# CONGRUENCE PROPERTIES OF ENDO-CLASSES AND THE LOCAL JACQUET–LANGLANDS CORRESPONDENCE

*by*

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These are notes from a lecture I gave at RIMS, Kyoto, for the conference *Analytic, geometric and p-adic aspects of automorphic forms and L-functions*, from 20 to 24 January, 2020. I wish to thank the organizers of this conference. These notes are intended to be a survey of recent results that have been published in [**31**].

## Notation

Let us fix some notation:

- $F$ is a non-Archimedean locally compact field of residual characteristic $p$,
- $\overline{F}$ is a separable closure of $F$,
- $\mathcal{W}_F$ is the Weil group of $\overline{F}/F$,
- $\mathcal{I}_F$ is the inertia subgroup of $\mathcal{W}_F$, that is its unique maximal compact subgroup,
- $\mathcal{P}_F$ is the wild inertia subgroup of $\mathcal{W}_F$, that is its unique maximal pro-$p$-subgroup.

## 1. Irreducible representations of the Weil group

Given an irreducible smooth complex representation $\sigma$ of $\mathcal{W}_F$, its restriction to $\mathcal{P}_F$ decomposes into a direct sum of finitely many irreducible representations of $\mathcal{P}_F$. These irreducible representations form a single $\mathcal{W}_F$-orbit under the conjugacy action, denoted $\mathcal{O}_F(\sigma)$. All representations in this orbit occur in the restriction of $\sigma$ to $\mathcal{P}_F$ with the same multiplicity $m = m(\sigma)$.

Fix $\alpha \in \mathcal{O}_F(\sigma)$. Its stabilizer in $\mathcal{W}_F$ is equal to $\mathcal{W}_T$ for a uniquely determined tamely ramified, finite extension $T$ of $F$ contained in $\overline{F}$. The representation $\alpha$ extends (non-canonically) to an irreducible representation $\rho$ of $\mathcal{W}_T$. The representation $\sigma$ can then be written:

$$\sigma \simeq \mathrm{Ind}_{T/F}(\rho \otimes \tau)$$

where $\tau$ is an irreducible representation of $\mathcal{W}_T$ trivial on $\mathcal{P}_T = \mathcal{P}_F$, of dimension $m$, uniquely determined up to isomorphism, and where $\mathrm{Ind}_{T/F}$ denotes induction from $\mathcal{W}_T$ to $\mathcal{W}_F$.

## 2. Cuspidal representations of $\mathrm{GL}_n(F)$

On the other hand, consider cuspidal irreducible smooth complex representations of the group $G = \mathrm{GL}_n(F)$ for some $n \geqslant 1$. Bushnell and Kutzko have constructed in [**8**] an explicit family of pairs $(\boldsymbol{J}, \boldsymbol{\lambda})$, where $\boldsymbol{J}$ is an open, compact mod centre subgroup of $G$ and $\boldsymbol{\lambda}$ is an irreducible representation of $\boldsymbol{J}$, such that:

– the compact induction of $\boldsymbol{\lambda}$ from $\boldsymbol{J}$ to $G$ is irreducible and cuspidal for all pairs $(\boldsymbol{J}, \boldsymbol{\lambda})$,

– any cuspidal irreducible representation of $G$ occurs this way, for a pair $(\boldsymbol{J}, \boldsymbol{\lambda})$, uniquely determined up to $G$-conjugacy.

These pairs have the following properties:

(1) The group $\boldsymbol{J}$ has a unique maximal compact subgroup $\boldsymbol{J}^0$ and a unique maximal normal pro-$p$-subgroup $\boldsymbol{J}^1$.

(2) The restriction of $\boldsymbol{\lambda}$ to $\boldsymbol{J}^1$ is a multiple of a single irreducible representation $\eta$.

