GROWTH OF PETERSSON INNER PRODUCTS OF FOURIER-JACOBI COEFFICIENTS OF SIEGEL CUSP FORMS

BIPLAB PAUL

1. Introduction

In this article we would like to describe some recent progress towards the generalized Ramanujan-Petersson conjecture (see subsection 2.4 for the statement of the conjecture) for the Petersson norm of Fourier-Jacobi coefficients of a Siegel cusp form. Next section is preliminary section on Siegel modular forms and Jacobi forms (of integral index) which mainly serves the purpose of fixing the notations and we state the precise conjecture at the end of this section. Section 3 is the main part of the article and is divided into three subsections. Besides proving the conjecture on average for Siegel cusp forms of arbitrary degree, subsection 3.1 discusses proof of the conjecture for the Saito-Kurokawa lifts and for the Duke-Imamoglu-Ikeda lifts. In subsection 3.2 we describe Omega results and lower bounds for the Petersson norm of these Fourier-Jacobi coefficients. Some related results for the Petersson scalar products of Fourier-Jacobi coefficients of two distinct Siegel cusp forms are discussed in the last subsection. Some of the results in this subsection are conditional. We end this subsection as well as the article by stating some open questions in this theory.

There are certain applications of these results in the arithmetic theory of Fourier coefficients of Siegel cusp forms. But we do not discuss any applications of our results in this article. Some of these applications are under preparation and the details will appear elsewhere.

2. Preliminaries

2.1. **Siegel modular forms.** For any positive integer $n \geq 1$, we denote the Siegel modular group of degree n by Γ_n which is defined as follows:

$$\Gamma_n := \left\{ M := \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_{2n}(\mathbb{Z}) \mid M^t J_n M = J_n \right\},$$

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where $J_n := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. The Siegel upper half-space of degree n will be denoted by $\mathcal{H}_n$ and is defined by

$$\mathcal{H}_n \ := \ \left\{ Z \in M_n(\mathbb{C}) \mid Z^t = Z, \Im(Z) > 0 \text{ (positive definite)} \right\}.$$

Note that $\mathcal{H}_1 = \mathcal{H}$ and $\Gamma_1 = \mathrm{SL}_2(\mathbb{Z})$. The natural action of Γ_n on $\mathcal{H}_n$ is given by

$$\begin{aligned} \Gamma_n \times \mathcal{H}_n \ &\rightarrow \ \mathcal{H}_n \\ \left(\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, Z \right) \ &\mapsto \ \gamma Z := (AZ + B)(CZ + D)^{-1}. \end{aligned}$$

Definition 1. *A holomorphic function $F : \mathcal{H}_n \to \mathbb{C}$ is said to be a Siegel modular form of weight k and of degree n for Γ_n if*

1. $F(\gamma Z) = \det(CZ + D)^k F(Z)$ *for all $\gamma \in \Gamma_n$ and $Z \in \mathcal{H}_n$.*
2. *when $n = 1$, the function F is bounded in any domain $\{Z \in \mathcal{H}_1 \mid \Im(Z) > c\}$ for any $c > 0$.*

By $M_k(\Gamma_n)$ we denote the space of all Siegel modular forms of weight k for Γ_n. Any $F \in M_k(\Gamma_n)$ has a Fourier expansion

$$F(Z) \ := \ \sum_{T \in \mathbb{E}_n} a(T) e^{2\pi i tr(TZ)},$$

where $\mathbb{E}_n := \{T = (t_{ij})_{n \times n} \mid T^t = T, t_{ii}, 2t_{ij} \in \mathbb{Z}, T \geq 0\}$. Write $S_k(\Gamma_n)$ for the space of Siegel cusp forms of weight k for Γ_n which is defined by

$$S_k(\Gamma_n) \ := \ \{F \in M_k(\Gamma_n) \mid a(T) = 0 \text{ unless } T > 0\}.$$

For further details on the basic facts of Siegel modular forms we refer to [1, 8].

