Triple product p-adic L-functions attached to p-adic families of modular forms

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1 Introduction

In this paper, we present the result [Fuk19, Theorem 5.2.1]. Let p be an odd prime. In [Hsi17], Hsieh constructed three-variable p-adic triple product L-functions attached to triples of Hida families. We generalize the result [Hsi17, (1) of Theorem 7.1] axiomatically and construct three-variable p-adic triple product L-functions in the unbalanced case attached to triples $(F, G^{(2)}, G^{(3)})$. Here, F is a Hida family and $G^{(i)}$ is a more general p-adic family for $i = 2, 3$. For example, we can take Hida families, Coleman families or CM-families as $G^{(i)}$.

To state our theorem precisely, we prepare some notation. We denote by $\mathbb{Q}$, $\mathbb{Q}_p$ and $\mathbb{C}$ the fields of rational numbers, p-adic rational numbers and complex numbers respectively. Let $\mathbb{Z}$ and $\mathbb{Z}_p$ be the rings of integers and p-adic integers respectively. Throughout this paper, we fix an isomorphism $i_p : \overline{\mathbb{Q}}_p \cong \mathbb{C}$ over $\overline{\mathbb{Q}}$. Here, $\overline{\mathbb{Q}}$ and $\overline{\mathbb{Q}}_p$ are the algebraic closures of the fields $\mathbb{Q}$ and $\mathbb{Q}_p$ respectively. We denote by $\mathbb{A}$ the adele over $\mathbb{Q}$. Let A be a ring. We denote by $a(n, f)$ the n-th coefficient of a formal power series $f \in A[\![q]\!]$, where n is a non-negative integer. Let ω_p be the Teichmüler character mod p. Let (N_1, N_2, N_3) be a triple of positive integers which are prime to p and (ψ_1, ψ_2, ψ_3) a triple of Dirichlet characters of modulo (N_1p, N_2p, N_3p) which satisfies the following hypothesis.

Hypothesis (1). *There exists an integer $a \in \mathbb{Z}$ such that $\psi_1\psi_2\psi_3 = \omega_p^{2a}$.*

Let K be a finite extension of $\mathbb{Q}_p$ and $\mathcal{O}_K$ the ring of integers of K. We denote by $\Lambda_K := \mathcal{O}_K[\![\Gamma]\!]$ the Iwasawa algebra over $\mathcal{O}_K$, where $\Gamma := 1+p\mathbb{Z}_p$. Let $\mathbf{I}_i$ be a normal finite flat extension of Λ_K for $i = 1, 2, 3$. We fix a set of non-zero $\mathcal{O}_K$-algebraic homomorphisms

$$\mathfrak{X}^{(i)} := \{Q_m^{(i)} : \mathbf{I}_i \to \overline{\mathbb{Q}}_p\}_{m \geq 1}$$

for $i = 1, 2, 3$. Let $G^{(i)} \in \mathbf{I}_i[\![q]\!]$ be a formal series such that the specialization

$$G^{(i)}(m) := \sum Q_m^{(i)}(a(n, G^{(i)}))q^n \in \overline{\mathbb{Q}}_p[\![q]\!]$$

is the Fourier expansion of a normalized cuspidal Hecke eigenform of weight $k^{(i)}(m)$, level $N_ip^{e^{(i)}(m)}$ and Nebentypus $\psi_i\omega_p^{-k^{(i)}(m)}\epsilon_m^{(i)}$ which is primitive outside of p for each positive integer m. Here, $k^{(i)}(m)$ and $e^{(i)}(m)$ are positive integers and $\epsilon_m^{(i)}$ is a finite character of Γ. Let $\mathfrak{X}_{\mathbf{I}_1}$ be the set of arithmetic points Q with weight $k_Q \geq 2$ and a finite part ϵ_Q defined in Definition 2.0.1. We take the pair $(\mathfrak{X}^{(1)}, G^{(1)})$ to be the pair $(\mathfrak{X}_{\mathbf{I}_1}, F)$, where F is a primitive Hida family F of tame level N_1 and Nebentypus ψ_1 defined in Definition 2.0.3. We denote by F_Q the specialization of F at Q for each $Q \in \mathfrak{X}_{\mathbf{I}_1}$. Let $R := \mathbf{I}_1\widehat{\otimes}_{\mathcal{O}_K}\mathbf{I}_2\widehat{\otimes}_{\mathcal{O}_K}\mathbf{I}_3$ be the complete tensor product of $\mathbf{I}_1, \mathbf{I}_2$ and $\mathbf{I}_3$ over $\mathcal{O}_K$. We define an unbalanced domain of interpolation points of R to be

$$\mathfrak{X}_R^F := \left\{ \underline{Q} = \left(Q_1, Q_{m_2}^{(2)}, Q_{m_3}^{(3)}\right) \in \mathfrak{X}_{\mathbf{I}_1} \times \mathfrak{X}^{(2)} \times \mathfrak{X}^{(3)} \;\middle|\; \begin{matrix} k_{Q_1} + k^{(2)}(m_2) + k^{(3)}(m_3) \equiv 0 \pmod 2, \\ k_{Q_1} \geq k^{(2)}(m_2) + k^{(3)}(m_3) \end{matrix} \right\}.$$

For each $\underline{Q} = \left(Q_1, Q^{(2)}_{m_2}, Q^{(3)}_{m_3}\right) \in \mathfrak{X}^F_R$, we denote by $(F, G^{(2)}, G^{(3)})(\underline{Q})$ the specialization of the triple $(F, G^{(2)}, G^{(3)})$ at $\underline{Q}$. We define a representation $\Pi'_{\underline{Q}} = \pi_{Q_1} \boxtimes \pi_{Q^{(2)}_{m_2}} \boxtimes \pi_{Q^{(3)}_{m_3}}$ of $(\mathrm{GL}_2(\mathbb{A}))^3$, where $(\pi_{Q_1}, \pi_{Q^{(2)}_{m_2}}, \pi_{Q^{(3)}_{m_3}})$ is the triple of automorphic representation attached to the triple $(F, G^{(2)}, G^{(3)})(\underline{Q})$. Let $(\chi_{\underline{Q}})_{\mathbb{A}}$ be the adelization of the following Dirichlet character

$$\chi_{\underline{Q}} := \omega_p^{\frac{1}{2}(2a - k_{Q_1} - k^{(2)}(m_2) - k^{(3)}(m_3))} (\epsilon_{Q_1} \epsilon^{(2)}_{m_2} \epsilon^{(3)}_{m_3})^{\frac{1}{2}}$$

for each $\underline{Q} = \left(Q_1, Q^{(2)}, Q^{(3)}\right) \in \mathfrak{X}^F_R$. We set $\Pi_{\underline{Q}} = \Pi'_{\underline{Q}} \otimes (\chi_{\underline{Q}})_{\mathbb{A}}$ for each $\underline{Q} \in \mathfrak{X}^F_R$. Let $\epsilon_l(s, \Pi_{\underline{Q}})$ be the local epsilon factor of $\Pi_{\underline{Q}}$ defined in [Ike92, page 227] for each finite prime l. We set $N = N_1 N_2 N_3$. Let $\mathbf{m}_1$ be the unique maximal ideal of $\mathbf{I}_1$. We summarize some hypotheses to state Main Theorem.

Hypothesis (2)**.** *The residual Galois representation* $\overline{\rho}_F := \rho_F \bmod \mathbf{m}_1 : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ *attached to* F *is absolutely irreducible as* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*-module and* p*-distinguished in the sense that the semi-simplification of* $\overline{\rho}_F$ *restricted to* $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$*-module is a sum of two different characters.*

Hypothesis (3)**.** *The number* $\gcd(N_1, N_2, N_3)$ *is square free.*

Hypothesis (4)**.** *For each* $\underline{Q} \in \mathfrak{X}^F_R$ *and for each prime* $l|N$*, we have* $\epsilon_l(1/2, \Pi_{\underline{Q}}) = 1$.

