INFINITE DIMENSIONAL HARMONIC ANALYSIS

Transactions of a Japanese-German Symposium

held from September 20th to 24th, 1999
at the University of Kyoto

Editors

Herbert Heyer
Mathematisches Institut
Universität Tübingen

Takeshi Hirai
Faculty of Science
Kyoto University

Nobuaki Obata
Graduate School of Mathematics
Nagoya University

Kyoto 1999

INFINITE DIMENSIONAL HARMONIC ANALYSIS

Transactions of a Japanese-German Symposium

held from September 20th to 24th, 1999
at the University of Kyoto

Editors

Herbert Heyer
Mathematisches Institut
Universität Tübingen

Takeshi Hirai
Faculty of Science
Kyoto University

Nobuaki Obata
Graduate School of Mathematics
Nagoya University

Kyoto 1999

Druck D. + M. Gräbner, 96146 Altendorf 2000, info@graebner.de

Cover design by Katrin Heyer, Zweibrücken 2000

PREFACE

The Proceedings of the 2nd Japanese-German Symposium on
Infinite Dimensional Harmonic Analysis
held from September 20th to September 24th 1999 at the Department of Mathematics of Kyoto University reflect the progress of research in harmonic analysis and probability theory achieved by a group of Japanese and German mathematicians whose exchange and collaboration proved to be vivid and productive over a number of years. In fact, the 1st Japanese-German Symposium on the cited topic took place in Tübingen in 1995, and there is a significant enthusiasm to follow the 4-years cycle and to meet in Tübingen next time.

The main contributors to the present volume are the participants who gave one-hour lectures at the symposium. Since the symposium had been conceived as an open meeting it attracted an additional number of Japanese mathematicians who took part in the scientific activities. Some of the invited speakers who were unable to participate also kindly submitted their papers for the Proceedings.

The topics discussed during the symposium and dealt with in this volume ranged from traditional potential theory to harmonic analysis on manifolds, from classical probability theory to quantum stochastic analysis, and from representation theory of locally compact groups to spectral analysis of noncommutative structures. Unifying view points became apparent whenever algebraic-topological structures including semigroups, groups and vector spaces were applied to make probabilistic phenomena more transparent. In the discussions following the talks and in individual conversations new aspects of cooperation developed.

Similar to the Tübingen Symposium of 1995 the Kyoto Symposium of 1999 has been organized within the "German-Japanese Cooperative Science Promotion Program" set up by the Japan Society for the Promotion of Science (JSPS) and the German Research Society (DFG). The generous support of these two agencies is greatly appreciated. Thankfully we also acknowledge the financial allowances granted by the Ministry of Science and Research of the Land Baden-Württemberg, the Friends of the University of Tübingen and the German-Eastasian Science Forum at Tübingen, and the technical help offered sur place by the Department of Mathematics of Kyoto University and by the Kyoto Convention Bureau.

All contributions to these Proceedings have been refereed. We are grateful to the referees for their help, in particular to S.G. Dani, C.F. Dunkl, J. Leslie, G. Ritter, and G. Pap.

The organizers of the symposium who are identical with the editors of these Proceedings extend their heartfelt thanks to all participants, in particular to the contributors to this publication which will certainly serve as a reference to current studies in infinite dimensional harmonic analysis, but hopefully also as a stimulation for further enrichment of the theory.

Herbert Heyer, Tübingen
Takeshi Hirai, Kyoto
Nobuaki Obata, Nagoya May 2000

はじめに

この論文集は, 1999 年 9 月 20 日から 9 月 24 日の日程で京都大学理学研究科数学教室において開催された

第 2 回日独セミナー「無限次元調和解析」

の成果をもとに編集されたものである。同セミナーは, 第 1 回日独セミナー「無限次元調和解析」が 1995 年にチュービンゲンにおいて開催されて以来, 調和解析と確率論の関連分野における日独両国の研究グループによる活発な共同研究と研究交流をうけて企画されたものであり, 次回開催を 4 年周期で期待する熱気とともに幕を閉じた。

本書の主な著者は, セミナーにおいて 1 時間講演をおこなった方々である。それに加え, このセミナーに対する日本人数学者の関心の高まりを反映している。さらに, セミナーへの出席がかなわなかった招待講演者で, 本書に論文を寄せていただいた方々もある。

本書の収録論文からもわかるように, セミナーにおいて議論された話題は, 伝統的なポテンシャル理論から多様体上の調和解析, 古典確率論から量子確率解析, 局所コンパクト群の表現論から非可換構造のスペクトル解析, というような広がりを見せた。半群・群・ベクトル空間などがもつ代数的・位相的構造を応用することで, 確率現象をより深く理解するための統一的な観点が浮かび上がってきた。講演後の討論や合間の議論は, 異分野協同から生まれる新しいアイデアを発展させる上で有効であった。

1995 年のチュービンゲンにおけるセミナーと同様に, 今回 1999 年の京都セミナーは, 日本学術振興会 (JSPS) とドイツ研究協会 (DFG) による日独科学協力事業・セミナーの一環として実施された。この 2 つの機関から受けたすべての援助に対して深い謝意を表したい。さらに, バーデンヴュルテンベルグ州科学研究省・チュービンゲン大学後援会・ドイツ東アジア科学フォーラムからの財政援助, 及び京都大学理学研究科数学教室・京都コンベンションビューローの協力に感謝するものである。

本書の収録論文はすべて査読されている。特に, S. G. Dani, C. F. Dunkl, J. Leslie, G. Ritter, G. Pap の各氏の協力に感謝する。

セミナーの主催者は本論文集の編集も務めた。ここに, あらためて, セミナーに参加されたすべての方々と, 特に, 本書に論文を寄せていただいた方々に心からの感謝の意を表したい。本書が, 無限次元調和解析における最新の研究動向を知る上で貴重な文献となること, さらには, この研究領域から多くの実りを得るための一助となることを期待したい。

2000 年 5 月

ハーバート ハイヤー (チュービンゲン)
平井 武 (京都)
尾畑 伸明 (名古屋)

INVITED SPEAKERS

Sergio Albeverio	Institut für Angewandte Mathematik, Abteilung für Stochastik, Universität Bonn, Wegelerstr. 6, 53115 Bonn, Deutschland
Asai Arai	Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan
Hitoshi Arai	Graduate School of Mathematical Sciences, The University of Tokyo, Tokyo 153-8914, Japan
Nobuhiro Asai	International Institute for Advanced Studies, Kizu, Soraku-gun, Kyoto 619-0225, Japan
Viacheslav P. Belavkin	Mathematics Department, University of Nottingham, Nottingham NG7 2RK, UK
Yukihiro Hashimoto	Graduate School of Mathematics, Nagoya University, Nagoya 464-8602, Japan
Wilfried Hazod	Institut für Mathematik, Universität Dortmund, Campus Nord, Vogelpothsweg 87, 44227 Dortmund, Deutschland
Herbert Heyer	Mathematisches Institut, Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Deutschland
Takeyuki Hida	Department of Mathematics, Meijo University, Nagoya 468-8502, Japan
Juri Hinz	Mathematisches Institut, Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Deutschland
Akihito Hora	Department of Environmental and Mathematical Sciences, Okayama University, Okayama 700-8530, Japan
Un Cig Ji	Graduate School of Mathematics, Nagoya University, Nagoya 464-8602, Japan
Joe Kamimoto	Department of Mathematics, Kumamoto University, Kumamoto 860-8555, Japan

Eberhard Kaniuth	Fachbereich 17, Mathematik, Universität-Gesamthochschule Paderborn, Warburger Str. 100, 33098 Paderborn, Deutschland
Takeshi Kawazoe	Department of Mathematics, Keio University at Fujisawa, Fujisawa 252-8520 Japan
Izumi Kubo	Department of Mathematics, Faculty of Science, Hiroshima University, 739-8526 Higashi-Hiroshima, Japan
Burkhard Kümmerer	Mathematisches Institut A, Universität Stuttgart, Pfaffenwaldring 57 ,70569 Stuttgart, Deutschland
Aloys Krieg	Lehrstuhl A für Mathematik, Technische Hochschule Aachen, Templergraben 55, 52062 Aachen, Deutschland
Jean Ludwig	Département de Mathématiques, Université de Metz, Ile du Saulcy, 57045 Metz Cedex, France
Kyo Nishiyama	Faculty of Integrated Human Studies, Kyoto University, Kyoto 606-8501, Japan
Nobuaki Obata	Graduate School of Polymathematics, Nagoya University, Nagoya 464-01, Japan
Margit Rösler	Zentrum Mathematik, Technische Universität München, Arcisstr. 21, 80290 München, Deutschland
Kimiaki Saito	Department of Mathematics, Meijo University, Nagoya 468-8502, Japan
Michael Schürmann	Institut für Mathematik und Informatik, Universität Greifswald, Jahnstr. 15a, 17487 Greifswald, Deutschland
Hiroaki Shimomura	Department of Mathematics, Fukui University, Fukui 910-8507, Japan
Roland Speicher	Mathematisches Institut, Universität Heidelberg, Im Neuenheimer Feld 288, 69120 Heidelberg, Deutschland
Michael Voit	Mathematisches Institut, Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Deutschland
Hiroshi Yamaguchi	Department of Mathematics, Josai University, Saitama 350-0290, Japan
Hiroshi Yamashita	Division of Mathematics, Graduate School of Science, Hokkaido University, Sapporo 060-0810, Japan

CONTENTS

ANALYSIS AND GEOMETRY ON MARKED CONFIGURATION SPACES

SERGIO ALBEVERIO, YURI KONDRATIEV

EUGENE LYTVYNOV, AND GEORGI US

Abstract

We carry out analysis and geometry on a marked configuration space Ω_X^M over a Riemannian manifold X with marks from a space M. We suppose that M is a homogeneous space M of a Lie group G. As a transformation group $\mathfrak{A}$ on Ω_X^M we take the "lifting" to Ω_X^M of the action on $X \times M$ of the semidirect product of the group $\mathrm{Diff}_0(X)$ of diffeomorphisms on X with compact support and the group G^X of smooth currents, i.e., all C^∞ mappings of X into G which are equal to the identity element outside of a compact set. The marked Poisson measure π_σ on Ω_X^M with Lévy measure σ on $X \times M$ is proven to be quasiinvariant under the action of $\mathfrak{A}$. Then, we derive a geometry on Ω_X^M by a natural "lifting" of the corresponding geometry on $X \times M$. In particular, we construct a gradient ∇^Ω and a divergence div^Ω. The associated volume elements, i.e., all probability measures μ on Ω_X^M with respect to which ∇^Ω and div^Ω become dual operators on $L^2(\Omega_X^M;\mu)$, are identified as the mixed marked Poisson measures with mean measure equal to a multiple of σ. As a direct consequence of our results, we obtain marked Poisson space representations of the group $\mathfrak{A}$ and its Lie algebra $\mathfrak{a}$. We investigate also Dirichlet forms and Dirichlet operators connected with (mixed) marked Poisson measures.

1991 *AMS Mathematics Subject Classification.* Primary 60G57. Secondary 57S10, 54H15

0 Introduction

In recent years, stochastic analysis and differential geometry on configuration spaces have been considerably developed in a series of papers [5–8], see also [37, 2, 3]. It has been shown, in particular, that the geometry of the configuration space Γ_X over a Riemannian manifold X can be constructed via a simple "lifting procedure" and is completely determined by the Riemannian structure of X. The mixed Poisson measures are then exhibited as the "volume elements" corresponding to the differential geometry introduced on Γ_X. Intrinsic Dirichlet forms and operators, their canonical processes, as well as Gibbs measures on configuration spaces, their characterization by integration by parts, and the corresponding stochastic dynamics are among the problems which have been treated in the above framework.

A starting point for this analysis, more exactly, for the definition of differentiation on the configuration space, was the representation of the group of diffeomorphisms $\mathrm{Diff}_0(X)$ on X with compact support that was constructed by G. A. Goldin et al. [18] and A. M. Vershik et al. [42] (see also [34, 38, 20]). The construction of this representation used, in turn, the fact, following from the Skorokhod theorem, that the Poisson measure is quasiinvariant with respect to the group $\mathrm{Diff}_0(X)$.

On the other hand, starting with the same work [42], many researchers consider representations also on marked (in particular, compound) Poisson spaces. In statistical mechanics of continuous systems, marked Poisson measures and their Gibbsian perturbations are used for the description of many concrete models, see e.g. [1]. Hence, it is natural to ask about geometry and analysis on marked Poisson spaces. The first work in this direction was the paper [26], in which, just as in the case of the usual Poisson measure, the action of the group $\mathrm{Diff}_0(X)$ was used for the definition of the differentiation. However, this group proved to be too small for reconstructing mixed marked Poisson measures as "volume elements," which means that $\mathrm{Diff}_0(X)$ is to be extended in a proper way, which we will describe in the present paper.

Let us recall that the configuration space Γ_X is defined as the space of all locally finite subsets (configurations) in X. Then, the marked configuration space Ω_X^M over X with marks from, generally speaking, a manifold M is defined as

$$\Omega_X^M := \{ (\gamma, s) \mid \gamma \in \Gamma_X,\ s \in M^\gamma \},$$

where M^γ stands for the set of all maps $\gamma \ni x \mapsto s_x \in M$. Let $\widetilde{\sigma}$ be a Radon measure on $X \times M$ such that $\widetilde{\sigma}(K \times M) < \infty$ for each compact $K \subset X$ and $\widetilde{\sigma}$ is nonatomic in X, i.e., $\widetilde{\sigma}(\{x\} \times M) = 0$ for each $x \in X$. Then, one can define on Ω_X^M a marked Poisson measure $\pi_{\widetilde{\sigma}}$ with Lévy measure $\widetilde{\sigma}$.

Of course, one could consider $\pi_{\widetilde{\sigma}}$ as a usual Poisson measure on the configuration space $\Gamma_{X\times M}$ over the Cartesian product of the underlying manifold X and the space of marks M, and study the properties of this measure using the results of [2–5]. However, such an approach does not distinguish between the two different natures of X and M and the different roles that these play in physics. Thus, our aim is to introduce and study such transformations of the marked configuration space which do "feel" this difference and lead to an appropriate stochastic analysis and differential geometry.

In our previous paper [24], we were concerned with the model case $M = \mathbb{R}_+$, which corresponds, in fact, to the case of a compound Poisson measure. As has been promised in [24], we generalize in the present paper the results of [24] to the case where M is a homogeneous space of a Lie group G. This situation is natural from the physical point of view. For example, one can take $X = \mathbb{R}^3$ and M to be the unit sphere S^2 in $\mathbb{R}^3$, and consider any marked configuration $(\gamma, s) = \{(x, s_x)_{x\in\gamma}\} \in \Omega_X^M$ as a system of particles in $\mathbb{R}^3$ situated at the points x of γ and having spin s_x at $x \in \gamma$. One has then to take G as the rotation group, see e.g. [13].

Let G^X denote the group of smooth currents, i.e., all C^∞ mappings $X \ni x \mapsto \eta(x) \in G$ which are equal to the identity element of G outside of a compact set (depending on η). We define the group $\mathfrak{A}$ as the semidirect product of the groups $\mathrm{Diff}_0(X)$ and G^X: for $a_1 = (\psi_1, \eta_1)$ and $a_2 = (\psi_2, \eta_2)$, where $\psi_1, \psi_2 \in \mathrm{Diff}_0(X)$ and $\eta_1, \eta_2 \in G^X$, the multiplication of a_1 and a_2 is given by

$$a_1 a_2 = (\psi_1 \circ \psi_2, \eta_1(\eta_2 \circ \psi_1^{-1})).$$

The group $\mathfrak{A}$ acts in $X \times M$ as follows: for any $a = (\psi, \eta) \in \mathfrak{A}$

$$X \times M \ni (x, m) \mapsto a(x, m) = (\psi(x), \eta(\psi(x))m) \in X \times M,$$

where, for $g \in G$ and $m \in M$, gm denotes the action of g on m. Since each $\omega \in \Omega_X^M$ can be interpreted as a subset of $X \times M$, the action of $\mathfrak{A}$ can be lifted to an action in Ω_X^M. The marked Poisson measure $\pi_{\widetilde{\sigma}}$ is proven to be quasiinvariant under it. Thus, we can easily construct, in particular, a representation of $\mathfrak{A}$ in $L^2(\pi_{\widetilde{\sigma}})$. It should be stressed, however, that our representation of $\mathfrak{A}$ is reducible, because so is the regular representation of $\mathfrak{A}$ in $L^2(\widetilde{\sigma})$, see subsec. 3.5 in [24] for details.

Having introduced the action of the group $\mathfrak{A}$ on Ω_X^M, we proceed to derive analysis and geometry on Ω_X^M in a way parallel to the works [7, 24], dealing with the usual configuration space Γ_X and the marked configuration space $\Omega_X^{\mathbb{R}_+}$, respectively. In particular, we note that the Lie algebra $\mathfrak{a}$ of the group $\mathfrak{A}$ is given by $\mathfrak{a} = V_0(X) \times C_0^\infty(X; \mathfrak{g})$, where $V_0(X)$ is the algebra of C^∞ vector fields on X having compact support and $C_0^\infty(X; \mathfrak{g})$ is the algebra of C^∞ compactly supported functions from X into the Lie algebra $\mathfrak{g}$ of the group G. For each $(v, u) \in \mathfrak{a}$, we define the notion of a directional derivative of a function $F\colon \Omega_X^M \to \mathbb{R}$ along (v, u), which is denoted by $\nabla^\Omega_{(v,u)} F$. We obtain an explicit form of this derivative on the special set $\mathcal{F}C_{\mathrm{b}}^\infty(\mathfrak{D}, \Omega_X^M)$ of smooth cylinder functions on Ω_X^M, which, in turn, motivates our definition of a tangent bundle $T(\Omega_X^M)$ of Ω_X^M, and of a gradient $\nabla^\Omega F$. We note only that the tangent space $T_\omega(\Omega_X^M)$ to the marked configuration space Ω_X^M at a point $\omega = (\gamma, s) \in \Omega_X^M$ is given by

$$T_\omega(\Omega_X^M) := L^2(X \to T(X) \dotplus \mathfrak{g}; \gamma),$$

where $\dotplus$ means direct sum.

Next, we derive an integration by parts formula on Ω_X^M, that is, we get an explicit formula for the dual operator $\operatorname{div}^\Omega$ of the gradient ∇^Ω on Ω_X^M. We prove that the probability measures on Ω_X^M for which ∇^Ω and $\operatorname{div}^\Omega$ become dual operators (with respect to $\langle \cdot, \cdot \rangle_{T(\Omega_X^M)}$) are exactly the mixed marked Poisson measures

$$\mu_{\varkappa,\widetilde{\sigma}} = \int_{\mathbb{R}_+} \pi_{z\widetilde{\sigma}}\, \varkappa(dz),$$

where $\varkappa$ is a probability measure on $\mathbb{R}_+$ (with finite first moment) and $\pi_{z\widetilde{\sigma}}$ is the marked Poisson measure on Ω_X^M with Lévy measure $z\widetilde{\sigma}$, $z \geq 0$. This means that the mixed marked Poisson measures are exactly the "volume elements" corresponding to our differential geometry on Ω_X^M.

Thus, having identified the right volume elements on Ω_X^M, we introduce for each measure $\mu_{\varkappa,\widetilde{\sigma}}$ the first order Sobolev space $H_0^{1,2}(\Omega_X^M, \mu_{\varkappa,\widetilde{\sigma}})$ by closing the corresponding Dirichlet form

$$\mathcal{E}^\Omega_{\mu_{\varkappa,\widetilde{\sigma}}}(F, G) = \int_{\Omega_X^M} \langle \nabla^\Omega F, \nabla^\Omega G \rangle_{T(\Omega_X^M)}\, d\pi_{\varkappa,\widetilde{\sigma}}, \qquad F, G \in \mathcal{F}C_{\mathrm{b}}^\infty(\mathfrak{D}, \Omega_X^M),$$

on $L^2(\Omega_X^M, \mu_{\varkappa,\widetilde{\sigma}})$. Just as in the analysis on the usual configuration space, this is the step where we really start doing real infinite dimensional analysis. The corresponding Dirichlet operator is denoted by $H^\Omega_{\mu_{\varkappa,\widetilde{\sigma}}}$; it is a positive definite selfadjoint operator on $L^2(\Omega_X^M, \mu_{\varkappa,\widetilde{\sigma}})$. The heat semigroup $\big(\exp(-tH^\Omega_{\mu_{\varkappa,\widetilde{\sigma}}})\big)_{t\geq 0}$ generated by it is calculated explicitly. The results on the ergodicity of this semigroup are absolutely analogous to the corresponding results of [7]. Particularly, we have ergodicity if and only if $\mu_{\varkappa,\widetilde{\sigma}} = \pi_{z\widetilde{\sigma}}$ for some $z > 0$, i.e., $\mu_{\varkappa,\widetilde{\sigma}}$ is a (pure) marked Poisson measure.

We also clarify the relation between the intrinsic geometry on Ω_X^M we have constructed with another kind of extrinsic geometry on Ω_X^M which is based on fixing the marked Poisson measure $\pi_{\widetilde{\sigma}}$ and considering the unitary isomorphism between $L^2(\Omega_X^M, \pi_{\widetilde{\sigma}})$ and the corresponding Fock space

$$\mathcal{F}(L^2(X \times M; \widetilde{\sigma})) = \bigoplus_{n=0}^{\infty} \hat{L}^2((X \times M)^n, n!\,\widetilde{\sigma}^{\otimes n}),$$

where $\hat{L}^2((X \times M)^n, n!\,\widetilde{\sigma}^{\otimes n})$ is the subspace of symmetric functions from $L^2((X \times M)^n, n!\,\widetilde{\sigma}^{\otimes n})$. Our main result here is to prove that $H^{\Omega}_{\pi_{\widetilde{\sigma}}}$ is unitarily equivalent (under the above isomorphism) to the second quantization operator of the Dirichlet operator $H^{X\times M}_{\widetilde{\sigma}}$ on the $L^2(X \times M; \widetilde{\sigma})$ space.

As a consequence of the results of this paper, we obtain a representation on the marked Poisson space $L^2(\pi_{\widetilde{\sigma}})$ not only of the group $\mathfrak{A}$, but also of its Lie algebra $\mathfrak{a}$. Let us remark that the groups of smooth (as well as measurable and continuous) currents are classical objects in representation theory, see e.g. [4, 41, 11, 12, 43, 20] and references therein for different representations of these groups. On the other hand, different representations of the group $\mathfrak{A}$ and its Lie algebra $\mathfrak{a}$, in the special case $G = \mathfrak{g} = \mathbb{R}$, were constructed and studied by G. Goldin et al. [17, 19, 16] from the point of view of nonrelativistic quantum mechanics.

Finally, we note that, in a way parallel to the work [8], the results of the present paper can be generalized to the interaction case where, instead of the Poisson measure $\pi_{\widetilde{\sigma}}$, describing a system of free particles, one takes its Gibbsian perturbation—more exactly, a marked Gibbs measure on Ω_X^M of Ruelle type (see [28, 29]).

1 Marked Poisson measures

1.1 Marked configuration space

Let X be a connected, oriented C^∞ non-compact Riemannian manifold. The configuration space Γ_X over X is defined as the set of all locally finite subsets in X:

$$\Gamma_X := \{\, \gamma \subset X \mid \#(\gamma \cap K) < \infty \text{ for each compact } K \subset X \,\},$$

where $\#(\cdot)$ denotes the cardinality of a set. One can identify any $\gamma \in \Gamma_X$ with the positive integer-valued Radon measure

$$\sum_{x\in\gamma} \varepsilon_x \in \mathcal{M}(X),$$

where $\sum_{x\in\varnothing} \varepsilon_x :=$ zero measure and $\mathcal{M}(X)$ denotes the set of all positive Radon measures on $\mathcal{B}(X)$.

Let also M be a connected oriented C^∞ (compact or non-compact) Riemannian manifold. The marked configuration space Ω_X^M over X with marks from M is defined as

$$\Omega_X^M := \{\, \omega = (\gamma, s) \mid \gamma \in \Gamma_X,\ s \in M^\gamma \,\},$$

where M^γ stands for the set of all maps $\gamma \ni x \mapsto m \in M$. Equivalently, we can define Ω_X^M as the collection of subsets in $X \times M$ having the following properties:

$$\Omega_X^M = \left\{ \omega \subset X \times M \;\middle|\; \begin{array}{c} \text{a)}\, \forall (x,m), (x',m') \in \omega : (x,m) \neq (x',m') \Rightarrow x \neq x' \\ \text{b)}\ \mathrm{Pr}_X\, \omega \in \Gamma_X \end{array} \right\},$$

where Pr_X denotes the projection of the Cartesian product of X and M onto X. Again, each $\omega \in \Omega_X^M$ can be identified with the measure

$$\sum_{(x,m)\in\omega} \varepsilon_{(x,m)} \in \mathcal{M}(X \times M).$$

It is worth noting that, for any bijection ϕ: $X \times M \to X \times M$, the image of the measure $\omega(\cdot)$ under the mapping ϕ, $(\phi^*\omega)(\cdot)$, coincides with $(\phi(\omega))(\cdot)$, i.e.,

$$(\phi^*\omega)(\cdot) = (\phi(\omega))(\cdot), \qquad \omega \in \Omega_X^M,$$

where $\phi(\omega) = \{\phi(x,m) \mid (x,m) \in \omega\}$ is the image of ω as a subset of $X \times M$.

Let $\mathcal{B}_c(X)$ and $\mathcal{O}_c(X)$ denote the families of all Borel, resp. open subsets of X that have compact closure. Let also $\mathcal{B}_c(X \times M)$ denote the family of all Borel subsets of $X \times M$ whose projection on X belongs to $\mathcal{B}_c(X)$.

Denote by $C_{0,\mathrm{b}}(X \times M)$ the set of real-valued bounded continuous functions f on $X \times M$ such that $\operatorname{supp} f \in \mathcal{B}_c(X \times M)$. As usually, we set for any $f \in C_{0,\mathrm{b}}(X \times M)$ and $\omega \in \Omega_X^M$

$$\langle f, \omega\rangle = \int_{X\times M} f(x,m)\, \omega(dx, dm) = \sum_{(x,m)\in\omega} f(x,m).$$

We note that, because of the definition of Ω_X^M, there are only a finite number of addends in the latter series.

Now, we are going to discuss the measurable structure of the space Ω_X^M. We will use a "localized" description of the Borel σ-algebra $\mathcal{B}(\Omega_X^M)$ over Ω_X^M.

For $\Lambda \in \mathcal{O}_c(X)$, define

$$\Omega_\Lambda^M := \{\, \omega \in \Omega_X^M \mid \mathrm{Pr}_X\, \omega \subset \Lambda \,\}$$

and for $n \in \mathbb{Z}_+ = \{0, 1, 2, \dots\}$

$$\Omega_\Lambda^M(n) := \{\, \omega \in \Omega_\Lambda^M \mid \#(\omega) = n \,\}.$$

It is obvious that

$$\Omega_\Lambda^M = \bigsqcup_{n=0}^{\infty} \Omega_\Lambda^M(n).$$

Let $\Lambda_{\mathrm{mk}} := \Lambda \times M$ (i.e., Λ_{mk} is the set of all "marked" elements of Λ) and let

$$\widetilde{\Lambda}^n_{\mathrm{mk}} := \big\{((x_1,m_1),\dots,(x_n,m_n)) \in \Lambda^n_{\mathrm{mk}} \mid x_j \neq x_k \text{ if } j \neq k\big\}.$$

There is a bijection

$$\mathcal{L}^{(n)}_\Lambda \colon \widetilde{\Lambda}^n_{\mathrm{mk}}/\mathfrak{S}_n \mapsto \Omega^M_\Lambda(n) \tag{1.1}$$

given by

$$\mathcal{L}^{(n)}_\Lambda \colon ((x_1,m_1),\dots,(x_n,m_n)) \mapsto \{(x_1,m_1),\dots,(x_n,m_n)\} \in \Omega^M_\Lambda(n),$$

where $\mathfrak{S}_n$ is the permutation group over $\{1,\dots,n\}$. On $\Lambda^n_{\mathrm{mk}}/\mathfrak{S}_n$ one introduces the related metric

$$\begin{aligned}&\delta\big[((x_1,m_1),\dots,(x_n,m_n)),((x'_1,m'_1),\dots,(x'_n,m'_n))\big]\\ &\quad= \inf_{\sigma\in\mathfrak{S}_n} d^n\big[((x_1,m_1),\dots,(x_n,m_n)),((x'_{\sigma(1)},m'_{\sigma(1)}),\dots,(x'_{\sigma(n)},m'_{\sigma(n)}))\big],\end{aligned}$$

where d^n is the metric on Λ^n_{mk} driven from the original metrics on X and M. Then, $\widetilde{\Lambda}^n_{\mathrm{mk}}/\mathfrak{S}_n$ becomes an open set in $\Lambda^n_{\mathrm{mk}}/\mathfrak{S}_n$ and let $\mathcal{B}(\widetilde{\Lambda}^n_{\mathrm{mk}}/\mathfrak{S}_n)$ be the trace σ-algebra on $\widetilde{\Lambda}^n_{\mathrm{mk}}/\mathfrak{S}_n$ generated by $\mathcal{B}(\Lambda^n_{\mathrm{mk}}/\mathfrak{S}_n)$. Let then $\mathcal{B}(\Omega^M_\Lambda(n))$ be the image σ-algebra of $\mathcal{B}(\widetilde{\Lambda}^n_{\mathrm{mk}}/\mathfrak{S}_n)$ under the bijection $\mathcal{L}^{(n)}_\Lambda$ and let $\mathcal{B}(\Omega^M_\Lambda)$ be the σ-algebra on Ω^M_Λ generated by the usual topology of (disjoint) union of topological spaces.

For any $\Lambda \in \mathcal{O}_c(X)$, there is a natural restriction map $p_\Lambda \colon \Omega^M_X \mapsto \Omega^M_\Lambda$ defined by

$$\Omega^M_X \ni \omega \mapsto p_\Lambda(\omega) := \omega \cap \Lambda_{\mathrm{mk}} \in \Omega^M_\Lambda.$$

The topology on Ω^M_X is defined as the weakest topology making all the mappings p_Λ continuous. The associated σ-algebra is denoted by $\mathcal{B}(\Omega^M_X)$.

For each $B \in \mathcal{B}_c(X \times M)$, we introduce a function $N_B \colon \Omega^M_X \to \mathbb{Z}_+ = \{0,1,2,\dots\}$ such that

$$N_B(\omega) := \#(\omega \cap B), \qquad \omega \in \Omega^M_X. \tag{1.2}$$

Then, it is not hard to see that $\mathcal{B}(\Omega^M_X)$ is the smallest σ-algebra on Ω^M_X such that all the functions N_B are measurable.

1.2 Marked Poisson measure

In order to construct a marked Poisson measure, we fix:

(i) an intensity measure σ on the underlying manifold X, which is supposed to be a nonatomic Radon one,

(ii) a non-negative function

$$X \times \mathcal{B}(M) \ni (x,\Delta) \mapsto p(x,\Delta) \in \mathbb{R}_+$$

such that, for σ-a.a. $x \in X$, $p(x,\cdot)$ is a finite measure on M.

Now, we define a measure $\widetilde{\sigma}$ on $(X \times M, \mathcal{B}(X \times M))$ as follows:

$$\widetilde{\sigma}(A) = \int_A p(x, dm)\, \sigma(dx), \qquad A \in \mathcal{B}(X \times M). \tag{1.3}$$

We will suppose that the measure $\widetilde{\sigma}$ is infinite and for any $\Lambda \in \mathcal{B}_c(X)$

$$\widetilde{\sigma}(\Lambda_{\mathrm{mk}}) = \int_X 1_\Lambda(x) p(x, M)\, \sigma(dx) < \infty, \tag{1.4}$$

i.e., $p(x, M) \in L^1_{\mathrm{loc}}(\sigma)$.

Now, we wish to introduce a marked Poisson measure on Ω_X^M (cf. e.g. [23, 22]). To this end, we take first the measure $\widetilde{\sigma}^{\otimes n}$ on $(X \times M)^n$, and for any $\Lambda \in \mathcal{O}_c(X)$, $\widetilde{\sigma}^{\otimes n}$ can be considered as a finite measure on Λ^n_{mk}. Since σ is nonatomic, we get

$$\widetilde{\sigma}^{\otimes n}(\Lambda^n_{\mathrm{mk}} \setminus \widetilde{\Lambda}^n_{\mathrm{mk}}) = 0$$

and we can consider $\widetilde{\sigma}^{\otimes n}$ as a measure on $(\widetilde{\Lambda}^n_{\mathrm{mk}}/\mathfrak{S}_n, \mathcal{B}(\widetilde{\Lambda}^n_{\mathrm{mk}}/\mathfrak{S}_n))$ such that

$$\widetilde{\sigma}^{\otimes n}(\widetilde{\Lambda}^n_{\mathrm{mk}}/\mathfrak{S}_n) = \widetilde{\sigma}(\Lambda_{\mathrm{mk}})^n.$$

Denote by $\widetilde{\sigma}_{\Lambda,n} := \widetilde{\sigma}^{\otimes n} \circ (\mathcal{L}_\Lambda^{(n)})^{-1}$ the image measure on $\Omega_\Lambda^M(n)$ under the bijection (1.1). Then, we can define a measure $\lambda_{\widetilde{\sigma}}^\Lambda$ on Ω_Λ^M by

$$\lambda_{\widetilde{\sigma}}^\Lambda := \sum_{n=0}^{\infty} \frac{1}{n!} \widetilde{\sigma}_{\Lambda,n},$$

where $\widetilde{\sigma}_{\Lambda,0} := \varepsilon_\varnothing$ on $\Omega_\Lambda^M(0) = \{\varnothing\}$. The measure $\lambda_{\widetilde{\sigma}}^\Lambda$ is finite and $\lambda_{\widetilde{\sigma}}^\Lambda(\Omega_\Lambda^M) = e^{\widetilde{\sigma}(\Lambda_{\mathrm{mk}})}$. Hence, the measure

$$\pi_{\widetilde{\sigma}}^\Lambda := e^{-\widetilde{\sigma}(\Lambda_{\mathrm{mk}})} \lambda_{\widetilde{\sigma}}^\Lambda$$

is a probability measure on $\mathcal{B}(\Omega_\Lambda^M)$. It is not hard to check the consistency property of the family $\{\pi_{\widetilde{\sigma}}^\Lambda \mid \Lambda \in \mathcal{O}_c(X)\}$ and thus to obtain a unique probability measure $\pi_{\widetilde{\sigma}}$ on $\mathcal{B}(\Omega_X^M)$ such that

$$\pi_{\widetilde{\sigma}}^\Lambda = p_\Lambda^* \pi_{\widetilde{\sigma}}, \qquad \Lambda \in \mathcal{O}_c(X).$$

This measure $\pi_{\widetilde{\sigma}}$ will be called a marked Poisson measure with Lévy measure $\widetilde{\sigma}$.

For any function $\varphi \in C_{0,\mathrm{b}}(X \times M)$, it is easy to calculate the Laplace transform of the measure $\pi_{\widetilde{\sigma}}$

$$\ell_{\pi_{\widetilde{\sigma}}}(\varphi) := \int_{\Omega_X^M} e^{\langle \varphi, \omega \rangle}\, \pi_{\widetilde{\sigma}}(d\omega) = \exp\left(\int_{X \times M} (e^{\varphi(x,m)} - 1)\, \widetilde{\sigma}(dx, dm) \right). \tag{1.5}$$

Example 1.1 Let $p(x,\cdot) \equiv \varepsilon_m(\cdot)$, where m is some fixed point of M and $x \in X$. Then, $\widetilde{\sigma} = \sigma \otimes \varepsilon_m$ and $\pi_{\widetilde{\sigma}} = \pi_\sigma$ is just the Poisson measure on $(\Gamma_X, \mathcal{B}(\Gamma_X))$ with intensity σ.

Example 1.2 Let $p(x,\cdot) \equiv \tau(\cdot)$, $x \in X$, where τ is a finite measure on $(M, \mathcal{B}(M))$. Now, $\widetilde{\sigma} = \hat{\sigma} = \sigma \otimes \tau$ and $\pi_{\widetilde{\sigma}}$ coincides with the marked Poisson measure under consideration in [26] (in the case where M is a manifold). Notice that the choice of $\widetilde{\sigma} = \hat{\sigma}$ as a product measure means a position-independent marking, while the choice of a general $\widetilde{\sigma}$ of the form (1.3) leads to a position-depending marking.

2 Transformations of the marked Poisson measure

2.1 Group of transformations of the marked configuration space

We are looking for a natural group $\mathfrak{A}$ of transformations of Ω_X^M such that

(i) $\pi_{\widetilde{\sigma}}$ is $\mathfrak{A}$-quasiinvariant;

(ii) $\mathfrak{A}$ is big enough to reconstruct $\pi_{\widetilde{\sigma}}$ by the Radon–Nikodym density $\dfrac{da^*\pi_{\widetilde{\sigma}}}{d\pi_{\widetilde{\sigma}}}$, where a runs through $\mathfrak{A}$.

Let us recall that in the work [26] the group $\mathrm{Diff}_0(X)$ was taken as $\mathfrak{A}$, just in the same way as in the case of the usual Poisson measure [7]. Here, $\mathrm{Diff}_0(X)$ stands for the group of diffeomorphisms of X with compact support, i.e., each $\psi \in \mathrm{Diff}_0(X)$ is a diffeomorphism of X that is equal to the identity outside a compact set (depending on ψ). The group $\mathrm{Diff}_0(X)$ satisfies (i). However, unlike the case of the Poisson measure, the condition (ii) is not satisfied, because, for example, in the case where $\widetilde{\sigma} = \sigma \otimes \tau$, there is no information about the measure τ that is contained in $\dfrac{d\psi^*\pi_{\widetilde{\sigma}}}{d\pi_{\widetilde{\sigma}}}$, see [26]. Therefore, just as in the case of [24], we need a proper extension of the group $\mathrm{Diff}_0(X)$.

In what follows, we will suppose that M is a homogeneous space of a Lie group G (see e.g. [10]). Let us recall that this means the existence of a C^∞ mapping $\theta\colon G \times M \to M$ satisfying the following conditions:

(i) If e is the unity element of the group G, then

$$\theta(e, m) = m \qquad \text{for all } m \in M;$$

(ii) If $g_1, g_2 \in G$, then

$$\theta(g_1, \theta(g_2, m)) = \theta(g_1 g_2, m) \qquad \text{for all } m \in M;$$

(iii) For arbitrary $m_1, m_2 \in M$, there exists $g \in G$ such that $\theta(g, m_1) = m_2$.

For any $g \in G$, we will denote by $\theta_g\colon M \to M$ the mapping given by $\theta_g(m) := \theta(g, m)$; then θ_g defines a diffeomorphism of M.

Let us fix an arbitrary point $m_0 \in M$ and let H be the isotropy group of M:

$$H := \{ g \in G \mid \theta_g(m_0) = m_0 \}.$$

Then, the homogeneous space M can always be identified with the factor space G/H (endowed with the unique corresponding C^∞ manifold structure), i.e., $M = G/H$.

Let us consider the group of *smooth currents*, i.e., all C^∞ mappings $X \ni x \mapsto \eta(x) \in G$, which are equal to e outside a compact set (depending on η). A multiplication $\eta_1\eta_2$ in this group is defined as the pointwise multiplication of the mappings η_1 and η_2. In the representation theory this group is denoted by G^X, or $C_0^\infty(X;G)$.

