, we have $F(z\gamma gk) = F(g)$,
3. F is of moderate growth.

We will take $\Gamma = \mathrm{GL}_2(\mathcal{O})$, where $\mathcal{O}$ is any maximal order in B. Let $\mathcal{O}'$ be the dual of $\mathcal{O}$ with respect to trace map on B. For $F \in \mathcal{M}(\mathrm{GL}_2(\mathcal{O}), r)$, we have the Fourier expansion

$$F(n(x)a_y) = u(y) + \sum_{\beta \in \mathcal{O}' \setminus \{0\}} A(\beta) y^2 K_{\sqrt{-1}r}(2\pi|\beta|y) e^{2\pi\sqrt{-1}\mathrm{tr}(\beta x)}$$

3 The Maass lift

Let $f \in S(\Gamma_0(p), \frac{r^2+1}{4})$ be an Atkin Lehner eigenfunction with eigenvalue $\epsilon \in \{-1, 1\}$. Let $\{c(n) : n \in \mathbb{Z} - \{0\}\}$ be the Fourier coefficients of f. Let us define the primitive elements of $\mathcal{O}'$ by

$$\mathcal{O}'_{\mathrm{prim}} := \{\beta \in \mathcal{O}' : \frac{1}{n}\beta \notin \mathcal{O}' \text{ for all positive integers } n\}.$$

Write $\beta \in \mathcal{O}'$ as

$$\beta = p^u n \beta_0, \qquad u \geq 0, n > 0, p \nmid n \text{ and } \beta_0 \in \mathcal{O}'_{\mathrm{prim}}.$$

Set

$$\delta = \begin{cases} 0 & \text{if } \beta_0 \notin \mathcal{O}; \\ 1 & \text{if } \beta_0 \in \mathcal{O}. \end{cases}$$

Define

$$A_f(\beta) := |\beta| \sum_{t=0}^{2u+\delta} \sum_{d|n} c(\frac{-|\beta|^2}{p^{t-1}d^2})(-\epsilon)^t. \tag{1}$$

The main theorem is the following.

3.1 Theorem. *Let $f \in S(\Gamma_0(p), \frac{r^2+1}{4})$ be an Atkin Lehner eigenfunction with eigenvalue $\epsilon \in \{-1, 1\}$ with Fourier coefficients $\{c(n)\}$. For $\beta \in \mathcal{O}'$, define $A_f(\beta)$ as above. Then the function $F_{f,\mathcal{O}}$ on $\mathrm{GL}_2(\mathbb{H})$ with Fourier coefficients $A_f(\beta)$ is a cusp form in $\mathcal{M}(\mathrm{GL}_2(\mathcal{O}), r)$.*

One way to prove the automorphy is to use the converse theorem due to Maass.

3.2 Theorem. (Maass [3]) *F given by the Fourier expansion is in $\mathcal{M}(\Gamma_{\mathcal{O}}, r)$ if and only if a family of twisted Dirichlet series are "nice".*

Here, $\Gamma_{\mathcal{O}} = \langle \left[\begin{smallmatrix} 1 & \beta \\ & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right] : \beta \in \mathcal{O} \rangle$. Unfortunately, we have $\mathrm{GL}_2(\mathcal{O}) = \Gamma_{\mathcal{O}}$ if and only if $p = 2, 3, 5$. We have used the Maass converse theorem to prove automorphy for $p = 2$ in joint paper with Muto-Narita [4]. For general p, the strategy is to use Borcherds theta lifts.

4 Borcherds Theta lifts

In a nutshell, the idea for the theta lift is given by

$$\Phi(n(x)a_y) \sim \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{h}} f(\tau)\Theta(\tau, n(x)a_y)d\tau.$$

To execute the strategy we have to do the following two things.