Hypothesis (5)**.** *Let* $i = 2, 3$ *and* n *a positive integer which is prime to* p*. There exits an element* $\langle n \rangle^{(i)} \in \mathbf{I}_i$ *which satisfies*

$$Q^{(i)}_m(\langle n \rangle^{(i)}) = \epsilon^{(i)}_m(n)(n\omega_p^{-1}(n))^{k^{(i)}(m)}$$

for each positive integer m*.*

Hypothesis (6)**.** *Let* $i = 2, 3$*. We have* $a(p, G^{(i)}(m)) \neq 0$ *or* $G^{(i)}(m)$ *is primitive for each positive integer* m*.*

Hypothesis (7)**.** *For each prime* $l|N$*, the* l*-th Fourier coefficients of* $F, G^{(2)}$ *and* $G^{(3)}$ *are non-zero.*

Let $L(s, \Pi_{\underline{Q}})$ be the triple product L-function attached to $\Pi_{\underline{Q}}$ defined in §3. Let $\Omega_{F_{Q_1}}$ be the canonical period defined in [Hsi17, (1.3)] and $\mathcal{E}_{F_{Q_1},p}(\Pi_{\underline{Q}})$ the modified p-Euler factor defined in [Hsi17, (1.2)]. Our main theorem is as follows.

Main Theorem. *Let us assume Hypotheses (1)∼(7). Then, there exists an element* $\mathcal{L}^F_{G^{(2)},G^{(3)}} \in R$ *such that we have the interpolation property :*

$$(\mathcal{L}^F_{G^{(2)},G^{(3)}}(\underline{Q}))^2 = \mathcal{E}_{F_{Q_1},p}(\Pi_{\underline{Q}}) \cdot \frac{L(\frac{1}{2}, \Pi_{\underline{Q}})}{(\sqrt{-1})^{2k_{Q_1}} \Omega^2_{F_{Q_1}}}$$

for every $\underline{Q} = (Q_1, Q^{(2)}_{m_2}, Q^{(3)}_{m_3}) \in \mathfrak{X}^F_R$.

Let $\langle\ \rangle_{\Lambda_K} : \mathbb{Z}_p^\times \to \Lambda_K^\times$ be a group homomorphism defined by $\langle z \rangle_{\Lambda_K} = [z\omega_p^{-1}(z)]$, where $[z\omega_p^{-1}(z)]$ is the group-like element of $z\omega_p(z)^{-1} \in \Gamma$ in $\Lambda_K^\times$. Let n be a positive integer which is prime to p. We have $Q(\langle n \rangle_{\Lambda_K}) = \epsilon_Q(n)(n\omega_p^{-1}(n))^{k_Q}$ for each arithmetic point $Q \in \mathfrak{X}_{\mathbf{I}_1}$. Then, if we take a Hida family as $G^{(i)}$, $\langle n \rangle_{\Lambda_K}$ satisfies the Hypothesis (5).

2 p-adic families of modular forms

Let K be a finite extension of $\mathbb{Q}_p$ and $\mathcal{O}_K$ the ring of integers of K. Let $\mathbf{I}$ be a normal finite flat extension of the Iwasawa algebra Λ_K over $\mathcal{O}_K$. In this section, we recall the definitions of ordinary $\mathbf{I}$-adic cusp forms, primitive Hida families and congruence numbers attached to Hida families. Let N be a positive integer which is prime to p. Throughout this section, we assume that $\mathbb{Q}_p(\chi) \subset K$ for each Dirichlet character χ modulo Np. Let A be a subring of $\overline{\mathbb{Q}}$. We denote by $\mathcal{S}_k(M, \psi, A)$ the A-module of cusp forms of weight k, level M and Nebentypus ψ whose Fourier coefficients at ∞ are included in A, where k, M are positive integers and ψ is a Dirichlet character modulo M. We set $\mathcal{S}_k(M, \psi, B) := \mathcal{S}_k(M, \psi, A) \otimes_A B$ for each A-algebra B.

Definition 2.0.1. *We call a continuous $\mathcal{O}_K$-algebra homomorphism $Q : \mathbf{I} \to \overline{\mathbb{Q}}_p$ an arithmetic point of weight $k_Q \geq 2$ and a finite part $\epsilon_Q : \Gamma \to \overline{\mathbb{Q}}_p^\times$ if the restriction $Q|_\Gamma : \Gamma \to \overline{\mathbb{Q}}_p^\times$ is given by $Q(x) = x^{k_Q}\epsilon_Q(x)$ for each $x \in \Gamma$. Here, $\epsilon_Q : \Gamma \to \overline{\mathbb{Q}}_p^\times$ is a finite character.*

Let $\mathfrak{X}_\mathbf{I}$ be the set of arithmetic points of $\mathbf{I}$. We denote by e the ordinary projection defined in [Hid85, (4.3)]. We recall the definition of ordinary $\mathbf{I}$-adic cusp forms defined in [Wil88].

Definition 2.0.2. *Let χ be a Dirichlet character modulo Np. We call a formal power series $\mathbf{f} \in \mathbf{I}[\![q]\!]$ an ordinary $\mathbf{I}$-adic cusp form of tame level N and Nebentypus χ if the specialization $\mathbf{f}_Q := \sum_{n\geq 0} Q(a(n, \mathbf{f}))q^n \in Q(\mathbf{I})[\![q]\!]$ of $\mathbf{f}$ is the Fourier expansion of an element of $e\mathcal{S}_{k_Q}(Np^{e_Q}, \chi\omega_p^{-k_Q}\epsilon_Q, Q(\mathbf{I}))$ with $e_Q \geq 1$ for all but a finite number of $Q \in \mathfrak{X}_\mathbf{I}$.*

Let $\mathbf{S}^{\mathrm{ord}}(N, \chi, \mathbf{I})$ be the $\mathbf{I}$-module consisting of ordinary $\mathbf{I}$-adic cusp forms of tame level N and Nebentypus χ. Next, we recall the definition of the Hecke algebra of $\mathbf{S}^{\mathrm{ord}}(N, \chi, \mathbf{I})$. For each prime $l \nmid Np$, we define the Hecke operator $T_l \in \mathrm{End}_\mathbf{I}(\mathbf{S}^{\mathrm{ord}}(N, \chi, \mathbf{I}))$ at l to be

$$T_l(f) = \sum_{n\geq 1} a(n, T_l(f))q^n$$

for each $f \in \mathcal{S}^{\mathrm{ord}}(N, \chi, \mathbf{I})$, where

$$a(n, T_l(f)) = \sum_{b|(n,l)} \langle b \rangle_{\Lambda_K} \chi(b) b^{-1} a(ln/b^2, f).$$

For each prime $l|Np$, we define the Hecke operator $T_l \in \mathrm{End}_\mathbf{I}(\mathbf{S}^{\mathrm{ord}}(N, \chi, \mathbf{I}))$ at l to be

$$T_l(f) = \sum_{n\geq 1} a(ln, f)q^n$$

for each $f \in \mathcal{S}^{\mathrm{ord}}(N, \chi, \mathbf{I})$. The Hecke algebra $\mathbf{T}^{\mathrm{ord}}(N, \chi, \mathbf{I})$ is defined by the sub-algebra of $\mathrm{End}_\mathbf{I}(\mathbf{S}^{\mathrm{ord}}(N, \chi, \mathbf{I}))$ generated by T_l for all primes l. Next, we recall the definition of primitive Hida families.