The group $\mathrm{Diff}_0(X)$ acts in G^X by automorphisms: for each $\psi \in \mathrm{Diff}_0(X)$,

$$G^X \ni \eta \overset{\alpha}{\mapsto} \alpha(\psi)\eta := \eta \circ \psi^{-1} \in G^X.$$

Thus, we can endow the Cartesian product of $\mathrm{Diff}_0(X)$ and G^X with the following multiplication: for $a_1 = (\psi_1, \eta_1)$, $a_2 = (\psi_2, \eta_2)$ from $\mathrm{Diff}_0(X) \times G^X$

$$a_1 a_2 = (\psi_1 \circ \psi_2, \eta_1(\eta_2 \circ \psi_1^{-1}))$$

and obtain a semidirect product

$$\mathrm{Diff}_0(X) \underset{\alpha}{\times} G^X =: \mathfrak{A}$$

of the groups $\mathrm{Diff}_0(X)$ and G^X.

The group $\mathfrak{A}$ acts in $X \times M$ in the following way: for any $a = (\psi, \eta) \in \mathfrak{A}$

$$X \times M \ni (x, m) \mapsto a(x,m) = (\psi(x), \theta(\eta(\psi(x)), m)) \in X \times M. \qquad (2.1)$$

If id denotes the identity diffeomorphism of X and e is the function identically equal to e on X, then we will just identify ψ with (ψ, e) and η with (id, η). The action (2.1) of an arbitrary $a = (\psi, \eta)$ can be represented as

$$(x, m) \mapsto a(x, m) = \eta\psi(x, m),$$

where

$$\begin{aligned} \psi(x,m) &= (\psi(x), m), \\ \eta(x,m) &= (x, \theta(\eta(x), m)). \end{aligned}$$

For any $a = (\psi, \eta) \in \mathfrak{A}$, denote $K_a := K_\psi \cup K_\eta$, where K_ψ and K_η are the minimal closed sets in X outside of which $\psi = \mathrm{id}$ and $\eta = e$, respectively. Evidently, $K_a \in \mathcal{B}_c(X)$,

$$a(K_a)_{\mathrm{mk}} = (K_a)_{\mathrm{mk}},$$

and a is the identity transformation outside $(K_a)_{\mathrm{mk}}$.

Now, let us recall some known facts concerning quasiinvariant measures on homogeneous spaces (see e.g. [45, 44]).

Theorem 2.1 *Suppose G is a Lie group and H its subgroup, and let dg, δ_G and dh, δ_H be fixed Haar measures and modular functions on G and H, respectively. Then:*

(i) *for every measure μ on G/H that is quasiinvariant with respect to the action of G on G/H, there exists a measurable positive function ξ on G verifying*

$$\xi(gh) = \frac{\delta_H(h)}{\delta_G(h)}\,\xi(g), \qquad g \in G,\, h \in H, \tag{2.2}$$

and

$$\int_G f(g)\xi(g)\,dg = \int_{G/H} \mu(d\,gH) \int_H f(gh)\,dh, \qquad f \in C_0(G), \tag{2.3}$$

where $C_0(G)$ denotes the set of continuous functions on G with compact support; for each $g \in G$ the Radon–Nikodym density is given by

$$p_g^\mu(\widetilde{g}H) := \frac{dg^*\mu}{d\mu}(\widetilde{g}H) = \frac{\xi(g^{-1}\widetilde{g})}{\xi(\widetilde{g})}, \qquad \widetilde{g}H \in G/H;$$

(ii) *there exists a quasiinvariant measure λ on G/H such that the function*

$$p^\lambda(g, \widetilde{g}H) := p_g^\lambda(\widetilde{g}H)$$

is differentiable on $G \times G/H$.

Remark 2.1 We recall that the modular function $\delta_G(\cdot)$ of a Lie group G is defined from the equality $r_{\widetilde{g}}^*\,dg = \delta_G(\widetilde{g})\,dg$, where dg is the Haar measure on G (i.e., a fixed left-invariant measure on G) and r_g denotes the right translation on G, i.e., $\widetilde{g} \mapsto r_g\widetilde{g} = g\widetilde{g}$.

We fix the measure λ on $M = G/H$ from Theorem 2.1, (ii). As easily seen from Theorem 2.1 (i), any quasiinvariant measure on M in equivalent to λ.

Remark 2.2 If $H = \{e\}$, i.e., $M = G$, then we can choose λ to be the Haar measure dg on G. Moreover, if $\delta_G(h) = \delta_H(h)$ for all $h \in H$ (and only in this case) there exists a λ being invariant with respect to the action of G on M. The latter condition holds automatically if G is unimodular, that is, $\delta_G(g) \equiv 1$ for all $g \in G$. This, in turn, holds for all compact and simple Lie groups.

In what follows, we will suppose that the measure σ is equivalent to the Riemannian volume ν on X: $\sigma(dx) = \rho(x)\,\nu(dx)$ with $\rho > 0$ ν-a.s., and that for ν-a.a. $x \in X$ $p(x,\cdot)$ is equivalent to the measure λ:

$$p(x, dm) = p(x,m)\,\lambda(dm) \quad \text{with } p(x,m) > 0 \ \lambda\text{-a.a. } m \in M.$$

Thus, the measure $\widetilde{\sigma}$ can be written in the form

$$\widetilde{\sigma}(dx, dm) = \rho(x)p(x,m)\,\nu(dx)\,\lambda(dm).$$

The condition $\widetilde{\sigma}(\Lambda_{\mathrm{mk}}) < \infty$, $\Lambda \in \mathcal{B}_c(X)$, implies that the function

$$q(x,m) := \rho(x)p(x,m)$$

satisfies

$$q^{1/2} \in L^2_{\mathrm{loc}}(X;\nu) \otimes L^2(M;\lambda). \tag{2.4}$$

Noting that

$$a^{-1}(x,m) = (\psi,\eta)^{-1}(x,m) = (\psi^{-1}(x), \theta(\eta^{-1}(x),m)),$$

we easily deduce the following

Proposition 2.1 *The measure $\widetilde{\sigma}$ is $\mathfrak{A}$-quasiinvariant and for any $a = (\psi,\eta) \in \mathfrak{A}$ the Radon–Nikodym density is given by*

$$\begin{cases} p_a^{\widetilde{\sigma}}(x,m) := \dfrac{d(a^*\widetilde{\sigma})}{d\widetilde{\sigma}}(x,m) = \dfrac{q(\psi^{-1}(x), \theta(\eta^{-1}(x),m))}{q(x,m)} p^\lambda(\eta(x),m)\, J_\nu^\psi(x), \\ \text{if } (x,m) \in \{0 < q(x,m) < \infty\} \cap \{0 < q(\psi^{-1}(x), \theta(\eta^{-1}(x),m)) < \infty\}, \\ p_a^{\widetilde{\sigma}}(x,m) = 1, \qquad \text{otherwise,} \end{cases}$$

where J_ν^ψ is the Jacobian determinant of ψ (w.r.t. the Riemannian volume ν).

We give two examples of the above construction, which are important from the point of view of the marked configuration space analysis. We refer the reader to e.g. [44, 45] for further examples.

Example 2.1 Let $G = \mathbb{R}_+$ be the dilation group (e.g. [15]), i.e., the multiplication in this group is given by the usual multiplication of numbers. As a homogeneous space M we take G itself, by identifying the action of the group with the multiplication in it. As a quasiinvariant measure λ on M we can take the restriction to $\mathbb{R}_+$ of the Lebesgue measure on $\mathbb{R}$.

The analysis and geometry on the marked configuration space $\Omega_X^{\mathbb{R}_+}$ were studied in our previous work [24]. Here we only mention that the choice $M = \mathbb{R}_+$ leads (via a natural isomorphism) to the class of compound Poisson measures. In other words, each mark $s_x \in \mathbb{R}_+$ corresponding to $x \in X$ describes the charge of the measure

$$\omega = (\gamma, s) = \sum_{x \in X} s_x \varepsilon_x \in \mathcal{M}(X)$$

at the point x (or, in the case where $X = \mathbb{R}$, the value of the jump of the process at x).

Example 2.2 Let $G = O(d+1)$ be the $(d+1)$-dimensional orthogonal group and let $M = S^d$ be the d-dimensional unit sphere in $\mathbb{R}^{d+1}$ with the natural action of the group $O(d+1)$ on S^d, see e.g. [13, 44, 45]. As λ we take the surface measure on S^d, which is invariant w.r.t. the action of $O(d+1)$. From the point of view of statistical mechanics, a mark $s_x \in S^d$ describes in this example the spin of the particle at the point x.

2.2 $\mathfrak{A}$-quasiinvariance of the marked Poisson measure

Any $a \in \mathfrak{A}$ defines by (2.1) a transformation of $X \times M$, and, consequently, a has the following "lifting" from $X \times M$ to Ω_X^M:

$$\Omega_X^M \ni \omega \mapsto a(\omega) = \{\, a(x,m) \mid (x,m) \in \omega \,\} \in \Omega_X^M. \tag{2.5}$$

(Note that, for a given $\omega \in \Omega_X^M$, $a(\omega)$ indeed belongs to Ω_X^M and coincides with ω for all but a finite number of points.) The mapping (2.5) is obviously measurable and we can define the image $a^*\pi_{\tilde{\sigma}}$ as usually. The following proposition is an analog of a corresponding fact about Poisson measures.

Proposition 2.2 *For any $a \in \mathfrak{A}$, we have*

$$a^*\pi_{\tilde{\sigma}} = \pi_{a^*\tilde{\sigma}}.$$

Proof. The proof is the same as for the usual Poisson measure π_σ with intensity σ and $\psi \in \mathrm{Diff}_0(X)$ (e.g., [7]), one has just to calculate the Laplace transform of the measure $a^*\pi_{\tilde{\sigma}}$ for any $f \in C_{0,\mathrm{b}}(X \times M)$ and to use the formula (1.5). ■

Proposition 2.3 *The marked Poisson measure $\pi_{\tilde{\sigma}}$ is quasiinvariant w.r.t. the group $\mathfrak{A}$, and for any $a \in \mathfrak{A}$ we have*

$$\frac{d(a^*\pi_{\tilde{\sigma}})}{d\pi_{\tilde{\sigma}}}(\omega) = \prod_{(x,m)\in\omega} p_a^{\tilde{\sigma}}(x,m). \tag{2.6}$$

Proof. The result follows from Skorokhod theorem on absolute continuity of Poisson measures (see, e.g., [39, 40]). ■

Remark 2.3 Notice that only a finite (depending on ω) number of factors in the product on the right hand side of (2.6) are not equal to one.

3 The differential geometry of marked configuration spaces

3.1 The tangent bundle of Ω_X^M

Let us denote by $V_0(X)$ the set of C^∞ vector fields on X (i.e., smooth sections of $T(X)$) that have compact support. Let $\mathfrak{g}$ denote the Lie algebra of G and let $C_0^\infty(X;\mathfrak{g})$ stand for the set of all C^∞ mappings of X into $\mathfrak{g}$ that have compact support. Then

$$\mathfrak{a} := V_0(X) \times C_0^\infty(X;\mathfrak{g})$$

can be thought of as a Lie algebra corresponding to the Lie group $\mathfrak{A}$. More precisely, for any fixed $v \in V_0(X)$ and for any $x \in X$, the curve

$$\mathbb{R} \ni t \mapsto \psi_t^v(x) \in X$$

is defined as the solution of the following Cauchy problem

$$\begin{cases} \frac{d}{dt}\psi_t^v(x) = v(\psi_t^v(x)), \\ \quad \psi_0^v(x) = x. \end{cases} \tag{3.1}$$

Then, the mappings $\{\psi_t^v,\ t \in \mathbb{R}\}$ form a one-parameter subgroup of diffeomorphisms in $\mathrm{Diff}_0(X)$ (see, e.g., [10]):

$$1)\forall t \in \mathbb{R} \quad \psi_t^v \in \mathrm{Diff}_0(X),$$
$$2)\forall t_1, t_2 \in \mathbb{R} \quad \psi_{t_1}^v \circ \psi_{t_2}^v = \psi_{t_1+t_2}^v.$$

Next, for each function $u \in C_0^\infty(X;\mathfrak{g})$, $x \in X$, and $t \in \mathbb{R}$, we set $\eta_t^u(x) := \exp(tu(x))$, where $\mathfrak{g} \ni Y \mapsto \exp Y \in G$ is the exponential mapping (see, e.g., [45]). Hence, for a fixed $x \in X$, $\{\eta_t^u(x),\ t \in \mathbb{R}\}$ is a one-parameter subgroup of G and

$$\begin{aligned} \eta_0^u(x) &= e, \\ \frac{d}{dt}\,\eta_t^u(x)\big|_{t=0} &= u(x). \end{aligned} \tag{3.2}$$

Let us recall a fundamental theorem in the theory of Lie groups.

Theorem 3.1 *There exists a neighborhood U of the zero in $\mathfrak{g}$ and a neighborhood O of the unit element e in G such that* $\exp\colon U \to O$ *is an analytic diffeomorphism.*

¿From this theorem, we conclude that, for each fixed $u \in C_0^\infty(X;\mathfrak{g})$, there exists $\varepsilon > 0$ such that for any $t \in (-\varepsilon, \varepsilon)$ the mapping $X \ni x \mapsto \eta_t^u(x) \in G$ belongs to G^X, which yields, in turn, that $\eta_t^u \in G^X$ for all $t \in \mathbb{R}$, and moreover η_t^u is a one-parameter subgroup of G^X.

Thus, for an arbitrary $(v,u) \in \mathfrak{a}$, we can consider the curve $\{(\psi_t^v, \eta_t^u),\ t \in \mathbb{R}\}$ in $\mathfrak{A}$. Hence, to any $\omega \in \Omega_X^M$ there corresponds the following curve in Ω_X^M:

$$\mathbb{R} \ni t \mapsto (\psi_t^v, \eta_t^u)\omega \in \Omega_X^M.$$

Define now for a function $F\colon \Omega_X^M \to \mathbb{R}$ the directional derivative of F along (v,u) as

$$(\nabla_{(v,u)}^\Omega F)(\omega) := \frac{d}{dt}F((\psi_t^v, \eta_t^u)\omega)\big|_{t=0},$$

provided the right hand side exists. We will also denote by ∇_v^Ω and ∇_u^Ω the directional derivatives along $(v,0)$ and $(0,u)$, respectively.

Absolutely analogously, one defines for a function $\varphi\colon X \times M \to \mathbb{R}$ the directional derivative of φ along (v,u):

$$(\nabla_{(v,u)}^{X\times M}\varphi)(x,m) = \frac{d}{dt}\varphi((\psi_t^v, \eta_t^u)(x,m))\big|_{t=0}. \tag{3.3}$$

Then, for a continuously differentiable function φ, we have from (2.1), (3.1), (3.2), and (3.3)

$$\begin{aligned}(\nabla^{X\times M}_{(v,u)}\varphi)(x,m) &= \frac{d}{dt}\,\varphi((\psi^v_t(x),\theta(\eta^u_t(\psi^v_t(x)),m)\big|_{t=0}\\ &= \frac{d}{dt}\varphi(\psi^v_t(x),m)\big|_{t=0}+\frac{d}{dt}\varphi(x,\theta(\eta^u_t(x),m))\big|_{t=0}\\ &\quad+\frac{d}{dt}\varphi(x,\theta(\eta^u_0(\psi^v_t(x)),m))\big|_{t=0}\\ &= \langle\nabla^X\varphi(x,m),v(x)\rangle_{T_x(X)}+\langle\nabla^G\varphi(x,\theta(e,m)),u(x)\rangle_{\mathfrak{g}}\\ &= \langle\nabla^{X\times M}\varphi(x,m),(v(x),u(x))\rangle_{T_{(x,m)}(X\times M)}.\end{aligned}\tag{3.4}$$

Here, $T_{(x,m)}(X\times M) := T_x(X)\dot{+}\mathfrak{g}$ and $\nabla^{X\times M} := (\nabla^X,\widetilde{\nabla}^M)$, where ∇^X denotes the gradient on X and

$$\begin{aligned}\widetilde{\nabla}^M f(m) &= \nabla^G\hat{f}(e,m),\\ \hat{f}(g,m) &:= f(\theta(g,m)),\qquad g\in G,\ m\in M,\end{aligned}\tag{3.5}$$

∇^G being the gradient on G.

Remark 3.1 Notice that upon (3.5) we have, for a fixed $u\in\mathfrak{g}$,

$$\begin{aligned}\langle\widetilde{\nabla}^M f(m),u\rangle_{\mathfrak{g}} &= \langle\nabla^G f(\theta(e,m)),u\rangle_{\mathfrak{g}}\\ &= \frac{d}{dt}f(\theta(e^{tu},m))\big|_{t=0}\\ &= \langle\nabla^M f(m),(Ru)(m)\rangle_{T_m(M)},\end{aligned}\tag{3.6}$$

where ∇^M denotes the usual gradient on M, and the vector field Ru on M is given by

$$M\ni m\mapsto (Ru)(m) := \frac{d}{dt}\theta(e^{tu},m)\big|_{t=0}.\tag{3.7}$$

Let us introduce a special class of "nice functions" on Ω^M_X. Denote by $\mathfrak{D}$ the set of all C^∞-functions φ on $X\times M$ such that the support of φ is in $\mathcal{B}_c(X\times M)$, and φ and all its $\nabla^{X\times M}$ derivatives are bounded. Next, let $C^\infty_b(\mathbb{R}^N)$ stand for the space of all C^∞-functions on $\mathbb{R}^N$ which together with all their derivatives are bounded. Then, we can introduce $\mathcal{F}C^\infty_b(\mathfrak{D},\Omega^M_X)$ as the set of all functions $F\colon\Omega^M_X\mapsto\mathbb{R}$ of the form

$$F(\omega)=g_F(\langle\varphi_1,\omega\rangle,\dots,\langle\varphi_N,\omega\rangle),\qquad \omega\in\Omega^M_X,\tag{3.8}$$

where $\varphi_1,\dots,\varphi_N\in\mathfrak{D}$ and $g_F\in C^\infty_b(\mathbb{R}^N)$ (compare with [7]). $\mathcal{F}C^\infty_b(\mathfrak{D},\Omega^M_X)$ will be called the set of smooth cylinder functions on Ω^M_X.

For any $F\in\mathcal{F}C^\infty_b(\mathfrak{D},\Omega^M_X)$ of the form (3.8) and a given $(v,u)\in\mathfrak{a}$, we have, just as in [7],

$$\begin{aligned}F((\psi^v_t,\eta^u_t)\omega) &= g_F(\langle\varphi_1,(\psi^v_t,\eta^u_t)\omega\rangle,\dots,\langle\varphi_N,(\psi^v_t,\eta^u_t)\omega\rangle)\\ &= g_F(\langle\varphi_1\circ(\psi^v_t,\eta^u_t),\omega\rangle,\dots,\langle\varphi_N\circ(\psi^v_t,\eta^u_t),\omega\rangle),\end{aligned}$$

and therefore

$$(\nabla^{\Omega}_{(v,u)}F)(\omega) = \sum_{j=1}^{N} \frac{\partial g_F}{\partial r_j}(\langle\varphi_1,\omega\rangle,\dots,\langle\varphi_N,\omega\rangle)\langle\nabla^{X\times M}_{(v,u)}\varphi_j,\omega\rangle. \tag{3.9}$$

In particular, we conclude from (3.9) that

$$\nabla^{\Omega}_{(v,u)} = \nabla^{\Omega}_{v} + \nabla^{\Omega}_{u}. \tag{3.10}$$

The expression of $\nabla^{\Omega}_{(v,a)}$ on smooth cylinder functions motivates the following definition.

Definition 3.1 The tangent space $T_\omega(\Omega_X^M)$ to the marked configuration space Ω_X^M at a point $\omega = (\gamma, s) \in \Omega_X^M$ is defined as the Hilbert space

$$\begin{aligned} T_\omega(\Omega_X^M) : &= L^2(X \to T(X) \dotplus \mathfrak{g};\gamma) \\ &= L^2(X \to T(X);\gamma) \oplus L^2(X \to \mathfrak{g};\gamma) \\ &= \bigoplus_{x\in\gamma} [T_x(X) \oplus \mathfrak{g}] \end{aligned}$$

with scalar product

$$\begin{aligned} \langle V^1_\omega, V^2_\omega\rangle_{T_\omega(\Omega_X^M)} &= \int_X \left(\langle V^1_\omega(x)_{T_x(X)}, V^2_\omega(x)_{T_x(X)}\rangle_{T_x(X)} + \langle V^1_\omega(x)_{\mathfrak{g}}, V^2_\omega(x)_{\mathfrak{g}}\rangle_{\mathfrak{g}}\right)\gamma(dx) \\ &= \sum_{x\in\gamma} \left(\langle V^1_\omega(x)_{T_x(X)}, V^2_\omega(x)_{T_x(X)}\rangle_{T_x(X)} + \langle V^1_\omega(x)_{\mathfrak{g}}, V^2_\omega(x)_{\mathfrak{g}}\rangle_{\mathfrak{g}}\right), \end{aligned} \tag{3.11}$$

where $V^1_\omega, V^2_\omega \in T_\omega(\Omega_X^M)$ and $V_\omega(x)_{T_x(X)}$ and $V_\omega(x)_{\mathfrak{g}}$ denote the projection of $V_\omega(x) \in T_x(X) \dotplus \mathfrak{g}$ onto $T_x(X)$ and $\mathfrak{g}$, respectively. (Notice that the tangent space $T_\omega(\Omega_X^M)$ depends only on the γ coordinate of ω.) The corresponding tangent bundle is

$$T(\Omega_X^M) = \bigcup_{\omega\in\Omega_X^M} T_\omega(\Omega_X^M).$$

As usually in Riemannian geometry, having directional derivatives and a Hilbert space as a tangent space, we can introduce a gradient.

Definition 3.2 We define the intrinsic gradient ∇^Ω of a function $F\colon \Omega_X^M \to \mathbb{R}$ as the mapping

$$\Omega_X^M \ni \omega \mapsto (\nabla^\Omega F)(\omega) \in T_\omega(\Omega_X^M)$$

such that, for any $(v,u) \in \mathfrak{a}$,

$$(\nabla^{\Omega}_{(v,u)}F)(\omega) = \langle(\nabla^\Omega F)(\omega), (v,u)\rangle_{T_\omega(\Omega_X^M)}.$$

By (3.9) and (3.4) we have, for an arbitrary $F \in \mathcal{F}C_b^\infty(\mathfrak{D}, \Omega_X^M)$ of the form (3.8) and each $\omega = (\gamma, s) \in \Omega_X^M$,

$$(\nabla^\Omega F)(\omega;x) = \sum_{j=1}^{N} \frac{\partial g_F}{\partial r_j}(\langle\varphi_1,\omega\rangle,\dots,\langle\varphi_N,\omega\rangle)\nabla^{X\times M}\varphi_j(x,s_x), \qquad x\in\gamma. \tag{3.12}$$

3.2 Integration by parts and divergence on the marked Poisson space

Let the marked configuration space Ω_X^M be equipped with the marked Poisson measure $\pi_{\widetilde{\sigma}}$. We strengthen the condition (2.4) by demanding that

$$q^{1/2} \in H_0^{1,2}(X \times M). \tag{3.13}$$

Here, $H_0^{1,2}(X \times M)$ denotes the local Sobolev space of order 1 constructed with respect to the gradient $\nabla^{X \times M}$ in the space $L^2_{\mathrm{loc}}(X;\nu) \otimes L^2(M;\lambda)$, i.e., $H_0^{1,2}(X \times M)$ consists of functions f defined on $X \times M$ such that, for any set $A \in \mathcal{B}_c(X \times M)$, the restriction of f to A coincides with the restriction to A of some function φ from the Sobolev space $H^{1,2}(X \times M)$ constructed as the closure of $\mathfrak{D}$ with respect to the norm

$$\|\varphi\|^2_{1,2} := \int_{X \times M} \Big(|\nabla^X \varphi(x,m)|^2_{T_x(X)} + |\widetilde{\nabla}^M \varphi(x,m)|^2_{\mathfrak{g}} + |\varphi(x,s)|^2 \Big) \nu(dx)\, \lambda(dm).$$

Additionally, we will suppose that, for each $\Lambda \in \mathcal{B}_c(X)$,

$$|\nabla^G p^\lambda(e,\cdot)|_{\mathfrak{g}} \in L^1(\Lambda_{\mathrm{mk}}, \widetilde{\sigma}), \tag{3.14}$$

where, as before,

$$p^\lambda(g,m) = \frac{dg^*\lambda}{d\lambda}(m).$$

The set $\mathcal{F}C_{\mathrm{b}}^\infty(\mathfrak{D}, \Omega_X^M)$ is a dense subset in the space

$$L^2(\Omega_X^M, \mathcal{B}(\Omega_X^M), \pi_{\widetilde{\sigma}}) =: L^2(\pi_{\widetilde{\sigma}}).$$

For any $(v,u) \in \mathfrak{a}$, we have a differential operator in $L^2(\pi_{\widetilde{\sigma}})$ on the domain $\mathcal{F}C_{\mathrm{b}}^\infty(\mathfrak{D}, \Omega_X^M)$ given by

$$\mathcal{F}C_{\mathrm{b}}^\infty(\mathfrak{D}, \Omega_X^M) \ni F \mapsto \nabla^\Omega_{(v,u)} F \in L^2(\pi_{\widetilde{\sigma}}).$$

Our aim now is to compute the adjoint operator $\nabla^{\Omega *}_{(v,u)}$ in $L^2(\pi_{\widetilde{\sigma}})$. This corresponds, of course, to the deriving of an integration by parts formula with respect to the measure $\pi_{\widetilde{\sigma}}$.

But first we present the corresponding formula on $X \times M$.

Definition 3.3 For any $(v,u) \in \mathfrak{a}$, the logarithmic derivative of the measure $\widetilde{\sigma}$ along (v,u) is defined as the following function on $X \times M$:

$$\beta^{\widetilde{\sigma}}_{(v,u)} := \beta^{\widetilde{\sigma}}_v + \beta^{\widetilde{\sigma}}_u$$

with

$$\beta^{\widetilde{\sigma}}_v(x,m) = \left\langle \frac{\nabla^X q(x,m)}{q(x,m)}, v(x) \right\rangle_{T_x(X)} + \operatorname{div}^X v(x),$$

$\operatorname{div}^X = \operatorname{div}^X_\nu$ being the divergence on X w.r.t. ν, and

$$\beta^{\widetilde{\sigma}}_u(x,m) = \left\langle \frac{\widetilde{\nabla}^M q(x,m)}{q(x,m)}, u(x) \right\rangle_{\mathfrak{g}} + \langle \nabla^G p^\lambda(e,m), -u(x) \rangle_{\mathfrak{g}}.$$

Upon (3.13), we conclude that, for each $(v,u)\in\mathfrak{a}$, the function $\nabla^{X\times M}_{(v,u)}\log q$ is quadratically integrable with respect to the measure $\widetilde{\sigma}$, and therefore, since the support of $\nabla^{X\times M}_{(v,u)}\log q$ belongs to $\mathcal{B}_c(X\times M)$, this function is from $L^1(X\times M,\widetilde{\sigma})$. Thus, in virtue of the condition (3.14), we get the inclusion $\beta^{\widetilde{\sigma}}_{(v,u)}\in L^1(X\times M,\widetilde{\sigma})$.

By using standard arguments, one shows the following

Lemma 3.1 (Integration by parts formula on $X\times M$) *For all $\varphi_1,\varphi_2\in\mathfrak{D}$, we have*

$$\begin{aligned}\int_{X\times M}(\nabla^{X\times M}_{(v,u)}\varphi_1)(x,m)\varphi_2(x,m)\,\widetilde{\sigma}(dx,dm) &=\\ =-\int_{X\times M}\varphi_1(x,m)(\nabla^{X\times M}_{(v,u)}\varphi_2)(x,m)\,\widetilde{\sigma}(dx,dm)&\\ -\int_{X\times M}\varphi_1(x,s)\varphi_2(x,s)\beta^{\widetilde{\sigma}}_{(v,u)}(x,m)\,\widetilde{\sigma}(dx,dm).&\end{aligned}$$

Remark 3.2 The function $\langle\nabla^G p^\lambda(e,m),-u(x)\rangle_{\mathfrak{g}}$, which appears in the definition of $\beta^{\widetilde{\sigma}}_u$ is, for each fixed $x\in X$, the divergence on M with respect to the measure λ of the vector field $Ru(x)$ on M defined by (3.7), see Remark 3.1. Indeed, for any $u\in\mathfrak{g}$ and for an arbitrary f from $C_0^\infty(M)$—the space of all C^∞ functions on M with compact support, we have

$$\begin{aligned}\int_M\widetilde{\nabla}^M_u f(m)\,\lambda(dm)&=\int_M\langle\nabla^M f(m),(Ru)(m)\rangle_{T_m(M)}\,\lambda(dm)\\ &=\int_M\frac{d}{dt}f(\theta(\exp(tu),m))\big|_{t=0}\,\lambda(dm)\\ &=\int_M f(m)\,\frac{d}{dt}p^\lambda(\exp(tu),m)\big|_{t=0}\,\lambda(dm)\\ &=\int_M f(m)\langle\nabla^G p^\lambda(e,m),u\rangle_{\mathfrak{g}}\,\lambda(dm).\end{aligned}$$

Definition 3.4 For any $(v,u)\in\mathfrak{a}$, the logarithmic derivative of the marked Poisson measure $\pi_{\widetilde{\sigma}}$ along (v,u) is defined as the following function on Ω^M_X:

$$\Omega^M_X\ni\omega\mapsto B^{\pi_{\widetilde{\sigma}}}_{(v,u)}(\omega):=\langle\beta^{\widetilde{\sigma}}_{(v,u)},\omega\rangle. \tag{3.15}$$

A motivation for this definition is given by the following theorem.

Theorem 3.2 (Integration by parts formula) *For all $F_1,F_2\in\mathcal{F}C^\infty_b(\mathfrak{D},\Omega^M_X)$ and each $(v,u)\in\mathfrak{a}$, we have*

$$\begin{aligned}\int_{\Omega^M_X}(\nabla^\Omega_{(v,u)}F_1)(\omega)F_2(\omega)\,\pi_{\widetilde{\sigma}}(d\omega)=-\int_{\Omega^M_X}F_1(\omega)(\nabla^\Omega_{(v,u)}F_2)(\omega)\,\pi_{\widetilde{\sigma}}(d\omega)&\\ -\int_{\Omega^M_X}F_1(\omega)F_2(\omega)B^{\pi_{\widetilde{\sigma}}}_{(v,u)}(\omega)\,\pi_{\widetilde{\sigma}}(d\omega),&\end{aligned}\tag{3.16}$$

or

$$\nabla^{\Omega *}_{(v,u)} = -\nabla^{\Omega}_{(v,u)} - B^{\pi_{\tilde{\sigma}}}_{(v,u)}(\omega) \tag{3.17}$$

as an operator equality on the domain $\mathcal{F}C_b^\infty(\mathfrak{D}, \Omega_X^M)$ *in* $L^2(\pi_{\tilde{\sigma}})$.

Proof. Because of (3.10), the formula (3.17) will be proved if we prove it first for the operator ∇^Ω_v, i.e., when $u(x) \equiv 0$, and then for the operator ∇^Ω_u, i.e., when $v(x) = 0 \in T_x(X)$ for all $x \in X$. We present below only the proof for ∇^Ω_u, since the proof for ∇^Ω_v is basically the same as that of the integration by parts formula in case of Poisson measures [7].

By Proposition 2.2, we have for all $u \in C_0^\infty(X; \mathfrak{g})$

$$\int_{\Omega_X^M} F_1(\eta_t^u(\omega)) F_2(\omega)\, \pi_{\tilde{\sigma}}(d\omega) = \int_{\Omega_X^M} F_1(\omega) F_2(\eta_{-t}^u(\omega))\, \pi_{\eta_t^u * \tilde{\sigma}}(d\omega).$$

Differentiating this equation with respect to t, interchanging d/dt with the integrals and setting $t = 0$, the l.h.s. becomes the l.h.s. of (3.16). To see that the r.h.s. then also coincides with the r.h.s. of (3.16), we note that

$$\frac{d}{dt} F_2(\eta_{-t}^u(\omega))\big|_{t=0} = -(\nabla^\Omega_u F_2)(\omega),$$

and by Proposition 2.3

$$\frac{d}{dt}\left[\frac{d\pi_{\eta_t^u * \tilde{\sigma}}}{d\pi_{\tilde{\sigma}}}(\omega)\right]\Big|_{t=0} = \sum_{(x,m)\in\omega} \frac{d}{dt} p^{\tilde{\sigma}}_{\eta_t^u}(x,m)\Big|_{t=0}$$
$$= -\langle \beta_u^{\tilde{\sigma}}, \omega\rangle = -B_u^{\pi_{\tilde{\sigma}}}(\omega). \quad \blacksquare$$

Definition 3.5 For a vector field

$$V : \Omega_X^M \ni \omega \mapsto V_\omega \in T_\omega(\Omega_X^M),$$

the divergence $\operatorname{div}^\Omega_{\pi_{\tilde{\sigma}}} V$ is defined via the duality relation

$$\int_{\Omega_X^M} \langle V_\omega, \nabla^\Omega F(\omega)\rangle_{T_\omega(\Omega_X^M)}\, \pi_{\tilde{\sigma}}(d\omega) = -\int_{\Omega_X^M} F(\omega)(\operatorname{div}^\Omega_{\pi_{\tilde{\sigma}}} V)(\omega)\, \pi_{\tilde{\sigma}}(d\omega)$$

for all $F \in \mathcal{F}C_b^\infty(\mathfrak{D}, \Omega_X^M)$, provided it exists (i.e., provided

$$F \mapsto \int_{\Omega_X^M} \langle V_\omega, \nabla^\Omega F(\omega)\rangle_{T_\omega(\Omega_X^M)}\, \pi_{\tilde{\sigma}}(d\omega)$$

is continuous on $L^2(\pi_{\tilde{\sigma}})$).

A class of smooth vector fields on Ω_X^M for which the divergence can be computed in an explicit form is described in the following proposition.

Proposition 3.1 *For any vector field*

$$V_\omega(x) = \sum_{j=1}^{N} F_j(\omega)(v_j(x), u_j(x)), \qquad \omega \in \Omega_X^M,\ x \in X,$$

with $F_j \in \mathcal{F}C_b^\infty(\mathfrak{D}, \Omega_X^M)$, $(v_j, u_j) \in \mathfrak{a}$, $j = 1, \dots, N$, *we have*

$$\begin{aligned}(\operatorname{div}^\Omega_{\pi_{\widetilde{\sigma}}} V)(\omega) &= \sum_{j=1}^{N} \big(\nabla^\Omega_{(v_j,u_j)} F_j\big)(\omega) + \sum_{j=1}^{N} B^{\pi_{\widetilde{\sigma}}}_{(v_j,u_j)}(\omega) F_j(\omega)\\ &= \sum_{j=1}^{N} \langle \nabla^\Omega F_j(\omega), (v_j, u_j)\rangle_{T_\omega(\Omega_X^M)} + \sum_{j=1}^{N} \langle \beta^{\widetilde{\sigma}}_{(v_j,u_j)}, \omega\rangle F_j(\omega).\end{aligned}$$

Proof. Due to the linearity of ∇^Ω, it is sufficient to consider the case $N = 1$, i.e., $V_\omega(x) = F_1(\omega)(v(x), u(x))$. By Theorem 3.2, we have for all $F_2 \in \mathcal{F}C_b^\infty(\mathfrak{D}, \Omega_X^M)$

$$\begin{aligned}&-\int_{\Omega_X^M} \langle V_\omega, \nabla^\Omega F_2(\omega)\rangle_{T_\omega(\Omega_X^M)} \pi_{\widetilde{\sigma}}(d\omega) = -\int_{\Omega_X^M} F_1(\omega) \nabla^\Omega_{(v,u)} F_2(\omega)\, \pi_{\widetilde{\sigma}}(d\omega)\\ &= \int_{\Omega_X^M} \big(\nabla^\Omega_{(v,u)} F_1\big)(\omega) F_2(\omega)\, \pi_{\widetilde{\sigma}}(d\omega) + \int_{\Omega_X^M} F_1(\omega) F_2(\omega) B^{\pi_{\widetilde{\sigma}}}_{(v,u)}(\omega)\, \pi_{\widetilde{\sigma}}(d\omega),\end{aligned}$$

which yields

$$\begin{aligned}(\operatorname{div}^\Omega_{\pi_{\widetilde{\sigma}}} V)(\omega) &= \nabla^\Omega_{(v,u)} F_1(\omega) + B^{\pi_{\widetilde{\sigma}}}_{(v,u)}(\omega) F_1(\omega)\\ &= \langle \nabla^\Omega F_1(\omega), (v,u)\rangle_{T_\omega(\Omega_X^M)} + \langle \beta^{\widetilde{\sigma}}_{(v,u)}, \omega\rangle F_1(\omega). \quad \blacksquare\end{aligned}$$

Remark 3.3 Extending the definition of $B^{\pi_{\widetilde{\sigma}}}$ in (3.15) to the class of vector fields $V = \sum_{j=1}^N F_j \otimes (v_j, u_j)$ by

$$B_V^{\pi_{\widetilde{\sigma}}}(\omega) := \sum_{j=1}^{N} \langle \beta^{\widetilde{\sigma}}_{(v_j,u_j)}, \omega\rangle F_j(\omega) + \sum_{j=1}^{N} \big(\nabla^\Omega_{(v_j,u_j)} F_j\big)(\omega),$$

we obtain that

$$\operatorname{div}^\Omega_{\pi_\sigma} \bullet = B^{\pi_{\widetilde{\sigma}}}_\bullet.$$

In particular, if $(v, u) \in \mathfrak{a}$, it follows, for the "constant" vector field $V_\omega \equiv (v, u)$ on Ω_X^M, that

$$\operatorname{div}^\Omega_{\pi_{\widetilde{\sigma}}}(v,u)(\omega) = \langle \operatorname{div}^{X\times M}_{\widetilde{\sigma}}(v,u), \omega\rangle,$$

where $\operatorname{div}^{X\times M}_{\widetilde{\sigma}}(v,u) = \beta^{\widetilde{\sigma}}_{(v,u)}$ is the divergence on $X \times M$ of (v, u) w.r.t. $\widetilde{\sigma}$:

$$\begin{aligned}&\int_{X\times M} \langle \nabla^{X\times M} \varphi(x,m), (v(x), u(x))\rangle_{T_{(x,m)}(X\times M)}\, \widetilde{\sigma}(dx, dm)\\ &\qquad = -\int_{X\times M} \varphi(x,m) \big(\operatorname{div}^{X\times M}_{\widetilde{\sigma}}(v,u)\big)(x,m)\, \widetilde{\sigma}(dx, dm), \qquad \varphi \in \mathfrak{D}.\end{aligned}$$

3.3 Integration by parts characterization

In the works [7, 8] it was shown that the mixed Poisson measures are exactly the "volume elements" corresponding to the differential geometry on the configuration space Γ_X. Now, we wish to prove that an analogous statement holds true in our case of Ω_X^M for mixed marked Poisson measures.

We start with a lemma that describes $\widetilde{\sigma}$ as the unique (up to a constant) measure on $X \times M$ with respect to which the divergence $\operatorname{div}_{\widetilde{\sigma}}^{X\times M}$ is the dual operator of the gradient $\nabla^{X\times M}$.