2.2. **Jacobi forms.** Let $H_{1,n}(\mathbb{R})$ be the real Heisenberg group of characteristic $(1, n)$, that is,

$$H_{1,n}(\mathbb{R}) \ := \ \mathbb{R}^{2n} \times \mathbb{R} = \left\{ [X, \kappa] \mid X \in \mathbb{R}^{2n}, \kappa \in \mathbb{R} \right\}$$

with the group structure:

$$[X_1, \kappa_1] \star [X_2, \kappa_2] := [X_1 + X_2, \kappa_1 + \kappa_2 + X_1 J_n X_2^t]$$

for any $[X_i, \kappa_i] \in H_{1,n}(\mathbb{R})$ $(i = 1, 2)$. By $G_n^J := Sp_n(\mathbb{R}) \ltimes H_{1,n}(\mathbb{R})$ we denote the Jacobi group of characteristic $(1, n)$ and the group law is given by: for $g_i := (M_i, [X_i, \kappa_i]) \in G_n^J$ $(i = 1, 2)$,

$$g_1 g_2 \ := \ \left(M_1 M_2, \left[X_1 M_2 + X_2, \kappa_1 + \kappa_2 + X_1 M_2 J_n X_2^t \right] \right).$$

The group G_n^J acts on $\mathcal{H}_n \times \mathbb{C}^n$ as follows:

$$\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, [X, \kappa] \right) \circ (\tau, z) \ := \ \left((A\tau + B)(C\tau + D)^{-1}, (z + \lambda\tau + \mu)(C\tau + D)^{-1} \right),$$

where $X = (\lambda, \mu) \in \mathbb{R}^n \times \mathbb{R}^n$. Let us put $\Gamma_n^J := \Gamma_n \ltimes H_{1,n}(\mathbb{Z})$. With these notations we ready to define Jacobi forms.

Definition 2. *Let $m \geq 0, k$ be integers. A holomorphic function $\phi : \mathcal{H}_n \times \mathbb{C}^n \to \mathbb{C}$ is said to be a Jacobi form of degree n, index m and weight k if it satisfies the following conditions:*

(1) *for any $\gamma := \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, [(\lambda, \mu), \kappa] \right) \in \Gamma_n^J$ we have*

$$\begin{aligned} \phi(\gamma \circ (\tau, z)) &= \mathit{exp}\left(2\pi i m \left((C\tau + D)^{-1} C \left[(z + \lambda\tau + \mu)^t\right] - \tau\left[\lambda^t\right] - 2\lambda z^t\right)\right) \\ &\qquad \times \det(C\tau + D)^k \phi(\tau, z). \end{aligned}$$

Here for matrices X of size $m \times r$ and A of size $m \times m$ we write $A[X]$ for $X^t A X$.

(2) *the function ϕ has a Fourier expansion of the form*

$$\phi(\tau, z) = \sum_{T \in \mathfrak{N}_n, r \in \mathbb{Z}^n} a_\phi(T, r) \mathit{exp}\big(2\pi i (tr(T\tau) + r z^t)\big)$$

with $a_\phi(T, r) = 0$ unless $4mT - r^t r \geq 0$. Here $\mathfrak{N}_n$ is given by

$$\mathfrak{N}_n := \{T = (t_{ij})_{n \times n} \mid T^t = T, t_{ii}, 2t_{ij} \in \mathbb{Z}\}.$$

The function ϕ is called a Jacobi cusp form if $a_\phi(T, r) = 0$ unless $4mT - r^t r > 0$.

The complex vector space of Jacobi cusp forms of degree n, weight k and index m is denoted by $J_{k,m}^{cusp}(\Gamma_n^J)$.

Let $\phi, \psi \in J_{k,m}^{cusp}(\Gamma_n^J)$. Then the Petersson inner product of ϕ and ψ is defined by

$$\langle \phi, \psi \rangle := \int_{\Gamma_n^J \backslash (\mathcal{H}_n \times \mathbb{C}^n)} \phi(\tau, z) \overline{\psi(\tau, z)} (\det v)^{k-n-2} \exp\left(-4\pi m v^{-1}\left[y^t\right]\right) du dv dx dy,$$

where $\tau = u + iv \in \mathcal{H}_n$, $z = x + iy \in \mathbb{C}^n$. It is known that the Petersson inner product defines a hermitian inner product on $J_{k,m}^{cusp}(\Gamma_n^J)$. For basic facts on Jacobi forms we refer to [3, 22].