Definition 2.0.3. *We call an element $\mathbf{f} \in \mathbf{S}^{\mathrm{ord}}(N, \chi, \mathbf{I})$ a primitive Hida family of tame level N and Nebentypus χ if the specialization $\mathbf{f}_Q$ is the Fourier expansion of an ordinary p-stabilized cuspidal newform for all but a finite number of $Q \in \mathfrak{X}_\mathbf{I}$.*

Next, we recall the definition of the congruence number. Let $F \in \mathbf{S}^{\mathrm{ord}}(N, \chi, \mathbf{I})$ be a primitive Hida family which satisfies Hypothesis (2). Let $\lambda_F : \mathbf{T}^{\mathrm{ord}}(N, \chi, \mathbf{I}) \to \mathbf{I}$ be an $\mathbf{I}$-algebra homomorphism defined by $\lambda_F(T) = a(1, T(F))$ for each $T \in \mathbf{T}^{\mathrm{ord}}(N, \chi, \mathbf{I})$. Let $\mathbf{m}_F$ be a unique maximal ideal of $\mathbf{T}^{\mathrm{ord}}(N, \chi, \mathbf{I})$ which contains $\mathrm{Ker}\lambda_F$. Let $\mathbf{T}^{\mathrm{ord}}(N, \chi, \mathbf{I})_{\mathbf{m}_F}$ be the localization of

$\mathbf{T}^{\mathrm{ord}}(N,\chi,\mathbf{I})$ by $\mathbf{m}_F$. Let $\lambda_{\mathbf{m}_F}:\mathbf{T}^{\mathrm{ord}}(N,\chi,\mathbf{I})_{\mathbf{m}_F}\to\mathbf{I}$ be the restriction of λ_F to $\mathbf{T}^{\mathrm{ord}}(N,\chi,\mathbf{I})_{\mathbf{m}_F}$. By [Hid88a, Corollary 3.7], there exists a finite dimensional $\mathrm{Frac}\mathbf{I}$-algebra B and an isomorphism

$$\lambda:\mathbf{T}^{\mathrm{ord}}(N,\chi,\mathbf{I})_{\mathbf{m}_F}\otimes_{\mathbf{I}}\mathrm{Frac}\mathbf{I}\cong\mathrm{Frac}\mathbf{I}\oplus B$$

such that $(\mathrm{pr}_{\mathrm{Frac}\mathbf{I}}\circ\lambda)|_{\mathbf{T}^{\mathrm{ord}}(N,\chi,\mathbf{I})_{\mathbf{m}_F}}=\lambda_{\mathbf{m}_F}$, where $\mathrm{pr}_{\mathrm{Frac}\mathbf{I}}:\mathrm{Frac}\mathbf{I}\oplus B\to\mathrm{Frac}\mathbf{I}$ is the projection to the first part.

Definition 2.0.4. *Let* $\mathrm{pr}_{\mathrm{Frac}\mathbf{I}}$ *(resp.* pr_B*) be the projection from* $\mathrm{Frac}\mathbf{I}\oplus B$ *to* $\mathrm{Frac}\mathbf{I}$ *(resp.* B*). We put* $h(\mathrm{Frac}\mathbf{I}):=\mathrm{pr}_{\mathrm{Frac}\mathbf{I}}\circ\lambda(\mathbf{T}^{\mathrm{ord}}(N,\chi,\mathbf{I})_{\mathbf{m}_F})$ *and* $h(B):=\mathrm{pr}_B\circ\lambda(\mathbf{T}^{\mathrm{ord}}(N,\chi,\mathbf{I})_{\mathbf{m}_F})$*. We define the module of congruence for* F *to be*

$$C(F):=h(\mathrm{Frac}\mathbf{I})\oplus h(B)/\lambda(\mathbf{T}^{\mathrm{ord}}(N,\chi,\mathbf{I})_{\mathbf{m}_F}).$$

Let

$$1_F\in\mathbf{T}^{\mathrm{ord}}(N,\chi,\mathbf{I})_{\mathbf{m}_F}\otimes_{\mathbf{I}}\mathrm{Frac}\mathbf{I}$$

be the idempotent element corresponded to $(1,0)\in\mathrm{Frac}\mathbf{I}\oplus B$ by λ. Let $\mathrm{Ann}(C(F)):=\{a\in\mathbf{I}\mid aC(F)=\{0\}\}$ be the annihilator of $C(F)$. By [Wil95, Corollary 2, page 482], $\mathbf{T}^{\mathrm{ord}}(N,\chi,\mathbf{I})_{\mathbf{m}_F}$ is a Gorenstein ring. Hence, by [Hid88b, Theorem 4.4], the annihilator $\mathrm{Ann}(C(F))$ is generated by an element.

Definition 2.0.5. *We call a generator* η_F *of* $\mathrm{Ann}(C(F))$ *a congruence number of* F*.*

Next, we introduce general p-adic families of modular forms. We fix a set of non-zero continuous $\mathcal{O}_K$-algebraic homomorphisms

$$\mathfrak{X}:=\{Q_m:\mathbf{I}\to\overline{\mathbb{Q}}_p\}_{m\geq1}.$$

Then, we define the specialization of an element $G=\sum_{n\geq0}a(n,G)q^n\in\mathbf{I}[\![q]\!]$, at $Q_m\in\mathfrak{X}$ to be $G_{Q_m}:=\sum_{n\geq0}Q_m(a(n,G))q^n\in Q_m(\mathbf{I})[\![q]\!]$. Let χ be a Dirichlet character modulo Np.

Definition 2.0.6. *We call an element* $G\in\mathbf{I}[\![q]\!]$ *a primitive* p*-adic families of tame level* N *and Nebentypus* χ *attached to* $\mathfrak{X}$ *if* G_{Q_m} *is the Fourier expansion of a cuspidal Hecke eigenform of weight* k_{Q_m}*, level* $Np^{e_{Q_m}}$ *and Nebentypus* $\chi\omega_p^{-k_{Q_m}}\epsilon_{Q_m}$ *which is primitive outside of* p *for each positive integer* $m\geq1$*. Here,* k_{Q_m} *and* e_{Q_m} *are positive integers and* ϵ_{Q_m} *is a finite character of* Γ*.*

3 Triple product L-functions

Let (g_1,g_2,g_3) be a triple of primitive forms of weight (k_1,k_2,k_3), level (M_1,M_2,M_3) and Nebentypus (χ_1,χ_2,χ_3). We assume that there exists a Dirichlet character χ such that $\chi_1\chi_2\chi_3=\chi^2$. Let (π_1,π_2,π_3) be a triple of automorphic representations of $\mathrm{GL}_2(\mathbb{A})$ attached to (g_1,g_2,g_3). In this section, we recall the definition of the triple product L-function attached to the automorphic representation

$$\Pi:=\pi_1\otimes(\chi)_{\mathbb{A}}\boxtimes\pi_2\boxtimes\pi_3,$$

where $(\chi)_{\mathbb{A}}$ is the adelization of χ. We define the triple product L-function $L(s,\Pi)$ to be

$$L(s,\Pi)=\prod_{v:\mathrm{place}}L_v(s,\Pi),\ \mathrm{Re}(s)>1,$$

where $L_v(s,\Pi)$ is the GCD local triple product L-function defined in [PSR87] and [Ike92]. Let l be a prime. The local L-function $L_l(s,\Pi)$ at l can be written by the form $1/P(p^{-s})$, where

$P(T) \in \mathbb{C}[T]$ such that $P(0) = 1$. By the result of [Ike98], the archimedean factor $L_\infty(s, \Pi)$ can be written by the form

$$L_\infty(s,\Pi) := \Gamma_{\mathbb{C}}(s+\frac{w}{2})\prod_{i=1}^{3}\Gamma_{\mathbb{C}}(s+1-k_i^*),$$

where $w = k_1 + k_2 + k_3 - 2$, $k_i^* = \frac{k_1+k_2+k3}{2} - k_i$ and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. By [Ike92, Proposition 2.5], the function $L(s,\Pi)$ is continued to the entire $\mathbb{C}$-plane analytically and by [Ike92, Proposition 2.4], the function $L(s,\Pi)$ satisfies the functional equation

$$L(s,\Pi) = \epsilon(s,\Pi)L(1-s,\Pi),$$

where $\epsilon(s,\Pi)$ is the global epsilon factor defined in [Ike92, page 230]. The epsilon factor $\epsilon(s,\Pi)$ can be decomposed by the product of the local epsilon factors

$$\epsilon(s,\Pi) = \prod_{v:\text{place}} \epsilon_v(s,\Pi)$$

and it is known that $\epsilon_v(\frac{1}{2},\Pi) \in \{\pm 1\}$.