1. Replace f by a vector valued modular form with respect to $\mathrm{SL}_2(\mathbb{Z})$.
2. Define the theta kernel.

Let us first define the vector valued modular forms. Define the discriminant form $D = \mathcal{O}'/\mathcal{O} \simeq (\mathbb{Z}/p\mathbb{Z}) \times (\mathbb{Z}/p\mathbb{Z})$. The group algebra $\mathbb{C}[D]$ is a $\mathbb{C}$-vector space generated by the formal basis vectors $\{e_\mu : \mu \in D\}$ with product defined by $e_\mu e_{\mu'} = e_{\mu+\mu'}$. Let $\mathrm{SL}_2(\mathbb{Z})$ act on $\mathbb{C}[D]$ via the representation ρ_D as follows:

$$\rho_D(\left[\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right])e_\mu = e(|\mu|^2)e_\mu, \rho_D(\left[\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right])e_\mu = -\frac{1}{p}\sum_{\mu' \in D} e(-(\mu, \mu'))e_{\mu'}.$$

Here $e(x) = \exp(2\pi i x)$. Now, given $f \in S(\Gamma_0(p), \frac{r^2+1}{4})$, define $\mathcal{L}_D(f) : \mathfrak{h} \to \mathbb{C}[D]$ by

$$\big(\mathcal{L}_D(f)\big)(\tau) = \sum_{\Gamma_0(p)\backslash \mathrm{SL}_2(\mathbb{Z})} f(M\langle\tau\rangle)\rho_D(M)^{-1}(e_0).$$

The main result is

4.1 Proposition. *Let $f \in S(\Gamma_0(p), \frac{r^2+1}{4})$ be an Atkin Lehner eigenfunction with eigenvalue $\epsilon \in \{-1, 1\}$ with Fourier coefficients $\{c(n)\}$.*

1. *For all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, we have*
$$\mathcal{L}_D(f)|\gamma = \rho_D(\gamma)\mathcal{L}_D(f).$$

2. *Write* $\mathcal{L}_D(f) = \sum_{\mu \in D} f_\mu e_\mu$. *Let* $c_\mu(n)$ *be the Fourier coefficients of* f_μ. *Then we have*

$$c_\mu(n) = \begin{cases} c(n) - \epsilon c(np) & \text{if } \mu = 0; \\ -\epsilon c(n) & \text{if } \mu \neq 0, n \equiv |\mu|^2 \pmod p; \\ 0 & \text{otherwise.} \end{cases}$$

Next, let us define the theta kernel. Let $(\mathcal{O}, |\cdot|^2) \simeq (\mathbb{Z}^4, A_0)$. Set $L := [\mathbb{Z}, \mathcal{O}, \mathbb{Z}]^t \simeq (\mathbb{Z}^6, A)$ with $A = \begin{bmatrix} & & 1 \\ & -A_0 & \\ 1 & & \end{bmatrix}$. Let $V = (\mathbb{R}^6, Q_A) = L \otimes \mathbb{R} \simeq \mathbb{R}^{1,5}$. We have that the connected component of $\mathrm{SO}(V) \simeq \mathrm{SO}(1,5)$ is isomorphic to $\mathrm{GL}_2(\mathbb{H})/Z$. Let $\mathcal{D}$ be the Grassmanian of positive oriented lines in the quadratic space V. We can identify the 5-dimensional hyperbolic space $\mathbb{H}_5$ with the connected component $\mathcal{D}^+$ via

$$\begin{aligned} \mathbb{H}_5 \ni (x,y) \mapsto \nu(x,y) &:= \frac{1}{\sqrt{2}}{}^t(y + y^{-1}Q_{A_0}(x), -y^{-1}x, y^{-1}) \\ &\mapsto \mathbb{R} \cdot \nu(x,y) \in \mathcal{D}^+. \end{aligned}$$

Every $\nu := \nu(x,y)$ defines an isometry

$$\iota_\nu : V \to \mathbb{R} \cdot \nu \oplus (\nu^\perp, Q_{A_0}|_{\nu^\perp}) \simeq \mathbb{R}^{1,5}, \quad \lambda \to (\lambda_\nu, \lambda_{\nu^\perp}).$$

Let $p : \mathbb{R}^6 \to \mathbb{R}$ be the polynomial given by $p(x_1, \cdots, x_6) = -2^{-2}x_1^2$. For $\tau = u + iv \in \mathfrak{h}, (x,y) \in \mathbb{H}_5$, define the theta function

$$\Theta_L(\tau, \nu(x,y), p) := \sum_{\mu \in D} \Big(\sum_{\lambda \in L + \mu} \big(exp(\frac{-\Delta}{8\pi v})(p) \big)(\iota_\nu(\lambda)) e(Q_A(\lambda_\nu)\tau + Q_A(\lambda_{\nu^\perp})\bar{\tau}) \Big) e_\mu.$$

Here, Δ is the Laplacian on $\mathbb{R}^{1,5}$.