2.3. **Fourier-Jacobi expansion of Siegel modular forms.** Let $Z = \begin{pmatrix} \tau & z \\ z^t & \tau' \end{pmatrix} \in \mathcal{H}_n$, where $\tau \in \mathcal{H}_{n-1}, z \in M_{n-1,1}(\mathbb{C})$ and $\tau' \in \mathcal{H}$. Then for any $F \in S_k(\Gamma_n)$ one has

$$F(Z) =: \sum_{m=1}^{\infty} \phi_m(\tau, z) e^{2\pi i m \tau'}.$$

This is expansion is referred to as Fourier-Jacobi expansion of F. It is known that ϕ_m is a Jacobi cusp form of weight k, index m and of degree $n-1$.

Observations: When $n = 1$: Fourier expansion of F and Fourier-Jacobi expansion of F coincide and ϕ_m is the m-th Fourier coefficient of F. In this case, by famous Ramanujan-Petersson conjecture (which is a theorem by Deligne's work), one knows

$$|\phi_m|^2 \ll_{F,\epsilon} m^{k-1+\epsilon} \quad \text{for any } \epsilon > 0.$$

2.4. **Generalizations of Ramanujan-Petersson conjecture.** There are three conjectures in literature which can be viewed as generalizations of the Ramanujan-Petersson conjecture in the theory of Siegel modular form.

(1) **Conjecture for Hecke eigenvalues:** When $F \in S_k(\Gamma_n)$ with $(k > n \geq 2)$ is a Hecke eigenform with Hecke eigenvalues $\lambda_F(m)$ for any positive integer m. For any $\epsilon > 0$, one expects

$$|\lambda_F(p^r)| \ll_\epsilon (p^r)^{\frac{nk}{2} - \frac{n(n+1)}{4} + \epsilon}$$

for all $r \geq 1$. This conjecture is originally stated in terms of Satake parameters and this equivalent form can be found in [18].

(2) **Conjecture for Fourier coefficients:** Let $F \in S_k(\Gamma_n)$ be a Siegel cusp form with Fourier coefficients $a(T)$. Then for any $\epsilon > 0$ we expect

$$|a(T)| \ll_{F,\epsilon} (|\det T|)^{\frac{k}{2} - \frac{n+1}{4} + \epsilon}.$$

This conjecture is due to H. L. Resnikoff and R. L. Saldaña [20, Conjecture IV].

(3) **Conjecture for Fourier-Jacobi coefficients:** Let $F \in S_k(\Gamma_n)$ be a cusp form with the Fourier-Jacobi coefficient ϕ_m for $m \geq 1$. Then for any $\epsilon > 0$ we expect

$$\| \phi_m \|^2 := \langle \phi_m, \phi_m \rangle \ll_{F,\epsilon} m^{k-1+\epsilon}.$$

This conjecture was made by W. Kohnen in [10, p.134] for degree $n = 2$ and in [11, p.718] for higher degrees.

The main aim of the lecture was to report on some recent progress in the direction of the **Conjecture 3** for Fourier-Jacobi coefficients and its related questions. We shall not discuss other two conjectures in this note.

3. Results: old and new

3.1. **Results towards generalized Ramanujan-Petersson conjecture.** Let $F, G \in S_k(\Gamma_n)$ be two Siegel cusp forms with the Fourier-Jacobi coefficients ϕ_m and ψ_m respectively. A variant of classical Hecke's argument shows (see [12, 14, 21]) that

$$\langle \phi_m, \psi_m \rangle \ll_{F,G} m^k. \tag{1}$$

Recently, W. Kohnen and J. Sengupta [13] showed that Conjecture 3 is true on average when degree $n = 2$. More precisely:

Theorem 1 ([13], particular case). *Let $F \in S_k(\Gamma_2)$ be a cusp form with Fourier-Jacobi coefficients $\phi_m (m \geq 1)$. Then*

$$\sum_{m \leq N} \| \phi_m \|^2 \ll_F N^k.$$

Remark 2. *The result of W. Kohnen and J. Sengupta is for cusp forms on the subgroup*

$$\Gamma_{2,0}(M) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_2 \mid C \equiv 0 \bmod M \right\}.$$

To keep the exposition uniform we stated the theorem only for the group Γ_2.