4 Construction of p-adic triple product L-functions

Let K be a finite extension of $\mathbb{Q}_p$ and $\mathbf{I}_i$ a normal finite flat extension of Λ_K for $i = 1, 2, 3$. We fix a triple of Dirichlet characters (ψ_1, ψ_2, ψ_3) of modulo (N_1p, N_2p, N_3p), where N_i is a positive integer which is prime to p for $i = 1, 2, 3$. Let $F \in \mathcal{S}^{\text{ord}}(N_1, \psi_1, \mathbf{I}_1)$ be a primitive Hida family defined in Definition 2.0.3. Let $G^{(i)} \in \mathbf{I}_i[\![q]\!]$ be a p-adic family of tame level N_i and Nebentypus ψ_i attached to

$$\mathfrak{X}^{(i)} := \{Q_m^{(i)} : \mathbf{I}_i \to \overline{\mathbb{Q}}_p\}_{m\geq 1}$$

for $i = 2, 3$. In this section, we prove Main theorem and construct the p-adic triple product L-function attached to $(F, G^{(2)}, G^{(3)})$. For simplicity, we assume $N_1 = N_2 = N_3 = 1$. Further, we assume that the triple $(F, G^{(2)}, G^{(3)})$ satisfies Hypothesis (1)$\sim$(7). We set $R := \mathbf{I}_1\widehat{\otimes}_{\mathcal{O}_K}\mathbf{I}_2\widehat{\otimes}_{\mathcal{O}_K}\mathbf{I}_3$ and

$$\mathfrak{X}_R^F := \left\{ \underline{Q} = \left(Q_1, Q_{m_2}^{(2)}, Q_{m_3}^{(3)}\right) \in \mathfrak{X}_{\mathbf{I}_1} \times \mathfrak{X}^{(2)} \times \mathfrak{X}^{(3)} \;\middle|\; \begin{array}{l} k_{Q_1} + k^{(2)}(m_2) + k^{(3)}(m_3) \equiv 0 \pmod 2, \\ k_{Q_1} \geq k^{(2)}(m_2) + k^{(3)}(m_3) \end{array} \right\}.$$

We define a formal operator $\mathbf{U}_{R,p} \in \mathrm{End}_R(R[\![q]\!])$ to be

$$\mathbf{U}_{R,p}(f) = \sum_{n\geq 0} a(pn, f)q^n$$

for each $f = \sum_{n\geq 0} a(n,f)q^n \in R[\![q]\!]$. Let $\Theta : \mathbb{Z}_p^\times \to R^\times$ be a character defined by

$$\Theta(z) = \psi_1\omega_p^{-a}(z)\langle z\rangle_{\mathbf{I}_1}{}^{\frac{1}{2}}(\langle z\rangle^{(2)}\langle z\rangle^{(3)})^{-\frac{1}{2}},$$

for each $z \in \mathbb{Z}_p^\times$, where $\langle z\rangle_{\mathbf{I}_1}$ is the image of $\langle z\rangle_{\Lambda_K}$ by the natural inclusion $\Lambda_K \hookrightarrow \mathbf{I}_1$. For each $f \in \sum_{n\geq 0} a(n,f)q^n \in R[\![q]\!]$, we define a Θ-twisted form $f|[\Theta] \in R[\![q]\!]$ to be

$$f|[\Theta] = \sum_{p\nmid n} \Theta(n) \cdot a(n,f)q^n.$$

We set $d := \frac{d}{dq}$. For each $\underline{Q} = (Q_1, Q^{(2)}_{m_2}, Q^{(3)}_{m_3}) \in \mathfrak{X}^F_R$, we have $f|[\Theta](\underline{Q}) = d^{r_{\underline{Q}}}(f(\underline{Q})|[\Theta_{\underline{Q}}])$ with the Dirichlet character

$$\Theta_{\underline{Q}} = \psi_1 \omega_p^{-a-r_{\underline{Q}}} \epsilon_{Q_1}^{\frac{1}{2}} \epsilon^{(2)\,-\frac{1}{2}}_{m_2} \epsilon^{(3)\,-\frac{1}{2}}_{m_3},$$

where $r_{\underline{Q}} = \frac{1}{2}(k_{Q_1} - k^{(2)}(m_2) - k^{(3)}(m_3))$. Here, $f(\underline{Q})|[\Theta_{\underline{Q}}]$ is the twisted cusp form by the Dirichlet character $\Theta_{\underline{Q}}$. We regard $G^{(2)}$ and $G^{(3)}$ as elements of $R[\![q]\!]$ by natural embeddings $\mathbf{I}_2 \hookrightarrow R$ and $\mathbf{I}_3 \hookrightarrow R$. We set $H := G^{(2)} \cdot (G^{(3)}|[\Theta]) \in R[\![q]\!]$. We define the Maass-Shimura differential operator δ_k to be

$$\delta_k := \frac{1}{2\pi\sqrt{-1}} \left(\frac{\partial}{\partial z} + \frac{k}{2\sqrt{-1}\mathrm{Im}(z)} \right)$$

for each non-negative integer k. Further, we set $\delta_k^m := \delta_{k+2m-2} \dots \delta_{k+2}\delta_k$, where m is a non-negative integer. We denote by $\mathcal{H}$ the holomorphic projection from the space of nearly holomorphic modular forms to modular forms defined in [Shi76]. Let $\mathbf{m}_R$ be the maximal ideal of R.

Lemma 4.0.1. *Let $\underline{Q} = (Q_1, Q^{(2)}_{m_2}, Q^{(3)}_{m_3}) \in \mathfrak{X}^F_R$. We fix a finite extension L of K such that $\mathcal{O}_L$ contains $Q_1(\mathbf{I}_1), Q^{(2)}_{m_2}(\mathbf{I}_2)$ and $Q^{(3)}_{m_3}(\mathbf{I}_3)$. Then, the sequence $\{U^{n!}_{\mathbf{R},p} H(\underline{Q})\}_{n\geq 1}$ converges in $\mathcal{O}_L[\![q]\!]$ by the $\mathbf{m}_R$-adic topology and the limit of the sequence equals to the Fourier expansion of $e\mathcal{H}(G^{(2)}(m_2)\delta^{r_{\underline{Q}}}_{k^{(3)}(m_3)} G^{(3)}(m_3)|\Theta_{\underline{Q}}) \in eS_{k_{Q_1}}(p^{e_{Q_1}}, \psi_1\omega_p^{k_{Q_1}}\epsilon_{Q_1}, L)$, with $e_{Q_1} := \max\{1, m_{\epsilon_{Q_1}}\}$. Here, $m_{\epsilon_{Q_1}}$ is the p-power of the conductor of ϵ_{Q_1}.*

Proof. It is known that $H(\underline{Q})$ is a Fourier expansion of a p-adic modular form and by [Hid85, Lemma 5.2], we have

$$H(\underline{Q}) = \mathcal{H}(G^{(2)}(m_2)\delta^{r_{\underline{Q}}}_{k^{(3)}(m_3)} G^{(3)}(m_3)|\Theta_{\underline{Q}}) + d(g'_{\underline{Q}}) \in L[\![q]\!],$$

where $g'_{\underline{Q}} \in L[\![q]\!]$ is a p-adic modular form. By [Hid85, (6.12)], $ed = 0$ and we have $eH(\underline{Q}) = e\mathcal{H}(G^{(2)}(m_2)\delta^{r_{\underline{Q}}}_{k^{(3)}(m_3)} G^{(3)}(m_3)|\Theta_{\underline{Q}})$. Further, by [Hid85, (4.3)], the sequence $\{U^{n!}_{R,p} H(\underline{Q})\}_{n\geq 1}$ converges in $\mathcal{O}_L[\![q]\!]$ by the $\mathbf{m}_R$-adic topology and the limit of the sequence equals to $eH(\underline{Q})$. We have completed the proof. □

To construct a triple product p-adic L-function $L^F_{G^{(2)},G^{(3)}} \in R$, we prove the following lemma and proposition.