(3) The representation $\eta$ extends (non-canonically) to a representation $\boldsymbol{\kappa}$ of $\boldsymbol{J}$, and $\boldsymbol{\lambda}$ is isomorphic to $\boldsymbol{\kappa} \otimes \boldsymbol{\xi}$ for a uniquely determined irreducible representation $\boldsymbol{\xi}$ of $\boldsymbol{J}$ trivial on $\boldsymbol{J}^1$.

(4) The representation $\eta$ is constructed as a Heisenberg representation from a character $\theta$ of a smaller open subgroup $H^1 \subseteq \boldsymbol{J}^1$ such that $\boldsymbol{J}^1/H^1$ is a finite-dimensional $\mathbb{F}_p$-symplectic space.

We denote by $\mathcal{C}(G)$ the set of all characters $\theta$ obtained this way when one varies the pairs $(\boldsymbol{J}, \boldsymbol{\lambda})$, and by $\mathcal{C}(F)$ the union of $\mathcal{C}(\mathrm{GL}_n(F))$ for all $n \geqslant 1$. Bushnell and Henniart ([**4**]) equipped the set $\mathcal{C}(F)$ with an equivalence relation called *endo-equivalence*. Equivalence classes for this relation are called *endo-classes*. The set $\mathcal{E}(F)$ of endo-classes only depends on $F$.

***Remark 2.1***. — When one fixes $n$, two characters $\theta, \theta' \in \mathcal{C}(\mathrm{GL}_n(F))$ defined on $H^1$, $H'^1$ respectively are endo-equivalent if and only if there is a $g \in \mathrm{GL}_n(F)$ such that $\theta' = \theta^g$ on $H'^1 \cap H^{1g}$.

The endo-class of any character $\theta$ as above occurring in a cuspidal irreducible representation $\pi$ of $G$ will be denoted $\boldsymbol{\Theta}(\pi)$.

## 3. The Ramification Theorem

We now consider the local Langlands correspondence $\pi \mapsto {}^L\pi$ (see [**21, 17, 18**]) between isomorphism classes of cuspidal irreducible representations of $\mathrm{GL}_n(F)$ and isomorphism classes of irreducible $n$-dimensional representations of $\mathcal{W}_F$.

***Theorem 3.1*** **([6])**. — *Let $\pi_1$, $\pi_2$ be cuspidal irreducible representations of* $\mathrm{GL}_{n_1}(F)$, $\mathrm{GL}_{n_2}(F)$ *respectively, and set $\sigma_1 = {}^L\pi_1$ and $\sigma_2 = {}^L\pi_2$. Then:*

$$\mathcal{O}_F(\sigma_1) \cap \mathcal{O}_F(\sigma_2) \neq \varnothing \quad \Longleftrightarrow \quad \boldsymbol{\Theta}(\pi_1) = \boldsymbol{\Theta}(\pi_2).$$

## 4. Inner forms and the local Jacquet-Langlands correspondence

More generally, the classification of cuspidal representations by compact induction, as well as the notion of endo-classes, also works for inner forms of general linear groups ([**2, 16, 26, 27,**

**28, 29, 3**]). It is natural to ask whether there exists, for inner forms, an analogue of Bushnell–Henniart's Ramification Theorem 3.1.

Let $H = \mathrm{GL}_m(D)$ be an inner form of $G = \mathrm{GL}_n(F)$, where $D$ is a central division $F$-algebra of reduced degree $d$ such that $md = n$. The local Jacquet–Langlands correspondence is a bijection $\pi \mapsto {}^{JL}\pi$ between the discrete series of $H$ and $G$, characterized by a character identity at elliptic regular conjugacy classes ([**19, 25, 13, 1**]).

Given a discrete series representation $\pi$ of $H$, it occurs as a subquotient of the parabolic induction of a cuspidal representation of the form:

$$\rho\chi_1 \otimes \cdots \otimes \rho\chi_r$$

where $r$ is a divisor of $m$, $\rho$ is a cuspidal irreducible representation of $\mathrm{GL}_{m/r}(D)$ and $\chi_1, \ldots, \chi_r$ are unramified characters of $\mathrm{GL}_{m/r}(D)$. Define $\boldsymbol{\Theta}(\pi) = \boldsymbol{\Theta}(\rho)$.