The proof of Theorem 1 is elementary and the main idea is to use Parseval's formula and clever choice of parameters. The proof works in a similar way as that of elliptic modular forms. In a joint work with B. Kumar [16], we generalized their method to show that the average Ramanujan-Petersson conjecture for Fourier-Jacobi coefficients is true for arbitrary degree.

Theorem 3. [16] *Let $F \in S_k(\Gamma_n)$ with the Fourier-Jacobi coefficients $\{\phi_m\}_{m \in \mathbb{N}}$. Then*

$$\sum_{m \leq N} \| \phi_m \|^2 \ll_F N^k.$$

In the above mentioned article [13], W. Kohnen and J. Sengupta also proved Conjecture 3 for an Hecke eigenform $F \in S_k(\Gamma_2)$ lying in the Maass subspace.

Theorem 4. [13] *Let $F \in S_k(\Gamma_2)$ be a Hecke eigenform of even weight k lying in the Maass subspace. Then*

$$\| \phi_m \|^2 \ll_{F,\epsilon} m^{k-1+\epsilon}.$$

One of the main ingredients to prove this theorem was a result of W. Kohnen and N.P. Skoruppa [12] which states

$$\| \phi_m \|^2 = \lambda(m) \| \phi_1 \|^2,$$

where $\lambda(m)$ is the mth Hecke eigenvalue of F. By the Saito-Kurokawa conjecture (which is a theorem due to the works of H. Maass, A. N. Andrianov and D. Zagier; see [3]), one knows that there exists a normalized Hecke eigenform $f(z) = \sum_{m \geq 1} a_f(m) e^{2\pi i m z} \in S_{2k-2}(\Gamma_1)$ such that the following holds:

$$\sum_{m=1}^{\infty} \frac{\lambda(m)}{m^s} = \frac{\zeta(s-k+1)\zeta(s-k+2)}{\zeta(2s-2k+4)} \cdot L(s,f) \quad (\Re(s) \gg 1). \tag{2}$$

Here $L(s,f) := \sum_{m \geq 1} a_f(m) m^{-s}$ is the Hecke L-function attached to f. Exploring the above relations and using Deligne bound $|a_f(m)| \ll_\epsilon m^{k-3/2+\epsilon}$ (for any $\epsilon > 0$), one can drive the result.

In fact, more careful analysis will lead to the following theorem.

Theorem 5. [16] *Let k be a positive even integer and $F \in S_k(\Gamma_2)$ be a non-zero cusp form lying in the Maass subspace and having ϕ_m $(m \geq 1)$ as the Fourier-Jacobi coefficients. Then there is an absolute positive constant c such that*

$$\| \phi_m \|^2 \ll_F \; m^{k-1} exp\left(c\frac{\sqrt{\log m}}{\log\log m} \right).$$

Further, B. Kumar and the author proved a strong version of Conjecture 3 for Duke-Imamoglu-Ikeda lifts (see T. Ikeda [6] for the details about this lift). Precisely:

Theorem 6. [16] *Let n, k be even positive integers such that $n \geq 4$ and $k > n+1$. Let $F \in S_k(\Gamma_n)$ be a Hecke eigenform which is a Duke-Imamoglu-Ikeda lift and having Fourier-Jacobi coefficients ϕ_m $(m \geq 1)$. Then*

$$\| \phi_m \|^2 \ll_F \; m^{k-1}.$$

The proof makes use of the following explicit relation due to H. Katsurada and H. Kawamura [7].