Lemma 4.0.2. *There exists a unique element $H^{\mathrm{ord}} \in R[\![q]\!]$ such that the specialization of H^{ord} at each $\underline{Q} = (Q_1, Q^{(2)}_{m_2}, Q^{(3)}_{m_3}) \in \mathfrak{X}^F_R$ equals to the Fourier expansion of the modular form $e\mathcal{H}(G^{(2)}(m_2)\delta^{r_{\underline{Q}}}_{k^{(3)}(m_3)} G^{(3)}(m_3)|\Theta_{\underline{Q}})$.*

Proof. Let $I_{\underline{Q}}$ be the ideal of R generalized by $\mathrm{Ker}Q_1$,$\mathrm{Ker}Q^{(2)}_{m_2}$ and $\mathrm{Ker}Q^{(3)}_{m_3}$ for each $\underline{Q} = (Q_1, Q^{(2)}_{m_2}, Q^{(3)}_{m_3}) \in \mathfrak{X}^F_R$. We denote by $\mathfrak{B}$ the set of finite intersections of $I_{\underline{Q}}$ for $\underline{Q} \in \mathfrak{X}^F_R$. Then, we can easily check that $\cap_{J\in\mathfrak{B}} J = \{0\}$. Further, we have the natural isomorphism $R \cong \varprojlim_{J\in\mathfrak{B}} (R/J)$. In particular, we have

$$R[\![q]\!] \cong \varprojlim_{J\in\mathfrak{B}} R[\![q]\!] \otimes_R (R/J).$$

For each $J = \cap_{i=1}^m I_{\underline{Q}_i} \in \mathfrak{B}$, it suffices to prove that there exists a unique element $H^{\mathrm{ord}}_J \in R[\![q]\!] \otimes_R (R/J)$ such that the image of H^{ord}_J by the natural embedding $i_J : R[\![q]\!] \otimes_R (R/J) \hookrightarrow$

$\prod_{i=1}^{m}(R[\![q]\!] \otimes_R R/I_{\underline{Q}_i})$ equals to $\left[e(H(\underline{Q}_i))\right]_{i=1}^{m}$. The uniqueness of H_J^{ord} is trivial. We prove the existence of H_J^{ord}.

Let $p_J : R[\![q]\!] \to R \otimes_R (R/J)$ be the natural projection. If $J = I_{\underline{Q}}$ for $\underline{Q} \in \mathfrak{X}_R^F$, we have $\lim_{n\to\infty} p_J(U_{R,p}^{n!}H) = e\mathcal{H}(\underline{Q})$ by Lemma 4.0.1. We assume that there exist elements $H_J^{\mathrm{ord}} = \lim_{n\to\infty} p_J(U_{R,p}^{n!}H) \in R[\![q]\!] \otimes (R/J)$ and $H_{J'}^{\mathrm{ord}} = \lim_{n\to\infty} p_{J'}(U_{R,p}^{n!}H) \in R[\![q]\!] \otimes (R/J')$ for a pair $(J, J') \in \mathcal{B} \times \mathcal{B}$. We define the R-linear map:

$$\begin{array}{ccc} (R[\![q]\!] \otimes_R (R/J)) \times (R[\![q]\!] \otimes_R (R/J')) & \overset{i_{J,J'}}{\longrightarrow} & (R[\![q]\!] \otimes_R (R/J + J')) \\ \cup & & \cup \\ (a, b) & \longmapsto & a - b \end{array}.$$

Then, we have $i_{J,J'}(H_J^{\mathrm{ord}}, H_{J'}^{\mathrm{ord}}) = \lim_{n\to\infty} i_{J,J'}(p_J(U_{R,p}^{n!}H), p_{J'}(U_{R,p}^{n!}H)) = 0$. Further, since $\mathrm{Ker}\ i_{J,J'} \cong R[\![q]\!] \otimes_R (R/J \cap J')$, there exists a unique element $H_{J\cap J'}^{\mathrm{ord}} \in R[\![q]\!] \otimes_R (R/J \cap J')$ such that the image of $H_{J\cap J'}^{\mathrm{ord}}$ in $(R[\![q]\!] \otimes_R (R/J)) \times (R[\![q]\!] \otimes_R (R/J'))$ equals to $(H_J^{\mathrm{ord}}, H_{J'}^{\mathrm{ord}})$. In particular, we have $H_{J\cap J'}^{\mathrm{ord}} = \lim_{n\to\infty} p_{J\cap J'}(U_{R,p}^{n!}H)$. Then, for each $J = \cap_{i=1}^{m} I_{\underline{Q}_i} \in \mathcal{B}$, there exists a unique element $H_J^{\mathrm{ord}} \in R[\![q]\!] \otimes_R (R/J)$ such that the image of H_J^{ord} by the natural embedding $i_J : R[\![q]\!] \otimes_R (R/J) \hookrightarrow \prod_{i=1}^{m}(R[\![q]\!] \otimes_R R/I_{\underline{Q}_i})$ equals to $\left[e(H(\underline{Q}_i))\right]_{i=1}^{m}$. We have completed the proof. □

Proposition 4.0.3. *The power series H^{ord} is an element of $\mathbf{S}^{\mathrm{ord}}(N, \psi_1, \mathbf{I}_1)\widehat{\otimes}_{\mathbf{I}_1} R$.*

Proof. We identify the Iwasawa algebra Λ_K with $\mathcal{O}_K[\![X]\!]$ by the isomorphism $[1+p] \mapsto 1 + X$ and we regard $\mathbf{I}_i$ as the normal finite flat extension of $\mathcal{O}_K[\![X_i]\!]$ for $i = 1, 2, 3$. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be a base of R over $R_0 = \mathcal{O}_K[\![X_1, X_2, X_3]\!]$. We put

$$H^{\mathrm{ord}} = \sum_{i=1}^{n} H^{(i)}\alpha_i,$$

where $H^{(i)} \in R_0[\![q]\!]$ for each $i = 1, \ldots, n$. We put $L = \mathrm{Frac}R$ and $L_0 = \mathrm{Frac}R_0$. Let $\mathrm{Tr}_{L/L_0} : L \to L_0$ be the trace map and $\alpha_1^*, \alpha_2^*, \ldots, \alpha_n^*$ be the dual base of $\alpha_1, \alpha_2 \ldots, \alpha_n$ with respect to Tr_{L/L_0}.Then, we have

$$H^{(i)}(\underline{Q}) = \mathrm{Tr}(H(\underline{Q})\alpha_i^*(\underline{Q}))$$

for all but a finite number of $\underline{Q} = (Q_1, Q_{m_2}^{(2)}, Q_{m_3}^{(3)}) \in \mathfrak{X}_R^F$. Further, $\mathrm{Tr}(H(\underline{Q})\alpha_i^*(\underline{Q}))$ is the Fourier expansion of an element of $eS_{k_{Q_1}}(Np^{e_{Q_1}}, \epsilon_{Q_1}\psi_1\omega_p^{-k_{Q_1}}, \underline{Q}(R))$. It suffices to prove

$$H^{(i)} \in \mathbf{S}^{\mathrm{ord}}(1, \psi_1, \mathcal{O}_K[\![X_1]\!])\widehat{\otimes}_{\mathcal{O}_K[\![X_1]\!]} R_0$$

for each $i = 1, \ldots, n$.

For each positive integers m_2, m_3, let $H_{m_2,m_3}^{(i)} \in \mathcal{O}_K[b_{m_2}^{(2)}, b_{m_3}^{(3)}][\![X_1]\!][\![q]\!]$ be the specialization of $H^{(i)}$ at $(Q_{m_2}^{(2)}, Q_{m_3}^{(3)})$, where $b_{m_2}^{(2)} := Q_{m_2}^{(2)}(X_2)$ and $b_{m_3}^{(3)} := Q_{m_3}^{(3)}(X_3)$. First, we prove $H_{m_2,m_3}^{(i)} \in \mathbf{S}^{\mathrm{ord}}(1, \psi_1, \mathcal{O}_K[b_{m_2}^{(2)}, b_{m_3}^{(3)}][\![X_1]\!])$. We define a subset $\mathfrak{X}_{m_2,m_3}^F$ of arithmetic points of $\mathbf{I}_1$ to be

$$\mathfrak{X}_{m_2,m_3}^F := \left\{ Q \in \mathfrak{X}_{\mathbf{I}_1} \;\middle|\; (Q, Q_{m_2}^{(2)}, Q_{m_3}^{(3)}) \in \mathfrak{X}_R^F \right\}.$$