Lemma 3.2 *Let the conditions* (3.13) *and* (3.14) *hold. Then, for every* $\Lambda \in \mathcal{O}_c(X)$ *the measures* $z\widetilde{\sigma}$, $z > 0$, *are the only positive Radon measures* ξ *on* Λ_{mk} *such that* $\operatorname{div}_{\widetilde{\sigma}}^{X\times M}$ *is the dual operator on* $L^2(\Lambda_{\mathrm{mk}};\xi)$ *of* $\nabla^{X\times M}$ *when considered with the domains* $V_0(\Lambda) \times C_0^\infty(\Lambda;\mathfrak{g})$, *resp.* $C_{0,\mathrm{b}}^\infty(\Lambda_{\mathrm{mk}})$ *(i.e., the set of all* $(v,u) \in \mathfrak{a}$, *resp.* $\varphi \in \mathfrak{D}$ *with support in* Λ, *resp.* Λ_{mk}*).*

Proof. In virtue of the conditions (3.13) and (3.14), the lemma is obtained in complete analogy with Remark 4.1 (iii) in [8]. Indeed, let $q_1(x,m)$ and $q_2(x,m)$ be two densities w.r.t. $\nu \otimes \lambda$ for which the logarithmic derivatives coincide. Then, we get

$$\begin{aligned} \nabla_v^X \log q_1(x,m) &= \nabla_v^X \log q_2(x,m), \qquad && v \in V_0(X),\\ \widetilde{\nabla}_u^M \log q_1(x,m) &= \widetilde{\nabla}_u^M \log q_2(x,m), \qquad && u \in C_0^\infty(\Lambda;\mathfrak{g}),\ \nu\otimes\lambda\text{-a.s.}, \end{aligned}$$

which yields respectively

$$\begin{aligned} q_1(x,m) &= q_2(x,m)c(m),\\ q_1(x,m) &= q_2(x,m)\widetilde{c}(x) \quad \nu\otimes\lambda\text{-a.s.} \end{aligned}$$

Therefore, $q_1(x,m) = \mathrm{const}\, q_2(x,m)$ $\nu\otimes\lambda$-a.s. ■

Let $\varkappa$ be a probability measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$. Then, we define a mixed marked Poisson measure as follows:

$$\mu_{\varkappa,\widetilde{\sigma}} = \int_{\mathbb{R}_+} \pi_{z\widetilde{\sigma}}\, \varkappa(dz). \tag{3.18}$$

Here, $\pi_{0\widetilde{\sigma}}$ denotes the Dirac measure on Ω_X^M with mass in $\omega = \{\varnothing\}$. Let $\mathcal{M}_l(\Omega_X^M)$, $l \in [1,\infty)$, denote the set of all probability measures on $(\Omega_X^M, \mathcal{B}(\Omega_X^M))$ such that

$$\int_{\Omega_X^M} |\langle f,\omega\rangle|^l\, \mu(d\omega) < \infty \quad \text{for all } f \in C_{0,\mathrm{b}}(X\times M),\ f \geq 0.$$

Clearly, $\mu_{\varkappa,\widetilde{\sigma}} \in \mathcal{M}_l(\Omega_X^M)$ if and only if

$$\int_{\mathbb{R}_+} z^l\, \varkappa(dz) < \infty. \tag{3.19}$$

We define $(\mathrm{IbP})^{\widetilde{\sigma}}$ to be the set of all $\mu \in \mathcal{M}_1(\Omega_X^M)$ with the property that $\omega \mapsto \langle \beta^{\widetilde{\sigma}}_{(v,u)}, \omega\rangle$ is μ-integrable for all $(v,u) \in \mathfrak{a}$ and which satisfy (3.16) with μ replacing $\pi_{\widetilde{\sigma}}$ for all $F_1, F_2 \in \mathcal{F}C_{\mathrm{b}}^{\infty}(\mathfrak{D}, \Omega_X^M)$, $(v,a) \in \mathfrak{g}$. We note that (3.16) makes sense only for such measures and that $B^{\pi_{\widetilde{\sigma}}}_{(v,u)}$ depends only on $\widetilde{\sigma}$ not on $\pi_{\widetilde{\sigma}}$. Obviously, since $\nabla^{X\times M}_{(v,u)}$ obeys the product rule for all $(v,u) \in \mathfrak{a}$, we can always take $F_2 \equiv 1$. Furthermore, $(\mathrm{IbP})^{\widetilde{\sigma}}$ is convex.

Theorem 3.3 *Let the condition* (3.13) *and* (3.14) *be satisfied. Then, the following conditions are equivalent:*
(i) $\mu \in (\mathrm{IbP})^{\widetilde{\sigma}}$;
(ii) $\mu = \mu_{\varkappa,\widetilde{\sigma}}$ *for some probability measure* $\varkappa$ *on* $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ *satisfying* (3.19) *with* $l = 1$.

Proof. The part (ii)⇒(i) is trivial. The proof of (i)⇒(ii) goes along absolutely analogously to that in the particular case where $G = M = \mathbb{R}_+$, see [24]. ■

As a direct consequence of Theorem 3.3, we obtain

Corollary 3.1 *The extreme points of* $(\mathrm{IbP})^{\widetilde{\sigma}}$ *are exactly* $\pi_{z\widetilde{\sigma}}$, $z \geq 0$.

3.4 A lifting of the geometry

Just as in the case of the geometry on the configuration space, we can present an interpretation of the formulas obtained in subsections 3.1–3.3 via a simple "lifting rule."

Suppose that $f \in C_{0,\mathrm{b}}(X \times M)$, or more generally f is an arbitrary measurable function on $X \times M$ for which there exists (depending on f) $\Lambda \in \mathcal{B}_{\mathrm{c}}(X)$ such that $\operatorname{supp} f \subset \Lambda_{\mathrm{mk}}$. Then, f generates a (cylinder) function on Ω_X^M by the formula

$$L_f(\omega) := \langle f, \omega\rangle, \qquad \omega \in \Omega_X^M.$$

We will call L_f the lifting of f.

As before, any vector field $(v,u) \in \mathfrak{a}$,

$$(v,u)\colon X \ni x \mapsto (v(x), u(x)) \in T_{(x,m)}(X \times M) = T_x(X) \dotplus \mathfrak{g},$$

can be considered as a vector field on Ω_X^M (the lifting of (v,u)), which we denote by $L_{(v,u)}$:

$$L_{(v,u)}\colon \Omega_X^M \ni \omega = \{\gamma, s\} \mapsto \{x \mapsto (v(x), u(x))\} \in T_\omega(\Omega_X^M) = L^2(X \to T(X) \dotplus \mathfrak{g}\,; \gamma).$$

For $(v_1,u_1), (v_2,u_2) \in \mathfrak{a}$, the formula (3.11) can be written as follows:

$$\left\langle L_{(v_1,u_1)}, L_{(v_2,u_2)}\right\rangle_{T_\omega\left(\Omega_X^M\right)} = L_{\langle (v_1,u_1),(v_2,u_2)\rangle_{T(X\times M)}}(\omega),$$

i.e., the scalar product of lifted vector fields is computed as the lifting of the scalar product

$$\langle (v_1(x), u_2(x)), (v_2(x), u_2(x))\rangle_{T_{(x,s_x)}(X\times M)} = f(x).$$

This rule can be used as a definition of the tangent space $T_\omega(\Omega_X^M)$.

The formula (3.9) has now the following interpretation:

$$(\nabla^{\Omega}_{(v,u)}L_\varphi)(\omega) = L_{\nabla^{X\times M}_{(v,u)}\varphi}(\omega), \qquad \varphi \in \mathfrak{D},\ \omega \in \Omega^M_X, \tag{3.20}$$

and the "lifting rule" for the gradient is given by

$$(\nabla^{\Omega}L_\varphi)(\gamma, s)\colon \gamma \ni x \mapsto \nabla^{X\times M}\varphi(x, s_x). \tag{3.21}$$

As follows from (3.15), the logarithmic derivative $B^{\pi_{\tilde{\sigma}}}_{(v,u)}\colon \Omega^M_X \to \mathbb{R}$ is obtained via the lifting procedure of the corresponding logarithmic derivative $\beta^{\tilde{\sigma}}_{(v,u)}\colon X \times M \to \mathbb{R}$, namely,

$$B^{\pi_{\tilde{\sigma}}}_{(v,u)}(\omega) = L_{\beta^{\tilde{\sigma}}_{(v,u)}}(\omega),$$

or equivalently, one has for the divergence of a lifted vector field:

$$\operatorname{div}^{\Omega}_{\pi_{\tilde{\sigma}}} L_{(v,a)} = L_{\operatorname{div}^{X\times M}_{\tilde{\sigma}}}(v, a). \tag{3.22}$$

We underline that by (3.20) and (3.21) one recovers the action of $\nabla^{\Omega}_{(v,a)}$ and ∇^{Ω} on all functions from $\mathcal{F}C^\infty_b(\mathfrak{D}, \Omega^M_X)$ algebraically from requiring the product or the chain rule to hold. Also, the action of $\operatorname{div}^{\Omega}_{\pi_{\tilde{\sigma}}}$ on more general cylindrical vector fields follows as in Remark 3.3 if one assumes the usual product rule for $\operatorname{div}_{\pi^{\Omega}_{\tilde{\sigma}}}$ to hold.

4 Representations of the Lie algebra $\mathfrak{a}$ of the group $\mathfrak{A}$

Using the $\mathfrak{A}$-quasiinvariance of $\pi_{\tilde{\sigma}}$, we can define the unitary representation of the group $\mathfrak{A} = \mathrm{Diff}_0(X) \underset{\alpha}{\times} G^X$ in the space $L^2(\pi_{\tilde{\sigma}})$. Namely, for $a \in \mathfrak{A}$, we define the unitary operator

$$\big(V_{\pi_{\tilde{\sigma}}}(a)F\big)(\omega) := F(a(\omega))\sqrt{\frac{da^{-1*}\pi_{\tilde{\sigma}}}{d\pi_{\tilde{\sigma}}}(\omega)}, \qquad F \in L^2(\pi_{\tilde{\sigma}}).$$

Then, we have

$$V_{\pi_{\tilde{\sigma}}}(a_1)V_{\pi_{\tilde{\sigma}}}(a_2) = V_{\pi_{\tilde{\sigma}}}(a_1a_2), \qquad a_1, a_2 \in \mathfrak{A}.$$

As has been noted in Introduction, this representation is reducible, cf. [24]

As in subsec. 3.1, to any vector field $v \in V_0(X)$ there corresponds a one-parameter subgroup of diffeomorphisms ψ^v_t, $t \in \mathbb{R}$. It generates a one-parameter unitary group

$$V_{\pi_{\tilde{\sigma}}}(\psi^v_t) := \exp[itJ_{\pi_{\tilde{\sigma}}}(v)], \qquad t \in \mathbb{R},$$

where $J_{\pi_{\tilde{\sigma}}}(v)$ denotes the selfadjoint generator of this group. Analogously, to a subgroup η^u_t, $u \in C^\infty_0(X; \mathfrak{g})$, there corresponds a one-parameter unitary group

$$V_{\pi_{\tilde{\sigma}}}(\eta^u_t) := \exp[itI_{\pi_{\tilde{\sigma}}}(u)]$$

with a generator $I_{\pi_{\tilde{\sigma}}}(u)$.

Proposition 4.1 *For any $v \in V_0(X)$ and $u \in C_0^\infty(X; \mathfrak{g})$, the following operator equalities on the domain $\mathcal{F}C_b^\infty(\mathfrak{D}, \Omega_X^M)$ hold:*

$$J_{\pi_{\bar{\sigma}}}(v) = \frac{1}{i}\,\nabla_v^\Omega + \frac{1}{2i}\,B_v^{\pi_{\bar{\sigma}}},$$
$$I_{\pi_{\bar{\sigma}}}(u) = \frac{1}{i}\,\nabla_u^\Omega + \frac{1}{2i}\,B_u^{\pi_{\bar{\sigma}}}.$$

Proof. These equalities follow immediately from the definition of the directional derivatives ∇_v^Ω and ∇_a^Ω, Theorem 3.2, and the form of the operators $V_{\pi_{\bar{\sigma}}}(\psi_t^v)$ and $V_{\pi_{\bar{\sigma}}}(\theta_t^u)$. ■

For any $(v, u) \in \mathfrak{a}$, define an operator

$$\mathcal{R}_{\pi_{\bar{\sigma}}}(v, u) := J_{\pi_{\bar{\sigma}}}(v) + I_{\pi_{\bar{\sigma}}}(u).$$

By Proposition 4.1,

$$\mathcal{R}_{\pi_{\bar{\sigma}}}(v, u) = \frac{1}{i}\nabla_{(v,u)}^\Omega + \frac{1}{2i}B_{(v,u)}^{\pi_{\bar{\sigma}}}.$$

We wish to derive now a commutation relation between these operators.

Lemma 4.1 *The Lie-bracket $[(v_1, u_1), (v_2, u_2)]$ of the vector fields (v_1, u_1), $(v_2, u_2) \in \mathfrak{a}$, i.e., a vector field from $\mathfrak{a}$ such that*

$$\nabla^{X\times M}_{[(v_1,u_1),(v_2,u_2)]} = \nabla^{X\times M}_{(v_1,u_1)}\nabla^{X\times M}_{(v_2,u_2)} - \nabla^{X\times M}_{(v_2,u_2)}\nabla^{X\times M}_{(v_1,u_1)} \quad \text{on } \mathfrak{D},$$

is given by

$$[(v_1, u_1), (v_2, u_2)] = ([v_1, v_2], \nabla_{v_1}^X u_2 - \nabla_{v_2}^X u_1 + [u_1, u_2]),$$

where $[v_1, v_2]$ is the Lie-bracket of the vector fields v_1, v_2 on X,

$$[u_1, u_2](x) = [u_1(x), u_2(x)]$$

(the latter being the Lie-bracket on $\mathfrak{g}$ of $u_1(x), u_2(x) \in \mathfrak{g}$), and $\nabla_v^X u$ is the derivative in direction v of a $\mathfrak{g}$-valued function u on X.

Proof. First, we have on $\mathfrak{D}$:

$$\nabla_{v_1}^X\nabla_{v_2}^X - \nabla_{v_2}^X\nabla_{v_1}^X = \nabla_{[v_1,v_2]}^X, \qquad v_1, v_2 \in V_0(X). \tag{4.1}$$

Next, using (3.5),

$$\widetilde{\nabla}_u^M f(x, m) = \langle \nabla^G \hat{f}(x, e, m), u(x)\rangle_{\mathfrak{g}}, \qquad \hat{f}(x, g, m) := f(x, \theta(g, m)),$$

and so

$$\begin{aligned}(\widetilde{\nabla}_{u_1}^M\widetilde{\nabla}_{u_2}^M - \widetilde{\nabla}_{u_2}^M\widetilde{\nabla}_{u_1}^M)f(x, m) \\ = \langle \nabla^G \hat{f}(x, e, m), [u_1(x), u_2(x)]\rangle_{\mathfrak{g}} \\ = \widetilde{\nabla}_{[u_1,u_2]}^M f(x, m), \qquad u_1, u_2 \in C_0^\infty(X; \mathfrak{g}).\end{aligned} \tag{4.2}$$

Finally,

$$
\begin{aligned}
&(\nabla_v^X \widetilde{\nabla}_u^M - \widetilde{\nabla}_u^M \nabla_v^X) f(x,m)\\
&= \langle \nabla^X \langle \nabla^G \hat{f}(x,e,m), u(x)\rangle_{\mathfrak{g}}, v(x)\rangle_{T_x(X)}\\
&\quad - \langle \nabla^G \langle \nabla^X \hat{f}(x,e,m), v(x)\rangle_{T_x(X)}, u(x)\rangle_{\mathfrak{g}}\\
&= \langle \nabla^X \nabla^G \hat{f}(x,e,m), v(x)\otimes u(x)\rangle_{T_x(X)\otimes\mathfrak{g}} + \langle \nabla^G \hat{f}(x,e,m), \nabla_v^X u(x)\rangle_{\mathfrak{g}}\\
&\quad - \langle \nabla^G \nabla^X \hat{f}(x,e,m), u(x)\otimes v(x)\rangle_{\mathfrak{g}\otimes T_x(X)}\\
&= \langle \nabla^G \hat{f}(x,e,m), \nabla_v^X u(x)\rangle_{\mathfrak{g}} = \widetilde{\nabla}^M_{\nabla_v^X u} f(x,m),\\
&\quad v\in V_0(X),\ u\in C_0^\infty(X;\mathfrak{g}).
\end{aligned}
\tag{4.3}
$$

The equalities (4.1)–(4.3) yield the lemma. ■

Proposition 4.2 *For arbitrary* (v_1,u_1), $(v_2,u_2)\in\mathfrak{a}$, *the following operator equality holds on* $\mathcal{F}C_{\mathrm{b}}^\infty(\mathfrak{D},\Omega_X^M)$:

$$[\mathcal{R}_{\pi_{\tilde\sigma}}(v_1,u_1),\mathcal{R}_{\pi_{\tilde\sigma}}(v_2,u_2)] = \mathcal{R}_{\pi_{\tilde\sigma}}([(v_1,u_1),(v_2,u_2)]).$$

In particular,

$$
\begin{aligned}
[J_{\pi_{\tilde\sigma}}(v_1), J_{\pi_{\tilde\sigma}}(v_2)] &= -iJ_{\pi_{\tilde\sigma}}([v_1,v_2]), && v_1,v_2\in V_0(X),\\
[I_{\pi_{\tilde\sigma}}(u_1), I_{\pi_{\tilde\sigma}}(u_2)] &= -I_{\pi_{\tilde\sigma}}([u_1,u_2]), && u_1,u_2\in C_0^\infty(X;\mathfrak{g}),\\
[J_{\pi_{\tilde\sigma}}(v), I_{\pi_{\tilde\sigma}}(u)] &= -iI_{\pi_{\tilde\sigma}}(\nabla_v^X u), && v\in V_0(X),\ u\in C_0^\infty(X;\mathfrak{g}).
\end{aligned}
$$

Proof. First we note that Lemma 4.1 and (3.9) immediately imply

$$\nabla^\Omega_{(v_1,u_1)}\nabla^\Omega_{(v_2,u_2)} - \nabla^\Omega_{(v_2,u_2)}\nabla^\Omega_{(v_1,u_1)} = \nabla^\Omega_{[(v_1,u_1),(v_2,u_2)]} \quad \text{on } \mathcal{F}C_{\mathrm{b}}^\infty(\mathfrak{D},\Omega_X^M).$$

Therefore, by using the chain rule, we conclude that the lemma will be proved if we show that

$$\nabla^\Omega_{(v_1,u_1)} B^{\pi_{\tilde\sigma}}_{(v_2,u_2)} - \nabla^\Omega_{(v_2,u_2)} B^{\pi_{\tilde\sigma}}_{(v_1,u_1)} = B^{\pi_{\tilde\sigma}}_{[(v_1,u_1),(v_2,u_2)]} \quad \pi_{\tilde\sigma}\text{-a.e.} \tag{4.4}$$

But upon the representation

$$B^{\pi_{\tilde\sigma}}_{(v,u)}(\omega) = \langle \nabla_v^X \log q + \widetilde{\nabla}_u^M \log q + \operatorname{div}^X v + \langle \nabla^G p^\lambda(e,m), -u(x)\rangle_{\mathfrak{g}}, \omega\rangle$$

and Remark 3.2, we easily derive (4.4) again from Lemma 4.1. ■

Thus, the operators $\mathcal{R}_{\pi_{\tilde\sigma}}(v,u)$, $(v,u)\in\mathfrak{a}$, give a marked Poisson space representation of the Lie algebra $\mathfrak{a}$ of the group $\mathfrak{A}$.

5 Intrinsic Dirichlet forms on marked Poisson spaces

5.1 Definition of the intrinsic Dirichlet form

¿From now on, the underlying space of "nice functions" on $X \times M$ will be instead of $\mathfrak{D}$ the space $\mathfrak{D}_0 := C_0^\infty(X \times M)$ consisting of all C^∞ functions with compact support in $X \times M$. Evidently, $\mathfrak{D}_0$ is a subset of $\mathfrak{D}$ and in the case where M is itself compact $\mathfrak{D}_0 = \mathfrak{D}$. Absolutely analogously to $\mathcal{F}C_b^\infty(\mathfrak{D}, \Omega_X^M)$ one constructs the set $\mathcal{F}C_b^\infty(\mathfrak{D}_0, \Omega_X^M)(\subset \mathcal{F}C_b^\infty(\mathfrak{D}, \Omega_X^M))$, which is dense in $L^2(\pi_{\tilde{\sigma}})$. By $\mathcal{FP}(\mathfrak{D}_0, \Omega_X^M)$ we denote the set of all cylinder functions of the form (3.8) in which the functions $\varphi_1, \dots, \varphi_N$ belong to $\mathfrak{D}_0$ and the generating function g_F is a polynomial on $\mathbb{R}^N$, i.e., $g_F \in \mathcal{P}(\mathbb{R}^N)$. Finally, in the same way we introduce $\mathcal{F}C_p^\infty(\mathfrak{D}_0, \Omega_X^M)$ where $g_F \in C_p^\infty(\mathbb{R}^N)$ (:=the set of all C^∞-functions f on $\mathbb{R}^N$ such that f and its partial derivatives of any order are polynomially bounded).

We have obviously

$$\mathcal{F}C_b^\infty(\mathfrak{D}_0, \Omega_X^M) \subset \mathcal{F}C_p^\infty(\mathfrak{D}_0, \Omega_X^M),$$
$$\mathcal{FP}(\mathfrak{D}_0, \Omega_X^M) \subset \mathcal{F}C_p^\infty(\mathfrak{D}_0, \Omega_X^M),$$

and these are algebras with respect to the usual operations. The existence of the Laplace transform $\ell_{\pi_{\tilde{\sigma}}}(f)$ for each $f \in C_0(X \times M)$ implies, in particular, that $\mathcal{F}C_p^\infty(\mathfrak{D}_0, \Omega_X^M) \subset L^2(\pi_{\tilde{\sigma}})$.

Definition 5.1 For $F_1, F_2 \in \mathcal{F}C_p^\infty(\mathfrak{D}_0, \Omega_X^M)$, we introduce a pre-Dirichlet form as

$$\mathcal{E}_{\pi_{\tilde{\sigma}}}^\Omega(F_1, F_2) = \int_{\Omega_X^M} \langle \nabla^\Omega F_1(\omega), \nabla^\Omega F_2(\omega) \rangle_{T_\omega(\Omega_X^M)} \, \pi_{\tilde{\sigma}}(d\omega). \tag{5.1}$$

Note that, for all $F \in \mathcal{F}C_p^\infty(\mathfrak{D}_0, \Omega_X^M)$, the formula (3.12) is still valid and therefore, for $F_1 = g_{F_1}(\langle \varphi_1, \cdot \rangle, \dots, \langle \varphi_N, \cdot \rangle)$ and $F_2 = g_{F_2}(\langle \xi_1, \cdot \rangle, \dots, \langle \xi_K, \cdot \rangle)$ from $\mathcal{F}C_p^\infty(\mathfrak{D}_0, \Omega_X^M)$, we have

$$\begin{aligned}
&\langle \nabla^\Omega F_1(\omega), \nabla^\Omega F_2(\omega) \rangle_{T_\omega(\Omega_X^M)} = \\
&= \sum_{j=1}^N \sum_{k=1}^K \frac{\partial g_{F_1}}{\partial r_j}(\langle \varphi_1, \omega \rangle, \dots, \langle \varphi_N, \omega \rangle) \frac{\partial g_{F_2}}{\partial r_k}(\langle \xi_1, \omega \rangle, \dots, \langle \xi_K, \omega \rangle) \times \\
&\quad \times \int_X \langle \nabla^{X \times M} \varphi_j(x, s_x), \nabla^{X \times M} \xi_k(x, s_x) \rangle_{T_{(x,s_x)}(X \times M)} \, \gamma(dx) \\
&= \sum_{j=1}^N \sum_{k=1}^K \frac{\partial g_{F_1}}{\partial r_j}(\langle \varphi_1, \omega \rangle, \dots, \langle \varphi_N, \omega \rangle) \frac{\partial g_{F_2}}{\partial r_k}(\langle \xi_1, \omega \rangle, \dots, \langle \xi_K, \omega \rangle) \times \\
&\quad \times \langle \langle \nabla^{X \times M} \varphi_j, \nabla^{X \times M} \xi_k \rangle_{T(X \times M)}, \omega \rangle.
\end{aligned} \tag{5.2}$$

Since for $\varphi, \xi \in \mathfrak{D}_0$, the function

$$\begin{aligned}
&\langle \nabla^{X \times M} \varphi(x, m), \nabla^{X \times M} \xi(x, m) \rangle_{T_{(x,m)}(X \times M)} = \\
&= \langle \nabla^X \varphi(x, m), \nabla^X \xi(x, m) \rangle_{T_x(X)} + \langle \tilde{\nabla}^M \varphi(x, m), \tilde{\nabla}^M \xi(x, m) \rangle_{\mathfrak{g}}
\end{aligned}$$

belongs to $\mathfrak{D}_0$, we conclude that

$$\langle \nabla^\Omega F_1(\cdot), \nabla^\Omega(\cdot) F_2(\cdot)\rangle_{T(\Omega_X^M)} \in L^1(\pi_{\tilde\sigma}), \qquad F_1, F_2 \in \mathcal{F}C_{\mathrm{p}}^\infty(\mathfrak{D}_0, \Omega_X^M),$$

and so (5.1) is well defined.

We will call $\mathcal{E}_{\pi_{\tilde\sigma}}^\Omega$ the intrinsic pre-Dirichlet form corresponding to the marked Poisson measure $\pi_{\tilde\sigma}$ on Ω_X^M. In the next subsection we will prove the closability of $\mathcal{E}_{\pi_{\tilde\sigma}}^\Omega$.

5.2 Intrinsic Dirichlet operators

We start with introducing the pre-Dirichlet operator corresponding to the measure $\tilde\sigma$ on $X \times M$ and to the gradient $\nabla^{X\times M}$:

$$\mathcal{E}_{\tilde\sigma}^{X\times M}(\varphi,\xi) := \int_{X\times M} \langle \nabla^{X\times M}\varphi(x,m), \nabla^{X\times M}\xi(x,m)\rangle_{T_{(x,m)}(X\times M)}\,\tilde\sigma(dx,dm), \tag{5.3}$$

where φ, $\xi \in \mathfrak{D}_0$. This form is associated with the Dirichlet operator

$$H_{\tilde\sigma}^{X\times M} := H_{\tilde\sigma}^X + H_{\tilde\sigma}^M \tag{5.4}$$

on $\mathfrak{D}_0$ which satisfies

$$\mathcal{E}_{\tilde\sigma}^{X\times M}(\varphi,\xi) = (H_{\tilde\sigma}^{X\times M}\varphi,\xi)_{L^2(\tilde\sigma)}, \qquad \varphi,\ \xi \in \mathfrak{D}_0. \tag{5.5}$$

Here, $H_{\tilde\sigma}^X$ and $H_{\tilde\sigma}^M$ are the Dirichlet operators of ∇^X and $\tilde\nabla^M$, respectively. Evidently,

$$H_{\tilde\sigma}^X\varphi(x,m) = -\Delta^X\varphi(x,m) - \langle \nabla^X \log q(x,m), \nabla^X\varphi(x,m)\rangle_{T_x(X)}, \tag{5.6}$$

where Δ^X denotes the Laplace-Beltrami operator corresponding to ∇^X.

Let us calculate the operator $H_{\tilde\sigma}^M$. Suppose $f \in \mathfrak{D}_0$ and $W \in C_0(X\times M; \mathfrak{g})$. Analogously to Remark 3.1, we conclude

$$\langle \tilde\nabla^M f(x,m), W(x,m)\rangle_{\mathfrak{g}} = \langle \nabla^M f(x,m), (RW)(x,m)\rangle_{T_m(M)}, \tag{5.7}$$

where $RW \in C_0^\infty(X\times M; TM)$ is given by

$$X\times M \ni (x,m) \mapsto (RW)(x,m) := \frac{d}{dt}\theta(\exp(tW(x,m)),m)\Big|_{t=0} \in T_mM. \tag{5.8}$$

Therefore, using the integration by parts formula on M for a vector field with a compact support, we get

$$\begin{aligned}
&\int_{X\times M} \langle \tilde\nabla^M f(x,m), W(x,m)\rangle_{\mathfrak{g}}\,\tilde\sigma(dx,dm)\\
&\quad= -\int_{X\times M} f(x,m)\big[\operatorname{div}^M(RW)(x,m)\\
&\qquad+ \langle \nabla^M \log q(x,m), (RW)(x,m)\rangle_{T_m(M)}\big]\,\tilde\sigma(dx,dm)\\
&= -\int_{X\times M} f(x,m)\big[\operatorname{div}^M(RW)(x,m) + \langle \tilde\nabla^M \log q(x,m), W(x,m)\rangle_{\mathfrak{g}}\big]\,\tilde\sigma(dx,dm),
\end{aligned}$$

where div^M is the divergence on M with respect to the usual gradient ∇^M and the measure λ. Thus, the divergence $\widetilde{\mathrm{div}}_{\widetilde{\sigma}}^M$ on $X \times M$ w.r.t. the gradient $\widetilde{\nabla}^M$ and the measure $\widetilde{\sigma}$ is given by

$$\widetilde{\mathrm{div}}_{\widetilde{\sigma}}^M W(x,m) = \mathrm{div}^M(RW)(x,m) + \langle \widetilde{\nabla}^M \log q(x,m), W(x,m)\rangle_{\mathfrak{g}}.$$

In particular, the divergence $\widetilde{\mathrm{div}}^M$ w.r.t. the measure $\nu(dx)\,\lambda(dm)$ equals

$$\widetilde{\mathrm{div}}^M W(x,m) = \mathrm{div}^M(RW)(x,m). \tag{5.9}$$

It is easy to see that, for $f \in \mathfrak{D}_0$, $W = \widetilde{\nabla}^M f \in C_0^\infty(X \times M; \mathfrak{g})$, and so we have finally

$$H_{\widetilde{\sigma}}^M f = \widetilde{\mathrm{div}}^M \widetilde{\nabla}^M f = -\widetilde{\Delta}^M f - \langle \widetilde{\nabla}^M \log q, \widetilde{\nabla}^M f\rangle_{\mathfrak{g}}, \qquad f \in \mathfrak{D}_0, \tag{5.10}$$

where

$$\widetilde{\Delta}^M f = \widetilde{\mathrm{div}}^M \widetilde{\nabla}^M f := \mathrm{div}^M(R(\widetilde{\nabla}^M f)). \tag{5.11}$$

The closure of the form $\mathcal{E}_{\widetilde{\sigma}}^{X\times M}$ on

$$L^2(X \times M; \widetilde{\sigma}) =: L^2(\widetilde{\sigma})$$

is denoted by $(\mathcal{E}_{\widetilde{\sigma}}^{X\times M}, D(\mathcal{E}_{\widetilde{\sigma}}^{X\times M}))$. This form generates a positive selfadjoint operator in $L^2(\widetilde{\sigma})$ (the so-called Friedrichs extension of $H_{\widetilde{\sigma}}^{X\times M}$, see e.g. [9]). For this extension we preserve the notation $H_{\widetilde{\sigma}}^{X\times M}$ and denote the domain by $D(H_{\widetilde{\sigma}}^{X\times M})$.

Let us introduce a differential operator $H_{\pi_{\widetilde{\sigma}}}^\Omega$ on the domain $\mathcal{F}C_{\mathrm{b}}^\infty(\mathfrak{D}_0, \Omega_X^M)$ which is given on any $F \in \mathcal{F}C_{\mathrm{b}}^\infty(\mathfrak{D}_0, \Omega_X^M)$ of the form (3.8) by the formula

$$\begin{aligned}(H_{\pi_{\widetilde{\sigma}}}^\Omega F)(\omega) := &- \sum_{j,k=1}^N \frac{\partial^2 F}{\partial r_j \partial r_k}(\langle\varphi_1,\omega\rangle,\dots,\langle\varphi_n,\omega\rangle)\langle\langle\nabla^{X\times M}\varphi_j, \nabla^{X\times M}\varphi_k\rangle_{T(X\times M)},\omega\rangle \\ &+ \sum_{j=1}^N \frac{\partial F}{\partial r_j}(\langle\varphi_1,\omega\rangle,\dots,\langle\varphi_n,\omega\rangle)\langle H_{\widetilde{\sigma}}^{X\times M}\varphi_j,\omega\rangle. \end{aligned} \tag{5.12}$$

Since

$$\langle \nabla^{X\times M}\log q, \nabla^{X\times M}\varphi_j\rangle_{T(X\times M)} \in L^2(\widetilde{\sigma}) \cap L^1(\widetilde{\sigma})$$

(see condition (3.13)), the r.h.s. of (5.12) is well defined as an element of $L^2(\pi_{\widetilde{\sigma}})$. The following theorem implies, in particular, that $H_{\pi_{\widetilde{\sigma}}}^\Omega$ is well defined as a linear operator on $\mathcal{F}C_{\mathrm{b}}^\infty(\mathfrak{D}_0, \Omega_X^M)$, i.e., independently of the representation of F as in (3.8).

Theorem 5.1 *The operator $H_{\pi_{\widetilde{\sigma}}}^\Omega$ is associated with the intrinsic Dirichlet form $\mathcal{E}_{\pi_{\widetilde{\sigma}}}^\Omega$ in the sense that, for all $F_1, F_2 \in \mathcal{F}C_{\mathrm{b}}^\infty(\mathfrak{D}_0, \Omega_X^M)$*

$$\mathcal{E}_{\pi_{\widetilde{\sigma}}}^\Omega(F_1,F_2) = (H_{\pi_{\widetilde{\sigma}}}^\Omega F_1, F_2)_{L^2(\pi_{\widetilde{\sigma}})}, \tag{5.13}$$

or

$$H_{\pi_{\widetilde{\sigma}}}^\Omega = -\,\mathrm{div}_{\pi_{\widetilde{\sigma}}}^\Omega \nabla^\Omega \quad \text{on } \mathcal{F}C_{\mathrm{b}}^\infty(\mathfrak{D}_0, \Omega_X^M).$$

We call $H_{\pi_{\widetilde{\sigma}}}^\Omega$ the intrinsic Dirichlet operator of the measure $\pi_{\widetilde{\sigma}}$.

Lemma 5.1 *For any $\varphi \in \mathfrak{D}_0$ and $W \in C_0^\infty(X \times M; \mathfrak{g})$, we have*

$$\widetilde{\operatorname{div}}^M(\varphi W)(x,m) = \langle \widetilde{\nabla}^M \varphi(x,m), W(x,m)\rangle_{\mathfrak{g}} + \varphi(x,m)\widetilde{\operatorname{div}}^M W(x,m).$$

Proof. By (5.7), (5.8), and (5.9)

$$\begin{aligned}
\widetilde{\operatorname{div}}^M(\varphi W)(x,m) &= \operatorname{div}^M \left[\frac{d}{dt}\,\theta(\exp(t\varphi(x,m)W(x,m)),m)\big|_{t=0}\right] \\
&= \operatorname{div}^M \left[\varphi(x,m)\,\frac{d}{dt}\,\theta(\exp(tW(x,m)),m)\big|_{t=0}\right] \\
&= \langle \nabla^M \varphi(x,m), \frac{d}{dt}\,\theta(\exp(tW(x,m)),m)\big|_{t=0}\rangle_{T_m(M)} \\
&\quad + \varphi(x,m)\operatorname{div}^M \left[\frac{d}{dt}\theta(\exp(tW(x,m)),m)\big|_{t=0}\right] \\
&= \frac{d}{dt}\,\varphi(x,\theta(\exp(tW(x,m)),m))\big|_{t=0} + \varphi(x,m)\widetilde{\operatorname{div}}^M W(x,m) \\
&= \langle \widetilde{\nabla}^M \varphi(x,m), W(x,m)\rangle_{\mathfrak{g}} + \varphi(x,m)\widetilde{\operatorname{div}}^M W(x,m). \quad \blacksquare
\end{aligned}$$

Proof of Theorem 5.1. For shortness of notations we will prove the formula (5.13) in the case where $F_1, F_2 \in \mathcal{F}C_b^\infty(\mathfrak{D}_0, \Omega_X^M)$ are of the form

$$F_1 = g_{F_1}(\langle\varphi,\omega\rangle), \quad F_2 = g_{F_2}(\langle\xi,\omega\rangle).$$

However, it is a trivial step to generalize the proof to general F_1, F_2.

Let $\Lambda \in \mathcal{O}_c(X)$ be chosen so that the supports of the functions φ and ξ are in Λ_{mk}. Then, by (5.1), (5.2), and the construction of the marked Poisson measure

$$\begin{aligned}
\mathcal{E}_{\pi_{\tilde\sigma}}^{\Omega}(F_1,F_2) &= \int_{\Omega_X^M} g'_{F_1}(\langle\varphi,\omega\rangle)g'_{F_2}(\langle\xi,\omega\rangle)\langle\langle\nabla^{X\times M}\varphi, \nabla^{X\times M}\xi\rangle_{T(X\times M)},\omega\rangle\,\pi_{\tilde\sigma}(d\omega) \\
&= -e^{\tilde\sigma(\Lambda_{\mathrm{mk}})}\sum_{n=1}^\infty \frac{1}{n!}\int_{\Lambda_{\mathrm{mk}}^n} g'_{F_1}(\varphi(x_1,m_1)+\cdots+\varphi(x_n,m_n)) \\
&\qquad \times g'_{F_2}(\xi(x_1,m_1)+\cdots+\xi(x_n,m_n)) \\
\times&\left[\sum_{i=1}^n \langle\nabla^{X\times M}\varphi(x_i,m_i), \nabla^{X\times M}\xi(x_i,m_i)\rangle_{T_{(x_i,m_i)}(X\times M)}\right]\tilde\sigma(dx_1,dm_1)\cdots\tilde\sigma(dx_1,dm_1) \\
&= e^{-\tilde\sigma(\Lambda_{\mathrm{mk}})}\sum_{n=1}^\infty \frac{1}{n!}\int_{\Lambda_{\mathrm{mk}}^n}\sum_{i=1}^n\langle\nabla_i^{X\times M} g_{F_1}(\varphi(x_1,m_1)+\cdots+\varphi(x_n,m_n)), \\
&\nabla_i^{X\times M} g_{F_2}(\xi(x_1,m_1)+\cdots+\xi(x_n,m_n))\rangle_{T_{(x_i,m_i)}(X\times M)}\,\tilde\sigma(dx_1,dm_1)\cdots\tilde\sigma(dx_n,dm_n),
\end{aligned}$$

where $\nabla_i^{X\times M}$ denotes the $\nabla^{X\times M}$ gradient in the (x_i,m_i) variables. Therefore, by using (5.10) and Lemma 5.1, we proceed in the calculation of $\mathcal{E}_{\pi_{\tilde\sigma}}^{\Omega}(F_1,F_2)$ as follows:

$$= e^{-\tilde\sigma(\Lambda_{\mathrm{mk}})}\sum_{n=1}^\infty \frac{1}{n!}\int_{\Lambda_{\mathrm{mk}}^n}\left[\sum_{i=1}^n H_{\tilde\sigma}^{(X\times M)_i} g_{F_1}(\varphi(x_1,m_1)+\cdots+\varphi(x_n,m_n))\right]\times$$

$$\times g_{F_1}(\xi(x_1,m_1)+\cdots+\xi(x_n,m_n))\,\widetilde{\sigma}(x_1,m_1)\cdots\widetilde{\sigma}(dx_n,dm_n)$$

$$= -e^{\widetilde{\sigma}(\Lambda_{mk})}\sum_{n=1}^{\infty}\frac{1}{n!}\int_{\Lambda_{mk}^n}\left[\sum_{i=1}^{n} g''_{F_1}(\varphi(x_1,m_1)+\cdots+\varphi(x_n,m_n))\times\right.$$
$$\times\langle\nabla^{X\times M}\varphi(x_i,m_i),\nabla^{X\times M}\varphi(x_i,m_i)\rangle_{T_{(x_i,m_i)}(X\times M)}$$
$$\left.+\,g'_{F_1}(\varphi(x_1,m_1)+\cdots+\varphi(x_n,m_n))H_{\widetilde{\sigma}}^{X\times M}\varphi(x_i,m_i)\right]\times$$
$$\times g_{F_2}(\xi(x_1,m_1)+\cdots+\xi(x_n,m_n))\,\widetilde{\sigma}(dx_1,dm_1)\cdots\widetilde{\sigma}(dx_n,dm_n)$$
$$= \int_{\Omega_X^M} H_{\pi_{\widetilde{\sigma}}}^{\Omega}F_1(\omega)F_2(\omega)\,\pi_{\widetilde{\sigma}}(d\omega). \quad \blacksquare$$

Remark 5.1 The operator $H^{\Omega}_{\pi_{\widetilde{\sigma}}}$ can be naturally extended to cylinder functions of the form

$$F(\omega) := e^{\langle\varphi,\omega\rangle}, \qquad \varphi\in\mathfrak{D}_0,\ \omega\in\Omega_X^M,$$

since such F belong to $L^2(\pi_{\widetilde{\sigma}})$. We then have

$$H^{\Omega}_{\pi_{\widetilde{\sigma}}}e^{\langle\varphi,\omega\rangle} = \langle H_{\widetilde{\sigma}}^{X\times M}\varphi - |\nabla^{X\times M}\varphi|^2_{T(X\times M)},\omega\rangle\, e^{\langle\varphi,\omega\rangle}. \tag{5.14}$$

As an immediate consequence of Theorem 5.1 we obtain

Corollary 5.1 *$(\mathcal{E}^{\Omega}_{\pi_{\widetilde{\sigma}}},\mathcal{F}C_{\mathrm{b}}^{\infty}(\mathfrak{D}_0,\Omega_X^M))$ is closable on $L^2(\pi_{\widetilde{\sigma}})$. Its closure $(\mathcal{E}^{\Omega}_{\pi_{\widetilde{\sigma}}},D(\mathcal{E}^{\Omega}_{\pi_{\widetilde{\sigma}}}))$ is associated with a positive definite selfadjoint operator, the Friedrichs extension of $H^{\Omega}_{\pi_{\widetilde{\sigma}}}$, which we also denote by $H^{\Omega}_{\pi_{\widetilde{\sigma}}}$ (and its domain by $D(H^{\Omega}_{\pi_{\widetilde{\sigma}}})$).*

Clearly, ∇^{Ω} also extends to $D(\mathcal{E}^{\Omega}_{\pi_{\widetilde{\sigma}}})$. We denote this extension by ∇^{Ω}.