Theorem 7. [7, Main theorem] *Let $n, k \in 2\mathbb{N}$ be such that $k > n+1$ and $F \in S_k(\Gamma_n)$ be a cuspidal Hecke eigenform which is a Duke-Imamoglu-Ikeda lift of a normalized Hecke eigenform $f \in S_{2k-n}(\Gamma_1)$. Then we have*

$$\zeta(2s-2k+2n)\sum_{m=1}^{\infty} \| \phi_m \|^2 \, m^{-s} = \| \phi_1 \|^2 \, \zeta(s-k+1)\zeta(s-k+n)L(s,f) \quad (\Re(s) \gg 1). \tag{3}$$

Here $\zeta(s)$ denotes the Riemann zeta function.

Remark 8. *So far we know that Conjecture 3 is true on average for arbitrary degree and the conjecture is true for the lifts, that is, the Saito-Kurokawa lifts and the Duke-Imamogle-Ikeda lifts. But our knowledge towards Conjecture 3 for non-lifts is limited. To the best of author's knowledge, the best known unconditional result in this direction is*

$$\| \phi_m \|^2 \ll_{F,\epsilon} \; m^{k-\frac{2n}{4n+1}+\epsilon} \quad (\epsilon > 0)$$

see [9, 11]. *Hence the conjecture is widely open in general.*

3.2. **Omega results and lower bounds.** It seems to be natural to ask the following:

Question: Is the conjectural bound $\| \phi_m \|^2 \ll_F \; m^{k-1+\epsilon}$ $(\epsilon > 0)$ optimal, that is, does there exist any Omega result?

The main aim of this subsection is to discuss some results which answer this question. Consider the Dirichlet series:

$$D(s,F) := \sum_{m=1}^{\infty} \frac{\| \phi_m \|^2}{m^s} \quad \text{for} \quad \Re(s) \gg 1.$$

Hecke bound (1) implies that the series $D(s, F)$ converges absolutely in $\Re(s) > k+1$. It is known (see W. Kohnen and N. P. Skoruppa [12] for $n = 2$ and T. Yamazaki [21] (also see A. Krieg [14]) for arbitrary $n \geq 2$) that the function $D(s, F)$ can be analytically continued to whole complex plane as a meromorphic function and $D(s, F)$ has a simple pole at $s = k$. Hence the following bound is not possible:

$$\| \phi_m \|^2 \ll_F \; m^{k-\alpha}$$

for any $\alpha > 1$.

In fact, using analytic properties of $D(s, F)$ with a classical theorem of Landau (see, for example, [17, Theorem 1.7]), W. Kohnen [11] showed the following theorem.

Theorem 9. [11] *Let $F \in S_k(\Gamma_n)$ be a non-zero cusp form. Then there is an explicit constant $c_F > 0$ depending on F such that both the inequalities*

$$\langle \phi_m, \phi_m \rangle \; \geq \; c_F \, m^{k-1} \quad \textit{and} \quad \langle \phi_m, \phi_m \rangle \; \leq \; c_F \, m^{k-1}$$

hold for infinitely many $m \geq 1$.

In particular, we have

$$\limsup_{m \to \infty} \frac{\| \phi_m \|^2}{m^{k-1}} \; > \; 0.$$

We recall from the theory of elliptic modular form (that is, for $n = 1$) that R. A. Rankin [19] proved

$$\limsup_{m \to \infty} \frac{|\phi_m|^2}{m^{k-1}} \; = \; \infty.$$

The analogue of Rankin's result is not true in general for arbitrary degree $n \geq 2$. In fact, when $F \in S_k(\Gamma_n)$ ($n, k \in 2\mathbb{N}$ with $k > n+1$ and $n \geq 4$) is a Hekce eigenform which is a Duke-Imamoglu-Ikeda lift, Theorem 6 gives

$$\limsup_{m \to \infty} \frac{\| \phi_m \|^2}{m^{k-1}} \; < \; \infty.$$

On the contrary, when $n = 2$ and F is a Saito-Kurokawa lift, one has the following.