For each positive integer k, there exists an arithmetic point $Q \in \mathfrak{X}_{\mathbf{I}_1}$ with $k_Q = k$. Then, we have $\#\mathfrak{X}_{m_2,m_3}^F = \infty$. Let $\mathbf{S}_{m_2,m_3}^{\mathrm{ord}} \subset \mathcal{O}_K[b_{m_2}^{(2)}, b_{m_3}^{(3)}][\![X_1]\!][\![q]\!]$ be an $\mathcal{O}_K[b_{m_2}^{(2)}, b_{m_3}^{(3)}][\![X_1]\!]$-module consisting of elements $f \in \mathcal{O}_K[b^{(2)}_{m_2}, b^{(3)}_{m_3}][\![X_1]\!][\![q]\!]$ such that, for all but a finite number of $Q \in \mathfrak{X}^F_{m_2,m_3}$, $f(Q)$ equals to the specialization of an element of $\mathbf{S}^{\mathrm{ord}}(1, \psi_1, \mathcal{O}_K[b^{(2)}_{m_2}, b^{(3)}_{m_3}][\![X_1]\!])$ at Q. Then, we have $\mathbf{S}^{\mathrm{ord}}(1, \psi_1, \mathcal{O}_K[b^{(2)}_{m_2}, b^{(3)}_{m_3}][\![X_1]\!]) \subset \mathbf{S}^{\mathrm{ord}}_{m_2,m_3}$ and $H^{(i)}_{m_2,m_3} \in \mathbf{S}^{\mathrm{ord}}_{m_2,m_3}$. It suffices to prove that we have $\mathbf{S}^{\mathrm{ord}}(1, \psi_1, \mathcal{O}_K[b^{(2)}_{m_2}, b^{(3)}_{m_3}][\![X_1]\!]) = \mathbf{S}^{\mathrm{ord}}_{m_2,m_3}$. Let $g_1, \dots, g_d$ be elements of $\mathbf{S}^{\mathrm{ord}}_{m_2,m_3}$ which are $\mathcal{O}_K[b^{(2)}_{m_2}, b^{(3)}_{m_3}][\![X_1]\!]$-linear independent. Then, there are positive integers $m_1, \dots, m_d$ such that

$$d = \det(a(m_i, g_j))_{1 \le i,j \le d} \neq 0 \in \mathcal{O}_K[b^{(2)}_{m_2}, b^{(3)}_{m_3}][\![X_1]\!].$$

Since $\#\mathfrak{X}^F_{m_2,m_3} = \infty$, there exists an element $Q \in \mathfrak{X}^F_{m_2,m_3}$ such that $d(Q) \neq 0$. Then, we have

$$\mathrm{rank}_{\mathcal{O}_K[b^{(2)}_{m_2}, b^{(3)}_{m_3}][\![X_1]\!]} \mathbf{S}^{\mathrm{ord}}_{m_2,m_3} = \mathrm{rank}_{\mathcal{O}_K[b^{(2)}_{m_2}, b^{(3)}_{m_3}][\![X_1]\!]} \mathbf{S}^{\mathrm{ord}}(1, \psi_1, \mathcal{O}_K[b^{(2)}_{m_2}, b^{(3)}_{m_3}][\![X_1]\!]).$$

Then, if we take an element $f \in \mathbf{S}^{\mathrm{ord}}_{m_2,m_3}$, there exists an element $a \in \mathcal{O}_K[b^{(2)}_{m_2}, b^{(3)}_{m_3}][\![X_1]\!]\backslash\{0\}$ such that $af \in \mathbf{S}^{\mathrm{ord}}(1, \psi_1, \mathcal{O}_K[b^{(2)}_{m_2}, b^{(3)}_{m_3}][\![X_1]\!])$. Since a has only finite roots, we have $f \in \mathbf{S}^{\mathrm{ord}}(1, \psi_1, \mathcal{O}_K[b^{(2)}_{m_2}, b^{(3)}_{m_3}][\![X_1]\!])$. Then, we have $\mathbf{S}^{\mathrm{ord}}(1, \psi_1, \mathcal{O}_K[b^{(2)}_{m_2}, b^{(3)}_{m_3}][\![X_1]\!]) = \mathbf{S}^{\mathrm{ord}}_{m_2,m_3}$.

For each positive integer m_3, let $H^{(i),m_3} \in \mathcal{O}_K[b^{(3)}_{m_3}][\![X_1, X_2]\!]$ be the specialization of $H^{(i)}$ at $Q^{(3)}_{m_3}$. Next, we prove $H^{(i),m_3} \in \mathbf{S}^{\mathrm{ord}}(1, \psi_1, \mathcal{O}_K[b^{(3)}_{m_3}][\![X_1]\!]) \widehat{\otimes}_{\mathcal{O}_K[b^{(3)}_{m_3}]} \mathcal{O}_K[b^{(3)}_{m_3}][\![X_2]\!]$. We define an $\mathcal{O}_K[b^{(3)}_{m_3}][\![X_1, X_2]\!]$-module $\mathbf{S}^{\mathrm{ord}}_{m_3} \subset \mathcal{O}_K[b^{(3)}_{m_3}][\![X_1, X_2]\!]$ consisting of elements $f(X_1, X_2)$ such that $f(X_1, b^{(2)}_m) \in \mathbf{S}^{\mathrm{ord}}(1, \psi_1, \mathcal{O}_K[\![X_1]\!]) \otimes_{\mathcal{O}_K} \mathcal{O}_{\overline{\mathbb{Q}}_p}$ for each positive integer m. We have already proved that $H^{(i),m_3} \in \mathbf{S}^{\mathrm{ord}}_{m_3}$. It is clear that $\mathbf{S}^{\mathrm{ord}}(1, \psi_1, \mathcal{O}_K[b^{(3)}_{m_3}][\![X_1]\!]) \widehat{\otimes}_{\mathcal{O}_K[b^{(3)}_{m_3}]} \mathcal{O}_K[b^{(3)}_{m_3}][\![X_2]\!] \subset \mathbf{S}^{\mathrm{ord}}_{m_3}$. Further, if $g_1, \dots, g_d \in \mathbf{S}^{\mathrm{ord}}_{m_3}$ are linear independent, there exist positive integers $m_1, \dots, m_d$ such that

$$d = \det(a(m_i, g_j))_{1 \le i,j \le d} \neq 0 \in \mathcal{O}_K[b^{(3)}_{m_3}][\![X_1, X_2]\!].$$

We can take a positive integer m_2 such that $d(X_1, b^{(2)}_{m_2}) \neq 0$. Then, $\mathrm{rank}_{\mathcal{O}_K[b^{(3)}_{m_3}][\![X_1,X_2]\!]} \mathbf{S}^{\mathrm{ord}}_{m_3} = \mathrm{rank}_{\mathcal{O}_K[b^{(3)}_{m_3}][\![X_1]\!]} \mathbf{S}^{\mathrm{ord}}(1, \psi_1, \mathcal{O}_K[b^{(3)}_{m_3}][\![X_1]\!])$. We take an element $a \in \mathcal{O}_K[b^{(3)}_{m_3}][\![X_1, X_2]\!]\backslash\{0\}$ such that $aH^{(i),m_3} \in \mathbf{S}^{\mathrm{ord}}(1, \psi_1, \mathcal{O}_K[b^{(3)}_{m_3}][\![X_1]\!]) \widehat{\otimes}_{\mathcal{O}_K[b^{(3)}_{m_3}]} \mathcal{O}_K[b^{(3)}_{m_3}][\![X_2]\!]$. Since we have $a(X_1, p^m) \neq 0$ for almost all positive integers m, there exists a positive integer k_{m_3} such that $H^{(i),m_3}(X_1, p^{m'}) \in \mathbf{S}^{\mathrm{ord}}(1, \psi_1, \mathcal{O}_K[b^{(3)}_{m_3}][\![X_1]\!])$ for each positive integer $m' \ge k_{m_3}$.