Corollary 5.2 *Let*

$$\begin{gathered} F(\omega) := g_F(\langle\varphi_1,\omega\rangle,\ldots,\langle\varphi_N,\omega\rangle), \qquad \omega\in\Omega_X^M, \\ \varphi_1,\ldots,\varphi_N\in D(\mathcal{E}_{\widetilde{\sigma}}^{X\times M}),\ g_F\in C_{\mathrm{b}}^{\infty}(\mathbb{R}^N). \end{gathered} \tag{5.15}$$

Then $F\in D(\mathcal{E}^{\Omega}_{\pi_{\widetilde{\sigma}}})$ and

$$(\nabla^{\Omega}F)(\omega;x) = \sum_{j=1}^{N}\frac{\partial g_F}{\partial r_j}(\langle\varphi_1,\omega\rangle,\ldots,\langle\varphi_N,\omega\rangle)\nabla^{X\times M}\varphi_j(x,s_x).$$

Proof. By approximation this is an immediate consequence of (3.12) and the fact that, for all $1\le i\le N$,

$$\int\langle|\nabla^{X\times M}\varphi_i|^2_{T(X\times M)},\omega\rangle\,\pi_{\widetilde{\sigma}}(d\omega) = \mathcal{E}_{\widetilde{\sigma}}^{X\times M}(\varphi_i,\varphi_i). \tag{5.16}$$

Remark 5.2 Let $\mu_{\nu,\widetilde{\sigma}} \in \mathcal{M}_2(\Omega_X^M)$ be given as in (3.18). Then, by Theorem 3.2, (ii)⇒(i), all results above are valid with $\mu_{\nu,\widetilde{\sigma}}$ replacing $\pi_{\widetilde{\sigma}}$. By (5.12) we have

$$H^{\Omega}_{\pi_{\widetilde{\sigma}}} = H^{\Omega}_{\mu_{\nu,\widetilde{\sigma}}} \quad \text{on } \mathcal{F}C_b^\infty(\mathfrak{D}_0, \Omega_X^M).$$

We note that the r.h.s. of (5.12) only depends on $\widetilde{\sigma}$ and the Riemannian structure of $X \times M$. The respective Friedrichs extension on $L^2(\mu_{\nu,\widetilde{\sigma}})$ is again denoted by $H^{\Omega}_{\mu_{\nu,\widetilde{\sigma}}}$, however it does necessarily not coincide with $H^{\Omega}_{\pi_{\widetilde{\sigma}}}$.

5.3 The heat semigroup and ergodicity

The results of this subsection are obtained absolutely analogously to the corresponding results of the paper [7], so we omit the proofs.

For $\mu_{\varkappa,\widetilde{\sigma}} \in \mathcal{M}_2(\Omega_X^M)$ let $T^{\Omega}_{\mu_{\varkappa,\widetilde{\sigma}}}(t) := \exp(-tH^{\Omega}_{\mu_{\varkappa,\widetilde{\sigma}}})$, $t > 0$. Define

$$E(\mathfrak{D}_1, \Omega_X^M) = \text{l.h.}\,\{\, \exp(\langle \log(1+\varphi), \cdot\rangle) \mid \varphi \in \mathfrak{D}_1 \,\},$$

where l.h. means the linear hull and

$$\mathfrak{D}_1 := \{\, \varphi \in D(H_{\widetilde{\sigma}}^{X\times M}) \cap L^1(\widetilde{\sigma}) \mid H_{\widetilde{\sigma}}^{X\times M}\varphi \in L^1(\widetilde{\sigma}) \\ \text{and } -\delta \le \varphi \le 0 \text{ for some } \delta \in (0,1) \,\}.$$

Proposition 5.1 *Let $\mu_{\varkappa,\widetilde{\sigma}}$ be as in* (3.18). *Assume that $H_{\widetilde{\sigma}}^{X\times M}$ is conservative, i.e.,*

$$\int_{X\times M} (H_{\widetilde{\sigma}}^{X\times M}\varphi)(x,m)\, \widetilde{\sigma}(dx,dm) = 0$$

for all $\varphi \in D(H_{\widetilde{\sigma}}^{X\times M}) \cap L^1(\widetilde{\sigma})$ such that $H_{\widetilde{\sigma}}^{X\times M}\varphi \in L^1(\widetilde{\sigma})$, and suppose that $(H_{\widetilde{\sigma}}^{X\times M}, \mathfrak{D}_0)$ is essentially selfadjoint on $L^2(\widetilde{\sigma})$. Then

$$T^{\Omega}_{\mu_{\varkappa,\widetilde{\sigma}}}(t) \exp(\langle \log(1+\varphi), \cdot\rangle) = \exp(\langle \log(1 + e^{-tH_{\widetilde{\sigma}}^{X\times M}}\varphi), \cdot\rangle), \qquad \varphi \in \mathfrak{D}_1, \tag{5.17}$$

$E(\mathfrak{D}_1, \Omega_X^M) \subset D(H^{\Omega}_{\mu_{\varkappa,\widetilde{\sigma}}})$, and

$$H^{\Omega}_{\mu_{\varkappa,\widetilde{\sigma}}} \exp(\langle \log(1+\varphi), \cdot\rangle) \\ = \langle (1+\varphi)^{-1} H_{\widetilde{\sigma}}^{X\times M}\varphi, \cdot\rangle \exp(\langle \log(1+\varphi), \cdot\rangle), \qquad \varphi \in \mathfrak{D}_1.$$

Remark 5.3 (i) The condition of essential selfadjointness of $H_{\widetilde{\sigma}}^{X\times M}$ on $\mathfrak{D}_0$ is fulfilled if X is complete and $|\beta^{\widetilde{\sigma}}|_{T(X\times M)} \in L^p_{\text{loc}}(X \times M; m \otimes \lambda)$ for some $p \ge \dim(X) + 1$.

(ii) Since $(\exp(-tH_{\widetilde{\sigma}}^{X\times M}))_{t>0}$ is sub-Markovian (i.e., $0 \le \exp(-tH_{\widetilde{\sigma}}^{X\times M})\varphi \le 1$ for all $t > 0$ and $\varphi \in L^2(\widetilde{\sigma})$, $0 \le \varphi \le 1$), because $(\mathcal{E}_{\widetilde{\sigma}}^{X\times M}, D(\mathcal{E}_{\widetilde{\sigma}}^{X\times M}))$ is a Dirichlet form, by a simple approximation argument Proposition 5.1 implies that the equality (5.17) holds for $t > 0$ and all $\varphi \in L^1(\widetilde{\sigma})$, $-1 < \varphi \le 0$.

Theorem 5.2 *Let the conditions of Proposition* 5.1 *hold. Then* $E(\mathfrak{D}_1, \Omega_X^M)$ *is an operator core for the Friedrichs extension* $H^{\Omega}_{\mu_{\varkappa,\tilde{\sigma}}}$ *on* $L^2(\mu_{\varkappa,\tilde{\sigma}})$. *(In other words:* $(H^{\Omega}_{\mu_{\varkappa,\tilde{\sigma}}}, E(\mathfrak{D}_1, \Omega_X^M))$ *is essentially selfadjoint on* $L^2(\mu_{\varkappa,\tilde{\sigma}})$.*)*

Theorem 5.3 *Suppose that the conditions of Theorem* 3.3 *and Proposition* 5.1 *hold. Then the following assertions are equivalent:*

(i) $\mu_{\varkappa,\tilde{\sigma}} = \pi_{z\tilde{\sigma}}$ *for some* $z > 0$.

(ii) $(\mathcal{E}^{\Omega}_{\mu_{\varkappa,\tilde{\sigma}}}, D(\mathcal{E}^{\Omega}_{\mu_{\varkappa,\tilde{\sigma}}}))$ *is irreducible (i.e., for* $F \in D(\mathcal{E}^{\Omega}_{\mu_{\varkappa,\tilde{\sigma}}})$, $\mathcal{E}^{\Omega}_{\mu_{\varkappa,\tilde{\sigma}}}(F, F) = 0$ *implies that* $F = \mathrm{const}$).

(iii) $(T^{\Omega}_{\mu_{\varkappa,\tilde{\sigma}}}(t))_{t>0}$ *is irreducible (i.e., if* $G \in L^2(\mu_{\varkappa,\tilde{\sigma}})$ *such that* $T^{\Omega}_{\mu_{\varkappa,\tilde{\sigma}}}(t)(GF) = GT^{\Omega}_{\mu_{\varkappa,\tilde{\sigma}}}(t)F$ *for all* $F \in L^\infty(\mu_{\varkappa,\tilde{\sigma}})$, $t > 0$, *then* $G = \mathrm{const}$).

(iv) *If* $F \in L^2(\mu_{\varkappa,\tilde{\sigma}})$ *such that* $T^{\Omega}_{\mu_{\varkappa,\tilde{\sigma}}}(t)F = F$ *for all* $T > 0$, *then* $F = \mathrm{const}$.

(v) $T^{\Omega}_{\mu_{\varkappa,\tilde{\sigma}}}(t) \not\equiv 1$ *and ergodic (i.e.,*

$$\int \left(T^{\Omega}_{\mu_{\varkappa,\tilde{\sigma}}}(t)F - \int F \, d\mu_{\varkappa,\tilde{\sigma}} \right)^2 d\mu_{\varkappa,\tilde{\sigma}} \to 0 \quad \text{as t} \to 0$$

for all $F \in L^2(\mu_{\varkappa,\tilde{\sigma}})$*).*

(vi) *If* $F \in D(H^{\Omega}_{\mu_{\varkappa,\tilde{\sigma}}})$ *with* $H^{\Omega}_{\mu_{\varkappa,\tilde{\sigma}}} = 0$, *then* $F = \mathrm{const}$.

Remark 5.4 Let us consider the diffusion process P on $X \times M$ associated to the Dirichlet form $(\mathcal{E}^{X\times M}_{\tilde{\sigma}}, D(\mathcal{E}^{X\times M}_{\tilde{\sigma}}))$. This process can be interpreted as distorted Brownian motion on the manifold $X \times M$. More precisely, the diffusion of points $x \in X$ is associated to the Dirichlet form of the measure σ, so that it is distorted Brownian motion on X, and the diffusion of marks s_x, $x \in X$, is associated to the $\widetilde{\nabla}^M$-Dirichlet form of the measure $p(x, dm)$ on M.

The existence of a diffusion process $\mathbf{P}$ corresponding to the Dirichlet form $(\mathcal{E}^{\Omega}_{\mu_{\varkappa,\tilde{\sigma}}}, D(\mathcal{E}^{\Omega}_{\mu_{\varkappa,\tilde{\sigma}}}))$ follows from [31], and its identification with the independent infinite particle process (on $X \times M$) may be proved by the same arguments as in [7]. By analogy with the case of the process P on $X \times M$, one can call $\mathbf{P}$ distorted Brownian motion on Ω_X^M.

6 Intrinsic Dirichlet operator and second quantization

In this section, we want to describe the Fock space realization of the marked Poisson spaces and show that $H^{\Omega}_{\pi_{\tilde{\sigma}}}$ is the second quantization of the operator $H^{X\times M}_{\tilde{\sigma}}$.

6.1 Marked Poisson gradient and chaos decomposition

Let us define another "gradient" on functions $F \colon \Omega_X^M \to \mathbb{R}$, which has specific useful properties on the marked Poisson space.

Definition 6.1 For any $F \in \mathcal{F}C_{\mathrm{p}}^{\infty}(\mathfrak{D}_0, \Omega_X^M)$ we define the marked Poisson gradient ∇^{MP} as

$$(\nabla^{\mathrm{MP}} F)(\omega, (x, m)) := F(\omega + \varepsilon_{(x,m)}) - F(\omega), \qquad \omega \in \Omega_X^M, \ (x, m) \in X \times M.$$

Let us mention that the operation

$$\Omega_X^M \ni \omega \mapsto \omega + \varepsilon_{(x,m)} \in \Omega_X^M$$

is a $\pi_{\widetilde{\sigma}}$-a.e. well-defined map because of the property

$$\pi_{\widetilde{\sigma}}\big(\{\omega = (\gamma, s) \in \Omega_X^M \mid x \in \gamma\}\big) = 0$$

for an arbitrary $x \in X$ (which easily follows from the construction of $\pi_{\widetilde{\sigma}}$). We consider ∇^{MP} as a mapping

$$\nabla^{\mathrm{MP}} \colon \mathcal{F}C_{\mathrm{p}}^{\infty}(\mathfrak{D}_0, \Omega_X^M) \ni F \mapsto \nabla^{\mathrm{MP}} F \in L^2(\widetilde{\sigma}) \otimes L^2(\pi_{\widetilde{\sigma}})$$

that corresponds to using the Hilbert space $L^2(\widetilde{\sigma})$ as a tangent space at any point $\omega \in \Omega_X^M$. Thus, for any $\varphi \in \mathfrak{D}_0$, we can introduce the directional derivative

$$\begin{aligned}(\nabla_{\varphi}^{\mathrm{MP}} F)(\omega) &= \langle \nabla^{\mathrm{MP}} F(\omega), \varphi \rangle_{L^2(\widetilde{\sigma})} \\ &= \int_{X \times M} (F(\omega + \varepsilon_{(x,m)}) - F(\omega))\varphi(x,m)\, \widetilde{\sigma}(dx, dm).\end{aligned}$$

The most important feature of the marked Poisson gradient is that it produces (via a corresponding "integration by parts formula") the orthogonal system of Charlier polynomials on $(\Omega_X^M, \mathcal{B}(\Omega_X^M), \pi_{\widetilde{\sigma}})$. Below, we describe this construction in detail using the isomorphism between $L^2(\pi_{\widetilde{\sigma}})$ and the symmetric Fock space (see [21, 25, 30])

Let $\mathcal{F}(L^2(\widetilde{\sigma}))$ denote the symmetric Fock space over $L^2(\widetilde{\sigma})$:

$$\mathcal{F}(L^2(\widetilde{\sigma})) := \bigoplus_{n=0}^{\infty} \mathcal{F}_n(L^2(\widetilde{\sigma}))n!,$$

where

$$\mathcal{F}_n(L^2(\widetilde{\sigma})) := (L^2(\widetilde{\sigma}))^{\widehat{\otimes} n} = \hat{L}^2((X \times M)^n, \widetilde{\sigma}^{\otimes n}), \qquad n \in \mathbb{N},$$
$$\mathcal{F}_0(L^2(\widetilde{\sigma})) := \mathbb{R},$$

$\widehat{\otimes}$ denoting the symmetric tensor product. Thus, for each $F = (f^{(n)})_{n=0}^{\infty} \in \mathcal{F}(L^2(\widetilde{\sigma}))$

$$\|F\|^2_{\mathcal{F}(L^2(\widetilde{\sigma}))} = \sum_{n=0}^{\infty} |f^{(n)}|_{\hat{L}^2(\widetilde{\sigma}^{\otimes n})} n!.$$

By $\mathcal{F}_{\mathrm{fin}}(\mathfrak{D}_0)$ we denote the dense subset of $\mathcal{F}(L^2(\widetilde{\sigma}))$ consisting of finite sequences $(f^{(n)})_{n=0}^{N}$, $n \in \mathbb{Z}_+$, such that each $f^{(n)}$ belongs to $\mathcal{F}_n(\mathfrak{D}_0) := \mathrm{a}.\mathfrak{D}_0^{\widehat{\otimes} n}$, the n-th symmetric algebraic tensor power of $\mathfrak{D}_0$:

$$\mathrm{a}.\mathfrak{D}_0^{\widehat{\otimes} n} := \mathrm{l.\,h.}\{\varphi_1 \widehat{\otimes} \cdots \widehat{\otimes} \varphi_n \mid \varphi_i \in \mathfrak{D}_0\}.$$

In virtue of the polarization identity, the latter set is spanned just by the vectors of the form $\varphi^{\otimes n}$ with $\varphi \in \mathfrak{D}_0$.

Now, we define a linear mapping

$$\mathcal{F}_{\text{fin}}(\mathfrak{D}_0) \ni F = (f^{(n)})_{n=0}^{N} \mapsto IF = (IF)(\omega) = \sum_{n=0}^{N} Q_n(f^{(n)};\omega) \in \mathcal{FP}(\mathfrak{D}_0, \Omega_X^M) \tag{6.1}$$

by using the following recursion relation:

$$\begin{aligned} Q_{n+1}(\varphi^{\otimes(n+1)};\omega) &= Q_n(\varphi^{\otimes n};\omega)(\langle\omega,\varphi\rangle - \langle\varphi\rangle_{\tilde{\sigma}}) \\ &\quad - nQ_n(\varphi^{\otimes(n-1)}\widehat{\otimes}(\varphi^2),\omega) - nQ_{n-1}(\varphi^{\otimes(n-1)};\omega)\langle\varphi^2\rangle_{\tilde{\sigma}}, \\ Q_0(1,\omega) &= 1, \quad \varphi \in \mathfrak{D}_0. \end{aligned} \tag{6.2}$$

Here, we have set $\langle\varphi\rangle_{\tilde{\sigma}} := \int \varphi\, d\tilde{\sigma}$. Notice that, since $\mathfrak{D}_0$ is an algebra under pointwise multiplication of functions, the latter definition is correct.

It is not hard to see that the mapping (6.1) is one-to-one. Moreover, the following proposition holds:

Proposition 6.1 *The mapping* (6.1) *can be extended by continuity to a unitary isomorphism between the spaces* $\mathcal{F}(L^2(\tilde{\sigma}))$ *and* $L^2(\pi_{\tilde{\sigma}})$.

For each $\varphi \in \mathfrak{D}_0$, let us define the creation and annihilation operators in $\mathcal{F}(L^2(\tilde{\sigma}))$ by

$$a^+(\varphi)\psi^{\otimes n} = \varphi\widehat{\otimes}\psi^{\otimes n}, \qquad a^-(\varphi)\psi^{\otimes n} = n(\varphi,\psi)_{L^2(\tilde{\sigma})}\psi^{\otimes(n-1)}, \qquad \psi \in \mathfrak{D}_0.$$

We will denote by the same letters the images of these operators under the unitary I.

Proposition 6.2 *We have, for each* $\varphi \in \mathfrak{D}_0$,

$$a^-(\varphi) = \nabla_\varphi^{\mathrm{MP}}, \qquad a^+(\varphi) = \nabla_\varphi^{\mathrm{MP}*}.$$

In particular,

$$Q_n(\varphi_1\widehat{\otimes}\cdots\widehat{\otimes}\varphi_n;\omega) = (\nabla_{\varphi_1}^{\mathrm{MP}*}\cdots\nabla_{\varphi_n}^{\mathrm{MP}*}1)(\omega), \qquad \omega \in \Omega_X^M.$$

Finally, for each $\varphi \in \mathfrak{D}_0$ we introduce the Poisson exponential

$$e(\varphi;\cdot) := \sum_{n=0}^{\infty} \frac{1}{n!} Q_n(\varphi^{\otimes n};\cdot) = I(\operatorname{Exp}\varphi),$$

where

$$\operatorname{Exp}\varphi = \left(\frac{1}{n!}\varphi^{\otimes n}\right)_{n=0}^{\infty}.$$

Then, one can show that, for $\varphi > -1$,

$$e(\varphi;\omega) = \exp\left[\langle\log(1+\varphi),\omega\rangle - \langle\varphi\rangle_{\tilde{\sigma}}\right], \qquad \omega \in \Omega_X^M. \tag{6.3}$$

6.2 Second quantization on the marked Poisson space

Let B be a contraction on $L^2(\widetilde{\sigma})$, i.e., $B \in \mathcal{L}(L^2(\widetilde{\sigma}), L^2(\widetilde{\sigma}))$, $\|B\| \leq 1$. Then, we can define the operator $\operatorname{Exp} B$ as the contraction on $\mathcal{F}(L^2(\widetilde{\sigma}))$ given by

$$\operatorname{Exp} B \restriction \mathcal{F}_n(L^2(\widetilde{\sigma})) := B \otimes \cdots \otimes B \quad (n \text{ times}),\ n \in \mathbb{N},$$
$$\operatorname{Exp} B \restriction \mathcal{F}_0(L^2(\widetilde{\sigma})) := 1.$$

For any selfadjoint positive operator A in $L^2(\widetilde{\sigma})$, we have a contraction semigroup e^{-tA}, $t \geq 0$, and it is possible to introduce a positive selfadjoint operator $d\operatorname{Exp} A$ as the generator of the semigroup $\operatorname{Exp}(e^{-tA})$, $t \geq 0$:

$$\operatorname{Exp}(e^{-tA}) = \exp(-td \operatorname{Exp} A). \tag{6.4}$$

The operator $d\operatorname{Exp} A$ is called the second quantization of A. We denote by H_A^{MP} the image of the operator $d\operatorname{Exp} A$ in the marked Poisson space $L^2(\pi_{\widetilde{\sigma}})$.

Theorem 6.1 *Let $\mathfrak{D}_0 \subset \operatorname{Dom} A$. Then, the symmetric bilinear form corresponding to the operator H_A^{MP} has the following representation:*

$$(H_A^{\mathrm{MP}} F_1, F_2)_{L^2(\pi_{\widetilde{\sigma}})} = \int_{\Omega_X^M} (\nabla^{\mathrm{MP}} F_1, A \nabla^{\mathrm{MP}} F_2)_{L^2(\widetilde{\sigma})}\, \pi_{\widetilde{\sigma}}(d\omega) \tag{6.5}$$

for all $F_1, F_2 \in \mathcal{FP}(\mathfrak{D}_0, \Omega_X^M)$.

Remark 6.1 The bilinear form (6.5) uses the marked Poisson gradient ∇^{MP} and a coefficient operator $A > 0$. We will call

$$\mathcal{E}_{\pi_{\widetilde{\sigma}},A}^{\mathrm{MP}}(F_1, F_2) = \int_{\Omega_X^M} (\nabla^{\mathrm{MP}} F, A \nabla^{\mathrm{MP}} G)_{L^2(\widetilde{\sigma})}\, \pi_{\widetilde{\sigma}}(d\omega)$$

the marked Poisson pre-Dirichlet form with coefficient A.

Proof of Theorem 5.1. The proof is analogous to that of Theorem 5.1 in [7]. Using again the fact that $\mathfrak{D}_0$ is an algebra under pointwise multiplication, one easily concludes that, for any $F \in \mathcal{FP}(\mathfrak{D}_0, \Omega_X^M)$ and any $\omega \in \Omega_X^M$, the gradient $\nabla^{\mathrm{MP}} F(\omega, (x, m))$ is a function in $\mathfrak{D}_0$ and hence

$$(\nabla^{\mathrm{MP}} F, A \nabla^{\mathrm{MP}} G)_{L^2(\widetilde{\sigma})} \in \mathcal{FP}(\mathfrak{D}_0, \Omega_X^M),$$

so that the form (6.5) is well-defined. Then, one verifies the formula (6.5) by using Propositions 5.1, 5.2 and the explicit formula for $d\operatorname{Exp} A$ on $\mathcal{F}_n(\mathfrak{D}_0)$:

$$d\operatorname{Exp} A\, \varphi^{\otimes n} = n(A\varphi)\widehat{\otimes}\varphi^{\otimes(n-1)}, \qquad \varphi \in \mathfrak{D}_0. \quad \blacksquare$$

6.3 The intrinsic Dirichlet operator as a second quantization

The following two theorems are again analogous to the corresponding results (Theorems 5.2 and 5.3) in [7], so we omit their proofs.

Let us consider the special case of the second quantization operator $d\,\mathrm{Exp}\,A$ where the operator A coincides with the Dirichlet operator $H_{\tilde\sigma}^{X\times M}$.

Theorem 6.2 *We have the equality*

$$H^{\mathrm{MP}}_{H_{\tilde\sigma}^{X\times M}} = H^{\Omega}_{\pi_{\tilde\sigma}}$$

on the dense domain $\mathcal{F}C_{\mathrm{p}}^{\infty}(\mathfrak{D}_0,\Omega_X^M)$. In particular, for all $F_1,F_2\in\mathcal{F}C_{\mathrm{p}}^{\infty}(\mathfrak{D}_0,\Omega_X^M)$

$$\int_{\Omega_X^M}\langle\nabla^{\Omega}F_1(\omega),\nabla^{\Omega}F_2(\omega)\rangle_{T_\omega(\Omega_X^M)}\,\pi_{\tilde\sigma}(d\omega)$$
$$=\int_{\Omega_X^M}(\nabla^{\mathrm{MP}}F_1(\omega),H_{\tilde\sigma}^{X\times M}\nabla^{\mathrm{MP}}F_2(\omega))_{L^2(\tilde\sigma)}\,\pi_{\tilde\sigma}(d\omega),$$

or

$$\nabla^{\Omega*}\nabla^{\Omega}=\nabla^{\mathrm{MP}*}H_{\tilde\sigma}^{X\times M}\nabla^{\mathrm{MP}}$$

as an equality on $\mathcal{F}C_{\mathrm{p}}^{\infty}(\mathfrak{D}_0,\Omega_X^M)$.

Theorem 6.3 *Suppose that the operator $H_{\tilde\sigma}^{X\times M}$ is essentially selfadjoint on the domain $\mathfrak{D}_0\subset\mathrm{Dom}(H_{\tilde\sigma}^{X\times M})$. Then, the intrinsic Dirichlet operator $H^{\Omega}_{\pi_{\tilde\sigma}}$ is essentially selfadjoint on the domain $\mathcal{F}C_{\mathrm{b}}^{\infty}(\mathfrak{D}_0,\Omega_X^M)$.*

Remark 6.2 Notice that in Theorem 6.3 we do not suppose the operator $H_{\tilde\sigma}^{X\times M}$ to be conservative. So, this theorem is a generalization of Theorem 5.2 in the special case where $\mu_{\varkappa,\tilde\sigma}=\pi_{\tilde\sigma}$.

Corollary 6.1 *Suppose that the condition of Theorem 6.3 is satisfied and let $T^{\Omega}_{\pi_{\tilde\sigma}}(t)=\exp(-tH^{\Omega}_{\pi_{\tilde\sigma}})$, $t>0$. Then, for each $\varphi\in\mathfrak{D}_0$, $\varphi>-1$, we have*

$$T^{\Omega}_{\pi_{\tilde\sigma}}(t)\exp(\langle\log(1+\varphi),\cdot\rangle)=\exp\left[\langle\log(1+e^{-tH_{\tilde\sigma}^{X\times M}}\varphi),\cdot\rangle-\langle(e^{-tH_{\tilde\sigma}^{X\times M}}-1)\varphi\rangle_{\tilde\sigma}\right]. \quad (6.6)$$

Proof. The formula (6.6) follows from Proposition 6.1, (6.3), (6.4) and Theorems 6.2 and 6.3. ∎

Remark 6.3 If $H_{\tilde\sigma}^{X\times M}$ is conservative, then

$$\int(e^{-tH_{\tilde\sigma}^{X\times M}}-1)\varphi\,d\tilde\sigma=0\qquad\text{for all } t\geq 0,$$

and so in this case (6.6) coincides with (5.17) for $\varphi\in\mathfrak{D}_0$, $\varphi>-1$.

ACKNOWLEDGMENTS

The authors were partially supported by the SFB 256, Bonn University. Support by the DFG through Projects 436 113/39 and 436 113/43, and by the BMBF through Project UKR-004-99 is gratefully acknowledged.

References

[1] M. Aizenman, S. Goldstein, and J. L. Lebowitz, Conditional equilibrium and the equivalence of microcanonical and grandcanonical ensembles in the thermodynamic limit, *Comm. Math. Phys.* **62** (1978), 279–302.

[2] S. Albeverio, A. Daletskii, and E. Lytvynov, Laplace operators and diffusions in tangent bundles over Poisson spaces, Preprint SFB 256 No. 629, Universität Bonn, 1999, to appear in *Proc. KNAW.*

[3] S. Albeverio, A. Daletskii, and E. Lytvynov, Laplace operators on differential forms over configuration spaces, Preprint SFB 256 No. 646, Universität Bonn, 2000, to appear in *J. Geom. Phys.*

[4] S. Albeverio, R. J. Høegh-Krohn, J. A. Marion, D. H. Testard, B. S. Torrésani, "Noncommutative Distributions—Unitary Representations of Gauge Groups and Algebras," Marcel Dekker, New York, Basel, Hong Kong, 1993.

[5] S. Albeverio, Yu. G. Kondratiev, and M. Röckner, Differential geometry of Poisson spaces, *C.R. Acad. Sci. Paris* **323** (1996), 1129–1134.

[6] S. Albeverio, Yu. G. Kondratiev, and M. Röckner, Canonical Dirichlet operator and distorted Brownian motion on Poisson spaces, *C.R. Acad. Sci. Paris* **323** (1996), 1179–1184.

[7] S. Albeverio, Yu. G. Kondratiev, and M. Röckner, Analysis and geometry on configuration spaces, *J. Func. Anal.* **154** (1998), 444–500.

[8] S. Alberverio, Yu. G. Kondratiev, and M. Röckner, Analysis and geometry on configuration spaces: The Gibbsian case, *J. Func. Anal.* **157** (1998), 242–291.

[9] V. I. Bogachev, N. V. Krylov, and M. Röckner, Elliptic regularity and essential self-adjointness of Dirichlet operators on $\mathbb{R}^d$, to appear in *Ann. Scuola Norm. Pisa.*

[10] W. M. Boothby, "An Introduction to Differentiable Manifolds and Riemannian Geometry," Academic Press, San Diego, 1975.

[11] I. M. Gelfand, M. I. Graev, A. M. Vershik, Representations of the group of smooth mappings of a manifold X into a compact Lie group, *Compositio Math.* **35** (1977), 299–334.

[12] I. M. Gelfand, M. I. Graev, A. M. Vershik, Representations of the group of functions taking values in a compact Lie group, *Compositio Math.* **42** (1981), 217–243.

[13] I. M. Gelfand, R. A. Minlos, and Z. Ya. Shapiro, "Representations of the Rotation and Lorentz Groups and their Applications," Pergamon Press, Oxford, London, 1963.

[14] H. O. Georgii, "Canonical Gibbs Measures," LNM 760, Springer-Verlag, Berlin, 1979.

[15] G. A. Goldin, What we have learned about local relativistic current algebra, *in* "Local Currents and their Applications. Proceedings of an International Conference," pp. 100–114, North-Holland, Amsterdam, 1974.

[16] G. A. Goldin and R. Menikoff, Quantum-mechanical representations of the group of diffeomorphisms and local current algebra describing tightly bound composite particles, *J. Math. Phys.* **26** (1985), 1880–1884.

[17] G. A. Goldin, R. Menikoff, and D. H. Sharp, Particle statistics from induced representations of a local current group, *J. Math. Phys.* **21** (1980), 650–664.

[18] G. A. Goldin, J. Grodnik, R. T. Powers, and D. H. Sharp, Nonrelativistic current algebra in the N/V limit, *J. Math. Pgys.* **15** (1974), 88–100.

[19] G. A. Goldin and D. H. Sharp, Particle spin from representations of the diffeomorphism group, *Comm. Math. Phys.* **92** (1983), 217–228.

[20] R. S. Ismagilov, "Representations of Infinite-Dimensional Groups," American Math. Soc., Providence, Rhode Island, 1996.

[21] Y. Ito and I. Kubo, Calculus on Gaussian and Poisson white noises, *Nagoya Math. J.* **111** (1988), 41–84.

[22] O. Kallenberg, "Foundations of Modern Probability," Springer-Verlag, New York, Berlin, Heidelberg, 1997.

[23] J. F. C. Kingman, "Poisson Processes," Clarendon Press, Oxford, 1993.

[24] Yu. G. Kondratiev, E. W. Lytvynov, and G. F. Us, Analysis and geometry on $\mathbb{R}_+$-marked configuration spaces, *Meth. Func. Anal. and Geometry* **5** (1999), no. 1, 29–64.

[25] Yu. G. Kondratiev, J. L. Silva, and L. Streit, Generalized Appell systems, *Meth. Func. Anal. and Topol.* **3** (1997), no. 3, 28–61.

[26] Yu. G. Kondratiev, J. L. Silva, and L. Streit, Differential geometry on compound Poisson space, *Meth. Func. Anal. and Topol.* **4** (1998), no. 1, 32–58.

[27] Yu. G. Kondrtatiev, J. L. Silva, L. Streit, and G. F. Us, Analysis on Poisson and Gamma spaces, *Infinite Dimensional Analysis, Quantum Probability and Related Topics* **1** (1998), 91–117.

[28] Yu. G. Kondratiev, T. Kuna, and J. L. da Silva, Marked Gibbs measures via cluster expansion, *Meth. Func. Anal. Topol.* **4** (1998), no. 4, 50–82.

[29] T. Kuna, "Studies in Configuration Space Analysis and Applications," Ph.D. Thesis, Bonn University, 1999

[30] E. W. Lytvynov, A. L. Rebenko, and G. V. Shchepan'uk, Wick calculus on spaces of generalized functions of compound Poisson white noise, *Rep. Math. Phys.* **39** (1997), 219–248.

[31] Z.-M. Ma and M. Röckner, Construction of diffusions on configuration spaces, Preprint, Bielefeld, 1998, to appear in *Osaka J. Math.*

[32] K. Matthes, J. Kerstan, and J. Mecke, "Infinite Divisible Point Processes," Akademie-Verlag, Berlin, 1978.

[33] X. X. Nguyen and H. Zessin, Martin–Dynkin boundary of mixed Poisson processes. *Z. Wahrsch. verw. Gebiete* **39** (1977), 191–200.

[34] N. Obata, Configuration space and unitary representations of the group of diffeomorphisms, *RIMS Kokyoku* **615** (1987), 129–153.

[35] N. Privault, A transfer principle from Wiener to Poisson space and applications, *J. Func. Anal.* **132** (1995), 335–360.

[36] N. Privault, A pointwide equivalence of gradients on configuration spaces, *C. R. Acad. Sci. Paris* **327** (1998) 677-682.

[37] M. Röckner, Stochastic analysis on configuration spaces: Basic ideas and recent results, In *New Directions in Dirichlet Forms* (eds. J. Jost et al.), Studies in Advanced Mathematics, Vol. 8, American Math. Soc., 1998, 157–232.

[38] H. Shimomura, Poisson measures on the configuration space and unitary representations of the group of diffeomorphisms, *J. Math. Kyoto Univ.* **34** (1994), 599–614.

[39] A. V. Skorokhod, On the differentiability of measures which correspond to stochastic processes I. Processes with independent increments, *Theoria Verojat. Primen* **2** (1957), 418–444.

[40] Y. Takahashi, Absolute continuity of Poisson random fields, *Publ. RIMS Kyoto Univ.* **26** (1990), 629–647.

[41] A. M. Vershik, I. M. Gelfand, and M. I. Graev, Irreducible representations of the group G^X and cohomologies, *Funct. Anal. Appl.* **8** (1974), 151–153.

[42] A. M. Vershik, I. M. Gelfand, and M. I. Graev, Representations of the group of diffeomorphisms, *Russian Math. Surveys* **30** (1975), no. 6, 1–50.

[43] A. M. Vershik, I. M. Gelfand, and M. I. Graev, A commutative model of representation of the group of flows $SL(2, \mathbb{R})^X$ that is connected with a unipotent subgroup, *Funct. Anal. Appl.* **17** (1983), 137–139.

[44] N. Ja. Vilenkin and A. U. Klimyk, "Representations of Lie Groups and Special Functions," Vol. 1, Kluwer Acad. Publ., Dordrecht, 1991.

[45] A. Wawrzyńczyk, "Group Representations and Special Functions," Kluwer Acad. Publ., Dordrecht, 1983.

Institut für Angewandte Mathematik, Universität Bonn, Wegelerstr. 6, D 53115 Bonn; and
SFB 256, Univ. Bonn; and
CERFIM (Locarno); Acc. Arch. (USI); and
BiBoS, Univ. Bielefeld

Institut für Angewandte Mathematik, Universität Bonn, Wegelerstr. 6, D 53115 Bonn; and
SFB 256, Univ. Bonn; and
Institute of Mathematics, Kiev; and
BiBoS, Univ. Bielefeld

Institut für Angewandte Mathematik, Universität Bonn, Wegelerstr. 6, D 53115 Bonn; and
BiBoS, Univ. Bielefeld

Radolfzellerstr. 9, D-81243 München, Germany

On Arithmetic Quantum Field Theory

Asao Arai
Department of Mathematics, Hokkaido University
Sapporo 060-0810, Japan
E-mail: arai@math.sci.hokudai.ac.jp

Abstract

We review fundamental aspects of arithmetic quantum field theory.

1991 Mathematics Subject Classification: 81T10, 81T60, 11A25

1 Introduction

In recent developments of theoretical physics, it has been shown that number theory has connections with physics in various aspects (e.g., [23, 30]). Among others, "statistical mechanics" of numbers may be interesting, because it is related in a direct way to the Riemann zeta function and may give a key to solve the Riemann hypothesis ([17, 18, 20, 21, 22, 27, 28, 29] and references therein).

Spector [28] pointed out relationships between analytic number theory and a free supersymmetric quantum field theory, and further discussed these aspects with notions of partial supersymmetry and "duality"[29]. Motivated by these works of Spector, we started in [14] a research program developing analytic number theory as a field of infinite dimensional analysis or mathematically rigorous quantum field theory. We call this type of theory an *arithmetic quantum field theory.* In this paper we review some fundamental results in [14].