4.2 Proposition. (Borcherds [1]) *For* $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{Z})$, *we have*

$$\Theta_L(\frac{a\tau + b}{c\tau + d}, \nu(x,y), p) = |c\tau + d|^5 \rho_D(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]) \Theta_L(\tau, \nu(x,y), p).$$

For $(x,y) \in \mathbb{H}_5$, define

$$\Phi_{f,\mathcal{O}}(\nu(x,y)) := \int_{\mathrm{SL}_2(\mathbb{Z}) \backslash \mathfrak{h}} \langle \mathcal{L}_D(f)(\tau), \overline{\Theta_L(\tau, \nu(x,y), p)} \rangle v^{\frac{5}{2}} \frac{dudv}{v^2}.$$

4.3 Proposition. *For every* $\gamma \in \mathrm{GL}_2(\mathcal{O})$, *we have*

$$\Phi_{f,\mathcal{O}}(\gamma\nu(x,y)) = \Phi_{f,\mathcal{O}}(\nu(x,y)).$$

Proof. Θ_L is invariant under a subgroup of $\mathrm{GL}_2(\mathcal{O})$ that fixes $\mathcal{O}'/\mathcal{O}$. Action of $\mathrm{GL}_2(\mathcal{O})$ preserves norms on $\mathcal{O}'/\mathcal{O}$, and Fourier coefficients of $f_\mu, \mu \in \mathcal{O}'/\mathcal{O}$ only depend on $|\mu|^2$. ■

Borcherds gives explicit formula for the Fourier coefficients of $\Phi_{f,\mathcal{O}}(\nu(x,y))$. We compute this to show that the Fourier coefficients of $\Phi_{f,\mathcal{O}}(\nu(x,y))$ are exactly $A_f(\beta)$ defined in (1). Hence, we obtain

$$\begin{aligned}\Phi_{f,\mathcal{O}}(\nu(x,y)) &= \sum_{\beta\in\mathcal{O}'\setminus\{0\}} A_f(\beta)y^2K_{\sqrt{-1}r}(2\pi|\beta|y)e^{2\pi\sqrt{-1}\mathrm{tr}(\beta x)}\\ &= F_{f,\mathcal{O}}(n(x)a_y),\end{aligned}$$

which shows that $F_{f,\mathcal{O}} \in \mathcal{M}(\mathrm{GL}_2(\mathcal{O}), r)$. Cuspidality follows from the observation that the Fourier expansion of $\Phi_{f,\mathcal{O}}$ at a different cusp corresponds to the Fourier expansion of the Borcherds lift for a shift of $\mathcal{O}$. This completes the proof of Theorem 3.1.

If f is a non-zero even Hecke eigenform, then $c(-1) \neq 0$. Hence $A_f(1) \neq 0$, and we get non-vanishing of $F_{f,\mathcal{O}}$. To show that $F_{f,\mathcal{O}}$ is non-zero for a general f, we use the fact that the space of Maass forms f for a fixed p and r is finite dimensional. In addition, we need to show that $f \to F_{f,\mathcal{O}}$ is Hecke equivariant.

If a prime $\ell \neq p$, then $B \otimes \mathbb{Q}_\ell =: B_\ell \simeq M_2(\mathbb{Q}_\ell)$ and $\mathrm{GL}_2(B_\ell) \simeq \mathrm{GL}_4(\mathbb{Q}_\ell)$. Hence, we can use the well-known Hecke theory for GL_4 and show that if f is a Hecke eigenform, then $F_{f,\mathcal{O}}$ is also a Hecke eigenform.