We put $H^{(i),m_3}_0 := H^{(i),m_3}$ and $c_m = p^{k_{m_3}+m}$ for each non-negative integer m. We define a power series $H^{(i),m_3}_m \in \mathcal{O}_K[b^{(3)}_{m_3}][\![X_1, X_2]\!][\![q]\!]$ inductively for each positive integer m to be

$$H^{(i),m_3}_m(X_1, X_2) := (H^{(i),m_3}_{m-1}(X_1, X_2) - H^{(i),m_3}_{m-1}(X_1, c_m))(X_2 - c_m)^{-1} \in \mathcal{O}_K[b^{(3)}_{m_3}][\![X_1, X_2]\!][\![q]\!].$$

By the induction of m, we have $H^{(i),m_3}_m(X_1, c_l) \in \mathbf{S}^{\mathrm{ord}}(1, \psi_1, \mathcal{O}_K[b^{(3)}_{m_3}][\![X_1]\!])$ for each non-negative integer m and $l \ge m+1$. In particular, if we put $H^{(i),m_3}_{m,m+1} := H^{(i),m_3}_m(X_1, c_{m+1})$, we have

$$H^{(i),m_3} = \sum_{m=1}^{\infty} H^{(i),m_3}_{m,m+1} \prod_{j=1}^{m} (X_2 - c_j) \in \mathbf{S}^{\mathrm{ord}}(1, \psi_1, \mathcal{O}_K[b^{(3)}_{m_3}][\![X_1]\!]) \widehat{\otimes}_{\mathcal{O}_K[b^{(3)}_{m_3}]} \mathcal{O}_K[b^{(3)}_{m_3}][\![X_2]\!].$$

Next, we prove $H^{(i)} \in \mathbf{S}^{\mathrm{ord}}(1, \psi_1, \mathcal{O}_K[\![X_1]\!]) \widehat{\otimes}_{\mathcal{O}_K} \mathcal{O}_K[\![X_2, X_3]\!]$. By the same way as above, we can take a non-zero element $a \in \mathcal{O}_K[\![X_1, X_2, X_3]\!]\backslash\{0\}$ such that $aH^{(i)}$ is an element of $\mathbf{S}^{\mathrm{ord}}(1, \psi_1, \mathcal{O}_K[\![X_1]\!]) \widehat{\otimes}_{\mathcal{O}_K} \mathcal{O}_K[\![X_2, X_3]\!]$. Further, there exists a positive integer k which satisfies $H^{(i)}(X_1, X_2, p^m) \in \mathbf{S}^{\mathrm{ord}}(1, \psi_1, \mathcal{O}_K[\![X_1]\!]) \widehat{\otimes}_{\mathcal{O}_K} \mathcal{O}_K[\![X_2]\!]$ for each $m \ge k$. We put $H^{(i)}_0 := H^{(i)}$ and $c'_m = p^{k+m}$ for each non-negative integer m. We define a power series $H^{(i)}_m \in \mathcal{O}_K[\![X_1, X_2, X_3]\!][\![q]\!]$ inductively for each positive integer m to be

$$H^{(i)}_m := (H^{(i)}_{m-1}(X_1, X_2, X_3) - H^{(i)}_{m-1}(X_1, X_2, c'_m))(X_3 - c'_m)^{-1} \in \mathcal{O}_K[\![X_1, X_2, X_3]\!][\![q]\!].$$

Then, we have

$$H^{(i)} = \sum_{m=0}^{\infty} H_m^{(i)}(X_1, X_2, c'_{m+1}) \prod_{j=1}^{m} (X_3 - c'_j) \in \mathbf{S}^{\mathrm{ord}}(1, \psi_1, \mathcal{O}_K[\![X_1]\!]) \widehat{\otimes}_{\mathcal{O}_K} \mathcal{O}_K[\![X_2, X_3]\!].$$

We have completed the proof. □

Definition 4.0.4. *We define an element $L^F_{G^{(2)},G^{(3)}} \in R$ to be*

$$L^F_{G^{(2)},G^{(3)}} := a(1, \eta_F 1_F(H^{\mathrm{ord}})).$$

Here, 1_F is the idempotent element defined in §2 and η_F is the congruence number defined in Definition 2.0.5.

By [Hid85, Proposition 4.5] and [Ich08, Theorem 1.1] , we have the interpolation formula of $L_{G^{(2)},G^{(3)}}$. However, we omit the detail of the proof of the interpolation formula. Let $\Omega_{F_{Q_1}}$ be the canonical period defined in [Hsi17, (1.3)] and $\mathcal{E}_{F_{Q_1},p}(\Pi_{\underline{Q}})$ the modified p-Euler factor defined in [Hsi17, (1.2)].

Proposition 4.0.5. *We assume Hypotheses (1)∼(7). Then, there exists an element $\mathcal{L}^F_{G^{(2)},G^{(3)}} \in R$ such that we have the interpolation property :*

$$(\mathcal{L}^F_{G^{(2)},G^{(3)}}(\underline{Q}))^2 = \mathcal{E}_{F_{Q_1},p}(\Pi_{\underline{Q}}) \cdot \frac{L(\frac{1}{2}, \Pi_{\underline{Q}})}{(\sqrt{-1})^{2k_{Q_1}} \Omega^2_{F_{Q_1}}}$$

for every $\underline{Q} = (Q_1, Q^{(2)}_{m_2}, Q^{(3)}_{m_3}) \in \mathfrak{X}^F_R$.

5 Examples

In this subsection, we give examples of the triple $(\mathbf{I}_i, \mathfrak{X}^{(i)}, G^{(i)})$ which satisfy Hypothesis (5), (6) and (7). As a first example, we can take families of CM forms of weight 1. Let L be a quadratic imaginary extension of $\mathbb{Q}$ with a discriminant D. We assume that D is square-free and prime to p. Let $\mathfrak{f}$ be an integral ideal of $\mathcal{O}_L$ such that $\mathfrak{f}$ is prime to Dp. We assume that $\mathrm{N}(\mathfrak{f})$ is square-free, where N is the absolute norm. Let $\mathfrak{C}(\mathbf{f}(p)^j)$ be the class ray group modulo $\mathbf{f}(p)^j$ over L for each $j \geq 0$. By the class field theory, $\mathfrak{C}(\mathfrak{f}(p)^\infty) = \varprojlim_{j\geq 0} \mathfrak{C}(\mathfrak{f}(p)^j)$ is a $\mathbb{Z}_p$-module of rank 2. Let $\Delta_\mathfrak{f}$ be the torsion part of $\mathfrak{C}(\mathfrak{f}(p)^\infty)$ and $\chi : \Delta_\mathfrak{f} \to \mathbb{C}^\times$ be a primitive character. Here, a primitive character means that it is not induced by any character from $\Delta_{\mathfrak{f}'}$ for $\mathfrak{f} \subsetneq \mathfrak{f}'$. Let L^-_∞/L be the anticyclotomic extension of L. By the class field theory, the Galois group $\mathrm{Gal}(L^-_\infty/L)$ is a direct summand of the $\mathbb{Z}_p$-torsion free part of $\mathfrak{C}(\mathfrak{f}(p)^\infty)$. Let $\mathrm{pr}_\mathfrak{f} : \mathfrak{C}(\mathfrak{f}(p)^\infty) \to \Delta_\mathfrak{f}$ and $\mathrm{pr}_- : \mathfrak{C}(\mathfrak{f}(p)^\infty) \to \mathrm{Gal}(L^-_\infty/L)$ be the natural projections to $\Delta_\mathfrak{f}$ and $\mathrm{Gal}(L^-_\infty/L)$ respectively. Let E be a finite Galois extension of $\mathbb{Q}_p$ such that the image of $\Delta_\mathfrak{f}$ by χ is contained in E. We define a group homomorphism

$$\Psi : \mathfrak{C}(\mathfrak{f}(p)^\infty) \to \mathcal{O}_E[\![\mathrm{Gal}(L^-_\infty/L)]\!]^\times$$

to be $\Psi(a) = \chi(\mathrm{pr}_\mathfrak{f}(a))[\mathrm{pr}_-(a)]$ for $a \in \mathfrak{C}(\mathfrak{f}(p)^\infty)$. Let $\mathrm{J}_{\mathfrak{f}(p)}$ be the group which consists of fractional ideals $\mathfrak{a}$ of L which is prime to $\mathfrak{f}(p)$. For each finite prime ideal $\mathfrak{l}$, we denote by $L_\mathfrak{l}$ the completion of L by $\mathfrak{l}$. Let $\mathcal{O}_{L_\mathfrak{l}}$ be the integers of $L_\mathfrak{l}$ and $\pi_\mathfrak{l}$ a generator of the maximal ideal of $\mathcal{O}_{L_\mathfrak{l}}$. We define a group homomorphism

$$\Psi^* : \mathrm{J}_{\mathfrak{f}(p)} \to \mathcal{O}_E[\![\mathrm{Gal}(L^-_\infty/L)]\!]^\times$$

to be $\Psi^*(\mathfrak{a}) = \prod_{\mathfrak{l}\nmid\mathfrak{f}(p)} \Psi_{\mathfrak{l}}(\pi_{\mathfrak{l}}^{n_{\mathfrak{l}}})$, where $\Psi = \prod_{\mathfrak{l}} \Psi_l$ and $\mathfrak{a} = \prod_{\mathfrak{l}\nmid\mathfrak{f}(p)} \mathfrak{l}^{n_{\mathfrak{l}}}$. We put