2 Arithmetical Functions in Boson Fock spaces

2.1 Partition functions and correlation functions

Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space with inner product $(\cdot\,,\,\cdot)_{\mathcal{H}}$ (complex linear in the second variable) and $\otimes_s^n\mathcal{H}$ be the n-fold symmetric tensor product Hilbert space of $\mathcal{H}$ ($n = 0, 1, 2, \cdots$; $\otimes_s^0\mathcal{H} := \mathbf{C}$). Then the Boson Fock space over $\mathcal{H}$ is defined by $\mathcal{F}_B(\mathcal{H}) := \oplus_{n=0}^{\infty} \otimes_s^n \mathcal{H}$. Let A be a nonnegative self-adjoint operator on $\mathcal{H}$ and

$$H_B(A) := d\Gamma_B(A) \tag{2.1}$$

be the second quantization of A on $\mathcal{F}_{\mathrm{B}}(\mathcal{H})$ (e.g., [19, §5.2], [25, p. 302, Example 2]). We denote by N_{B} the number operator on $\mathcal{F}_{\mathrm{B}}(\mathcal{H})$: $N_{\mathrm{B}} := d\Gamma_{\mathrm{B}}(I)$, where I denotes identity.

For $s > 0$, we define

$$Z_{\mathrm{B}}(s;A) := \mathrm{Tr}\, e^{-sH_{\mathrm{B}}(A)}, \quad \tilde{Z}_{\mathrm{B}}(s;A) := \mathrm{Tr}\left\{(-1)^{N_{\mathrm{B}}} e^{-sH_{\mathrm{B}}(A)}\right\}, \tag{2.2}$$

provided that $e^{-sH_{\mathrm{B}}(A)}$ is trace class on $\mathcal{F}_{\mathrm{B}}(\mathcal{H})$, where Tr denotes trace.

Remark 2.1 In statistical mechanics of quantum fields, $Z_{\mathrm{B}}(s;A)$ is called the *partition function* of the Hamiltonian $H_{\mathrm{B}}(A)$ at temperature $1/s$ (physically s denotes an *inverse temperature*). The function $\tilde{Z}_{\mathrm{B}}(s;A)$ is not so standard. We call it the *graded partition function* of the Hamiltonian $H_{\mathrm{B}}(A)$ at temperature $1/s$. This type of partition function was considered in a concrete case by Spector [29].

To treat the partition functions in a unified way, we introduce a more general partition function

$$Z_{\mathrm{B}}(s,z;A) := \mathrm{Tr}\left(\Gamma_{\mathrm{B}}(z)e^{-sH_{\mathrm{B}}(A)}\right) \tag{2.3}$$

with

$$z \in D := \{w \in \mathbf{C} \mid |w| \le 1\}, \tag{2.4}$$

provided that $e^{-sH_{\mathrm{B}}(A)}$ is trace class on $\mathcal{F}_{\mathrm{B}}(\mathcal{H})$, where $\Gamma_{\mathrm{B}}(z) := \oplus_{n=0}^{\infty} z^n$ acting on $\mathcal{F}_{\mathrm{B}}(\mathcal{H})$. We have

$$Z_{\mathrm{B}}(s,1;A) = Z_{\mathrm{B}}(s;A), \quad Z_{\mathrm{B}}(s,-1;A) = \tilde{Z}_{\mathrm{B}}(s;A). \tag{2.5}$$

In what follows, we assume the following.

Hypothesis (A) *The operator A is strictly positive, self-adjoint and, for some $s > 0$, e^{-sA} is trace class on $\mathcal{H}$.*

Theorem 2.1 *Let $z \in D$. Then the operator $\Gamma_{\mathrm{B}}(z)e^{-sH_{\mathrm{B}}(A)}$ is trace class on $\mathcal{F}_{\mathrm{B}}(\mathcal{H})$ and*

$$Z_{\mathrm{B}}(s,z;A) = \frac{1}{\det\left(I - ze^{-sA}\right)}, \tag{2.6}$$

where $\det(I+S)$ is the determinant for $I+S$ with S a trace class operator [26, §XIII.17].

Using Theorem 2.1 and the product law of the determinant $\det(I+\cdot)$, we can derive relations of partition functions at different temperatures:

Theorem 2.2 *For all $n \in \mathbf{N}$ and $z \in D$,*

$$Z_{\mathrm{B}}(s,z;A) = \det\left(\sum_{k=0}^{n-1} z^k e^{-ksA}\right) Z_{\mathrm{B}}(ns,z^n;A) \tag{2.7}$$

and

$$Z_{\mathrm{B}}(s,z;A)Z_{\mathrm{B}}(s,-z;A) = Z_{\mathrm{B}}(2s,z^2;A). \tag{2.8}$$

Remark 2.2 In general, relationships among theories at different coupling constants are referred to as "duality" [29]. Eq.(2.8) is a duality relation, where the coupling constant is the inverse temperature.

In statistical mechanics, *correlation functions* are also important objects. We denote by $a_{\mathcal{H}}(f)$ $(f \in \mathcal{H})$ the annihilation operator on $\mathcal{F}_{\mathrm{B}}(\mathcal{H})$ (e.g., [19, §5.2], [25, §X.7]) ($a_{\mathcal{H}}(f)$ is antilinear in f). For all $t > s$ and $f, g \in D(A^{-1/2})$ ($D(A^{-1/2})$ denotes the domain of $A^{-1/2}$). We can define

$$R_{\mathrm{B}}(t, z; f, g; A) := \frac{\mathrm{Tr}\left(\Gamma_{\mathrm{B}}(z) a_{\mathcal{H}}(f)^* a_{\mathcal{H}}(g) e^{-tH_{\mathrm{B}}(A)}\right)}{Z_{\mathrm{B}}(t, z; A)}, \quad z \in D. \tag{2.9}$$

This is called a *two-point correlation function.* In the same manner as in [19, Proposition 5.2.28], we can show that

$$R_{\mathrm{B}}(t, z; f, g; A) = (g, ze^{-tA}(1 - ze^{-tA})^{-1} f)_{\mathcal{H}},. \tag{2.10}$$

2.2 Arithmetical aspects

By Hypothesis (A), the spectrum $\sigma(A)$ of A is purely discrete with

$$\sigma(A) = \{E_n(A)\}_{n=1}^{\infty}, \tag{2.11}$$

$0 < E_1(A) \leq E_2(A) \leq \cdots$, $E_n(A) \to \infty\, (n \to \infty)$, counted with algebraic multiplicity. There exists a complete orthonormal system (CONS) $\{\phi_n\}_{n=1}^{\infty}$ of $\mathcal{H}$ such that $\phi_n \in D(A)$, $A\phi_n = E_n(A)\phi_n$, $n \in \mathbf{N}$. We set

$$a_n := a_{\mathcal{H}}(\phi_n) \tag{2.12}$$

Then we have canonical commutation relations

$$[a_n, a_m^*] = \delta_{mn}, \quad [a_n, a_m] = 0, \quad [a_n^*, a_m^*] = 0, \quad n, m \geq 1, \tag{2.13}$$

on the finite particle subspace of $\mathcal{F}_{\mathrm{B}}(\mathcal{H})$.

We denote by

$$\mathcal{P} := \{p_n\}_{n=1}^{\infty} \tag{2.14}$$

the set of all prime numbers with $p_n < p_{n+1}$, $n \geq 1$ ($p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, p_5 = 11, \cdots$).

By definition, an arithmetical function is a complex-valued function on $\mathbf{N}$. An arithmetical function f is called *completely multiplicative* if it satisfies

$$f(1) = 1, \quad f(mn) = f(m)f(n), \quad m, n \in \mathbf{N}.$$

Let $N \geq 2$ be a natural namber. Then, by the fundamental theorem of arithmetic, there exists a unique set $\{i_1, \cdots, i_n, \alpha_1, \cdots, \alpha_n\} \subset \mathbf{N}$ $(i_1 < \cdots < i_n)$ such that

$$N = (p_{i_1})^{\alpha_1} \cdots (p_{i_n})^{\alpha_n}. \tag{2.15}$$

Then we define an arithmetical function $\gamma(N)$ by $\gamma(1) := 0$ and

$$\gamma(N) := \sum_{k=1}^{n} \alpha_k, \quad N \geq 2. \tag{2.16}$$

The arithmetical function defined by $\lambda(1) := 1$ and

$$\lambda(N) := (-1)^{\gamma(N)}, \quad N \geq 2. \tag{2.17}$$

is called the *Liouville fucntion* [1, §2.12]. This function is completely multiplicative.

Using the representation (2.15) of N, we can define a vector $\Psi_N \in \mathcal{F}_{\rm B}(\mathcal{H})$ by

$$\Psi_N := C_N(a_{i_1}^*)^{\alpha_1}\cdots(a_{i_n}^*)^{\alpha_n}\Omega_{\mathcal{H}}, \qquad (2.18)$$

where $\Omega_{\mathcal{H}} := \{1,0,0,\cdots\}$ is the Fock vacuum in $\mathcal{F}_{\rm B}(\mathcal{H})$ and $C_N := 1/\sqrt{\alpha_1!\cdots\alpha_n!}$ is a normalization constant so that $\|\Psi_N\| = 1$. We set $\Psi_1 := \Omega_{\mathcal{H}}$. A key fact is the following.

Lemma 2.3 [28] *The set* $\{\Psi_N\}_{N=1}^{\infty}$ *is a CONS of* $\mathcal{F}_{\rm B}(\mathcal{H})$.

Lemma 2.4 *For all* $N \in \mathbf{N}$, Ψ_N *is a unique eigenvector (up to constant multiples) of* $\Gamma_{\rm B}(z)$ *with eigenvalue* $z^{\gamma(N)}$.

We introduce a function $F_A : \mathbf{N} \to (0,\infty)$ as follows: $F_A(1) := 1$ and if $N \geq 2$ is represented as (2.15), then

$$F_A(N) := \prod_{k=1}^{n} e^{\alpha_k E_{i_k}(A)}. \qquad (2.19)$$

It is easy to see that F_A is completely multiplicative.

Lemma 2.5 *For all* $N \in \mathbf{N}$, Ψ_N *is a unique eigenvector (up to constant multiples) of* $H_{\rm B}(A)$ *with eigenvalue* $\log F_A(N)$.

By Lemmas 2.4 and 2.5, we have

$$Z_{\rm B}(s,z;A) = \sum_{N=1}^{\infty} \frac{z^{\gamma(N)}}{F_A(N)^s}, \quad z \in D. \qquad (2.20)$$

By this fact and Theorem 2.1, we obtain the following.

Theorem 2.6 *For all* $z \in D$,

$$\sum_{N=1}^{\infty} \frac{z^{\gamma(N)}}{F_A(N)^s} = \frac{1}{\prod_{n=1}^{\infty}(1 - ze^{-sE_n(A)})}. \qquad (2.21)$$

Remark 2.3 Formula (2.21) may be regarded as a general form unifying arithmetical formulas known under the name of *Euler products* [1, Chapter 11]. See Section 2.3 below.

We introduce a function $\varrho(N,m) : \mathbf{N} \times \mathbf{N} \to \{0\} \cup \mathbf{N}$ by

$$\varrho(1,m) := 0, \quad \varrho(N,m) := \sum_{k=1}^{n} \alpha_k \delta_{i_k m} \qquad (2.22)$$

if $N \geq 2$ is expressed as (2.15) ($N, m \in \mathbf{N}$).

Theorem 2.7 *Let* $t > s$. *Then, for all* $m \in \mathbf{N}$ *and* $z \in D$,

$$\sum_{N=1}^{\infty} \frac{z^{\gamma(N)}\varrho(N,m)}{F_A(N)^t} = \frac{z}{e^{tE_m(A)} - z} Z_{\rm B}(t,z;A). \qquad (2.23)$$

Let $N \geq 2$ be given as (2.15). Then, each divisor m of N is of the form

$$m = p_{i_1}^{r_1} \cdots p_{i_n}^{r_n} \tag{2.24}$$

with $0 \leq r_j \leq \alpha_j$, $j = 1, \cdots, n$. We define a vector $\Psi_{N,m} \in \mathcal{F}_{\rm B}(\mathcal{H})$ by

$$\Psi_{N,m} := C_{N,m} a_{i_1}^{*\ r_1} \cdots a_{i_n}^{*\ r_n} \Omega_{\mathcal{H}}, \tag{2.25}$$

where $C_{N,m} > 0$ is a normalization constant. For an $m \in \mathbf{N}$ and $N \in \mathbf{N}$, we mean by $m|N$ that m is a divisor of N. The set $\{\Psi_{N,m}\}_{m|N}$ of vectors is orthonormal. We introduce

$$\mathcal{F}_{\rm B}^{(N)}(\mathcal{H}) := \mathcal{L}\{\Psi_{N,m}\}_{m|N}, \tag{2.26}$$

where $\mathcal{L}\{\cdot\}$ means the subspace spanned algebraically by the vectors in the set $\{\cdot\}$. We set $\mathcal{F}_{\rm B}^{(1)} := \{\alpha\Omega_{\mathcal{H}} | \alpha \in \mathbf{C}\}$. We denote by P_N the orthogonal projection from $\mathcal{F}_{\rm B}(\mathcal{H})$ onto $\mathcal{F}_{\rm B}^{(N)}(\mathcal{H})$.

Proposition 2.8 *Let $z \in D$. Then, for all N,*

$$\operatorname{Tr}\left(P_N \Gamma_{\rm B}(z) e^{-sH_{\rm B}(A)} P_N\right) = \sum_{m|N} \frac{z^{\gamma(m)}}{F_A(m)^s}. \tag{2.27}$$

2.3 Connections with analytic number theory

A basic object in analytic number theory is the *Dirichlet series*

$$D(s,f) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \tag{2.28}$$

for an arithmetical function f and $s \in \mathbf{C}$, provided that the infinite series converges. The *Riemann zeta function*

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s > 1, \tag{2.29}$$

is a special case of $D(s,f)$. We first show that $\zeta(s)$ and $D(s,\lambda)$ can be represented as partition functions of $H_{\rm B}(A)$ with a suitable A. For this purpose, we consider the case where $\mathcal{H}$ is given by

$$\ell^2 := \oplus_{n=1}^{\infty}\mathbf{C} = \left\{\psi = \{\psi_n\}_{n=1}^{\infty} \middle| \psi_n \in \mathbf{C}, n \geq 1, \sum_{n=1}^{\infty} |\psi_n|^2 < \infty\right\}. \tag{2.30}$$

On this Hilbert space we define an operator $\omega_{\mathcal{P}}$ as follows:

$$D(\omega_{\mathcal{P}}) = \left\{\psi = \{\psi_n\}_{n=1}^{\infty} \in \ell^2 \middle| \sum_{n=1}^{\infty} |(\log p_n)\psi_n|^2 < \infty\right\}, \tag{2.31}$$

$$(\omega_{\mathcal{P}}\psi)_n = (\log p_n)\psi_n, \quad \psi \in D(\omega_{\mathcal{P}}),\ n \geq 1. \tag{2.32}$$

Then $\omega_{\mathcal{P}}$ is strictly positive and self-adjoint. Moreover, the spectrum of $\omega_{\mathcal{P}}$ is purely discrete with

$$\sigma(\omega_{\mathcal{P}}) = \{\log p_n\}_{n=1}^{\infty} \tag{2.33}$$

with the multiplicity of each eigenvalue $\log p_n$ being one. A normalized eigenvector of $\omega_{\mathcal{P}}$ with eigenvalue $\log p_n$ is given by

$$e_n := \{\delta_{nj}\}_{j=1}^{\infty} \in \ell^2. \tag{2.34}$$

Theorem 2.9 *For all $s > 1$ and $z \in D$,*

$$Z_{\mathrm{B}}(s, z; \omega_{\mathcal{P}}) = \sum_{N=1}^{\infty} \frac{z^{\gamma(N)}}{N^s}. \tag{2.35}$$

Applying Theorem 2.6 with $A = \omega_{\mathcal{P}}$, we obtain the following.

Corollary 2.10 *For all $s > 1$ and $z \in D$,*

$$\sum_{N=1}^{\infty} \frac{z^{\gamma(N)}}{N^s} = \frac{1}{\prod_{p \in \mathcal{P}} (1 - zp^{-s})}. \tag{2.36}$$

An application of Theprem 2.7 gives the following.

Corollary 2.11 *For all $s > 1$, $n \in \mathbf{N}$ and $z \in D$,*

$$\sum_{N=1}^{\infty} \frac{z^{\gamma(N)} \varrho(N, n)}{N^s} = \frac{z}{p_n^s - z} Z_{\mathrm{B}}(s, z; \omega_{\mathcal{P}}). \tag{2.37}$$

The operator $\omega_{\mathcal{P}}$ may be regarded as as a special case of a more general operator associated with a completely multiplicative function. Let f be a completely multiplicative function such that $0 < f(n) < 1$ for all $n \geq 2$ and

$$\sum_{n=1}^{\infty} f(p_n) < \infty, \tag{2.38}$$

and define an operator A_f on ℓ^2 by

$$D(A_f) = \left\{ \psi = \{\psi_n\}_{n=1}^{\infty} \middle| \sum_{n=1}^{\infty} |\log f(p_n)|^2 |\psi_n|^2 < \infty \right\}, \tag{2.39}$$

$$(A_f \psi)_n = [-\log f(p_n)] \psi_n, \quad \psi \in D(A_f),\ n \geq 1. \tag{2.40}$$

Then A_f is a strictly positive self-adjoint operator and e^{-A_f} is trace class on ℓ^2. It is easy to see that

$$F_{A_f}(N) = \frac{1}{f(N)}, \quad N \in \mathbf{N}. \tag{2.41}$$

Hence we have

$$Z_{\mathrm{B}}(1, z; A_f) = \sum_{n=1}^{\infty} z^{\gamma(n)} f(n), \quad z \in D. \tag{2.42}$$

Applying Theorem 2.6, we obtain the following fact.

Corollary 2.12 *Let f be as above. Then, for all $z \in D$,*

$$\sum_{n=1}^{\infty} z^{\gamma(n)} f(n) = \frac{1}{\prod_{p \in \mathcal{P}} (1 - z f(p))}. \tag{2.43}$$

Theorem 2.7 gives the following.

Corollary 2.13 *Let f be as above. Then, for all $n \in \mathbf{N}$ and $z \in D$,*

$$\sum_{N=1}^{\infty} z^{\gamma(N)} \varrho(N, n) f(N) = \frac{z f(p_n)}{1 - z f(p_n)} Z_{\mathrm{B}}(1, z; A_f). \tag{2.44}$$

Applying Proposition 2.8, we have for all $s > 1$

$$\mathrm{Tr}\left(P_N z^{N_{\mathrm{B}}} e^{-sH_{\mathrm{B}}(\omega_{\mathcal{P}})} P_N\right) = \sum_{m|N} \frac{z^{\gamma(m)}}{m^s}, \quad z \in D. \tag{2.45}$$

3 Arithmetical Functions in Fermion Fock Spaces

3.1 Partition functions and correlation functions

Let $\mathcal{K}$ be a separable infinite dimensional Hilbert space and $\otimes_{\rm as}^n \mathcal{K}$ be the n-fold antisymmetric tensor product Hilbert space of $\mathcal{K}$ ($n = 0, 1, 2, \cdots$; $\otimes_{\rm as}^0 \mathcal{K} := \mathbf{C}$). Then the Fermion Fock space over $\mathcal{K}$ is defined by $\mathcal{F}_{\rm F}(\mathcal{K}) := \oplus_{n=0}^\infty \otimes_{\rm as}^n \mathcal{K}$.

Let T be a nonnegative self-adjoint operator on $\mathcal{K}$ and

$$H_{\rm F}(T) := d\Gamma_{\rm F}(T). \tag{3.1}$$

be the second quantization of T in $\mathcal{F}_{\rm F}(\mathcal{K})$. The number operator on $\mathcal{F}_{\rm F}(\mathcal{K})$ is defined by $N_{\rm F} := d\Gamma_{\rm F}(I)$.

Let $s > 0$, $z \in D$ and

$$Z_{\rm F}(s, z; T) := \mathrm{Tr}\left(\Gamma_{\rm F}(z) e^{-sH_{\rm F}(T)}\right), \tag{3.2}$$

provided that $e^{-sH_{\rm F}(T)}$ is trace class on $\mathcal{F}_{\rm F}(\mathcal{H})$, where $\Gamma_{\rm F}(z) := \oplus_{n=0}^\infty z^n$ acting on $\mathcal{F}_{\rm F}(\mathcal{K})$.

In what follows, we assume the following.

Hypothesis (T) *For some $s > 0$, e^{-sT} is trace class on $\mathcal{K}$.*

Theorem 3.1 *For all $z \in D$, $\Gamma_{\rm F}(z)e^{-sH_{\rm F}(T)}$ is trace class on $\mathcal{F}_{\rm F}(\mathcal{K})$ and*

$$Z_{\rm F}(s, z; T) = \det(I + z e^{-sT}). \tag{3.3}$$

By Theorems 2.1 and 3.1, we have interesting relations between bosonic and fermionic partition functions:

Corollary 3.2 *Consider the case $\mathcal{H} = \mathcal{K}$ and A be an operator on $\mathcal{H}$ obeying Hypothesis (A) in Section 2. Then, for all $z \in D$,*

$$Z_{\rm B}(s, -z; A) = \frac{1}{Z_{\rm F}(s, z; A)}. \tag{3.4}$$

Theorem 3.3 *For all $n \in \mathbf{N}$ and $z \in D$,*

$$Z_{\rm F}(ns, -z^n; T) = \det\left(\sum_{k=1}^{n-1} z^k e^{-skT}\right) Z_{\rm F}(s, -z; T), \tag{3.5}$$

$$Z_{\rm F}(s, -z; T) Z_{\rm F}(s, z; T) = Z_{\rm F}(2s, -z^2; T). \tag{3.6}$$

Remark 3.1 Relation (3.6) is a form of *duality* of fermionic partition functions. A special case is discussed in [29].

Corollary 3.4 *Consider the case $\mathcal{H} = \mathcal{K}$ and A be an operator on $\mathcal{H}$ obeying Hypothesis (A). Then*

$$Z_{\rm B}(2s, z^2; A) Z_{\rm F}(s, z; A) = Z_{\rm B}(s, z; A) \tag{3.7}$$

Remark 3.2 Relation (3.7) is also a form of *duality* of fermionic and bosonic partition functions. For a special case, see [29].

Let $u, v \in \mathcal{K}$ and $z \in D$. Then a *fermionic two-point correlation function* is defined by

$$R_{\mathrm{F}}(s, z; u, v; T) := \frac{\mathrm{Tr}\left(\Gamma_{\mathrm{F}}(z) e^{-sH_{\mathrm{F}}(T)} b_{\mathcal{K}}(u)^* b_{\mathcal{K}}(v)\right)}{Z_{\mathrm{F}}(s, z; T)}. \tag{3.8}$$

where $b_{\mathcal{K}}(u)$ $(u \in \mathcal{K})$ the annihilation operator on $\mathcal{F}_{\mathrm{F}}(\mathcal{K})$ (e.g., [19, §5.2]). It is easy to see (e.g., cf. [19]) that

$$R_{\mathrm{F}}(s, z; u, v; T) = (v, ze^{-sT}(1 + ze^{-sT})^{-1}u)_{\mathcal{K}}. \tag{3.9}$$

3.2 Arithmetical aspects

By Hypothesis (T), the spectrum of T is purely discrete with

$$\sigma(T) = \{E_n(T)\}_{n=1}^{\infty}, \tag{3.10}$$

$0 < E_1(T) \le E_2(T) \le \cdots$, $E_n(T) \to \infty\,(n \to \infty)$, counted with algebraic multiplicity. There exists a CONS $\{u_n\}_{n=1}^{\infty}$ of $\mathcal{K}$ such that $u_n \in D(T)$, $Tu_n = E_n(T)u_n$, $n \in \mathbf{N}$. We set

$$b_n := b_{\mathcal{K}}(u_n). \tag{3.11}$$

Then we have canonical anti-commutation relations

$$\{b_n, b_m^*\} = \delta_{mn}, \quad \{b_n, b_m\} = 0, \quad \{b_n^*, b_m^*\} = 0, \quad n, m \ge 1, \tag{3.12}$$

where $\{X, Y\} := XY + YX$. In particular, $b_n^2 = 0$, $b_n^{*2} = 0$, $n \in \mathbf{N}$.

For $N \in \mathbf{N}$ we define $\nu(N)$ by $\nu(1) := 1$ and

$$\nu(N) = n, \quad N \ge 2, \tag{3.13}$$

if N is represented as (2.15) [1, p.247].

A natural number $m \ge 2$ is called *square free* if it is written as a product of mutually different prime numbers. As a convention, 1 is defined to be square free. We denote by $\mathcal{S}_0$ the set of square free elements in $\mathbf{N}$:

$$\mathcal{S}_0 := \{m \in \mathbf{N} | m \text{ is square free}\}. \tag{3.14}$$

For each $N \in \mathbf{N}$, we define a set $\mathcal{S}_0(N)$ as follows:

$$\mathcal{S}_0(1) := \{1\}, \tag{3.15}$$

$$\mathcal{S}_0(N) := \{m \in \mathcal{S}_0 | m \text{ is a divisor of } N\}, \quad N \ge 2. \tag{3.16}$$

Let $N \ge 2$ be given as (2.15). Then each element m of $\mathcal{S}_0(N)$ is of the form

$$m = p_{i_1}^{q_1} \cdots p_{i_n}^{q_n}, \tag{3.17}$$

where $q_j = 0$ or $q_j = 1$ $(j = 1, \cdots, n)$. Corresponding to this, we define a vector $\Phi_{N,m}$ by

$$\Phi_{N,m} := b_{i_1}^{*\,q_1} \cdots b_{i_n}^{*\,q_n} \Omega_{\mathcal{K}}, \tag{3.18}$$

where $\Omega_{\mathcal{K}} := \{1, 0, 0, \cdots\}$ is the Fock vacuum in $\mathcal{F}_{\mathrm{F}}(\mathcal{K})$.

Let

$$\mathcal{F}_{\mathrm{F}}^{(1)}(\mathcal{K}) := \{c\Omega_{\mathcal{K}} | c \in \mathbf{C}\}, \quad \mathcal{F}_{\mathrm{F}}^{(N)}(\mathcal{K}) := \mathcal{L}\{\Phi_{N,m} | m \in \mathcal{S}_0(N)\}, \quad N \geq 2. \tag{3.19}$$

Then $\mathcal{F}_{\mathrm{F}}^{(N)}(\mathcal{K})$ is finite dimensional with $\dim \mathcal{F}_{\mathrm{F}}^{(N)}(\mathcal{K}) = 2^{\nu(N)}$. We denote by R_N the orthogonal projection from $\mathcal{F}_{\mathrm{F}}(\mathcal{K})$ onto $\mathcal{F}_{\mathrm{F}}^{(N)}(\mathcal{K})$.

Let $N \geq 2$ be of the form (2.15),

$$\mathcal{K}_N := \mathcal{L}\{u_{i_k} | k = 1, \cdots, n\} \tag{3.20}$$

and T_N be the restriction of T to $\mathcal{K}_N$. Then we can show that

$$\operatorname{Tr}\left(R_N \Gamma_{\mathrm{F}}(z) e^{-sH_{\mathrm{F}}(T)} R_N\right) = \det(1 + z e^{-sT_N}). \tag{3.21}$$

Let $m \in \mathcal{S}_0, m \geq 2$ and

$$m = p_{i_1} \cdots p_{i_r} \tag{3.22}$$

be its factorization in prime numbers ($i_j \neq i_k$, $j \neq k$). Then we define a vector Φ_m in $\mathcal{F}_{\mathrm{F}}(\mathcal{K})$ by

$$\Phi_m := b_{i_1}^* \cdots b_{i_r}^* \Omega_{\mathcal{K}}. \tag{3.23}$$

For $m = 1$, we set $\Phi_1 := \Omega_{\mathcal{K}}$. For $m \notin \mathcal{S}_0$, we define $\Phi_m := 0$.

Lemma 3.5 [28] *The set $\{\Phi_m\}_{m \in \mathcal{S}_0}$ is a CONS of $\mathcal{F}_{\mathrm{F}}(\mathcal{K})$.*

The *Möbius function* $\mu : \mathbf{N} \to \{0, \pm 1\}$ is defined as follows: $\mu(1) := 1$, $\mu(m) := 0$ if $m \notin \mathcal{S}_0$ and $\mu(m) := (-1)^r$ if m is written as the product of mutually different r prime numbers. We have

$$\mu(m) = (-1)^{\gamma(m)}, \quad m \in \mathcal{S}_0. \tag{3.24}$$

Lemma 3.6 *For all $m \in \mathcal{S}_0$, Φ_m is an eigenvector of N_{F} with eigenvalue $\gamma(m)$.*

Lemma 3.7 *For all $m \in \mathcal{S}_0$, Φ_m is an eigenvector of $H_{\mathrm{F}}(T)$ with eigenvalue* $\log F_T(m)$, *where F_T is defined by (2.19) with $A = T$.*

It follows from Lemmas 3.6 and 3.7 that

$$Z_{\mathrm{F}}(s, z; T) = \sum_{m=1}^{\infty} \frac{z^{\gamma(m)} |\mu(m)|}{F_T(m)^s}, \quad z \in D, \tag{3.25}$$

where we have used that $\mu(m) = 0$ for all $m \notin \mathcal{S}_0$ and $|\mu(m)| = 1$ for all $m \in \mathcal{S}_0$. By (3.25) and Theorem 3.1, we obtain the following.

Theorem 3.8 *Let $z \in D$. Then*

$$\sum_{m=1}^{\infty} \frac{z^{\gamma(m)} |\mu(m)|}{F_T(m)^s} = \prod_{n=1}^{\infty} \left(1 + z e^{-sE_n(T)}\right). \tag{3.26}$$

Theorems 3.8 and 2.6 imply the following.

Corollary 3.9 *Let $z \in D$. Then,*

$$\sum_{m=1}^{\infty} \frac{z^{\gamma(m)}|\mu(m)|}{F_T(m)^s} = \frac{1}{\sum_{n=1}^{\infty} \frac{(-z)^{\gamma(n)}}{F_T(n)^s}}. \tag{3.27}$$

We introduce a function η on $\mathbf{N} \times \mathbf{N}$ by

$$\eta(1,n) := 0, \tag{3.28}$$

$$\eta(m,n) := \sum_{k=1}^{r} (-1)^{k-1} \delta_{i_k n} \tag{3.29}$$

if $m \in \mathcal{S}_0$ is expressed as (3.22). If $m \notin \mathcal{S}_0$, then $\eta(m,n) := 0$ for all $n \in \mathbf{N}$.

Theorem 3.10 *Let $z \in D$ and $n \in \mathbf{N}$. Then*

$$\sum_{m=1}^{\infty} \frac{z^{\gamma(m)}\eta(m,n)}{F_T(m)^s} = \frac{z}{e^{sE_n(T)} + z} Z_{\mathrm{F}}(s,z;T). \tag{3.30}$$

The left hand side of (3.21) is equal to $\sum_{m \in \mathcal{S}_0(N)} z^{\gamma(m)}/F_T(m)^s$. Hence we obtain

$$\sum_{m|N} \frac{z^{\gamma(m)}|\mu(m)|}{F_T(m)^s} = \det\left(1 + ze^{-sT_N}\right). \tag{3.31}$$

3.3 Connections with analytic number theory

Consider the case where $\mathcal{H} = \ell^2$ and $T = \omega_{\mathcal{P}}$. Let $z \in D$ and $s > 1$. Then we have

$$Z_{\mathrm{F}}(s,z;\omega_{\mathcal{P}}) = \sum_{m=1}^{\infty} \frac{z^{\gamma(m)}|\mu(m)|}{m^s}. \tag{3.32}$$

Let f be a completely multiplicative function as in Section 2.3 and $z \in D$. Then, by (2.41), we have

$$Z_{\mathrm{F}}(1,z;A_f) = \sum_{m=1}^{\infty} z^{\gamma(m)}|\mu(m)|f(m). \tag{3.33}$$

By Theorem 3.8, we obtain the following.

Corollary 3.11 *For all $z \in D$,*

$$\sum_{m=1}^{\infty} z^{\gamma(m)}|\mu(m)|f(m) = \prod_{p \in \mathcal{P}} (1 + zf(p)). \tag{3.34}$$

Theorem 3.10 gives the following.

Corollary 3.12 *For all $n \in \mathbf{N}$ and $z \in D$,*

$$\sum_{m=1}^{\infty} z^{\gamma(m)}\eta(m,n)f(m) = \frac{zf(p_n)}{1 + zf(p_n)} Z_{\mathrm{F}}(1,z;A_f). \tag{3.35}$$

Jordan's totient function $J_s(N)$ ($s \geq 0, N \in \mathbf{N}$) is defined by $J_s(1) := 1$ and, for $N \geq 2$,

$$J_s(N) = N^s \prod_{p|N;p\in\mathcal{P}} \left(1 - \frac{1}{p^s}\right) \tag{3.36}$$

[1, p.48]. The special case

$$\varphi(N) = J_1(N) \tag{3.37}$$

is *Euler's totient function* [1, p.25, p.27]. We have

$$\det\left(1 - e^{-s(\omega_{\mathcal{P}})_N}\right) = \prod_{p|N;p\in\mathcal{P}} \left(1 - \frac{1}{p^s}\right), \quad s \geq 0,\ N \geq 2. \tag{3.38}$$

Hence we obtain

$$J_s(N) = N^s \det\left(1 - e^{-s(\omega_{\mathcal{P}})_N}\right), \quad s \geq 0,\ N \geq 2, \tag{3.39}$$

which, together with (3.21), implies that

$$J_s(N) = N^s \mathrm{Tr}\left(R_N(-1)^{N_F} e^{-sH_F(\omega_{\mathcal{P}})} R_N\right), \quad s \geq 0,\ N \in \mathbf{N}. \tag{3.40}$$

This gives an expression of Jordan's totient function in terms of Fock space objects. Formula (3.31) implies the well known identity [1, p.48]:

$$J_s(N) = \sum_{m|N} \mu(m) \left(\frac{N}{m}\right)^s, \quad s \geq 0,\ N \in \mathbf{N}. \tag{3.41}$$

4 Arithmetical Aspects of Boson-Fermion Fock Spaces

4.1 Some general aspects

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces as before. Then the Boson-Fermion Fock space associated with the pair $\langle\mathcal{H},\mathcal{K}\rangle$ is defined by the tensor product Hilbert space

$$\mathcal{F}_{\mathrm{BF}}(\mathcal{H},\mathcal{K}) := \mathcal{F}_{\mathrm{B}}(\mathcal{H}) \otimes \mathcal{F}_{\mathrm{F}}(\mathcal{K}). \tag{4.1}$$

Let A and T be nonnegative self-adjoint operators on $\mathcal{H}$ and $\mathcal{K}$ respectively. Then the operator

$$H(A,T) := H_{\mathrm{B}}(A) \otimes I + I \otimes H_{\mathrm{F}}(T) \tag{4.2}$$

on $\mathcal{F}_{\mathrm{BF}}(\mathcal{H},\mathcal{K})$ is nonnegative and self-adjoint.

We assume the following.

Hypothesis (AT) *The operators A and T satisfy Hypothesis (A) in Section 2 and Hypothesis (T) in Section 3 respectively.*

Under this assumption, $e^{-sH(A,T)}$ is trace class and we can define a partition function

$$Z(s,z,w;A,T) := \mathrm{Tr}\left(\Gamma_{\mathrm{B}}(z) \otimes \Gamma_{\mathrm{F}}(w) e^{-sH(A,T)}\right), \quad z,w \in D. \tag{4.3}$$

We have

$$Z(s,z,w;A,T) = Z_{\mathrm{B}}(s,z;A) Z_{\mathrm{F}}(s,w;T), \quad z,w \in D. \tag{4.4}$$

If one can represent the left hand side of (4.4) in various ways, (4.4) may produce nontrivial arithmetical relations for eigenvalues of A and T. Moreover, different expressions of $\mathrm{Tr}\left(Xe^{-sH(A,T)}\right)$ with X an operator on $\mathcal{F}_{\mathrm{BF}}(\mathcal{H},\mathcal{K})$ may yield interesting arithmetical relations. These are basic ideas to search for arithmetical relations by quantum field theoretical methods.

We carry over the notation in the preceding sections. Let $N \geq 2$ be of the form (2.15) and $m \in \mathcal{S}_0(N)$. Then we can write

$$m = (p_{i_1})^{q_1} (p_{i_2})^{q_2} \cdots (p_{i_n})^{q_n}, \tag{4.5}$$

where $q_j = 0$ or $q_j = 1$. Based on these factorizations, we define a vector

$$\Omega_{N,m} := C_{N,m} \left[(a_{i_1}^*)^{\alpha_1 - q_1} \cdots (a_{i_n}^*)^{\alpha_n - q_n} \Omega_{\mathcal{H}}\right] \otimes \left[(b_{i_1}^*)^{q_1} \cdots (b_{i_n}^*)^{q_n} \Omega_{\mathcal{K}}\right], \tag{4.6}$$

where $C_{N,m} > 0$ is a normalization constant. For $N = 1$ and $m = 1$, we set $\Omega_{1,1} := \Omega_{\mathcal{H}} \otimes \Omega_{\mathcal{K}}$.

Lemma 4.1 [28] *The set $\{\Omega_{N,m} | N \geq 1,\, m \in \mathcal{S}_0(N)\}$ is a CONS of $\mathcal{F}_{\mathrm{BF}}(\mathcal{H},\mathcal{K})$.*

The following fact is easily proven.

Lemma 4.2 *Let $N \in \mathbf{N}$, $m \in \mathcal{S}_0(N)$ and $z,w \in D$. Then $\Omega_{N,m}$ is an eigenvector of $\Gamma_{\mathrm{B}}(z) \otimes \Gamma_{\mathrm{F}}(w)$ with eigenvalue $z^{\gamma(N)-\gamma(m)} w^{\gamma(m)}$.*

For each $N \in \mathbf{N}$, we define a function $Y_{A,T}(N,\cdot)$ on $\mathcal{S}_0(N)$ by

$$Y_{A,T}(N,m) := \prod_{k=1}^{n} e^{(\alpha_k - q_k)E_{i_k}(A) + q_k E_{i_k}(T)}, \quad m \in \mathcal{S}_0(N), \tag{4.7}$$

when N and m are represented as (2.15) and (4.5) respectively. Note that

$$Y_{A,T}(N,m) = F_A\left(\frac{N}{m}\right) F_T(m). \tag{4.8}$$

Lemma 4.3 *Let $N \in \mathbf{N}$ and $m \in \mathcal{S}_0(N)$. Then $\Omega_{N,m}$ is an eigenvector of $H(A,T)$ with eigenvalue* $\log Y_{A,T}(N,m)$.

Theorem 4.4 *Let $z,w \in D$. Then*

$$Z(s,z,w;A,T) = \sum_{N=1}^{\infty} \sum_{m|N} \frac{z^{\gamma(N)-\gamma(m)} w^{\gamma(m)} |\mu(m)|}{Y_{A,T}(N,m)^s}. \tag{4.9}$$

Corollary 4.5 *Let $z,w \in D$. Then*

$$\sum_{N=1}^{\infty} \sum_{m|N} \frac{z^{\gamma(N)-\gamma(m)} w^{\gamma(m)} |\mu(m)|}{Y_{A,T}(N,m)^s} = Z_{\mathrm{B}}(s,z;A) Z_{\mathrm{F}}(s,w;T). \tag{4.10}$$

Remark 4.1 If we put into the right hand side of (4.10) the formulas established in Sections 2 and 3, then we obtain explicit formulas, which are nontrivial.

Remark 4.2 By rescaling as $T \to tT/s$ $(t > 0)$ in (4.10), we can obtain relations at different temperatures $1/s$ and $1/t$. Hence (4.10) include "duality relations".