***Theorem 4.1*** **([32, 31, 14]).** — *The local Jacquet–Langlands correspondence preserves endo-classes.*

This result is one of the main steps in the explicit description of the local Jacquet–Langlands correspondence in terms of Bushnell–Kutzko's simple types ([**31, 14**]).

In the sequel, I will explain a crucial step in the proof of this theorem. Surprisingly, it involves the *modular representation theory* of $H$.

## 5. Modular representations of $p$-adic groups

Fix a prime number $\ell$ different from $p$ and an algebraic closure $\overline{\mathbb{F}}_\ell$ of $\mathbb{F}_\ell$, and consider smooth $\overline{\mathbb{F}}_\ell$-representations of $H$. As in the complex case, one has Haar measures, normalized parabolic induction and restriction functors, cuspidal representations and uniqueness of the cuspidal support for irreducible representations. Here, as in the complex case, a representation is cuspidal if all its proper Jacquet modules are zero or, equivalently, if all its matrix coefficients are compactly supported mod centre. However, there is a crucial difference: an irreducible cuspidal representation may occur as a subquotient of a parabolically induced representation ([**33**] Corollaire 5).

Say an irreducible representation of $H$ is supercuspidal when it does not occur as a subquotient of a parabolically induced representation. For irreducible representations of $H$, there is a well-behaved notion of supercuspidal support.

***Theorem 5.1*** **([23]).** — *Let $\pi$ be an irreducible $\overline{\mathbb{F}}_\ell$-representation of $H$.*

(1) *There are a Levi subgroup $M$ of $H$ and an irreducible supercuspidal representation $\rho$ of $M$ such that $\pi$ occurs as a subquotient of the parabolic induction of $\rho$ to $H$ with respect to any parabolic subgroup with Levi compotent $M$.*

(2) *The pair $(M, \rho)$ is unique up to $H$-conjugacy.*

***Remark 5.2.*** — Unlike uniqueness of the cuspidal support, Theorem 5.1 is non-trivial. Uniqueness of the supercuspidal support also holds for special linear $p$-adic groups [**9**] and small unramified unitary groups [**20**], but is false in general (see [**15, 12**] for a counterexample in the symplectic group $\mathrm{Sp}_8$).

Based on the well-defined notion of supercuspidal support for the group $H$, we have the following decomposition theorem.

***Theorem 5.3*** (**[30]**). — *The category* $\mathrm{Rep}_{\overline{\mathbb{F}}_\ell}(H)$ *of smooth* $\overline{\mathbb{F}}_\ell$*-representations of* $H$ *decomposes into blocks, which correspond bijectively to inertial classes of supercuspidal pairs of* $H$*.*

## 6. Reduction mod $\ell$ of representations over $\ell$-adic numbers

Now fix an algebraic closure $\overline{\mathbb{Q}}_\ell$ of the field of $\ell$-adic numbers and consider smooth $\overline{\mathbb{Q}}_\ell$-representations of $H$. Let $\overline{\mathbb{Z}}_\ell$ denote the ring of algebraic integers of $\overline{\mathbb{Q}}_\ell$. An irreducible representation $\pi$ of $H$ on a $\overline{\mathbb{Q}}_\ell$-vector space $V$ is said to be *integral* if it has an $H$-stable $\overline{\mathbb{Z}}_\ell$-lattice $L \subseteq V$. We have the following $p$-adic Brauer-Nesbitt principle.

***Proposition 6.1*** (**[34] I.9.6**). — *The* $\overline{\mathbb{F}}_\ell$*-representation* $L\otimes\overline{\mathbb{F}}_\ell$ *has finite length. Its semi-simplification only depends on the isomorphism class of* $\pi$*, and not on the choice of* $L$*.*

The semi-simplification of $L \otimes \overline{\mathbb{F}}_\ell$ will be denoted $\mathsf{r}_\ell(\pi)$ and called the reduction mod $\ell$ of $\pi$. Two integral irreducible representations are said to be *congruent* if they have the same reduction mod $\ell$.