Theorem 10. [16] *Let $F \in S_k(\Gamma_2)$ be a non-zero cusp form lying in the Maass subspace. Then there are positive constants c (absolute), c_1 (depending on F) and infinitely many $m \geq 1$ such that*

$$\| \phi_m \|^2 \geq \; c_1 \, m^{k-1} exp \left(c \frac{\sqrt{\log m}}{\log \log m} \right).$$

The proof uses a recent result of S. Gun, the author and J. Sengupta [5] about Hecke eigenvalues of the Saito-Kurokawa lifts, Kohnen–Skoruppa relation

$$\| \phi_m \|^2 = \; \lambda(m) \; \| \phi_1 \|^2$$

and an idea of R. A. Rankin.

For an arbitrary non-zero cusp form, the trivial lower bound for the Petersson norm of the Fourier-Jacobi coefficient is $\parallel \phi_m \parallel \geq 0$ for all $m \geq 1$. Further, there are non-zero cusp forms $F \in S_k(\Gamma_2)$ such that the first Fourier-Jacobi coefficient $\phi_1 = 0$. But this is not the case for lifts, that is, for the Saito-Kurokawa lifts and for the Duke-Imamoglu-Ikeda lifts. Exploring the explicit relations (2) and (3) one can deduce the following result.

Theorem 11. [16] *Let $k \geq 2$ be even and $F \in S_k(\Gamma_2)$ be a non-zero cusp form lying in the Maass subspace and having Fourier-Jacobi coefficients ϕ_m. Then there exist positive constants c_2, c_3 such that*

$$\parallel \phi_m \parallel^2 \geq \; c_2 \, m^{k-1} exp \left(-c_3 \sqrt{\frac{\log m}{\log \log m}} \right).$$

When $n \geq 4$ is even with $k > n+1$ and $F \in S_k(\Gamma_n)$ is a Hecke eigenform which is a Duke-Imamoglu-Ikeda lift. Then

$$\parallel \phi_m \parallel^2 \geq \; c_5 \, m^{k-1}$$

for some positive constant c_5 depending on F.

3.3. **Some related results and open questions.** S. Gun and N. Kumar [4] generalized Theorem 9 and proved similar results for arithmetic progressions.

Theorem 12. [4] *Let $F \in S_k(\Gamma_n), \; n > 1$ be a non-zero cusp form with Fourier-Jacobi coefficients ϕ_m. Then for any $q > 1, a \in \mathbb{N}$ with $(a, q) = 1$, there exist a positive constant $c_{F,q}$ and infinitely many $m \equiv a \bmod q$ such that*

$$\parallel \phi_m \parallel^2 > \; c_{F,q} \, m^{k-1}.$$

Theorem 13. [4] *Let $F \in S_k(\Gamma_n), \; n > 1$ be a non-zero cusp form with Fourier-Jacobi coefficients ϕ_m. Then for any $q > 1$, there are positive constant $c_{F,q}$ and $b, c \in \mathbb{N}$ with $(bc, q) = 1$ such that*

$$\parallel \phi_m \parallel^2 > \; qc_{F,q} \, m^{k-1}$$

holds for infinitely many $m \equiv b \bmod q$ and

$$\parallel \phi_m \parallel^2 < \; qc_{F,q} \, m^{k-1}$$

holds for infinitely many $m \equiv c \bmod q$.

The proof of Theorem 12 and Theorem 13 combines ideas from [11] and from S.Böcherer, J. H. Bruinier and W. Kohnen [2]. For further details see the article of S. Gun and N. Kumar. In a recent work with B. Kumar, the author further generalizes the above results and proves the following.