$$F_\Psi = \sum_{\mathfrak{a}\nmid\mathfrak{f}(p)} \Psi^*(\mathfrak{a}) q^{\mathrm{N}(\mathfrak{a})},$$

where $\mathfrak{a}$ runs through integral ideals of L which are prime to $\mathfrak{f}(p)$. Let $\epsilon : \mathrm{Gal}(L_\infty^-/L) \to \overline{\mathbb{Q}}^\times$ be a finite character. We denote by $P_\epsilon : \mathcal{O}_E[\![\mathrm{Gal}(L_\infty^-/L)]\!] \to \overline{\mathbb{Q}}_p$ the $\mathcal{O}_E$-algebra homomorphism defined by $P_\epsilon([w]) = \epsilon(w)$ for $w \in \mathrm{Gal}(L_\infty^-/L)$. It is known that for each finite character $\epsilon : \mathrm{Gal}(L_\infty^-/L) \to \overline{\mathbb{Q}}^\times$, the series $f_\epsilon := P_\epsilon(F_\Psi) \in P_\epsilon(\mathcal{O}_E[\![\mathrm{Gal}(L_\infty^-/L)]\!])[\![q]\!]$ is the Fourier expansion of a classical modular form of weight 1 and level $(-D)\mathrm{N}(\mathbf{f})p^{e_\epsilon}$, where e_ϵ is a positive integer (*cf.* [Miy06, Theorem 4.8.2]). By the definition, f_ϵ is the CM-form. We remark that the p-th coefficient $a(p, F_\Psi) \in \mathcal{O}_E[\![\mathrm{Gal}(L_\infty^-/L)]\!]$ of F_Ψ is zero by the definition. However, if $\epsilon : \mathrm{Gal}(L_\infty^-/L) \to \overline{\mathbb{Q}}^\times$ is primitive and the conductor is sufficiently large, it is known that f_ϵ is a primitive form (*cf.* [Miy06, Theorem 4.8.2]). Then, if we put $\mathfrak{X} := \{\mathrm{Ker}P_\epsilon \mid f_\epsilon \text{ is primitive}\}$, the cardinality of $\mathfrak{X}$ is not finite, and the triple $(\mathcal{O}_E[\![\mathrm{Gal}(L_\infty^-/L)]\!], \mathfrak{X}, F_\Psi)$ satisfies the condition (6). Further, it is not difficult to prove that the triple $(\mathcal{O}_E[\![\mathrm{Gal}(L_\infty^-/L)]\!], \mathfrak{X}, F_\Psi)$ satisfies the condition (5). Let $\mathrm{pr}_{\mathbb{A}^\times} : \mathbb{A}^\times \to \mathfrak{C}(\mathfrak{f}(p)^\infty)$ be the natural projection defined by the class field theory. We denote by $j_p : \mathbb{Q}_p^\times \hookrightarrow \mathbb{A}^\times$ the natural injection. If we put $\langle n \rangle = n\omega_p(n)^{-1}\Psi([\mathrm{pr}_{\mathbb{A}^\times} \circ j_p(n\omega_p(n)^{-1})])^{-1} \in \mathcal{O}_E[\![\mathrm{Gal}(L_\infty^-/L)]\!]^\times$ for each positive integer n which is prime to p, $\langle n \rangle$ satisfies the condition of (5). Since $D\mathrm{N}(\mathbf{f})$ is square-free, by [Miy06, Theorem 4.6.17], F_Ψ satisfies Hypothesis (7).

As a second example of $(\mathbf{I}_i, \mathfrak{X}^{(i)}, G^{(i)})$, we give Coleman families. For an element $x \in K$ and $\epsilon \in p^{\mathbb{Q}}$, we denote by $\mathcal{B}[x,\epsilon]_K$ the closed ball of radius ϵ and center x, seen as a K-affinoid space. We denote by $\mathcal{A}_{\mathcal{B}[x,\epsilon]_K}$ the ring of analytic functions on $\mathcal{B}[x,\epsilon]_K$ and by $\mathcal{A}^0_{\mathcal{B}[x,\epsilon]_K}$ the subring of power bounded elements of $\mathcal{A}_{\mathcal{B}[x,\epsilon]_K}$. We remark that if $\epsilon \in K$, the ring $\mathcal{A}^0_{\mathcal{B}[x,\epsilon]_K}$ is isomorphic to the ring

$$\mathcal{O}_K\langle \epsilon^{-1}(T-x)\rangle = \left\{ \sum_{n\geq 0} a_n \left(\epsilon^{-1}(T-x)\right)^n \in \mathcal{O}_K[\![\epsilon^{-1}(T-x)]\!] \,\middle|\, \lim_{n\to\infty} |a_n|_p = 0 \right\}.$$

Let M be a positive integer which is prime to p and square-free. Let ϵ_M be a Dirichlet character mod M. Let f be a p-stabilized newform of weight k_0, level Mp , slope $\alpha < k_0 - 1$ and Nebentypus $\epsilon_M\omega_p^{i-k_0}$ where $0 \leq i \leq p-1$. Further, we assume that $a(p,f)^2 \neq \epsilon_M(p)p^{k_0-1}$ if $i = 0$. Then, by Coleman in [Col97], there exists an element $\epsilon \in p^{\mathbb{Q}} \cap K$ and a series

$$G \in \mathcal{A}^0_{\mathcal{B}[k_0,\epsilon]_K}[\![q]\!]$$

such that the specialization $G(k)$ of G at k is the Fourier expansion of a normalized Hecke eigenform of weight k, level Mp, slope α and Nebentypus $\epsilon_M\omega_p^{i-k}$ for each positive integer $k \in \mathcal{B}[k_0,\epsilon]_K(K)$ which is greater than $\alpha + 1$. Further, we prove in [Fuk19, A2.7] that we can take a sufficiently small ϵ such that $G(k)$ is a p-stabilized newform for each positive integer $k \in \mathcal{B}[k_0,\epsilon]_K(K)$ which is greater than $\alpha + 1$. If we put $X = \epsilon^{-1}(T - k_0)$, we can regard the Coleman series G as a series $G(X)$ in $\mathcal{O}_K[\![X]\!]$. Let $k \in \mathcal{B}[k_0,\epsilon]_K(K)$ be a positive integer which is greater than $\alpha + 1$. If we put $b_k = \epsilon^{-1}(k - k_0)$, $G(b_k)$ is the Fourier expansion of a p-stabilized newform of weight k, level Mp, slope α and Nebentypus $\epsilon_M\omega_p^{i-k}$. We denote by $P_k : \mathcal{O}_K[\![X]\!] \to K$ the continuous $\mathcal{O}_K$-algebra homomorphism defined by $P_k(X) = b_k$. We define $\mathfrak{X}$ to be the set consisting of P_k for each positive integer $k \in \mathcal{B}[k_0,\epsilon]_K(K)$ which is greater than $\alpha + 1$. Then, the triple $(\mathcal{O}_K[\![X]\!], \mathfrak{X}, G(X))$ satisfies Hypothesis (6). We check that the triple $(\mathcal{O}_K[\![X]\!], \mathfrak{X}, G(X))$ satisfies Hypothesis (5). Let $\exp(x)$ and $\log(x)$ be the formal exponential series and log series in $K[\![x]\!]$ defined by

$$\exp(x) = \sum_{n \geq 0} \frac{1}{n!} x^n,$$
$$\log(x) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} x^n.$$

We fix an isomorphism $\Lambda_K \cong \mathcal{O}_K[\![X]\!]$ defined by $[1+p] \mapsto X+1$ and we define a formal series

$$\langle n \rangle' := \langle n \rangle_{\Lambda_K}((1+p)^{k_0} \exp(\epsilon X \log(1+p)) - 1)$$

for each positive integer n which is prime to p. We remark that since we have $|p^m|_p \leq |m!|_p$ for each positive integer m, the series $\langle n \rangle'$ is contained in $\mathcal{O}_K[\![X]\!]$. Further, for each positive integer n which is prime to p, the series $\langle n \rangle'$ satisfies the condition of Hypothesis (5). Since M is square-free, by [Miy06, Theorem 4.6.17], $G(X)$ satisfies Hypothesis (7).

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