4.2 Connections with analytic number theory

We consider the case where $\mathcal{H} = \mathcal{K} = \ell^2$ and $A = T = \omega_{\mathcal{P}}$. Then we have $Y_{\omega_{\mathcal{P}},\omega_{\mathcal{P}}}(N, m) = N$. Hence Corollary 4.5 gives

$$\sum_{N=1}^{\infty} \sum_{m|N} \frac{z^{\gamma(N)-\gamma(m)} w^{\gamma(m)} |\mu(m)|}{N^s} = Z_{\mathrm{B}}(s, z; \omega_{\mathcal{P}}) Z_{\mathrm{F}}(s, w; \omega_{\mathcal{P}}), \quad s > 1. \tag{4.11}$$

This yields well known relations

$$\sum_{N=1}^{\infty} \frac{2^{\nu(N)}}{N^s} = \frac{\zeta(s)}{D(s,\lambda)}, \quad \sum_{N=1}^{\infty} \frac{\lambda(N) 2^{\nu(N)}}{N^s} = \frac{D(s,\lambda)}{\zeta(s)}, \quad s > 1.$$

Let f be the completely multiplicative function considered in Section 2.3 and

$$H := H(A_f, A_f)$$

Then we have for all $s > 1$

$$\mathrm{Tr}\left(\Gamma_{\mathrm{F}} e^{-sH}\right) = 1, \quad \mathrm{Tr}\left(\Gamma_{\mathrm{B}} e^{-sH}\right) = 1, \tag{4.12}$$

which are supersymmetric identities [6, 28]. These relations imply the following:

$$\sum_{m=1}^{\infty} \mu(m) f(m) = \frac{1}{\sum_{n=1}^{\infty} f(n)}, \quad \sum_{m=1}^{\infty} |\mu(m)| f(m) = \frac{1}{\sum_{n=1}^{\infty} \lambda(n) f(n)}. \tag{4.13}$$

By Corollary 4.5 with rescaling $T \to tT/s$, we obtain

$$\sum_{N=1}^{\infty} \sum_{m|N} \frac{z^{\gamma(N)-\gamma(m)} w^{\gamma(m)} |\mu(m)|}{N^s m^{t-s}} = Z_{\mathrm{B}}(s, z; \omega_{\mathcal{P}}) Z_{\mathrm{F}}(t, w; \omega_{\mathcal{P}}), \quad t > s > 1. \tag{4.14}$$

Remark 4.3 General theories on Boson-Fermion Fock spaces have been developed in [3, 5, 6, 7, 9, 11, 13, 15, 16]. See also [2, 4, 8, 10] for related aspects. Applications of these theories to arithmetic quantum field theories may yield interesting results in analytic number theory.

References

[1] T. M. Apostol, " Introduction to Analytic Number Theory", Springer-Verlag, New York, 1976.

[2] A. Arai, Supersymmetry and singular perturbations, *J. Funct. Anal.* **60** (1985), 378–393.

[3] A. Arai, Path integral representation of the index of Kähler-Dirac operators on an infinite dimensional manifold, *J. Funct. Anal.* **82** (1989), 330–369.

[4] A. Arai, Supersymmetric embedding of a model of a quantum harmonic oscillator interacting with infinitely many bosons, *J. Math. Phys.* **30** (1989), 512–520.

[5] A. Arai, A general class of infinite dimensional Dirac operators and related aspects, in "Functional Analysis & Related Topics" (Ed. S. Koshi), World Scientific, Singapore, 1991.

[6] A. Arai, A general class of infinite dimensional Dirac operators and path integral representation of their index, *J. Funct. Anal.* **105** (1992), 342–408.

[7] A. Arai, Dirac operators in Boson-Fermion Fock spaces and supersymmetric quantum field theory, *J. Geom. Phys.* **11** (1993), 465–490.

[8] A. Arai, Supersymmetric extension of quantum scalar field theories, in "Quantum and Non-Commutative Analysis" (Ed. H.Araki et al), Kluwer Academic Publishers, Dordrecht, 1993.

[9] A. Arai, On self-adjointness of Dirac operators in Boson-Fermion Fock spaces, *Hokkaido Math. Jour.* **23** (1994), 319–353.

[10] A. Arai, Operator-theoretical analysis of a representation of a supersymmetry algebra in Hilbert space, *J. Math. Phys.* **36** (1995), 613–621.

[11] A. Arai, Supersymmetric quantum field theory and infinite dimensional analysis, *Sugaku Expositions* **9** (1996), 87–98.

[12] A. Arai, Trace formulas, a Golden-Thompson inequality and classical limit in Boson Fock space, *J. Funct. Anal.* **136** (1996), 510–547.

[13] A. Arai, Strong anticommutativity of Dirac operators on Boson-Fermion Fock spaces and representations of a supersymmetry algebra, *Math. Nachr.* **207** (1999), 61–77.

[14] A. Arai, Infinite dimensional analysis and analytic number theory, *Hokkaido University Preprint Series* #450, 1999.

[15] A. Arai and I. Mitoma, De Rham-Hodge-Kodaira decomposition in ∞-dimensions, *Math. Ann.* **291** (1991), 51–73.

[16] A. Arai, and I. Mitoma, Comparison and nuclearity of spaces of differential forms on topological vector spaces, *J. Funct. Anal.* **111** (1993), 278–294.

[17] I. Bakas and M. J. Bowick, Curiosities of arithmetic gases, *J. Math. Phys.* **32** (1991), 1881–1884.

[18] J-B. Bost and A. Connes, Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory, *Selecta Math.* (New Series) **1** (1995), 411–457.

[19] O. Bratteli and D. W. Robinson, " Operator Algebras and Quantum Statistical Mechanics 2", Second Edition, Springer-Verlag, Berlin, Heidelberg, 1997.

[20] P. Contucci and A. Knauf, The low activity phase of some Dirichlet series, *J. Math. Phys.* **37** (1996), 5458–5475.

[21] B. Julia, On the "statistics" of primes, *J. Phys. France* **50** (1989), 1371–1375.

[22] B. Julia, Statistical theory of numbers, in "Number Theory and Physics" (J.-M. Luck, P. Moussa and M. Waldschmidt, editors), Springer-Verlag, Berlin, Heidelberg, 1990.

[23] J.-M. Luck, P. Moussa and M. Waldschmidt (editors), "Number Theory and Physics", Springer-Verlag, Berlin, Heidelberg, 1990.

[24] M. Reed and B. Simon, "Methods of Modern Mathematical Physics Vol.I: Functional Analysis", Academic Press, New York 1972.

[25] M. Reed and B. Simon, "Methods of Modern Mathematical Physics Vol.II: Fourier Analysis, Self-adjointness", Academic Press, New York, 1975

[26] M. Reed and B. Simon, "Methods of Modern Mathematical Physics Vol.IV: Analysis of Operators", Academic Press, New York, 1978

[27] D. Spector, Multiplicative functions, Dirichlet convolution, and quantum systems, *Phys. Lett. A* **140** (1989), 311–316.

[28] D. Spector, Supersymmetry and the Möbius inversion function, *Commun. Math. Phys.* **127** (1990), 239–252.

[29] D. Spector, Duality, partial supersymmetry, and arithmetic number theory, *J. Math. Phys.* **39** (1998), 1919–1927.

[30] M. Waldschmidt, P. Moussa, J.-M. Luck and C. Itzykson (editors), "From Number Theory to Physics", Springer-Verlag, Berlin, Heidelberg, 1992.

Harmonic Analysis on Negatively Curved Manifolds

- Carleson measure, Brownian motion and a gradient estimate for harmonic functions -

Hitoshi ARAI

This paper is mainly a summary of recent work of the author on harmonic analysis on negatively curved manifolds, and we refer the reader to [10], [6] and [7] for details.

Let (M, g) be a complete, simply connected n dimensional Riemannian manifold whose sectional curvatures K_M satisfy

$$-\infty < -\kappa_1^2 \leq K_M \leq -\kappa_2^2 < 0,$$

where κ_1 and κ_2 are positve constants. In this paper we are concerned with Hardy spaces, BMO, Carleson measure and their probabilistic aspects. Further we give a gradient estimates for harmonic functions and its application to Bloch functions on negatively curved manifolds.

Notation Throughout this paper we fix a point o in M as a reference point. The constants depending only on g, n, κ_1, κ_2 and o will usually be denoted by C or C'. But C and C' may change in value from one occurrence to the next. For two nonnegative functions f and g defined on a set U, the notation $f \lessapprox g$ indicate that $f(x) \leq Cg(x)$ for all $x \in U$, and $f \approx g$ means that $f \lessapprox g$ and $g \lessapprox f$.

Acknowledgement The author thanks to Professor Heyer, Professor Obata and Professor Hirai for their invitation to the conference on "Infinite Dimensional Harmonic Analysis".

1 Background material

Before going to the main body of this report, let us give a brief review of results obtained by Anderson and Schoen ([3]), Cifuentes and Korányi ([18]), and the author ([6], [7]).

Let $S(\infty)$ be the sphere at infinity of M, and $\overline{M}$ Eberlein and O'Neill's compactification $M \cup S(\infty)$ of M (see [23]). The following theorem plays a fundamental and important role in our work:

Theorem AS1 (Anderson and Schoen [3]; [1], [31]) (1) *The Martin compactification of M with respect to the Laplacian Δ_g on M is homoemorphic to $\overline{M}$, and the Martin boundary consists only of minimal points.*

(2) *For every $z \in M$, there exists a unique function $K_z(x,Q)$ $(Q \in S(\infty), x \in \overline{M}\backslash\{Q\})$ such that for every $Q \in S(\infty)$,*

$$K_z(\cdot, Q) \text{ is positive harmonic on } M, \tag{1}$$

$$K_z(\cdot, Q) \text{ is continuous on } \overline{M} \setminus \{Q\}, \tag{2}$$

$$K_z(Q', Q) = 0 \text{ for all } Q' \in S(\infty) \setminus \{Q\}, \text{ and} \tag{3}$$

$$K_z(z, Q) = 1. \tag{4}$$

(This function is called the Poisson kernel normalized at z.)

(3) *For every $z \in M$ and for every positive harmonic function u on M, there exists a unique Borel measure m_u^z on $S(\infty)$ such that*

$$u(x) = \int_{S(\infty)} K_z(x,Q) f(Q) dm_u^z(Q), \quad x \in M \tag{5}$$

(The measure m_u^z is called the Martin representing measure relative to u and z.)

Throughout this paper, we write $K(x,Q) = K_o(x,Q)$, and denote by ω^x the Martin representing measure relative to the constant function 1 and $x \in M$. It is called the harmonic measure relative to x. In particular, let $\omega = \omega^o$. Note that $\omega^x(S(\infty)) = 1$ and $d\omega^x(Q) = K(x,Q)d\omega(Q)$, for all $x \in M$.

For notational simplicity, we denote

$$\tilde{f}(x) = \int_{S(\infty)} K(x,Q) f(Q) d\omega(Q), \quad x \in M,$$

for every $f \in L^1(S(\infty), \omega)$.

In their paper [3], Anderson and Schoen generalized to the manifold M Fatou's theorem on boundary behavior of bounded harmonic functions on the open unit disc. To describe their theorem we need some notation. For $x \in M$ and $y \in \overline{M}$ $(x \neq y)$, let γ_{xy} be the unit speed geodesic with $\gamma_{xy}(0) = x$ and $\gamma_{xy}(t) = y$ for some $t \in (0, +\infty]$. Since such a number t is uniquely determined, we denote it by t_{xy}. Anderson and Schoen defined the following analogue of the classical nontangential region: For $Q \in S(\infty)$ and $d > 0$, let

$$T_d(Q) = \bigcup_{t>0} B(\gamma_{oQ}(t), d), \tag{6}$$

where $B(x,r)$ is the geodesic ball with center x and radius r.

Theorem AS2 (Anderson and Schoen [3]) *Let u be a bounded harmonic function on M. Then for ω-a.e. $Q \in S(\infty)$, the nontangential limit*

$$\lim_{x \in T_d(Q)} u(x)$$

exists for all $d > 0$.

This result was extended by Ancona [1], Mouton [38] and the author [7]: Ancona proved an analogue of Fatou-Doob theorem, Mouton verified Calderón-Stein type theorem and the author obtained an analogue of a local version of Fatou-Doob theorem.

2 Admissible maximal functions and Hardy spaces

In [6], we studied another analogue to M of the classical nontangential region. In order to describe it, let us mention some terminologies: For $p \in M$, $v \in T_pM$ and $\delta > 0$, let $C(p, v, \delta)$ be the cone about the tangent vector v of angle δ defined by

$$C(p, v, \delta) := \{x \in \overline{M} : \angle_p(v, \dot{\gamma}_{px}(0)) < \delta\},$$

where $\angle_p$ denotes the angle in T_pM and $\dot{\gamma}_{px}(t)$ is its tangent vector at t.

For $z \in M \setminus \{o\}$ and $t \in \mathbf{R}$, we denote

$$C(z, t) = C(\gamma_{oz}(t_{oz} + t), \dot{\gamma}_{oz}(t_{oz} + t), \pi/4), \text{ and } z(t) = \gamma_{oz}(t_{oz} + t),$$

and let

$$\Delta(x, t) = C(x, t) \cap S(\infty).$$

Our analogue is the following:

Definition 2.1 ([6]) *For $Q \in S(\infty)$ and $\alpha \in \mathbf{R}$, let*

$$\Gamma_\alpha(Q) = \{z \in M : Q \in \Delta(z, \alpha)\}, \tag{7}$$

and we call this set an admissible region at Q.

Using this notion, we can define an analogue of nontangential maximal function, admissible maximal functions, as follows: For a function u on M, let

$$N_\alpha(u)(Q) = \sup_{x \in \Gamma_\alpha(Q)} |u(x)|, \quad Q \in S(\infty), \quad \alpha \in \mathbf{R}.$$

Furthermore we can define Hardy type spaces in terms of our maximal functions:

$$H^p_\alpha = \left\{ f \in L^1(S(\infty), \omega) : N_\alpha(\tilde{f}) \in L^p(S(\infty, \omega) \right\}, \quad 1 \le p \le \infty$$

and we denote

$$\|f\|_{H^p_\alpha} := \|N_\alpha(\tilde{f})\|_{L^p(\omega)}.$$

It is easy to prove that $(H^p_\alpha, \|\cdot\|_{H^p_\alpha})$ is a Banach space and that for every $\alpha, \beta \in \mathbf{R}$, $H^p_\alpha = H^p_\beta$, and moreover for every $f \in H^p_\alpha = H^p_\beta$,

$$C^{-1}_{\alpha,\beta}\|f\|_{H^p_\alpha} \le \|f\|_{H^p_\beta} \le C_{\alpha,\beta}\|f\|_{H^p_\alpha},$$

where $C_{\alpha,\beta}$ is a positive constant depending only on n, κ_1, κ_2, α and β (see [10]). Therefore in this paper we deal only with H^p_0, and we denote

$$H^p = H^p_0, \quad \text{and} \quad \|\cdot\|_{H^p} = \|\cdot\|_{H^p_\alpha}.$$

We study also atomic Hardy spaces in the sense of Coifman and Weiss and probabilistic versions of Hardy spaces. Let us describe them. First we are concerned with atomic Hardy spaces. For any $Q \in S(\infty)$, we define $\Delta_t(Q)$ to be the "ball" in $S(\infty)$ centered at Q of radius $\log(1/r)$,

$$\Delta_t(Q) := \Delta(\gamma_{oQ}(t), 0) \ (= C(\gamma_{oQ}(t), \dot{\gamma}_{oQ}(t), \pi/4) \cap S(\infty)),$$

It is easy to see that the function

$$\rho_0(Q,Q') := \left(\inf\{e^{-t} : Q' \in \Delta_t(Q)\} + \inf\{e^{-t} : Q \in \Delta_t(Q')\}\right)/2, \quad Q, Q' \in S(\infty)$$

is a quasi-distance in the sense of [19] such that $(S(\infty), \rho, \omega)$ is a space of homogeneous type. Therefore the abstract theory in [19] can be transplanted to our case. For instance, some covering lemmas, theorems on atomic Hardy spaces and BMO on spaces of homogeneous type hold true for $(S(\infty), \omega, \rho)$. Now let us mention the definition of atomic Hardy spaces on $S(\infty)$. In [19], atomic Hardy spaces and BMO on a space of homogeneous type are defined in terms of its quasi-distance. However in our case, we can prove that the family of balls defined by ρ is equivalent to $\{\Delta_t(Q)\}$, that is,

(8) $$\Delta_{\log(1/r)+k_1}(Q) \subset \{Q' : \rho(Q,Q') < r\} \subset \Delta_{\log(1/r)-k_2}(Q).$$

where k_1 and k_2 are positive constants depending only on M.

For this reason, one can define atomic Hardy spaces and BMO in terms of $\{\Delta_t(Q)\}$ which are equivalent to those defined by the quasi-distace ρ : a function a on $S(\infty)$ is called an atom if the support of a is contained in a "ball" $\Delta_r(Q)$, $\int_{S(\infty)} a d\omega = 0$, and $\|a\|_{L^\infty(\omega)} \le \omega(\Delta_r(Q))^{-1}$. Since $\omega(S(\infty)) = 1$, we regard also the constant function 1 as an atom. The atomic Hardy spaces H^1_{atom} is defined as the set of all functions h in $L^1(S(\infty), \omega)$ such that h has an atomic decomposition

(9) $$h = \sum_{j=1}^{\infty} \lambda_j a_j,$$

where $\lambda_j \in \mathbf{R}$, and a_j's are atoms and $\sum_{j=1}^{\infty} |\lambda_j| < \infty$. We set

$$\|h\|_{1,\text{atom}} = \inf\left\{\sum_{j=1}^{\infty} |\lambda_j| : h = \sum_{j=1}^{\infty} \lambda_j a_j,\ a_j\text{'s are atoms}\right\}$$

for $h \in H^1_{\text{atom}}$.

Let $\text{BMO}(\omega)$ be the set of all functions $f \in L^1(S(\infty), \omega)$ such that

$$\|f\|_{\text{BMO}} = \sup_{Q\in S(\infty), r\in\mathbf{R}} \frac{1}{\omega(\Delta_r(Q))} \int_{\Delta_r(Q)} |f - m_{\Delta_r(Q)} f| d\omega + \|f\|_{L^1(\omega)} < \infty,$$

where

$$m_{\Delta_r(Q)} f = \frac{1}{\omega(\Delta_r(Q))} \int_{\Delta_r(Q)} f d\omega.$$

Theorem CW ([19]). *The dual of H^1_{atom} is regarded as the space* $\text{BMO}(\omega)$ *in the following sense: If $h = \sum \lambda_j a_j \in H^1_{\text{atom}}$, then for each $\ell \in \text{BMO}(\omega)$*

$$\langle h, \ell\rangle := \lim_{m\to\infty} \lambda_j \int_X \ell a_j d\omega$$

is a well defined continuous linear functional and its norm is equivalent to $|\ell|_{\text{BMO}}$. Moreover, every linear continuous functional on H^1_{atom} has this form.

In this paper we will also deal with probabilistic analogues of Hardy spaces. To define them, we need to recall some facts on Brownian motion on M and its Markov properties: Let W be the set of all continuous maps from $[0, \infty)$ to M, and let $Z_t(w) = w(t)$, $w \in W$. Since by Yau [47] the life time of Brownian motion on M is equal to $+\infty$, so there exists a system of probability measures $\{P_x\}_{x\in M}$ on W such that (P_x, Z_t) is a Brownian motion starting at x. From Sullivan [43] or Kifer [31] it follows the following facts:

(I) There exists a limit $Z_\infty(w) := \lim_{t\to\infty} Z_t(w)$ for almost sure $w \in W$ with respect to P_x, $x \in M$. Moreover, $Z_\infty(w) \in S(\infty)$ for P_x-a.s. $w \in W$.

(II) For every $x \in M$ and for every Borel subset F of $S(\infty)$,

$$\omega^x(F) = P_x(\{w \in W : Z_\infty(w) \in F\}).$$

For every $f \in L^1(\omega)$, $\tilde{f}(x) = E_x[f(Z_\infty)]$ for all $x \in M$ and $\lim_{t\to\infty} \tilde{f}(Z_t) = f(Z_\infty)$ P_x-a.s., where $E_x[\ ]$ denotes the expectation with respect to P_x $(x \in M)$. We denote $P = P_o$ and $E[\ ] = E_o[\ ]$. Let

$$H^p_{\text{prob}} := \left\{ f \in L^p(\omega) : \|f\|_{H^p_{\text{prob}}} = E\left[\sup_{0\le t<\infty} \left|\tilde{f}(Z_t)\right|^p\right]^{1/p} < \infty \right\}, \quad 1 \le p < \infty.$$

Let $\mathcal{B}$ (resp. $\mathcal{B}_t$) be the smallest σ-field for which all random variables Z_s, $s \geq 0$ (resp. Z_s, $0 \leq s \leq t$) are measurable. For a probability Borel measure μ on M, let $P_\mu(A) = \int_{S(\infty)} P_x(A) d\mu(x)$, $A \subset W$. We denote by $(W, \mathcal{F}^\mu, \mathcal{F}^\mu_t, P_\mu)$ the usual P_μ augmentation of $(W, \mathcal{B}, \mathcal{B}_t, P_\mu)$ in the sense of [41, III 9]. In particular, $(W, \mathcal{F}^x, \mathcal{F}^x_t, P_x)$ denotes the P_x-augumentation of $(W, \mathcal{B}, \mathcal{B}_t, P_\mu)$. Put $\tilde{\mathcal{F}} := \bigcap \mathcal{F}^\mu$ and $\tilde{F}_t := \bigcap \mathcal{F}^\mu_t$, where the intersection is taken over all probability Borel measures μ on M. Then $(Z_t, W, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_t, P_x : x \in M)$ is a strong Markov process. If fact, considering that M is diffeomorphic to $\mathbf{R}^n$, it is a honest FD diffusion in the sense of [41, III 3, III 13].

It is known that the usual P_x-augumentation $(W, \mathcal{F}^x, \mathcal{F}^x_t, P_x)$ satisfies the so-called usual condition (see [41, III 9]). Moreover, for every harmonic function u on M, the process $u(Z_t)$ is a continuous local $(P_x, \mathcal{F}^x_t)$-martingale. Denote by $(W, \mathcal{F}, \mathcal{F}_t, P)$ the usual P_o-augumentation $(W, \mathcal{F}^o, \mathcal{F}^o_t, P_o)$. As usual, Hardy spaces of martingales are defined as follows:

$$\mathcal{M}^p := \left\{ X \in L^1(W, \mathcal{W}, P) : \|X\|_{\mathcal{M}^p} := E\left[\sup_{0 \leq t < \infty} |E[X|\mathcal{F}_t]|^p \right]^{1/p} < \infty \right\},$$

$(1 \leq p < \infty)$, where and always $E[\cdot|\mathcal{C}]$ denotes the conditional expectation with respect to P and a sub σ-field $\mathcal{C}$ of $\mathcal{F}$. Note that Meyer's previsibility theorem ([41, VI 15, Theorem 15.4]) implies that for every $X \in L^1(W, P)$, the process $(E[X/\mathcal{F}_t])_{t \geq 0}$ is an $(\mathcal{F}_t)$-continuous martingale.

For $X \in L^1(W, \mathcal{F}, P)$, let $\mathcal{N}'(X) := E[X|\sigma(Z_\infty)]$, where $\sigma(Z_\infty)$ is the sub σ-field of $\mathcal{F}$ generated by the random variable Z_∞. Then by (I) there exists a unique element $f \in L^1(\omega)$ such that $\mathcal{N}'(X) = f(Z_\infty)$, P-a.s. Denote the function f by $\mathcal{N}X$.

Now we can mention another probabilistic analogue of Hardy spaces:

$$H^p_{\text{mart}} := \{\mathcal{N}(X) : X \in \mathcal{M}^p\}, \quad 1 \leq p < \infty,$$

and as a norm on H^p_{mart}, we consider $\|\mathcal{N}(X)\|_{H^p_{\text{mart}}} := \|X\|_{\mathcal{M}^p}$.

For two normed spaces $(A, \|\ \|_A)$ and $(B, \|\ \|_B)$, we denote by $A \preceq B$ that $A \subset B$ and $\|x\|_B \leq C\|x\|_A$ for every $x \in A$, where C is a constant independent of x. Further we set $A \simeq B$ if $A \preceq B$ and $B \preceq A$.

In 1987, we announced in [6] the following Theorems 2.1 and 2.2 (see [10] for detailed proofs). :

Theorem 2.1 ([6]; see also [10])

$$H^1(\omega) \preceq H^1_{\text{prob}} \preceq H^1_{\text{mart}} \preceq H^1_{\text{atom}}(\omega)$$

Let k be a constant such that for every $Q_1, Q_2 \in S(\infty)$ and $r \in \mathbf{R}$, $\Delta_r(Q_1) \cap \Delta_r(Q_2) \neq \emptyset$ implies $\Delta_r(Q_2) \subset \Delta_{r-k}(Q_1)$. (This constant always exsits.)

Theorem 2.2 ([6]; see also [10]) *Consider the following geometric condition:*

(β) *For every $Q \in S(\infty)$, $t > k$ and $z \in C(\gamma_{oQ}(t), 0)$,*

$$\Delta_t(\gamma_{\zeta_z}(+\infty)) \bigcap \Delta_t(Q) \neq \emptyset.$$

If our manifold M satisfies the condition (β), we have $H^1_{\mathrm{atom}}(\omega) \preceq H^1(\omega)$.

When M is rotationally symmetric at o or the dimension of M is two, the condition (β) is satisfied. However recently, Cifuentes and Korányi proved the following

Theorem CK2 (Cifuentes and Korányi [18]) *The manifold M satisfies always the condition (β).*

Therefore combining our Theorems 2.1 and 2.2 with Theorem CK2, the following theorem is obtained:

Theorem 2.3 (Arai [6], Cifuentes and Korányi [18])

$$H^1(\omega) \simeq H^1_{\mathrm{atom}}(\omega) \simeq H^1_{\mathrm{prob}} \simeq H^1_{\mathrm{mart}}$$

3 Carleson measure

In this section we study a condition on a measure μ on M in order that the Martin integral operator,

$$K[f](z) = \int_{S(\infty)} K(z,Q) f(Q) d\omega(Q) \; (= \tilde{f}(z)), \; z \in M,$$

is bounded from $L^p(\omega)$ to $L^p(M, \mu)$. This problem was studied by L. Carleson in the classical Euclidean case, and he found a necessary and sufficient condition called now "Carleson condition". We study a version to M of "Carleson condition":

Definition 3.1 *For a set $A \subset S(\infty)$ and $r > 0$, let*

$$S_r[A] := \{z \in M \setminus B(o,r) : \Delta(z,0) \subset A\}.$$

A given complex Borel measure μ *on* M *is said to be a Carleson measure on* M *if for every* $r > 0$,

$$\|\mu\|_{c,r} := \sup_{Q \in S(\infty), t>1} \frac{|\mu|(S_r[\Delta_t(Q)])}{\omega(\Delta_t(Q))} + |\mu|(M) < \infty,$$

where $|\mu|$ *is the total variation of* μ. *We wirte* $\|\mu\|_c = \|\mu\|_{c,1}$.

As an analogue of the classical Carleson-Hörmander's theorem, we obtain the following

Theorem 3.1 ([10]) *Let* μ *be a complex Borel measure on* M. *Then the following are equivalent:*

(i) μ *is a Carleson measure on* M.

(ii) $\|\mu\|_{c,r} < \infty$ *for some* $r > 0$.

(iii) *For every* $1 \leq p < \infty$, *the Martin integral operator* K *is bounded from* $H^p(\omega)$ *to* $L^p(M, |\mu|)$.

(iv) *For every* $1 < p < \infty$, *the operator* K *is bounded from* $L^p(\omega)$ *to* $L^p(M, |\mu|)$.

(v) *For some* $1 < p < \infty$, *the operator* K *is bounded from* $L^p(\omega)$ *to* $L^p(M, |\mu|)$.

Furthermore, for every $r > 0$, *there is a constant* C'_r *depending only on* M, o *and* r *such that*

$$C_r'^{-1}\|\mu\|_{c,r} \leq \|\mu\|_c \leq C'_r\|\mu\|_{c,r}.$$

We give also a kind of an analytic characterization of Carleson measures. Let $G(x,y)$ be Green's function on M (see [3] or [4]). For a Borel measure μ on M, the function

$$G[\mu](x) = \int_M G(x,y)d\mu(y), \quad x \in M$$

is called the Green potential of μ. In this section we study boundary behavior of the Green potentials of the following weighted measures: for a nonnegative Borel measure μ on M, let

$$\mu_0(A) = \int_A \frac{1}{G(o,w)}d\mu(w), \quad A \subset M.$$

A nonnegative function f on M is said to be asymptotically bounded if there exists a positive constant $R > 0$ such that $\sup_{x \in M \setminus B(o,R)} f(x) < \infty$. Then we have the following

Theorem 3.2 ([10]) *Let* μ *be a nonnegative Borel measure on* M. *Suppose that* $\mu(H) < \infty$ *for every compact set* H *in* M. *Then the following statements are equivalent:*

(i) $G[\mu_0]$ *is asymptotically bounded on* M.

(ii) μ *is a Carleson measure and satisfies the following condition* (F)*:*

(F) *There exist positive constants* r *and* C *such that*

$$(10) \qquad \int_{B(z,1)} G(z,w)d\mu(w) \leq CG(o,z) \quad \text{for every } z \in M \setminus B(o,r).$$

For $f \in L^1(\omega)$, let

$$d\mu_f(w) = G(o,w)|\nabla\tilde{f}(w)|^2 dV(w),$$

where dV is the volume measure with respect to the metric g, and $|\nabla\tilde{f}(w)|$ is the norm of the gradient of $\tilde{f}$ with respect to g, that is, in a local coordinate neighborhood,

$$|\nabla\tilde{f}(w)|^2 = \sum_{ij} g^{ij}(w)\frac{\partial f(w)}{\partial x_i}\frac{\partial f(w)}{\partial x_j},$$

where $(g^{ij}(w))$ is the inverse matrix of the metric $(g_{ij}(w))$. This is an analogue to M of the classical Littlewood-Paley measure.

It is easy to see that for $f \in L^1(\omega)$, $\mu_f(M) < \infty$ if and only if $f \in L^2(\omega)$.

As a corollary of Theorem 3.2 we obtain the following characterization of BMO functions in terms of Carleson measures and Green potentials:

Theorem 3.3 ([10]) *Let* $f \in L^2(\omega)$. *Then the following are equivalent:*

(i) $f \in \mathrm{BMO}(\omega)$

(ii) μ_f *is a Carleson measure on* M.

(iii) *The Green potential*

$$G_f(x) := \int_M G(x,w)|\nabla\tilde{f}(w)|^2 dV(w)$$

is asymptotically bounded.

(iv) *The potential* G_f *defined in* (iii) *is bounded on* M.

Remark. As known, in the classical Euclidean case, the part "(i)$\Longleftrightarrow$ (ii)" was obtained by Fefferman and Stein [24]. In the case of the Bergman ball in $\mathbf{C}^n$, analogous results to Theorem 3.3 were proved in Jevtić [27]. See also [8] and [9].

4 A gradient estimate for harmonic functions and Bloch functions.

In this section we will apply Theorem 3.3 to Bloch function theory on Riemannian manifolds.

Classicaly Bloch functions were defined on the open unit disc D in $\mathbf{C}$ as follows: a holomorphic function f on D is said to be a Bloch function on D if

$$\sup_{z\in D}(1-|z|)|f'(z)| < \infty. \tag{11}$$

This means that f is a Bloch function if and only if the norm of gradient $|\nabla f|$ with respect to the Poincaré metric is bounded. Now the notion of Bloch functions is naturally extended to Riemannian manifold $(\mathcal{R}, h)$:

Definition 4.1 *Let f be a harmonic function on $\mathcal{R}$. Then f is said to be a harmonic Bloch function on $\mathcal{M}$ if*

$$\|f\|_B := \sup_{x\in\mathcal{R}} |\nabla f(x)| < \infty,$$

where $|\nabla f|$ is the norm of gradient of f with respect to the metric h, i.e. $|\nabla f(x)|^2 = \sum_{i,j} h^{ij}(x)(\partial f(x)/\partial x_i)(\partial f(x)/\partial x_j)$, where $(h^{ij}(x))$ is the inverse matrix of the Riemannian metric $(h_{ij}(x))$.

In particular, if $(\mathcal{R}, h)$ is a Kähler manifold, then a function u is said to be a holomorphic Bloch function on M if u is a harmonic Bloch function and holomorphic on $\mathcal{R}$.

In [32], Krantz and Ma defined Bloch functions on a bounded strongly pseudoconvex domain with smooth boundary. See Timoney [44] for Bloch functions on symmetric domains. If $(\mathcal{R}, h)$ is a bounded smoothly strongly pseudoconvex domain endowed with the Bergman metric, it is easy to see that our definition of Bloch functions is equivalent to one by Krantz and Ma.

If the Ricci curvature of $\mathcal{R}$ is nonnegative, then from Yau and Chen's results it follows that the class of Bloch functions is equal to the class of harmonic functions with linear order growth (see [34] and [30]).

Theorem 4.1 ([10]) *Suppose $f \in BMO(\omega)$. Then $\tilde{f}$ is a harmonic Bloch function on M. Indeed*

$$\sup_{x\in M} \|\nabla \tilde{f}(x)\| \le C\|f\|_{\mathrm{BMO}}, \tag{12}$$

where C is a positive constant depending only on M and o.

In particular, there exists a unbounded harmonic Bloch function on M.

Let $\mathbf{T}$ be the unit circle. Denote by BMOA($\mathbf{T}$) the set of all functions f in BMO($\mathbf{T}$) such that the Poisson integral of f is holomorphic in the open unit disc D. Then it is known that if $f \in$ BMOA($\mathbf{T}$), then its Poisson integral is a holomorphic Bloch function on D (cf. [40]). Krantz and Ma [32] extended this fact to bounded strongly pseudoconvex domains with smooth boundaries. Our proof of Theorem 4.1 is different from their proofs.

It should be noted that the inequality (12) is closely related to Jerison and Kenig [28, Lemma 9.9] for harmonic functions with respect to the Euclidean Laplacian.

Let $u(z) = \sum_{k=m}^{\infty} z^{15^k}$ ($z \in D$). Then u is a holomorphic Bloch function, and for large m,

$$\limsup_{r\to 1} \frac{|u(re^{i\theta})|}{\sqrt{\log(1-r)^{-1}\log\log\log(1-r)^{-1}}} > 0.685\|u\|_B \text{ a.e. } \theta \in [0, 2\pi)$$

(see [40, p.194]).

In 1985, Makarov proved the following

Theorem M (Makarov [36]; see also Pommerenke [40, p.186]) *Let u be a holomorphic Bloch function on D. Then for almost every $\theta \in [0, 2\pi)$,*

$$\limsup_{r\to 1} \frac{|u(re^{i\theta})|}{\sqrt{\log(1-r)^{-1}\log\log\log(1-r)^{-1}}} \le \|u\|_B.$$

Also a probabilistic version of Theorem M was obtained by Lyons [35]:

Theorem L (Lyons [35]) Let u be a holomorphic Bloch function on D. Let X_t be hyperbolic Brownian motion on D. Then

$$\limsup_{t\to\infty} \frac{|u(X_t)|}{\sqrt{\log(1-|X_t|)^{-1}\log\log\log(1-|X_t|)^{-1}}} \le \|u\|_B.$$

We will generalize Theorem L to our manifold M. We begin with characterizing Bloch functions in terms of Brownian motion:

Theorem 4.2 ([10]) *For a harmonic function u on M, the following* (i) *and* (ii) *are equivalent:*

(i) *u is a harmonic Bloch function on M.*

(ii) *The stochastic process $\{u(Z_t)\}_t$ satisfies that*

$$\|u\|_{B,\text{prob}}^2 := \sup_{x\in M} \left\{ \frac{E_x[|u(Z_T) - u(Z_0)|^2]}{E_x[T]} : T \in \mathcal{T}_x, E_x[T] > 0 \right\} < \infty,$$

where $\mathcal{T}_x$ is the set of all $(\mathcal{F}_t^x)$-stopping times. Furthermore, $\|u\|_B \le \|u\|_{B,\text{prob}} \le \sqrt{2}\|u\|_B$.

In the case of the open unit disc in $\mathbf{C}$, a martingale characterization of holomorphic Bloch functions was given in Muramoto [39]. We will prove Theorem 4.2 by simplifying and exploiting the method in [39] by combining an idea in Lyons [35].

Now we describe on our genealization of Thoerem L:

Theorem 4.3 ([10]) *Let u be a harmonic Bloch functions on M. Then*

$$\limsup_{t\to\infty} \frac{|u(Z_t)|}{\sqrt{d(o,Z_t)\log\log d(o,Z_t)}} \leq C\|u\|_B \quad P\text{-a.s.}$$

As an immediate consequence of Theorem 4.3 we have the following

Corollary 4.4 ([10]) *Let $M = \{x \in \mathbf{R}^n : |x| < 1\}$ and let g be the hyperbolic metric on M. Then for a harmonic Bloch function u on (M,g),*

$$\limsup_{t\to\infty} \frac{|u(Z_t) - u(o)|}{\sqrt{\log(1-|Z_t|)^{-1}\log\log\log(1-|Z_t|)^{-1}}} \leq C\|u\|_B \quad a.s.P^o$$

References

[1] A. Ancona, Negatively curved manifolds, elliptic operators, and the Martin boundary, Ann. of Math. 125 (1987), 495-536.

[2] M. T. Anderson, The Dirichlet problem at infinity for manifolds of negative curvature, J. Diff. Geometry, 18 (1983), 701–721.

[3] M. T. Anderson and R. Schoen, Positive harmonic functions on complete manifolds of negative curvatures, Ann. of Math. 121 (1985), 429-461.

[4] K. Aomoto, L'analyse harmonique sur les espaces riemannienes á courbure riemannienne negative I, J. Fac. Sci. Univ. Tokyo, 13 (1966), 85–105.

[5] H. Arai, Measure of Carleson type on filtrated probability spaces and the corona theorem on complex Brownian spaces, Proc. Amer. Math. Soc. 96 (1986), 643–647.

[6] H. Arai, Harmonic analysis on negatively curved manifolds I, Proc. Japan Acad. 63 (1987), 239–242.

[7] H. Arai, Boundary behavior of functions on complete manifolds of negative curvature, Tohoku Math. J. 41 (1989), 307–319.

[8] H. Arai, Kähler diffusions, Carleson measures and BMOA functions of several complex variables, Complex Variables, Theory and Appl. 22 (1993), 255–266.

[9] H. Arai, Singular elliptic operators related to harmonic analysis and complex analysis of several variables, *Trends in Probablity and Related Analysis*, 1–34, World Sci. Publ. 1999.

[10] H. Arai, Hardy spaces, Carleson measures and a gradient estimate for harmonic functions on negatively curved manifolds, submitted for publication.

[11] R. F. Bass, Probabilistic Techniques in Analysis, Springer-Verlag, New York, Berlin, Heidelberg, 1995

[12] D. L. Burkholder, R. F. Gundy, M. L. Silverstein, A maximal function characteization of the class H^p, Trans. Amer. Math. Soc. 157 (1971), 137–153.

[13] L. Carleson, An interpolation problem for bounded analytic functions, Amer. J. Math. 80 (1958), 921–930.

[14] L. Carleson, Interpolations by bounded analytic functions and the corona problem, Ann. of Math. 76 (1962), 547–559.

[15] P. S. Chee, On vanishing Carleson measures, Boll. U.M.I. 7, (1993) 431–435.

[16] P. Cifuentes, H^p-classes on rank one symmetric spaces of noncompact type, I. Nontangental and probabilistic maximal functions, Trans. Amer. Math. Soc. 294 (1986), 133–149.

[17] P. Cifuentes, H^p-classes on rank one symmetric spaces of noncompact type, II. Nontangential maximal function and area integral, Bull. Sci. math. 2^e série 108 (1984), 355–371.

[18] P. Cifuentes and A. Korányi, Admissible convergence in Cartan-Hadamard manifolds, submitted for publication.

[19] R. R. Coifman and G. Weiss, Extensions of Hardy spaces and their use in analysis, Bull. Amer. Math. Soc., 83 (1977), 569-645.