## 7. Congruence properties of the local Jacquet-Langlands correspondence

Now replace $\mathbb{C}$ by $\overline{\mathbb{Q}}_\ell$ in Paragraphs 3 and 4 by fixing a field isomorphism $\iota : \mathbb{C} \to \overline{\mathbb{Q}}_\ell$. Since cuspidal representations are characterized by the properties of the support of their matrix coefficients, base change from $\mathbb{C}$ to $\overline{\mathbb{Q}}_\ell$ perserves cuspidality. One thus gets a local Langlands correspondence between isomorphism classes of cuspidal irreducible $\overline{\mathbb{Q}}_\ell$-representations of $\mathrm{GL}_n(F)$ and isomorphism classes of irreducible $n$-dimensional $\overline{\mathbb{Q}}_\ell$-representations of $\mathcal{W}_F$. The congruence properties of this local Langlands correspondence have been studied by Vignéras, Bushnell–Henniart and Dat (**[35, 7, 10]**).

For the local Jacquet–Langlands correspondence, one first has to check that the set of $\overline{\mathbb{Q}}_\ell$-representations of $H$ coming from a discrete series complex representation by base change from $\mathbb{C}$ to $\overline{\mathbb{Q}}_\ell$ does not depends on the choice of $\iota$. This comes from the fact that discrete series complex representations can be characterized as Langlands quotients of certain induced representations.

***Theorem 7.1*** (**[11, 24, 31]**). — *Let* $\pi$*,* $\pi'$ *be integral discrete series* $\overline{\mathbb{Q}}_\ell$*-representations of* $H$*.*

(1) ${}^{JL}\pi$ *and* ${}^{JL}\pi'$ *are integral, and they are congruent if and only if* $\pi$*,* $\pi'$ *are congruent.*

(2) $\mathsf{r}_\ell({}^{JL}\pi)$ *and* $\mathsf{r}_\ell({}^{JL}\pi')$ *are in the same block of* $\mathrm{Rep}_{\overline{\mathbb{F}}_\ell}(G)$ *if and only if* $\mathsf{r}_\ell(\pi)$ *and* $\mathsf{r}_\ell(\pi')$ *are in the same block of* $\mathrm{Rep}_{\overline{\mathbb{F}}_\ell}(H)$*.*

## 8. Linked representations

Now go back to complex representations.

***Definition 8.1.*** — Let $\pi$, $\pi'$ be discrete series complex representations of $H$. They are:

(1) 1-*linked* if there are a prime number $\ell \neq p$ and an isomorphism of fields $\iota : \mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$ such that $\mathsf{r}_\ell(\iota^*\pi)$ and $\mathsf{r}_\ell(\iota^*\pi')$ are in the same block of $\mathrm{Rep}_{\overline{\mathbb{F}}_\ell}(H)$.

(2) *linked* if there are discrete series representations $\pi_0, \dots, \pi_r$ of $H$ such that $\pi_0 = \pi$, $\pi_r = \pi'$ and $\pi_i$, $\pi_{i-1}$ are 1-linked for all $i \in \{1, \dots, r\}$.

Any two linked discrete series representations of $H$ have the same endo-class. The following theorem says that the converse is true.