Theorem 14. [15] *Let $F, G \in S_k(\Gamma_n)$ with the Fourier-Jacobi coefficients $\{\phi_m\}_{m \in \mathbb{N}}$ and $\{\psi_m\}_{m \in \mathbb{N}}$ respectively and the Petersson inner products $\langle \phi_m, \psi_m \rangle$ are real for all $m \in \mathbb{N}$. Then we have the following.*

(1) *If* $\langle F, G\rangle \neq 0$*, then there is a non-zero constant* $c_{F,G}$ *such that* $\langle \phi_m, \psi_m\rangle > c_{F,G}m^{k-1}$ *for infinitely many* $m \in \mathbb{N}$ *and* $\langle \phi_m, \psi_m\rangle < c_{F,G}m^{k-1}$ *for infinitely many* $m \in \mathbb{N}$*.*

(2) *If* $\langle F, G\rangle = 0$ *and* $\langle \phi_{m_0}, \psi_{m_0}\rangle \neq 0$ *for some* $m_0 \in \mathbb{N}$*, then the sequence* $\{\langle \phi_m, \psi_m\rangle\}_{m\in\mathbb{N}}$ *changes sign infinitely often.*

Theorem 15. [15] *Let* $p \in \mathbb{N}$ *be a prime,* $a \in \mathbb{N}$ *with* $(a, p) = 1$ *and* $F, G \in S_k(\Gamma_n)$ *be non-zero cusp forms with Fourier-Jacobi coefficients* $\{\phi_m\}_{m\in\mathbb{N}}$ *and* $\{\psi_m\}_{m\in\mathbb{N}}$ *respectively. Further assume that there are no real zeros of* $L(s, \chi)$ *in the region* $(0, 1)$ *for any even non-trivial Dirichlet character* χ *and* $\langle \phi_m, \psi_m\rangle \in \mathbb{R}$ *for all* $m \in \mathbb{N}$*. Then one can define a constant* $c_{F,G,p}$ *depending on* F, G *and* p*. If* $c_{F,G,p} \neq 0$*, then both the inequalities*

$$\langle \phi_m, \psi_m\rangle > pc_{F,G,p}m^{k-1} \quad \text{and} \quad \langle \phi_m, \psi_m\rangle < pc_{F,G,p}m^{k-1}$$

hold for infinitely many m *with* $m \equiv \pm a \bmod p$*.*

In particular, Theorem 14 implies that if $\langle F, G\rangle \neq 0$ then $|\langle \phi_m, \psi_m\rangle| > |c_{F,G}|m^{k-1}$ for infinitely many m. But it does not seem to be the case when $\langle F, G\rangle = 0$. In fact, if $F, G \in S_k(\Gamma_2)$ (for k even) are Hecke eigenforms lying in the Maass subspace and $\langle F, G\rangle = 0$ then $\langle \phi_m, \psi_m\rangle = 0$ for all $m \geq 1$ (see [12, Theorem 2]). Further, there are examples of Hecke eigenforms $F, G \in S_k(\Gamma_2)$ with F lies in the Maass subspace and G lies in the orthogonal complement of Maass subspace such that

$$|\langle \phi_m, \psi_m\rangle| \ll_{F,G,\epsilon} m^{k-3/2+\epsilon}$$

for any $\epsilon > 0$ and this upper bound is optimal (see [16]).

Open question 1. Find an optimal upper bound for $|\langle \phi_m, \psi_m\rangle|$ when $\langle F, G\rangle = 0$ and both F, G lie in the orthogonal complement of the Maass subspace. Note that, for arbitrary degree, using Cauchy–Schwarz inequality one has

$$|\langle \phi_m, \psi_m\rangle| \ll_{F,G,\epsilon} m^{k-1+\epsilon} \quad (\epsilon > 0)$$

when one assumes Conjecture 3. But in view of above discussions it is not clear whether this bound is optimal when $\langle F, G\rangle = 0$.

Open question 2. Is it true that there exist infinitely many $m \equiv a \bmod p$ such that both the inequalities in Theorem 15 are true? Removing the hypothesis "there are no real zeros of $L(s, \chi)$ in the region $(0, 1)$ for any even non-trivial Dirichlet character χ" is another question.

Open question 3. As mentioned in Remark 8, Conjecture 3 is widely open in general.

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(Biplab Paul) FACULTY OF MATHEMATICS, KYUSHU UNIVERSITY, 744 MOTOOKA, NISHI-KU, FUKUOKA 819-0395, JAPAN.

E-mail address: bpaul@math.kyushu-u.ac.jp