[20] A. Debiard, Comparison des espaces H^p géométrique et probabilistes au dessus de l'espace hermitien hyperbolique, Bull. Sci. Math. 103 (1979), 305–351.

[21] A. Debiard, Espaces H^p au-dessus de l'espace hermitien hyperbolique de $\mathbf{C}^n$ $(n > 1)$ II, J. Functional Analysis 40 (1981), 185–265.

[22] R. Durrett, Brownian Motion and Martingales in Analysis, Wadsworth, Belmont, CA, 1984.

[23] P. Eberlein and B. O'Neill, Visibility manifolds, Pacific J. Math. 46 (1973), 45–109.

[24] C. Fefferman and E. M. Stein, H^p spaces of several variables, Acta Math. 129 (1972), 137–193.

[25] D. Geller, Some results in H^p theory for the Heisenberg group, Duke Math. J. 47 (1980), 365–390.

[26] R. K. Getoor and M. J. Sharpe, Conformal martingales, Invent. Math. 16 (1972), 271-308.

[27] M. Jevtić, On the Carleson measures characterization of BMO functions on the unit sphere, Proc. Amer. Math. Soc. 123 (1995), 3371–3377.

[28] D. S. Jerison and C. E. Kenig, Boundary behavior of harmonic functions in non-tangentially accessible domains, Adv. in Math. 46 (1982), 80-147.

[29] J. Kamimoto, Remarks on Bloch functions on weakly pseudoconvex domains, RIMS Kokyuroku 1067 (1998), 83–88.

[30] A. Kasue, Harmonic functions of polynomial growth on complete manifolds, I Proc. Sympos. Pure Math. 54 (1993), 281–290, II J. Math. Soc. Japan 47 (1995), 35–63.

[31] Y. Kifer, Brownian motion and positive harmonic functions on complete manifolds of non-positive curvature, in "From Local Times to Global Geometry, Control and Physics", Pitman Reseach Notes in math. Ser. 150 (1986), 187–232.

[32] S. G. Krantz and D. Ma, Bloch functions on strongly pseudoconvex domains, Indian Univ. Math. J. 37 (1988), 145–163.

[33] S. G. Krantz and S.-Y. Li, A note on Hardy spaces and functions of bounded mean oscillation on domains in $\mathbf{C}^n$, Michigan Math. J. 41 (1994), 51–71.

[34] P. Li and L. F. Tam, Linear growth harmonic functions on a complete manifolds, J. Diff. Geom. 29 (1989), 421–425.

[35] T. Lyons, A synthetic proof of Makarov's law of the iterated logarithm, Bull. London Math. Soc. 22 (1990), 159-162.

[36] N. G. Makarov, On the distortion of boundary sets under conformal mappings, Proc. London Math. Soc. 51 (1985), 369–384.

[37] P. A. Meyer, Le dual de $H^1(\mathbf{R}^\nu)$, Lect. Notes in Math. 581 81977), 132–195.

[38] F. Mouton, Comportement asymptotique des fonctions harmoniques en courbure négative, Comment. Math. Helv. 70 (1995), 475–505.

[39] K. Muramoto, Bloch functions on the unit disk and martingale, Math. J. Toyama Univ. 13 (1990), 45-50.

[40] Ch. Pommerenke, Boundary Behaviour of Conformal Maps, Springer-Verlag, 1992.

[41] L. Roger and D. Williams, Diffusions, Markov Processes, and Martingales, vol. 1 (2nd ed.), vol. 2, John Wiley & Sons, Chichester, New York, Brisbane, Tronto, Singapore, 1994, 1987.

[42] R. Schoen and S. T. Yau, Lectures on Differential Geometry, International Press, 1994.

[43] D. Sullivan, The Dirichlet problem at infinity for a negatively curved manifold, J. Diff. Geom. 18 (1983), 723–732.

[44] R. Timoney, Bloch functions in several variables, I Bull. London Math. Soc. 12 (1980), 241–267, II J. Reine Angew. Math. 319 (1980), 241–267.

[45] D. C. Urllich, A Bloch functions in the unit ball with no radial limits, Bull. London Math. Soc. 20 (1988), 337–341.

[46] N. Th. Varopoulos, A probabilistic proof of the Garnett-Jones theorem on BMO, Pacific J. Math., 90 (1980), 201-221.

[47] S.-T. Yau, On the heat kernel of a complete Riemannian manifold, J. Math. Pure Appl. 57 (1978), 191–201.

[48] S.-T. Yau, Harmonic functions on complete Riemannian manifolds, Comm. Pure Appl. Math. 28 (1975), 201–228.

Address :
Graduate School of Mathematical Sciences
The University of Tokyo
3-8-1 Komaba, Meguro-ku
Tokyo 153-8914, JAPAN
e-mail : harai@mint.ocn.ne.jp

CHARACTERIZATION OF HIDA MEASURES IN WHITE NOISE ANALYSIS

NOBUHIRO ASAI [1]
Graduate School of Mathematics
Nagoya University
Nagoya, 464-8602, JAPAN

IZUMI KUBO
Department of Mathematics
Graduate School of Science
Hiroshima University
Higashi-Hiroshima, 739-8526, JAPAN

and

HUI-HSIUNG KUO
Department of Mathematics
Louisiana State University
Baton Rouge, LA 70803, USA

1. INTRODUCTION

In the recent paper [3] by Asai et al., the growth order of holomorphic functions on a nuclear space has been considered. For this purpose, certain classes of growth functions u are introduced and many properties of Legendre transform of such functions are investigated. In [4], applying Legendre transform of u under the conditions (U0), (U2) and (U3) (see §2), the Gel'fand triple

$$[\mathcal{E}]_u \subset (L^2) \subset [\mathcal{E}]_u^*$$

associated with a growth function u is constructed.

The main purpose of this work is to prove Theorem 4.4, so-called, the characterization theorem of Hida measures (generalized measures). As examples of such measures, we shall present the Poisson noise measure and the Grey noise measure in Example 4.5 and 4.6, respectively.

The present paper is organized as follows. In §2, we give a quick review of some fundamental results in white noise analysis and introduce the notion of Legendre transform utilized by Asai et al. in [3],[4]. In §3, we simply cite some useful properties of the Legendre transform from [3]. In §4, we discuss the characterization of Hida measures (generalized measures).

[1] Current address: International Institute for Advanced Studies, Kizu, Kyoto, 619-0225, JAPAN.

Acknowledgements. N. Asai wants to thank the Daiko Foundation and the Kamiyama Foundation for research support.

2. Preliminaries

In this section, we will summarize well-known results in white noise analysis [9],[20],[22] and notions from Asai et al.[1],[2],[3],[4]. Complete details and further developments will be appeared in [5]. Some similar results have been obtained independently by Gannoun et al. [8].

Let $\mathcal{E}_0$ be a real separale Hilbert space with the norm $|\cdot|_0$. Suppose $\{|\cdot|_p\}_{p=0}^{\infty}$ is a sequence of densely defined inner product norms on $\mathcal{E}_0$. Let $\mathcal{E}_p$ be the completion of $\mathcal{E}$ with respect to the norm $|\cdot|_p$. In addition we assume

(a) There exists a constant $0 < \rho < 1$ such that $|\cdot|_0 \le \rho|\cdot|_1 \le \cdots \le \rho^p|\cdot|_p \le \cdots$.
(b) For any $p \ge 0$, there exists $q \ge p$ such that the inclusion $i_{q,p} : \mathcal{E}_q \hookrightarrow \mathcal{E}_p$ is a Hilbert-Schmidt operator.

Let $\mathcal{E}'$ and $\mathcal{E}_p'$ denote the dual spaces of $\mathcal{E}$ and $\mathcal{E}_p$, respectively. We can use the Riesz representation theorem to identify $\mathcal{E}_0$ with its dual space $\mathcal{E}_0'$. Let $\mathcal{E}$ be the projective limit of $\{\mathcal{E}_p;\ p \ge 0\}$. Then we get the following continuous inclusions:

$$\mathcal{E} \subset \mathcal{E}_p \subset \mathcal{E}_0 \subset \mathcal{E}_p' \subset \mathcal{E}', \qquad p \ge 0.$$

The above condition (b) says that $\mathcal{E}$ is a nuclear space and so $\mathcal{E} \subset \mathcal{E}_0 \subset \mathcal{E}'$ is a Gel'fand triple.

Let μ be the standard Gaussian measure on $\mathcal{E}'$ with the characteristic function given by

$$\int_{\mathcal{E}'} e^{i\langle x,\xi\rangle}\, d\mu(x) = e^{-\frac{1}{2}|\xi|_0^2}, \qquad \xi \in \mathcal{E}.$$

The probability space $(\mathcal{E}', \mu)$ is called a *white noise space* or *Gaussian space*. For simplicity, we will use (L^2) to denote the Hilbert space of μ-square integrable functions on $\mathcal{E}'$. By the Wiener-Itô theorem, each $\varphi \in (L^2)$ can be uniquely expressed as

$$\varphi(x) = \sum_{n=0}^{\infty} I_n(f_n)(x) = \sum_{n=0}^{\infty} \langle :x^{\otimes n}:, f_n\rangle, \qquad f_n \in \mathcal{E}_0^{\widehat{\otimes} n}, \tag{2.1}$$

where I_n is the multiple Wiener integral of order n and $:x^{\otimes n}:$ is the Wick tensor of $x \in \mathcal{E}'$ (see [20].) Moreover, the (L^2)-norm of φ is given by

$$\|\varphi\|_0 = \left(\sum_{n=0}^{\infty} n!|f_n|_0^2\right)^{1/2}. \tag{2.2}$$

Let $u \in C_{+,\frac{1}{2}}$ be the set of all positive continuous functions on $[0,\infty)$ satisfying

$$\lim_{r\to\infty} \frac{\log u(r)}{\sqrt{r}} = \infty.$$

In addition, we introduce conditions:

(U0) $\inf_{r\geq 0} u(r) = 1$.

(U1) u is increasing and $u(0) = 1$.

(U2) $\lim_{r\to\infty} r^{-1} \log u(r) < \infty$.

(U3) $\log u(x^2)$ is convex on $[0,\infty)$.

Obviously, (U1) is a stronger condition than (U0).

Let $C_{+,\log}$ denote the set of all positive continuous functions u on $[0,\infty)$ satisfying the condition:

$$\lim_{r\to\infty} \frac{\log u(r)}{\log r} = \infty.$$

It is easy to see $C_{+,\frac{1}{2}} \subset C_{+,\log}$.

The *Legendre transform* ℓ_u of $u \in C_{+,\log}$ is defined to be the function

$$\ell_u(t) = \inf_{r>0} \frac{u(r)}{r^t}, \qquad t \in [0,\infty).$$

Some useful properties of the Legendre transform will be refered in section 3.

¿From now on, we take a function $u \in C_{+,\frac{1}{2}}$ satisfying (U0) (U2) (U3). We shall constract a Gel'fand triple associated with u. For $\varphi \in (L^2)$ being represented by Equation (2.1) and $p \geq 0$, define

$$\|\varphi\|_{p,u} = \left(\sum_{n=0}^{\infty} \frac{1}{\ell_u(n)} |f_n|_p^2 \right)^{1/2}. \tag{2.3}$$

Let $[\mathcal{E}_p]_u = \{\varphi \in (L^2);\ \|\varphi\|_{p,u} < \infty\}$. Define the space $[\mathcal{E}]_u$ of *test functions* on $\mathcal{E}'$ to be the projective limit of $\{[\mathcal{E}_p]_u;\ p \geq 0\}$. The dual space $[\mathcal{E}]_u^*$ of $[\mathcal{E}]_u$ is called the space of *generalized functions* on $\mathcal{E}'$.

Choose an appropriate p_0 such that $c\rho^{2p_0}\sqrt{2} \leq 1$ for some c. Then two conditions (a) and (U2) imply that $[\mathcal{E}_p]_u \subset (L^2)$ for all $p \geq p_0$. Hence $[\mathcal{E}]_u \subset (L^2)$ holds. By identifying (L^2) with its dual space we get the following continuous inclusions:

$$[\mathcal{E}]_u \subset [\mathcal{E}_p]_u \subset (L^2) \subset [\mathcal{E}_p]_u^* \subset [\mathcal{E}]_u^*, \qquad p \geq p_0,$$

where $[\mathcal{E}_p]_u^*$ is the dual space of $[\mathcal{E}_p]_u$. Moreover, $[\mathcal{E}]_u$ is a nuclear space and so $[\mathcal{E}]_u \subset (L^2) \subset [\mathcal{E}]_u^*$ is a Gel'fand triple. Note that $[\mathcal{E}]_u^* = \cup_{p\geq 0} [\mathcal{E}_p]_u^*$ and for $p \geq p_0$, $[\mathcal{E}_p]_u^*$ is the completion of (L^2) with respect to the norm

$$\|\varphi\|_{-p,(u)} = \left(\sum_{n=0}^{\infty} (n!)^2 \ell_u(n)\, |f_n|_{-p}^2 \right)^{1/2}. \tag{2.4}$$

For ξ belonging to the complexification $\mathcal{E}_c$ of $\mathcal{E}$, the renormalized exponential function $:e^{\langle\cdot,\xi\rangle}:$ is defined by

$$:e^{\langle\cdot,\xi\rangle}: = \sum_{n=0}^{\infty} \frac{1}{n!} \langle :\cdot^{\otimes n}:, \xi^{\otimes n}\rangle.$$

Then we have the norm estimate,

$$\| :e^{\langle\cdot,\xi\rangle}: \|_{-q,(u)}^2 = \sum_{n=0}^{\infty} \ell_u(n) |\xi|_{-q}^{2n} =: \mathcal{L}_u(|\xi|_{-q}^2). \tag{2.5}$$

For later uses, let us define the notion of *equivalent functions* here.

Definition 2.1. Two positive functions f and g on $[0,\infty)$ are called *equivalent* if there exist constants $c_1, c_2, a_1, a_2 > 0$ such that

$$c_1 f(a_1 r) \le g(r) \le c_2 f(a_2 r), \qquad \forall\, r \in [0,\infty).$$

Example 2.2.

$$g_k(r) = \exp\left[2\sqrt{r \log_{k-1} \sqrt{r}}\right], \tag{2.6}$$

where $\log_k(r)$ is given by

$$\log_1(r) = \log(\max\{e, r\}), \quad \log_k(r) = \log_1(\log_{k-1}(r)), \quad k \ge 2.$$

Then the function g_k belongs to $C_{+,1/2}$ and satisfies conditions (U1) (U2) (U3). In the sense of Definition 2.1, the function g_k is equivalent to the function given by

$$u_k(r) = \sum_{n=0}^{\infty} \frac{1}{b_k(n) n!} r^n$$

where $b_k(n)$ is the *k-th order Bell number.* Hence we get the Gel'fand triple,

$$[\mathcal{E}]_{g_k} \subset (L^2) \subset [\mathcal{E}]_{g_k}^*$$

known as the *CKS-space associated with* g_k, which is the same as the one defined by the k-th order Bell number $b_k(n)$. See more details in [1],[2],[3], [4],[5],[6],[15],[16].

Example 2.3. For $0 \le \beta < 1$, let u be the function defined by

$$u(r) = \exp\left[(1+\beta) r^{\frac{1}{1+\beta}}\right].$$

It is easy to check that u belongs to $C_{+,1/2}$ and satisfies conditions (U1) (U2) (U3). Hence this Gel'fand triple,

$$(\mathcal{E})_\beta \subset (L^2) \subset (\mathcal{E})_\beta^*$$

which is well-known as the *Hida-Kubo-Takenaka space* for $\beta = 0$ [9],[10],[17], [18],[22] and the *Kondratiev-Streit space* for a general β [12], [20]. For $\beta = 1$ case, see [11],[13],[14].

Remark. We have the following chain of Gel'fand triples:

$(\mathcal{E})_1 \subset [\mathcal{E}]_{g_k} \subset [\mathcal{E}]_{g_l} \subset (\mathcal{E})_\beta \subset (\mathcal{E})_\gamma \subset (L^2) \subset (\mathcal{E})_\gamma^* \subset (\mathcal{E})_\beta^* \subset [\mathcal{E}]_{g_l}^* \subset [\mathcal{E}]_{g_k}^* \subset (\mathcal{E})_1^*$
where $0 \leq \gamma \leq \beta < 1$ and $2 \leq l \leq k$.

3. Properties of Legendre transforms

First we mention the following notions of concave and convex functions which will be used frequently.

Definition 3.1. A positive function f on $[0, \infty)$ is called

(1) *log-concave* if the function $\log f$ is concave on $[0, \infty)$;
(2) *log-convex* if the function $\log f$ is convex on $[0, \infty)$;
(3) *(log, exp)-convex* if the function $\log f(e^x)$ is convex on $\mathbb{R}$;
(4) *(log, x^2)-convex* if the function $\log f(x^2)$ is convex on $[0, \infty)$.

We will need the fact that if f is log-concave, then the sequence $\{f(n)\}_{n=0}^\infty$ is log-concave. To check this fact, note that for any $t_1, t_2 \geq 0$ and $0 \leq \lambda \leq 1$,

$$f(\lambda t_1 + (1-\lambda)t_2) \geq f(t_1)^\lambda f(t_2)^{1-\lambda}.$$

In particular, take $t_1 = n, t_2 = n+2$, and $\lambda = 1/2$ to get

$$f(n)f(n+2) \leq f(n+1)^2, \qquad \forall\, n \geq 0.$$

Hence the sequence $\{f(n)\}_{n=0}^\infty$ is log-concave.

The next theorem is from Lemma 3.4 in [3].

Theorem 3.2. *Let $u \in C_{+,\log}$. Then the Legendre transform ℓ_u is log-concave. (Hence ℓ_u is continuous on $[0, \infty)$ and the sequence $\{\ell_u(n)\}_{n=0}^\infty$ is log-concave.)*

¿From Theorem 2 (b) in [1] we have the fact: If $\{\alpha(n)/n!\}_{n=0}^\infty$ is log-concave and $\alpha(0) = 1$, then

$$\alpha(n+m) \leq \binom{n+m}{n} \alpha(n)\alpha(m), \qquad \forall\, n, m \geq 0.$$

By Theorem 3.2 the sequence $\{\ell_u(n)\}$ is log-concave. Hence we can apply the above fact to the sequence $\alpha(n) = n!\ell_u(n)/\ell_u(0)$ to get the next theorem.

Theorem 3.3. *Let $u \in C_{+,\log}$. Then for all integers $n, m \geq 0$, we have*

$$\ell_u(0)\ell_u(n+m) \leq \ell_u(n)\ell_u(m).$$

In the next theorem we state some properties of the Legendre transform ℓ_u of a (log, exp)-convex function u in $C_{+,\log}$. It is from Lemmas 3.6 and 3.7 in [3].

Theorem 3.4. *Let $u \in C_{+,\log}$ be (log, exp)-convex. Then*

(1) *$\ell_u(t)$ is decreasing for large t,*
(2) $\lim_{t\to\infty} \ell_u(t)^{1/t} = 0$,

(3) $u(r) = \sup_{t\geq 0} \ell_u(t) r^t$ *for all* $r \geq 0$.

On the other hand, for a (log, x^2)-convex function u in $C_{+,\log}$, its Legendre transform ℓ_u has the properties in the next theorem from Lemmas 3.9 and 3.10 in [3]. If in addition u is increasing, then u is also (log, exp)-convex and hence ℓ_u has the properties in the above Theorem 3.4.

Theorem 3.5. *Let* $u \in C_{+,\log}$. *We have the assertions:*

(1) *u is (log, x^2)-convex if and only if $\ell_u(t)t^{2t}$ is log-convex.*

(2) *If u is (log, x^2)-convex, then for any integers $n, m \geq 0$,*

$$\ell_u(n)\ell_u(m) \leq \ell_u(0) 2^{2(n+m)} \ell_u(n+m).$$

Now, suppose $u \in C_{+,\log}$ and assume that $\lim_{n\to\infty} \ell_u(n)^{1/n} = 0$. We define the *L-function* $\mathcal{L}_u$ of u by

$$\mathcal{L}_u(r) = \sum_{n=0}^{\infty} \ell_u(n) r^n. \tag{3.1}$$

Note that $\mathcal{L}_u$ is an entire function. By Theorem 3.4 (2), ℓ_u is defined for any (log, exp)-convex function u in $\in C_{+,\log}$. Moreover, we have the next theorem from Theorem 3.13 in [3].

Theorem 3.6. *(1) Let $u \in C_{+,\log}$ be (log, exp)-convex. Then its L-function $\mathcal{L}_u$ is also (log, exp)-convex and for any $a > 1$,*

$$\mathcal{L}_u(r) \leq \frac{ea}{\log a} u(ar), \qquad \forall r \geq 0.$$

(2) Let $u \in C_{+,\log}$ be increasing and (log, x^2)-convex. Then there exists a constant C such that

$$u(r) \leq C\mathcal{L}_u(2^2 r), \qquad \forall r \geq 0.$$

Recall from Proposition 2.3 (3) in [3]: If f is increasing and (log, x^2)-convex for some $k > 0$, then f is (log, exp)-convex. Hence the above Theorem 3.6 yields the next theorem.

Theorem 3.7. *Let $u \in C_{+,\log}$ be increasing and (log, x^2)-convex. Then the functions u and $\mathcal{L}_u$ are equivalent.*

In the next section 4, we will consider the characterization of Hida measures (generalized measures). We prepare two lemmas for this purpose. The proof of Lemma 3.8 is simple application of Theorem 3.5 so that we just state it without proof.

Lemma 3.8. *Suppose $u \in C_{+,\log}$ is (log, x^2)-convex. Then*

$$\mathcal{L}_u(r)^2 \leq \ell_u(0)\mathcal{L}_u(2^3 r), \qquad \forall r \in [0, \infty). \tag{3.2}$$

Remark. Note that $\mathcal{L}_u(r) \geq \ell_u(0)$ for all $r \geq 0$. Hence we have

$$\ell_u(0)\mathcal{L}_u(r) \leq \mathcal{L}_u(r)^2 \leq \ell_u(0)\mathcal{L}_u(2^3 r), \qquad \forall r \in [0, \infty).$$

Thus $\mathcal{L}_u$ and $\mathcal{L}_u^2$ are equivalent for any (log, x^2)-convex function $u \in C_{+,\log}$. If, in addition, u is increasing, then u and $\mathcal{L}_u$ are equivalent by Theorem 3.7. It follows that u and u^2 are equivalent for such a function u.

The next Lemma 3.9 can be obtained from Theorem 3.8 and Lemma 3.6.

Lemma 3.9. *Suppose $u \in C_{+,\log}$ is increasing and (log, x^2)-convex. Then for any $a > 1$, we have*

$$\mathcal{L}_u(r) \leq \sqrt{\ell_u(0)\frac{ea}{\log a}}\, u\big(a2^3 r\big)^{1/2}. \tag{3.3}$$

4. Characterization of Hida Measures

Before going to the main theorem, we need to introduce another equivalent family of norms on $[\mathcal{E}]_u$, i.e., $\{\|\cdot\|_{\mathcal{A}_{p,u}};\, p \geq 0\}$. This family of norms is intrinsic in the sense that $\|\varphi\|_{\mathcal{A}_{p,u}}$ is defined directly in terms of the analyticity and growth condition of φ.

First, it is well-known that each test function φ in $[\mathcal{E}]_u$ has a unique analytic extension (see §6.3 of [20]) given by

$$\varphi(x) = \langle\!\langle :e^{\langle\cdot,x\rangle}:, \Theta\varphi\rangle\!\rangle, \qquad x \in \mathcal{E}_c', \tag{4.1}$$

where Θ is the unique linear operator taking $e^{\langle\cdot,\xi\rangle}$ into $:e^{\langle\cdot,\xi\rangle}:$ for all $\xi \in \mathcal{E}_c$. By Theorem 6.2 in [20] with minor modifications, Θ is shown to be a continuous linear operator from $[\mathcal{E}]_u$ into itself. Note that we still assume conditions (U0), (U2) and (U3) on u given in section 2.

Now, let $p \geq 0$ be any fixed number. Choose $p_1 > p$ such that $2\rho^{2(p_1-p)} \leq 1$. Then use Equations (4.1), (2.5) and Theorem 3.6 to get

$$|\varphi(x)| \leq \|\Theta\varphi\|_{p_1,u}\, \|:e^{\langle\cdot,x\rangle}:\|_{-p_1,(u)} \leq \|\Theta\varphi\|_{p_1,u}\sqrt{\frac{2e}{\log 2}}\, u\big(2|x|_{-p_1}^2\big)^{1/2}.$$

Note that $2|x|_{-p_1}^2 \leq 2\rho^{2(p_1-p)}|x|_{-p}^2 \leq |x|_{-p}^2$ by the above choice of p_1. Since u is an increasing function, we see that

$$|\varphi(x)| \leq \|\Theta\varphi\|_{p_1,u}\sqrt{\frac{2e}{\log 2}}\, u\big(|x|_{-p}^2\big)^{1/2}.$$

But Θ is a continuous linear operator from $[\mathcal{E}]_u$ into itself. Hence there exist positive constants q and $K_{p,q}$ such that $\|\Theta\varphi\|_{p_1,u} \leq K_{p,q}\|\varphi\|_{q,u}$. Therefore,

$$|\varphi(x)| \leq C_{p,q}\|\varphi\|_{q,u}\, u\big(|x|_{-p}^2\big)^{1/2}, \qquad x \in \mathcal{E}_{p,c}', \tag{4.2}$$

where $C_{p,q} = K_{p,q}\sqrt{2e/\log 2}$. This is the growth condition for test functions.

Being motivated by Equation (4.2), we define

$$\|\varphi\|_{\mathcal{A}_{p,u}} = \sup_{x\in\mathcal{E}'_{p,c}} |\varphi(x)|\, u\big(|x|_{-p}^2\big)^{-1/2}. \tag{4.3}$$

Obviously, $\|\cdot\|_{\mathcal{A}_{p,u}}$ is a norm on $[\mathcal{E}]_u$ for each $p \geq 0$.

Theorem 4.1. *Suppose $u \in C_{+,1/2}$ satisfies conditions (U1) (U2) (U3). Then the families of norms $\{\|\cdot\|_{\mathcal{A}_{p,u}}; p \geq 0\}$ and $\{\|\cdot\|_{p,u}; p \geq 0\}$ are equivalent, i.e., they generate the same topology on $[\mathcal{E}]_u$.*

Remark. This theorem gives an alternative construction of test functions. This idea is due to Lee [21], see also §15.2 of [20]. For $p \geq 0$, let $\mathcal{A}_{p,u}$ consist of all functions φ on $\mathcal{E}'_c$ satisfying the conditions:

(a) φ is an analytic function on $\mathcal{E}'_{p,c}$.
(b) There exists a constant $C \geq 0$ such that

$$|\varphi(x)| \leq Cu\big(|x|_{-p}^2\big)^{1/2}, \qquad \forall x \in \mathcal{E}'_{p,c}.$$

For each $\varphi \in \mathcal{A}_{p,u}$, define $\|\varphi\|_{\mathcal{A}_{p,u}}$ by Equation (4.3). Then $\mathcal{A}_{p,u}$ is a Banach space with norm $\|\cdot\|_{\mathcal{A}_{p,u}}$. Let $\mathcal{A}_u$ be the projective limit of $\{\mathcal{A}_{p,u}; p \geq 0\}$. We can use the above theorem to conclude that $\mathcal{A}_u = [\mathcal{E}]_u$ as vector spaces with the same topology. Here the equality $\mathcal{A}_u = [\mathcal{E}]_u$ requires the use of analytic extensions of test functions in $[\mathcal{E}]_u$, which exists in view of Equation (4.1).

Proof. Let $p \geq 0$ be any given number. We have already shown that there exist constants $q > p$ and $C_{p,q} \geq 0$ such that Equation (4.2) holds. It follows that

$$\|\varphi\|_{\mathcal{A}_{p,u}} = \sup_{x\in\mathcal{E}'_{p,c}} |\varphi(x)|\, u\big(|x|_{-p}^2\big)^{-1/2} \leq C_{p,q}\|\varphi\|_{q,u}.$$

Hence for any $p \geq 0$, there exist constants $q > p$ and $C_{p,q} \geq 0$ such that

$$\|\varphi\|_{\mathcal{A}_{p,u}} \leq C_{p,q}\|\varphi\|_{q,u}, \qquad \forall \varphi \in [\mathcal{E}]_u. \tag{4.4}$$

To show the converse, first note that by condition (U2) there exist constants $c_1, c_2 > 0$ such that $u(r) \leq c_1 e^{c_2 r}$, $r \geq 0$. Next note that by Fernique's theorem (see [7], [19], [20]) we have

$$\int_{\mathcal{E}'} e^{2c_2|x|_{-\lambda}^2}\, d\mu(x) < \infty \qquad \text{for all large } \lambda.$$

Now, let $p \geq 0$ be any given number. Choose $q > p$ large enough such that

$$4e^2\|i_{q,p}\|_{HS}^2 < 1, \qquad \int_{\mathcal{E}'} e^{2c_2|x|_{-q}^2}\, d\mu(x) < \infty. \tag{4.5}$$

With this choice of q we will show below that

$$\|\varphi\|_{p,u} \leq L_{p,q}\|\varphi\|_{\mathcal{A}_{q,u}}, \qquad \forall \varphi \in [\mathcal{E}]_u, \tag{4.6}$$

where $L_{p,q}$ is the constant given by

$$L_{p,q} = \sqrt{c_1}\left(1 - 4e^2\|i_{q,p}\|_{HS}^2\right)^{-1/2} \int_{\mathcal{E}'} e^{2c_2|x|_{-q}^2}\, d\mu(x). \tag{4.7}$$

Observe that the theorem follows from Equations (4.4) and (4.6).

Finally, we prove Equation (4.6). Let $\varphi \in [\mathcal{E}]_u$. Then we can use an integral form of S-transform (see [20]) given by

$$F(\xi) = S\varphi(\xi) = \int_{\mathcal{E}'} \varphi(x+\xi)\, d\mu(x), \qquad \xi \in \mathcal{E}_c.$$

Hence for the above choice of q, we have

$$\begin{aligned}
|F(\xi)| &\le \int_{\mathcal{E}'} |\varphi(x+\xi)|\, d\mu(x) \\
&\le \int_{\mathcal{E}'} \left(|\varphi(x+\xi)|\, u\big(|x+\xi|_{-q}^2\big)^{-1/2}\right) u\big(|x+\xi|_{-q}^2\big)^{1/2}\, d\mu(x) \\
&\le \|\varphi\|_{\mathcal{A}_{q,u}} \int_{\mathcal{E}'} u\big(|x+\xi|_{-q}^2\big)^{1/2}\, d\mu(x).
\end{aligned}$$

Here by condition (U1), we have $u(r)^{1/2} \le u(r)$ for all $r \ge 0$. Therefore,

$$|F(\xi)| \le \|\varphi\|_{\mathcal{A}_{q,u}} \int_{\mathcal{E}'} u\big(|x+\xi|_{-q}^2\big)\, d\mu(x). \tag{4.8}$$

By condition (U3), we have

$$u\left(\left(\tfrac{1}{2}r_1 + \tfrac{1}{2}r_2\right)^2\right) \le u\big(r_1^2\big)^{1/2} u\big(r_2^2\big)^{1/2}, \qquad \forall r_1, r_2 \ge 0.$$

Put $r_1 = 2|x|_{-q}$ and $r_2 = 2|\xi|_{-q}$ to get

$$\begin{aligned}
u\big(|x+\xi|_{-q}^2\big) &\le u\left(\left(\tfrac{1}{2}2|x|_{-q} + \tfrac{1}{2}2|\xi|_{-q}\right)^2\right) \\
&\le u\big(4|x|_{-q}^2\big)^{1/2} u\big(4|\xi|_{-q}^2\big)^{1/2}.
\end{aligned}$$

Then integrate over $\mathcal{E}'$ to obtain the inequality:

$$\int_{\mathcal{E}'} u\big(|x+\xi|_{-q}^2\big)\, d\mu(x) \le u\big(4|\xi|_{-q}^2\big)^{1/2} \int_{\mathcal{E}'} u\big(4|x|_{-q}^2\big)^{1/2}\, d\mu(x). \tag{4.9}$$

Put Equation (4.9) into Equation (4.8) to get

$$|F(\xi)| \le \|\varphi\|_{\mathcal{A}_{q,u}} u\big(4|\xi|_{-q}^2\big)^{1/2} \int_{\mathcal{E}'} u\big(4|x|_{-q}^2\big)^{1/2}\, d\mu(x). \tag{4.10}$$

Now, by the inequality $u(r) \le c_1 e^{c_2 r}$, we have

$$\int_{\mathcal{E}'} u\big(4|x|_{-q}^2\big)^{1/2}\, d\mu(x) \le \sqrt{c_1} \int_{\mathcal{E}'} e^{2c_2|x|_{-q}^2}\, d\mu(x), \tag{4.11}$$

which is finite by the choice of q in Equation (4.5).

¿From Equations (4.10) and (4.11), we see that

$$|F(\xi)| \le \|\varphi\|_{\mathcal{A}_{q,u}} \sqrt{c_1} \left(\int_{\mathcal{E}'} e^{2c_2|x|_{-q}^2} \, d\mu(x) \right) u\big(4|\xi|_{-q}^2\big)^{1/2}, \qquad \xi \in \mathcal{E}_c.$$

With this inequality and the choice of q in Equation (4.5) we can apply Lemma 4.2 (see below) and Equation (2.3) to show that for any $\varphi \in [\mathcal{E}]_u$,

$$\|\varphi\|_{q,u} \le L_{p,q} \|\varphi\|_{\mathcal{A}_{q,u}},$$

where $L_{p,q}$ is given by Equation(4.7). Thus the inequality in Equation (4.6) holds and so the proof is completed. □

In the proof of the prevous theorem, we have used the next lemma from [3].

Lemma 4.2 ([3]). *Suppose $u \in C_{+,1/2}$ satisfies conditions (U1) (U2) (U3). Let F be a complex-valued function on $\mathcal{E}_c$ satisfying the conditions:*

(1) *For any $\xi, \eta \in \mathcal{E}_c$, the function $F(z\xi + \eta)$ is an entire function of $z \in \mathbb{C}$.*
(2) *There exist constants $K, a, p \ge 0$ such that*

$$|F(\xi)| \le K u\big(a|\xi|_{-p}^2\big)^{1/2}, \qquad \xi \in \mathcal{E}_c.$$

Let $q \in [0, p)$ be a number such that $ae^2 \|i_{p,q}\|_{HS}^2 < 1$. Then there exist functions $f_n \in \mathcal{E}_{q,\mathbb{C}}^{\widehat{\otimes}}$ such that $F(\xi) = \sum_{n=0}^{\infty} \langle f_n, \xi^{\widehat{\otimes}} \rangle$ and

$$|f_n|_q^2 \le K(ae^2 \|i_{p,q}\|_{HS}^2)^n \ell_u(n). \tag{4.12}$$

Definition 4.3. A measure ν on $\mathcal{E}'$ is called a *Hida measure* associated with u if $[\mathcal{E}]_u \subset L^1(\nu)$ and the linear functional $\varphi \mapsto \int_{\mathcal{E}'} \varphi(x)\, d\nu(x)$ is continuous on $[\mathcal{E}]_u$.

In this case, ν induces a generalized function, denoted by $\widetilde{\nu}$, in $[\mathcal{E}]_u^*$ such that

$$\langle\!\langle \widetilde{\nu}, \varphi \rangle\!\rangle = \int_{\mathcal{E}'} \varphi(x) \, d\nu(x), \qquad \varphi \in [\mathcal{E}]_u. \tag{4.13}$$

Theorem 4.4. *Suppose $u \in C_{+,1/2}$ satisfies conditions (U1) (U2) (U3). Then a measure ν on $\mathcal{E}'$ is a Hida measure with $\widetilde{\nu} \in [\mathcal{E}]_u^*$ if and only if ν is supported by $\mathcal{E}_p'$ for some $p \ge 0$ and*

$$\int_{\mathcal{E}_p'} u\big(|x|_{-p}^2\big)^{1/2} \, d\nu(x) < \infty. \tag{4.14}$$

Remarks. (a) The integrability condition in the theorem can be replaced by

$$\int_{\mathcal{E}_p'} u\big(|x|_{-p}^2\big) \, d\nu(x) < \infty.$$

To verify this fact, just note that u and u^2 are equivalent (from the Remark of Lemma 3.8) and $|x|_{-q} \le \rho^{q-p} |x|_{-p}$ for $0 \le p \le q$ and $x \in \mathcal{E}_p'$.

(b) This theorem is due to Lee [21] for the case $u(r) = e^r$. See §15.2 of the book [20] for the case $u(r) = \exp\left[(1+\beta)r^{\frac{1}{1+\beta}}\right]$, $0 \le \beta < 1$. In the case of $\beta = 1$, we need special treatment since our Legendre transform method should be modified. In order to take care of $\beta = 1$ case, we have to remove the assumption

$$\lim_{r\to\infty} \frac{\log u(r)}{\sqrt{r}} = \infty$$

on u introduced in §2, for example. It will be discussed in the future. On the other hand, there are references [13],[14] discussed this case with a diffrent way from our point of view.

Proof. To prove the sufficiency, suppose ν is supported by $\mathcal{E}'_p$ for some $p \ge 0$ and Equation (4.14) holds. Then for any $\varphi \in [\mathcal{E}]_u$,

$$\begin{aligned}\int_{\mathcal{E}'} |\varphi(x)|\, d\nu(x) &= \int_{\mathcal{E}'_p} |\varphi(x)|\, d\nu(x) \\ &= \int_{\mathcal{E}'_p} \left(|\varphi(x)| u\big(|x|^2_{-p}\big)^{-1/2}\right) u\big(|x|^2_{-p}\big)^{1/2}\, d\nu(x) \\ &\le \|\varphi\|_{\mathcal{A}_{p,u}} \int_{\mathcal{E}'_p} u\big(|x|^2_{-p}\big)^{1/2}\, d\nu(x). \end{aligned} \tag{4.15}$$

By Theorem 4.1, $\{\|\cdot\|_{\mathcal{A}_{p,u}};\, p \ge 0\}$ and $\{\|\cdot\|_{p,u};\, p \ge 0\}$ are equivalent. Hence Equation (4.15) implies that $[\mathcal{E}]_u \subset L^1(\nu)$ and the linear functional

$$\varphi \longmapsto \int_{\mathcal{E}'} \varphi(x)\, d\nu(x), \qquad \varphi \in [\mathcal{E}]_u,$$

is continuous on $[\mathcal{E}]_u$. Thus ν is a Hida measure with $\widetilde{\nu}$ in $[\mathcal{E}]^*_u$.

To prove the necessity, suppose ν is a Hida measure inducing a generalized function $\widetilde{\nu} \in [\mathcal{E}]^*_u$. Then for all $\varphi \in [\mathcal{E}]_u$,

$$\langle\!\langle \widetilde{\nu}, \varphi \rangle\!\rangle = \int_{\mathcal{E}'} \varphi(x)\, d\nu(x). \tag{4.16}$$

Since $\{\|\cdot\|_{\mathcal{A}_{p,u}};\, p \ge 0\}$ and $\{\|\cdot\|_{p,u};\, p \ge 0\}$ are equivalent, the linear functional $\varphi \mapsto \langle\!\langle \widetilde{\nu}, \varphi \rangle\!\rangle$ is continuous with respect to $\{\|\cdot\|_{\mathcal{A}_{p,u}};\, p \ge 0\}$. Hence there exist constants $K, q \ge 0$ such that for all $\varphi \in [\mathcal{E}]_u$,

$$\big|\langle\!\langle \widetilde{\nu}, \varphi \rangle\!\rangle\big| \le K \|\varphi\|_{\mathcal{A}_{q,u}}. \tag{4.17}$$

Note that by continuity, Equations (4.16) and (4.17) also hold for all $\varphi \in \mathcal{A}_{q,u}$ defined in the Remark of Theorem 4.1.