***Theorem 8.2*** (**[31]**). — *Two complex discrete series representations of $H$ and linked if and only if they have the same endo-class.*

Putting all these results together, we get:

***Corollary 8.3***. — *Let $\pi$, $\pi'$ be discrete series complex representations of $H$. Then $\pi$, $\pi'$ have the same endo-class if and only if ${}^{JL}\pi$, ${}^{JL}\pi'$ have the same endo-class.*

It follows that, given a discrete series complex representation $\pi$ of $H$, the endo-class $\mathbf{\Theta}({}^{JL}\pi)$ only depends on $\mathbf{\Theta}(\pi)$. This defines a map from the set of endo-classes of discrete series representation of $H$ to that of endo-classes of discrete series representation of $G$, and one verifies that this map is bijective. Moreover, varying $n$ and the inner form, one gets a bijective map:

$$\boldsymbol{j}_F : \mathbf{\Theta} \mapsto {}^{JL}\mathbf{\Theta}$$

from $\mathcal{E}(F)$ to itself. Theorem 4.1 is then equivalent to saying that this map is the identity.

## 9. The restriction map

Let $K$ be a finite, tamely ramified extension of $F$ contained in $\overline{F}$. Associated with it, there is the set $\mathcal{E}(K)$ of $K$-endo-classes. It corresponds, via the Ramification Theorem 3.1, to the set of $\mathcal{W}_K$-conjugacy classes of irreducible representations of $\mathcal{P}_K = \mathcal{P}_F$. Bushnell-Henniart (**[4]**) have defined a surjective map:

$$\boldsymbol{\rho}_{K/F} : \mathcal{E}(K) \to \mathcal{E}(F)$$

called the *restriction map*. It corresponds, through the local Langlands correspondence, to passing from $\mathcal{W}_K$-conjugacy classes to $\mathcal{W}_F$-conjugacy classes of irreducible representations of $\mathcal{P}_F$.

## 10. The last step

Given an endo-class $\mathbf{\Theta} \in \mathcal{E}(F)$, there exists an $m \geqslant 1$ such that the group $H = \mathrm{GL}_m(D)$ has a cuspidal representation $\pi$ with the following properties:

(1) the endo-class of $\pi$ is equal to $\mathbf{\Theta}$,

(2) the transfer ${}^{JL}\pi$ is a cuspidal representation of $G = \mathrm{GL}_n(F)$, with $n = md$.

Let $K$ be the unique finite unramified extension of $F$ contained in $\overline{F}$ of degree the torsion number of $\pi$, that is the number of unramified characters $\chi$ of $H$ such that $\pi\chi \simeq \pi$. This number only depends on $n$ and $\mathbf{\Theta}$. Choose an endo-class $\mathbf{\Psi} \in \mathcal{E}(K)$ whose restriction to $\mathcal{E}(F)$ is equal to $\mathbf{\Theta}$.

Following Bushnell-Henniart's analysis of the traces of $\pi$ and ${}^{JL}\pi$ at certain well chosen elliptic regular conjugacy elements in terms of types ([**5**]), one get the following result:

***Proposition 10.1*** **([5] 6, [31] 7, [14] 4).** — *One has* $\boldsymbol{j}_F\boldsymbol{\Theta} = \boldsymbol{\rho}_{K/F}(\boldsymbol{j}_K\boldsymbol{\Psi})$.

Fix an $F$-embedding of $K$ in $\mathrm{M}_m(D)$, and let $H_K$ be the centralizer of $K$ in $H$. Define $G_K$ similarly. Let $\rho$ be a cuspidal representation of $H_K$ of endo-class $\boldsymbol{\Psi}$. The endo-class of its transfer to $G_K$ is thus $\boldsymbol{j}_K\boldsymbol{\Psi}$.

Dotto's idea ([**14**]) is to choose the integer $m \geqslant 1$ so that $G_K$ and $H_K$ are $K$-isomorphic. (For instance, choose $m$ large enough so that $G_K$ and $H_K$ are both split over $K$.) We thus may identify $G_K$ and $H_K$ so that $\rho$ is isomorphic to its transfer to $G_K$. This gives us $\boldsymbol{j}_K\boldsymbol{\Psi} = \boldsymbol{\Psi}$. Applying the restriction map and Proposition 10.1, we get the desired identity $\boldsymbol{j}_F\boldsymbol{\Theta} = \boldsymbol{\rho}_{K/F}(\boldsymbol{\Psi}) = \boldsymbol{\Theta}$.

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