Now, let $F_{f,\mathcal{O}}$ be a Hecke eigenform. Suppose $\pi_{F,\mathcal{O}} = \otimes\pi_\ell$ is the irreducible cuspidal automorphic representation of $\mathrm{GL}_2(B_\mathbb{A})$ corresponding to $F_{f,\mathcal{O}}$. Let $\sigma_f = \otimes\sigma_\ell$ be the irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ associated to f. Then, for $\ell \neq p$, the local representation π_ℓ is the spherical component of the induced representation $\mathrm{Ind}_{P_{2,2}(\mathbb{Q}_\ell)}^{\mathrm{GL}_4(\mathbb{Q}_\ell)}(|\det|^{-1/2}\sigma_\ell \times |\det|^{1/2}\sigma_\ell)$. We have

$$L(s, \pi_{F,\mathcal{O}}) = L(s+1/2, \sigma_f)L(s-1/2, \sigma_f),$$

i.e. $F_{f,\mathcal{O}}$ does not satisfy the generalized Ramanujan conjecture. Note that the strong multiplicity one theorem for $\mathrm{GL}_2(B_\mathbb{A})$ implies that, if $\mathcal{O}_1$ and $\mathcal{O}_2$ are two maximal orders in B, then $\pi_{F,\mathcal{O}_1} = \pi_{F,\mathcal{O}_2}$. Hence, $F_{f,\mathcal{O}_1}$ and $F_{f,\mathcal{O}_2}$ give two vectors in the same representation.

5 Maass space

Let us finish with the description of Maass space in the case $p = 2$. Let the Maass space $\mathcal{M}^*(\mathrm{GL}_2(\mathcal{O}), r)$ denote the subspace of cusp forms F in $\mathcal{M}(\mathrm{GL}_2(\mathcal{O}), r)$ with Fourier coefficients $A(\beta)$ satisfying the following.

1. If $\beta = \varpi_2^u n\beta_0$, then $A(\beta)$ depends only on $K := |\beta|^2$, u and n. We write $A(\beta)$ as $A(K, u, n)$.
2. $A(K, u, n)$ satisfy the recurrence relation
   - $A(K, u, n) = (-3\epsilon/\sqrt{2})A(K/2, u-1, n) - A(K/4, u-2, n)$ for some $\epsilon \in \{-1, 1\}$.
   - $A(K, u, n) = \sum_{d|n} dA(K/d^2, u, 1)$.

5.1 Theorem. (Wagh [5]) *$F \in \mathcal{M}^*(\mathrm{GL}_2(\mathcal{O}), r)$ if and only if $F = F_f$ for some $f \in S(\Gamma_0(2), \frac{r^2+1}{4})$.*

We plan to extend this theorem to $p > 2$ in the future.

References

[1] Borcherds, R. E.: *Automorphic forms with singularities on Grassmanians*. Invent. Math., **132** (1998), 491–562.

[2] Eichler, M., Zagier, D.: *The Theory of Jacobi Forms* Progress in Mathematics, 55. Birkhauser, Boston (1985)

[3] Maass, H.: *Automorphe Funktionen von meheren Veränderlichen und Dirchletsche Reihen.* Abh. Math. Sem. Univ. Hamburg, **16**, no. 3-4, (1949) 72–100.

[4] Muto, M., Narita, H., Pitale, A.: *Lifting to $GL(2)$ over a division quaternion algebra and an explicit construction of CAP representations.* Nagoya Math. J., **222**, issue 01, (2016) 137–185.

[5] Wagh, S.: *Maass space for lifts to* GL_2 *over a division quaternion algebra* Journal of Number Theory, **28**, (2020) 230–261

Department of Mathematics, Faculty of Science and Engineering, Waseda University, Tokyo, Japan
Email address: **hnarita@waseda.jp**

Department of Mathematics, University of Oklahoma, Norman, OK, 73019, USA
Email address: **apitale@ou.edu**

Department of Mathematics, University of Wisconsin, Madison, USA
Email address: **waghsiddhesh@gmail.com**