Now, with this q, we define a function θ on $\mathcal{E}'_{q,c}$ by

$$\theta(x) = \mathcal{L}_u\big(2^{-4}\langle x, x\rangle_{-q}\big), \qquad x \in \mathcal{E}'_{q,c},$$

where $\langle\cdot,\cdot\rangle_{-q}$ is the bilinear pairing on $\mathcal{E}'_{q,c}$. Obviously, θ is analytic on $\mathcal{E}'_{q,c}$. On the other hand, apply Lemma 3.9 with $a = k = 2$ to get

$$|\theta(x)| \le \mathcal{L}_u\big(2^{-4}|x|^2_{-q}\big) \le \sqrt{\frac{2e}{\log 2}}\, u\big(|x|^2_{-q}\big)^{1/2}, \qquad \forall x \in \mathcal{E}'_{q,c}.$$

This shows that $\theta \in \mathcal{A}_{q,u}$ and we have

$$\|\theta\|_{\mathcal{A}_{q,u}} \le \sqrt{\frac{2e}{\log 2}}.$$

Then apply Equation (4.17) to the function θ,

$$\big|\langle\!\langle \widetilde{\nu}, \theta\rangle\!\rangle\big| \le K\|\theta\|_{\mathcal{A}_{q,u}} \le K\sqrt{\frac{2e}{\log 2}}.$$

Therefore, from Equation (4.16) with $\varphi = \theta$ we conclude that

$$\left|\int_{\mathcal{E}'} \theta(x)\, d\nu(x)\right| \le K\sqrt{\frac{2e}{\log 2}}. \tag{4.18}$$

Note that $\theta(x) = \mathcal{L}_u\big(2^{-4}|x|^2_{-q}\big)$ for $x \in \mathcal{E}'$. Hence Equation (4.18) implies that

$$\int_{\mathcal{E}'} \mathcal{L}_u\big(2^{-4}|x|^2_{-q}\big)\, d\nu(x) < \infty.$$

But $u(r) \le C\mathcal{L}_u(4r)$ from Theorem 3.6 (2) with $k = 2$. Therefore,

$$\int_{\mathcal{E}'} u\big(2^{-6}|x|^2_{-q}\big)\, d\nu(x) < \infty.$$

Now, choose $p > q$ large enough such that $\rho^{2(p-q)} \le 2^{-6}$. Then $|x|^2_{-p} \le 2^{-6}|x|^2_{-q}$. Recall that u is increasing. Hence

$$\int_{\mathcal{E}'} u\big(|x|^2_{-p}\big)\, d\nu(x) < \infty.$$

Note that $u(r) \ge 1$ and so $u(r)^{1/2}(r) \le u(r)$. Thus we conclude that

$$\int_{\mathcal{E}'} u\big(|x|^2_{-p}\big)^{1/2}\, d\nu(x) < \infty.$$

This inequality implies that ν is supported by $\mathcal{E}'_p$ and Equation (4.14) holds. □

Example 4.5. (Poisson noise measure)
Let $\mathcal{P}$ be the Poisson measure on $\mathcal{E}^*$ given by

$$\exp\Big(\int_{\mathbb{R}} (e^{i\xi(t)} - 1)dt\Big) = \int_{\mathcal{E}^*} e^{i\langle x,\xi\rangle} \mathcal{P}(dx), \quad \xi \in \mathcal{E}^*.$$

It has been shown [6] that the Poisson noise measure induces a generalized function in $[\mathcal{E}]^*_{g_2}$. Thus by Theorem 4.4 and Example 2.2 we have the integrability condition

$$\int_{\mathcal{E}_p^*} \exp\Big(|x|_{-p}\sqrt{\log|x|_{-p}}\Big)\mathcal{P}(dx) < \infty$$

for some p.

Example 4.6. (Grey noise measure)
Let $0 < \lambda \leq 1$. The grey noise measure on $\mathcal{E}^*$ is the measure ν_λ having the characteristic function

$$L_\lambda(|\xi|_0^2) = \int_{\mathcal{E}^*} e^{i\langle x,\xi\rangle}\nu_\lambda(dx), \quad \xi \in \mathcal{E},$$

where $L_\lambda(t)$ is the Mittag-Leffler function with parameter λ;

$$L_\lambda(t) = \sum_{n=0}^{\infty} \frac{(-t)^n}{\Gamma(1+\lambda n)}.$$

Here Γ is the Gamma function. This measure was introduced by Schneider [23]. It is shown in [20] that ν_λ is a Hida measure which induces a generalized function Φ_{ν_λ} in $(\mathcal{E})_{1-\lambda}^*$. Therefore by Theorem 4.4 and Example 2.3 the grey noise measure ν_λ satisfies

$$\int_{\mathcal{E}_p^*} \exp\big(\frac{1}{2}(2-\lambda)|x|_{-p}^{\frac{2}{2-\lambda}}\big)\nu_\lambda(dx) < \infty$$

for some p.

References

[1] Asai, N., Kubo, I., and Kuo, H.-H.: Bell numbers, log-concavity, and log-convexity; in: *Recent Developments in Stochastic Analysis*, L. Accardi et al. (eds.) Kluwer Academic Publishers, 1999

[2] Asai, N., Kubo, I., and Kuo, H.-H.: Characterization of test functions in CKS-space; in: *Proc. International Conference on Mathematical Physics and Stochastic Processes*, S. Albeverio et al. (eds.) World Scientific, 1999

[3] Asai, N., Kubo, I., and Kuo, H.-H.: Log-concavity, log-convexity, and growth order in white noise analysis; *Madeira Preprint* **37** (1999)

[4] Asai, N., Kubo, I., and Kuo, H.-H.: CKS-space in terms of growth functions; in *Quantum Information II*, K. Saito et al. (eds.) World Scientific, 2000

[5] Asai, N., Kubo, I., and Kuo, H.-H.: General characterization theorems and intrinsic topologies in white noise analysis; *Preprint* (1999)

[6] Cochran, W. G., Kuo, H.-H., and Sengupta, A.: A new class of white noise generalized functions; *Infinite Dimensional Analysis, Quantum Probability and Related Topics* **1** (1998) 43–67

[7] Fernique, M. X.: Intégrabilité des vecteurs Gaussiens; *Académie des Sciences, Paris, Comptes Rendus* **270** Séries A (1970) 1698–1699

[8] Gannoun, R., Hachaichi, R., Ouerdiane, H., and Rezgui, A.: Un Thérème de Dualité Entre Espaces de Fonctions Holomorphes à Croissance Exponentielle; *BiBoS preprint*, no. 829 (1998)

[9] Hida, T., Kuo, H.-H., Potthoff, J., and Streit, L.: *White Noise: An Infinite Dimensional Calculus*. Kluwer Academic Publishers, 1993

[10] Kontratiev, Yu. G.: Nuclear spaces of entire functions in problems of infinite dimensional analysis; *Soviet Math. Dokl.* **22** (1980) 588–592

[11] Kontratiev, Yu. G., Leukert, P and Streit, L.: Wick calculus in Gaussian analysis; *Acta Appl. Math.* **44** (1996) 269–294

[12] Kondratiev, Yu. G. and Streit, L.: Spaces of white noise distributions: Constructions, Descriptions, Applications. I; *Reports on Math. Phys.* **33** (1993) 341–366

[13] Kondratiev, Yu. G., Streit, L., Westerkamp, W.: A note on positive distributions in Gaussian analysis; *Ukrainean Math. J.* **47** (1995) 749–759

[14] Kondratiev, Yu. G., Streit, L., Westerkamp, W. and Yan, J.-A: Generalized functions in infinite dimensional analysis; *Hiroshima math. J.* **28** (1998) 213–260

[15] Kubo, I.: Entire functionals and generalized functionals in white noise analysis; in:"*Analysis on Infinite-Dimensional Lie Groups and Algebras,*" H. Heyer and J. Marion (eds.) World Scientific, 1998

[16] Kubo, I., Kuo, H.-H., and Sengupta, A.: White noise analysis on a new space of Hida distributions; *Infinite Dimensional Analysis, Quantum Probability and Related Topics* **2** (1999) 315–335

[17] Kubo, I. and Takenaka, S.: Calculus on Gaussian white noise I; *Proc. Japan Acad.* **56A** (1980) 376–380

[18] Kubo, I. and Takenaka, S.: Calculus on Gaussian white noise II; *Proc. Japan Acad.* **56A** (1980) 411–416

[19] Kuo, H.-H.: *Gaussian Measures in Banach Spaces.* Lecture Notes in Math. **463**, Springer-Verlag, 1975

[20] Kuo, H.-H.: *White Noise Distribution Theory.* CRC Press, 1996

[21] Lee, Y.-J.: Analytic version of test functionals, Fourier transform and a characterization of measures in white noise calculus; *J. Funct. Anal.* **100** (1991) 359–380

[22] Obata, N.: *White Noise Calculus and Fock Space.* Lecture Notes in Math. **1577**, Springer-Verlag, 1994

[23] Schneider, W. R.: Grey noise; in: *Stochastic Processes, Physics and Geometry,* S. Albeverio et al. (eds.) World Scientific, 1990

ON STOCHASTIC GENERATORS OF POSITIVE DEFINITE EXPONENTS.

V. P. BELAVKIN.

ABSTRACT. A characterisation of quantum stochastic positive definite (PD) exponent is given in terms of the conditional positive definiteness (CPD) of their form-generator. The pseudo-Hilbert dilation of the stochastic form-generator and the pre-Hilbert dilation of the corresponding dissipator is found. The structure of quasi-Poisson stochastic generators giving rise to a quantum stochastic birth processes is studied.

1. INTRODUCTION

Quantum probability theory provides examples of positive-definite (PD) infinitely-divisible functions on non-Abelian groups which serve as characteristic functions of quantum chaotic states, generalizing the characteristic functions of classical stochastic processes with independent increments. The simplest examples are given by quantum point processes [1] which are characterized by analytical functions on the unit ball $B = \{y \in \mathcal{B} : \|y\| \leq 1\}$ of a non-commutative group C*-algebra. Such processes generate Markov quantum dynamics by one-parameter families $\phi = (\phi_t)_{t>0}$ of nonlinear completely positive maps $\phi_t : B \to \mathcal{A}$ on the unit ball of a C*-algebra $\mathcal{B}$, into an operator algebra $\mathcal{A}$ of a Hilbert space $\mathcal{H}$. As in the linear case, an analytical map ϕ_t is completely positive iff it is positive definite (PD),

$$(1.1) \qquad \sum_{x,z\in B} \langle \eta^x | \phi(x^\star z) \eta^z \rangle := \sum_{i,k} \langle \eta_i | \phi(y_i^\star y_k) \eta_k \rangle \geq 0, \quad \forall \eta_j \in \mathcal{H}, y_j \in B,$$

where $\eta^y = \eta_j \neq 0$ only for $y = y_j, j = 1, 2, \dots$. The simplest quantum point dynamics of this kind is given by the quantum Markov birth process which is described by the one-parameter semigroup

$$\phi_s(y)\phi_r(y) = \phi_{s+r}(y), \quad \phi_0(y) = 1, \quad y \in B$$

of infinitely divisible bounded PD functions $\phi_t : B \to \mathbb{C}$ with the normalization property $\phi_t(1) = 1$, where $1 \in B$ is (approximative) identity of $\mathcal{B}$. The continuity of the semigroup ϕ suggests the exponential form $\phi_t(y) = \exp[t\lambda(y)]$ of the functions ϕ_t . The corresponding analytic generator

$$\lambda(y) = \frac{1}{t}\ln \phi_t(y) := \lim_{t\searrow 0} \frac{1}{t}(\phi_t(y) - 1)$$

of such semigroup is conditionally completely definite (CPD), and this is equivalent to the PD property (1.1) for $\phi = \lambda$ under the condition $\sum_j \eta^j = 0$ and $\lambda(1) = 0$

Date: April 2000.

1991 *Mathematics Subject Classification.* Stochastic Analysis and Probability Theory.

Key words and phrases. Quantum Stochastic Processes, Independent Increments, Quantum Itô algebras, Infinitely Divisible States, Conditionally Positive Functionals..

This work was supported by Royal Society grant for the UK-Japan collaboration.

. The CPD functions have been studied in [2] and the corresponding dilations $\phi_t(y) = \langle \pi_t(y) \rangle$ to the multiplicative stochastic exponents $\pi_t(y) =: \exp \Lambda(t,y)$: of a quantum process $\Lambda(t,y)$ with independent increments and the vacuum mean $\langle \Lambda(t,y) \rangle = t\lambda(y)$ in Fock space were obtained in [3, 4]. The unital $\star$ -multiplicative property

$$\pi_t(x^\star z) = \pi_t(x)^\dagger \pi_t(z), \quad \pi_t(1) = I,$$

obviously implies the PD (1.1) of $\phi = \pi_t$, and the stationarity of the increments $\Lambda^s(t) = \Lambda(t+s) - \Lambda(s)$ implies the cocycle exponential property

$$\pi_s(y)\pi_r^s(y) = \pi_{r+s}(y), \quad \forall r, s > 0,$$

with respect to the natural time-shift $\pi \mapsto \pi^s$ in the Fock space of the representation π . The dilation of the CPD generators λ over the suggests their general form $\lambda(y) = \varphi(y) - \kappa$, where φ is a PD function on B with $\varphi(0) = 0$ and $\kappa = \varphi(1)$.

Here we shall extend this dilation theorem to the stochastic PD families ϕ satisfying the cocycle exponential property

$$\phi_s(y)\phi_r^s(y) = \phi_{r+s}(y), \quad \forall r, s > 0,$$

but not yet the unital multiplicative property. In particular, we shall obtain the structure of the stochastic form-generator for a family ϕ of PD functions $\phi_t(\omega)$: $B \to \mathbb{C}$, given as the adapted stochastic process $\phi_t(\omega, y)$ for each $y \in B$ with respect to a classical process $\omega = \{\omega(t)\}$ with independent increments, and having the cocycle exponential property with respect to the time-shift $\phi_t^s(\omega) = \phi_t(\omega^s)$, $\omega^s = \{\omega(t+s)\}$. Such stochastic functions can be unbounded, but they are usually normalized, $\phi_t(\omega, 1) = m_t(\omega)$, to a positive-valued process $m_t \geq 0$, having the martingale property

$$m_t(\omega) = \epsilon_t[m_s](\omega), \quad \forall s > t, \quad m_0(\omega) = 1,$$

where ϵ_t is the conditional expectation with respect to the history of the process ω up to time t . As follows from our dilation theorem, for example the stochastic exponent

$$\phi_t(y) = (1 + \alpha(y))^{p(t)} \exp[t\lambda(y)]$$

with respect to the standard Poisson process $p(t,\omega)$ is PD and normalized in the mean iff $1+\alpha$ and $\kappa + \lambda$ are PD for a $\kappa \geq 0$, and $\alpha(1) + \lambda(1) = 0$.

2. The Generators of Quantum Stochastic PD Exponents.

Let us consider a (noncommutative) Itô $\mathfrak{b}$ -algebra $\mathfrak{a}$ [4, 5], i.e. an associative $\star$ -algebra, identified with the algebra of quadruples $a = (a_\nu^\mu)_{\nu=+,\bullet}^{\mu=-,\bullet}$,

$$a_\bullet^\bullet = i(a), \quad a_+^\bullet = k(a), \quad a_\bullet^- = k^\bullet(a), \quad a_+^- = l(a),$$

under the product $ba = (b_\bullet^\mu a_\nu^\bullet)$ and the involution $a \mapsto b = a^\star \in \mathfrak{a}$, $b^\star = a$, represented by the quadruples $b = a^\flat$ with $b_{-\nu}^\mu = a_{-\mu}^{\nu\dagger}$, where $-\pm = \mp$, $-\bullet = \bullet$. Here $i(b)k(a) = k(ba)$ is the GNS $\star$ -representation $i(a^\star) = i(a)^\dagger$ associated with a linear positive $\star$ -functional $l : \mathfrak{a} \mapsto \mathbb{C}$, $l(a^\star) = l(a)^*$, and $k^\bullet(a^\star) = k(a)^\dagger$ is the linear functional on the pre-Hilbert space $\mathcal{K}$ of the Kolmogorov decomposition $l(a^\star a) = k(a)^\dagger k(a)$ of the functional l , separating $\mathfrak{a}$ in the sense $a = 0 \Leftrightarrow i(a) = k(a) = l(a) = 0$.

Let B denote a (noncommutative) semigroup with identity $1 \in B$ and involution $y \mapsto y^\star \in B$, $(x^\star z)^\star = z^\star x, \forall x, y, z \in B$, say, a (noncommutative) group with $y^\star = y^{-1}$, or the unital semigroup $B = 1 \oplus \mathfrak{b}$ of a $\star$-algebra $\mathfrak{b}$ with $(1 \oplus a)^\star (1 \oplus c) = 1 \oplus a \star c$, where $a \star c = c + a^\star c + a^\star$ for $a, c \in \mathfrak{b}$. The stochastically differentiable operator-valued exponent $\phi_t(y)$ over B with respect to a quantum stationary process, with independent increments $\Lambda^s(t) = \Lambda(t+s) - \Lambda(s)$ generated by a separable Itô algebra $\mathfrak{a}$ is described by the quantum stochastic equation

$$\text{(2.1)} \qquad \mathrm{d}\phi_t(y) = \phi_t(y)\,\alpha(y)\,\mathrm{d}A(t) := \phi_t(y) \sum_{\mu,\nu} \alpha^\mu_\nu(y)\,\mathrm{d}A^\nu_\mu, \qquad y \in B$$

with the initial condition $\phi_0(y) = I$, for all $y \in B$. Here $\alpha(y) \in \mathfrak{a}$ is given by the quadruple $\alpha^\bullet_\bullet = [\alpha^m_n]$, $\alpha^\bullet_- = [\alpha^m_+]$, $\alpha^-_\bullet = [\alpha^-_n]$, α^-_+ of complex functions $\alpha^\mu_\nu : B \to \mathbb{C}$, $\mu \in \{-,1,2,...\}$, $\nu \in \{+,1,2,...\}$ and $A = (A^\nu_\mu)^{\nu=+,\bullet}_{\mu=-,\bullet}$ is the quadruple of the canonical integrators given by the standard time $A^+_-(t) = tI$, annihilation $A^n_-(t)$, creation $A^+_m(t)$ and exchange $A^n_m(t)$ operators in Fock space over $L^2(\mathbb{R}_+ \times \mathbb{N})$ with $m, n \in \mathbb{N} = \{1,2,...\}$. The infinitesimal increments $\mathrm{d}A^\nu_\mu = A^{t\nu}_\mu(\mathrm{d}t)$ are formally defined by the Hudson-Parthasarathy multiplication table [6] and the $\flat$-property [4],

$$\text{(2.2)} \qquad \mathrm{d}A^\beta_\mu \mathrm{d}A^\nu_\gamma = \delta^\beta_\gamma \mathrm{d}A^\nu_\mu, \qquad \boldsymbol{A}^\flat = \boldsymbol{A},$$

where δ^β_γ is the usual Kronecker delta restricted to the indices $\beta \in \{-,1,2,...\}$, $\gamma \in \{+,1,2,...\}$ and $A^{\flat\mu}_{-\nu} = A^{\nu\dagger}_{-\mu}$ with respect to the reflection of the indices $(-,+)$ only. The structural functions α^μ_ν for the $*$-cocycles $\phi^*_t = \phi_t$, where $\phi^*_t(y) = \phi_t(y^\star)^\dagger$ should obviously satisfy the $\flat$-property $\alpha^\flat = \alpha$, where $\alpha^{\flat\nu}_{-\mu} = \alpha^{\mu *}_{-\nu}$, $\alpha^{\mu *}_\nu(y) = \alpha^\mu_\nu(y^\star)^\dagger$ even in the case of nonlinear α^μ_ν. The summation in (2.1) is defined as a quantum stochastic differential [4] if $\sum_{n=1}^\infty \alpha^-_n(y^\star)\alpha^n_+(y) < \infty$ and the matrix $[\alpha^m_n(y)], m, n \in \mathbb{N}$ represents a bounded operator in the Hilbert space $\ell^2_\mathbb{N} = \{\zeta^\bullet : \mathbb{N} \to \mathbb{C} | \sum_{n=1}^\infty |\zeta^n|^2 < \infty\}$ for each $y \in B$. If the coefficients α^μ_ν are independent of t, ϕ satisfies the cocycle property $\phi_s(y)\phi^s_r(y) = \phi_{s+r}(y)$, where ϕ^s_t is the solution to (1) with $A^\mu_\nu(t)$ replaced by $A^{s\mu}_\nu(t)$. Define the tensors $a^\mu_\nu = \alpha^\mu_\nu(y)$ also for $\mu = +$ and $\nu = -$, by

$$\alpha^+_\nu(y) = 0 = \alpha^\mu_-(y), \qquad \forall y \in B,$$

and then one can extend the summation in (2.1) to the trace of the quadratic matrices $\mathbf{a} = [a^\mu_\nu]$ so it is also over $\mu = +$, and $\nu = -$. By such an extension the multiplication table for $\mathrm{d}A(\mathbf{a}) = \mathrm{d}A^\nu_\mu a^\mu_\nu = \alpha \mathrm{d}A$ can be written as

$$\mathrm{d}A(\mathbf{b})\,\mathrm{d}A(\mathbf{a}) = \mathrm{d}A(\mathbf{ba}), \quad \mathbf{ba} = [b^\mu_\lambda a^\lambda_\nu]$$

in terms of the usual matrix product $b^\mu_\lambda a^\lambda_\nu = b^\mu_\bullet a^\bullet_\nu$ and the involution $\mathbf{a} \mapsto \mathbf{a}^\flat$ can be obtained by the pseudo-Hermitian conjugation $a^{\flat\nu}_\beta = g^{\nu\kappa} a^{\mu *}_\kappa g_{\mu\beta}$ respectively to the indefinite (Minkowski) metric tensor $\mathbf{g} = [g_{\mu\nu}]$ and its inverse $\mathbf{g}^{-1} = [g^{\mu\nu}]$, given by $g_{\mu\nu} = \delta^\mu_{-\nu} = g^{\mu\nu}$.

Let us prove that the "spatial" part $\lambda = (\lambda^\mu_\nu)^{\mu\neq+}_{\nu\neq-}$ of the quantum stochastic germ $\lambda^\mu_\nu(y) = \delta^\mu_\nu + \alpha^\mu_\nu(y)$ for a PD cocycle exponent ϕ must be conditionally PD in the following sense.

Theorem 1. *Suppose that the quantum stochastic equation (2.1) with $\phi_0(y) = y$ has a PD solution in the sense of positive definiteness (1.1) of the matrix $[\phi_t(y^\star_i y_k)]$,*

$\forall t>0$. *Then the germ-matrix* $\lambda = p+\alpha$ *to* $p=(\delta^\mu_\nu)^{\mu\neq+}_{\nu\neq-}$ *satisfies the CPD property*

$$\sum_j e\zeta_j = 0 \Rightarrow \sum_{i,k}\langle\zeta_i|\lambda(y_i^\star y_k)\zeta_k\rangle \geq 0.$$

Here $\zeta\in\mathbb{C}\oplus\ell^2_{\mathbb{N}}$, $e=(e^\mu_\nu)^{\mu\neq+}_{\nu\neq-}$, $e^\mu_\nu=\delta^+_\nu\delta^\mu_-$ *is the one-dimensional projector, written both with* λ *in the matrix form as*

$$\lambda = \begin{pmatrix}\lambda & \lambda_\bullet\\ \lambda^\bullet & \lambda^\bullet_\bullet\end{pmatrix}, \qquad e=\begin{pmatrix}1&0\\0&0\end{pmatrix}, \tag{2.3}$$

where $\lambda=\alpha^-_+$, $\lambda^m=\alpha^m_+$, $\lambda_n=\alpha^-_n$, $\lambda^m_n=\delta^m_n+\alpha^m_n$, *with* $\delta^m_n(y)=\delta^m_n$ *such that* $\lambda(y^\star)=\lambda(y)^\dagger$, $\lambda^n(y^\star)=\lambda_n(y)^\dagger$, $\lambda^m_n(y^\star)=\lambda^n_m(y)^\dagger$.

Proof. Let us denote by $\mathcal{D}$ the $\mathbb{C}$ -span $\left\{\sum_f\xi^f\otimes f^\otimes : \xi^f\in\mathbb{C}, f^\bullet\in\ell^2_{\mathbb{N}}\otimes L^2(\mathbb{R}_+)\right\}$ of coherent (exponential) functions $f^\otimes t(\tau)=\bigotimes_{t\in\tau}f^\bullet(t)$, given for each finite subset $\tau=\{t_1,...,t_n\}\subseteq\mathbb{R}_+$ by tensors $f^\otimes(\tau)=f^{n_1}(t_1)...f^{n_N}(t_N)$, where $f^n, n=\mathbb{N}$ are square-integrable complex functions on $\mathbb{R}_+$ and $\xi^f=0$ for almost all $f^\bullet=(f^n)$. The co-isometric shift T_s intertwining $A^s(t)$ with $A(t)=T_sA^s(t)T_s^\dagger$ is defined on $\mathcal{D}$ by $T_s(f^\otimes)(\tau)=f^\otimes(\tau+s)$. The PD property (1.1) of the quantum stochastic adapted map ϕ_t into the $\mathcal{D}$ -forms $\langle\eta|\phi_t(y)\eta\rangle$, for $\eta\in\mathcal{D}$ can be obviously written as

$$\sum_{i,k}\sum_{f,h}\bar{\xi}^f_i\phi_t(f^\bullet,y_i^\star y_k,h^\bullet)\xi^h_k\geq 0, \tag{2.4}$$

for any sequence $y_j\in B, j=1,2,\ldots$, where

$$\phi_t(f^\bullet,y,h^\bullet)=\langle f^\otimes|\phi_t(y)h^\otimes\rangle e^{-\int_t^\infty f^\bullet(s)^\dagger h^\bullet(s)\mathrm{d}s},$$

$\xi^f\neq 0$ only for a finite subset of $f^\bullet\in\{f^\bullet_i, i=1,2,...\}$. If the $\mathcal{D}$ -form $\phi_t(y)$ satisfies the stochastic equation (2.1), the complex function $\phi_t(f^\bullet,y,h^\bullet)$ satisfies the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln\phi_t(f^\bullet,y,h^\bullet)=f^\bullet(t)^\dagger h^\bullet(t)+\sum_{m,n=1}^\infty f^m(t)^*\alpha^m_n(y)h^n(t)$$
$$+\sum_{m=1}^\infty f^m(t)^*\alpha^m_+(y)+\sum_{n=1}^\infty\alpha^-_n(y)h^n(t)\phi+\alpha^-_+(y)$$

where $f^\bullet(t)^\dagger h^\bullet(t)=\sum_{n=1}^\infty f^n(t)^*h^n(t)$. The positive definiteness, (2.4), ensures the conditional positivity

$$\sum_j\sum_f\xi^f_j=0\Rightarrow\sum_{i,k}\sum_{f,h}\bar{\xi}^f_i\lambda_t(f^\bullet,y_i^\star y_k,h^\bullet)\xi^h_k\geq 0$$

of the form $\lambda_t(f^\bullet,y,h^\bullet)=\frac{1}{t}(\phi_t(f^\bullet,y,h^\bullet)-1)$ for each $t>0$ and any $y_j\in B$. This applies also for the limit λ_0 at $t\downarrow 0$, coinciding with the quadratic form

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_t(f^\bullet,y,h^\bullet)|_{t=0}=\sum_{m,n}\bar{a}^m\lambda^m_n(y)c^n+\sum_m\bar{a}^m\lambda^m(y)+\sum_n\lambda_n(y)c^n+\lambda(y),$$

where $a^\bullet=f^\bullet(0)$, $c^\bullet=h^\bullet(0)$, and the λ 's are defined in (2.3). Hence the form

$$\sum_{i,k}\sum_{\mu,\nu}\bar{\zeta}^\mu_i\lambda^\mu_\nu(y_i^\star y_k)\zeta^\nu_k:=\sum_{i,k}\bar{\zeta}_i\lambda(y_i^\star y_k)\zeta_k$$

$$+\sum_{i,k}\left(\sum_n \bar{\zeta}_i\lambda_n\left(y_i^\star y_k\right)\zeta_k^n+\sum_m \bar{\zeta}_i^m\lambda^m\left(y_i^\star y_k\right)\zeta_k+\sum_{m,n}\bar{\zeta}_i^m\lambda_n^m\left(y_i^\star y_k\right)\zeta_k^n\right)$$

with $\zeta=\sum_f \xi^f$, $\zeta^\bullet=\sum_f \xi^f a_f^\bullet$, where $a_f^\bullet=f^\bullet(0)$, is positive if $\sum_j \zeta_j=0$. The components ζ and $\zeta^\bullet$ of these vectors are independent because for any $\zeta\in\mathbb{C}$ and $\zeta^\bullet=(\zeta^1,\zeta^2,...)\in\ell^2_{\mathbb{N}}$ there exists such a function $a^\bullet\mapsto\xi^a$ on $\ell^2_{\mathbb{N}}$ with a finite support, that $\sum_a\xi^a=\zeta$, $\sum_a\xi^a a^\bullet=\zeta^\bullet$, namely, $\xi^a=0$ for all $a^\bullet\in\ell^2_{\mathbb{N}}$ except $a^\bullet=0$, for which $\xi^a=\zeta-\sum_{n=1}^\infty\zeta^n$ and $a^\bullet=e_n^\bullet$, the n -th basis element in $\ell^2_{\mathbb{N}}$, for which $\xi^a=\zeta^n$. This proves the complete positivity of the matrix form λ , with respect to the matrix orthoprojector p_0 defined in (2.3) on the ket-vectors $\zeta=(\zeta^\mu)$ □

3. A Dilation Theorem for the Form-Generator.

The CPD property of the germ-matrix λ with respect to the projective matrix p_0 (2.3) obviously implies the positivity of the dissipation form

$$\sum_{x,z}\langle\zeta^x|\Delta(x,z)\zeta^z\rangle:=\sum_{k,l}\sum_{\mu,\nu}\langle\zeta_k^\mu|\Delta_\nu^\mu(y_k,y_l)\zeta_l^\nu\rangle, \tag{3.1}$$

where $\zeta^-=\zeta=\zeta^+$ and $\zeta_j=\zeta^{y_j}$ for any (finite) sequence $y_j\in B$, $j=1,2,...$, corresponding to non-zero $\zeta_y\in\mathbb{C}\oplus\ell^2_{\mathbb{N}}$. Here $\Delta=(\Delta_\nu^\mu)_{\nu=+,\bullet}^{\mu=-,\bullet}$ is the stochastic dissipator

$$\Delta(x,z)=\lambda(x^\star z)-e\lambda(z)-\lambda(x^\star)e+e\lambda(1)e$$

with the elements

$$\begin{aligned}\Delta_n^m(x,z)&=\alpha_n^m(x^\star z)+\delta_n^m,\\ \Delta_n^-(x,z)&=\alpha_n^-(x^\star z)-\alpha_n^-(z)=\Delta_+^n(z,x)^\dagger,\\ \Delta_+^-(x,z)&=\alpha_+^-(x^\star z)-\alpha_+^-(z)-\alpha_+^-(x^\star)+d,\end{aligned}\tag{3.2}$$

where $d=\alpha_+^-(1)\le 0$ ($d=0$ for the case of the martingale $M_t=\phi_t(1)$). In particular the matrix-valued map $\lambda_\bullet^\bullet=[\lambda_n^m]$ is PD. If the functions λ^m , λ_n,λ have the form

$$\lambda^m(y)=\varphi^m(y)-c^m,\quad \lambda_n(y)=\varphi_n(y)-c_n,\quad \lambda(y)=\varphi(y)-c \tag{3.3}$$

such that $\varphi=\lambda-c$, is a PD map for a constant Hermitian matrix $c=(c_\nu^\mu)_{\nu\neq-}^{\mu\neq+}$, the CPD condition is fulfilled for λ .

In order to make the formulation of the following dilation theorem as concise as possible, we need the notion of the $\flat$ -representation of B in a pseudo-Hilbert space $\mathcal{E}=\mathbb{C}\oplus\mathcal{K}\oplus\mathbb{C}$ with respect to the indefinite metric

$$(\xi|\xi)=2\operatorname{Re}\bar{\xi}^-\xi^++\|\xi^\circ\|^2+|\xi^+|^2 d \tag{3.4}$$

for the triples $\xi=(\xi^-,\xi^\circ,\xi^+)\in\mathcal{E}$, where $\xi^-,\xi^+\in\mathbb{C}$, $\xi^\circ\in\mathcal{K}$, $\mathcal{K}$ is a pre-Hilbert space. The operators A in this space are given by the 3×3 -block-matrices $\mathbf{A}=[A_\nu^\mu]_{\nu=+,\circ,+}^{\mu=-,\circ,+}$, and the pseudo-Hermitian conjugation $(A^\flat\xi|\xi)=(\xi|A\xi)$ is given by the usual Hermitian conjugation $A_\nu^{\dagger\mu}=A_\mu^{\nu *}$ as $\mathbf{A}^\flat=\mathbf{G}^{-1}\mathbf{A}^\dagger\mathbf{G}$ respectively to the indefinite metric tensor $\mathbf{G} = [G_{\mu\nu}]$ and its inverse $\mathbf{G}^{-1} = [G^{\mu\nu}]$, given by

$$\mathbf{G} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & I_\circ^\circ & 0 \\ 1 & 0 & d \end{bmatrix}, \qquad \mathbf{G}^{-1} = \begin{bmatrix} -d & 0 & 1 \\ 0 & I_\circ^\circ & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{3.5}$$

with a real d , where $I_\circ^\circ$ is the identity operator in $\mathcal{K}$. The algebras of all operators A on $\mathcal{K}$ and $\mathcal{E}$ with $A^\dagger\mathcal{K} \subseteq \mathcal{K}$ and $A^\flat\mathcal{E} \subseteq \mathcal{E}$ are denoted by $\mathcal{A}(\mathcal{K})$ and $\mathcal{A}(\mathcal{E})$.

Theorem 2. *The following are equivalent:*

1. *The dissipator (3.2), defined by the* $\flat$ *-map* α *with* $\alpha_+^-(1) = d$ *, is positive definite:*

$$\sum_{x,z} \langle \zeta_x | \Delta(x,z) \zeta_z \rangle \geq 0$$

2. *There exist: a pre-Hilbert space* $\mathcal{K}$ *, a unital* $\dagger$ *- representation* j *in* $\mathcal{A}(\mathcal{K})$ *,*

$$j(x^\star z) = j(x)^\dagger j(z), \quad j(1) = I, \tag{3.6}$$

of the $\star$ *-multiplication structure of* B *, a* j *-cocycle on* B *,*

$$k(x^\star z) = j(x)^\dagger k(z) + k(x^\star), \tag{3.7}$$

having values in $\mathcal{K}$ *, and a function* $l : B \to \mathbb{C}$ *, having the coboundary property*

$$l(x^\star z) = l(z) + l(x^\star) + k^\bullet(x^\star) k(z), \tag{3.8}$$

with $k^\bullet(y^\star) = k(y)^\bullet, l(y^\star) = l(y)^\bullet$ *, such that* $\lambda(y) = l(y) + d$ *,*

$$\lambda_n(y^\star) = k(y)^\dagger L_n^\circ + L_n^- = \lambda^n(y)^\dagger,$$

and $\lambda_n^m(y) = L_m^{\circ\bullet} j(y) L_n^\circ$ *for some elements* $L_n^\circ \in \mathcal{K}$ *with the adjoints* $L_n^{\circ\bullet} = L_\circ^n : \mathcal{K} \to \mathbb{C}$ *and* $L_n^- \in \mathbb{C}$ *.*

3. *There exist a pseudo-Hilbert space,* $\mathcal{E}$ *, namely,* $\mathbb{C} \oplus \mathcal{K} \oplus \mathbb{C}$ *with the indefinite metric tensor* $\mathbf{G} = [G_{\mu\nu}]$ *given above for* $\mu, \nu = -, \circ, +$ *, and* $d = \lambda(1)$ *, a unital* $\flat$ *-representation* $\jmath = [\jmath_\nu^\mu]_{\nu=-,\circ,+}^{\mu=-,\circ,+}$ *of the* $\star$ *-multiplication structure of* B *on* $\mathcal{E}$ *:*

$$\jmath(x^\star z) = \jmath(x)^\flat \jmath(z), \quad \jmath(1) = \mathbf{I} \tag{3.9}$$

with $\jmath(y)^\flat = \mathbf{G}^{-1} \jmath(y)^\dagger \mathbf{G}$ *, given by the matrix elements*

$$\jmath_\circ^\circ = j, \quad \jmath_+^\circ = k, \quad \jmath_\circ^- = k^\bullet, \quad \jmath_+^- = l, \quad \jmath_-^- = 1 = \jmath_+^+$$

and all other $\jmath_\nu^\mu = 0$ *, and a linear operator* $\mathbf{L} : \mathbb{C} \oplus \ell_N^2 \to \mathcal{E}$ *, with the components* $[L^\mu, L_\bullet^\mu]$ *, where*

$$L^- = 0, \quad L^\circ = 0, \quad L^+ = 1, \quad L_\bullet^- = (L_n^-), \quad L_\bullet^\circ = (L_n^\circ), \quad L_\bullet^+ = 0,$$

and $\mathbf{L}^\flat = \begin{pmatrix} 1 & 0 & \delta \\ 0 & L_\circ^\bullet & L_+^\bullet \end{pmatrix} = \mathbf{L}^\dagger \mathbf{G}$ *, where* $L_\circ^\bullet = L_\bullet^{\circ\dagger}, L_+^\bullet = L_\bullet^{-\dagger}$ *, such that*

$$\mathbf{L}^\flat \jmath(y) \mathbf{L} = \lambda(y), \qquad \forall y \in B. \tag{3.10}$$

4. *The germ-matrix* $\lambda(y) = (\alpha_\nu^\mu(y) + \delta_\nu^\mu)_{\nu\neq-}^{\mu\neq+}$ *is CPD with respect to the ortho-projector* $\mathbf{e}$ *, defined in (2.3) :*

$$\sum_y \mathbf{e}\zeta^y = 0 \Rightarrow \sum_{x,z} \langle \zeta^x | \lambda(x^\star z) \zeta^z \rangle \geq 0.$$

Proof. Similar to the dilation theorem in [4], see also [7], [8], [9] □

4. Pseudo-Poisson processes and their generators.

Let us consider the case $B = 1 \oplus \mathfrak{b}$ of the unital semigroup for a $\star$ -algebra $\mathfrak{b}$ with $\lambda(1 \oplus b) = d + \gamma(b)$ given by a linear matrix -function

$$\gamma = \begin{pmatrix} \gamma & \gamma_\bullet \\ \gamma^\bullet & \gamma_\bullet^\bullet \end{pmatrix} = \lambda - d, \quad d = \begin{pmatrix} d & d_\bullet \\ d^\bullet & d_\bullet^\bullet \end{pmatrix} = \lambda(1)$$

of $b \in \mathfrak{b}$ for $y = 1 \oplus b$. Following [4], the linear quantum stochastic process $\Lambda(t)$: $b \mapsto \gamma(b) A(t)$ with independent increments, generating together with $A(t, \mathbf{d}) = A_\mu^\nu(t) d_\nu^\mu$ the stochastic PD exponent

$$\phi_t(1 \oplus b) =: \exp[A(t, \mathbf{d}) + \Lambda(t, b)] : \quad b \in \mathfrak{b}$$

as the solution of the equation (2.1), will be called the pseudo-Poissonian[4] over the algebra $\mathfrak{b}$.

If B is a unit ball of an operator algebra $\mathcal{B}$, the linear form-generator can be extended to the whole algebra. The structure (3.3) of the linear form-generator for PD cocycles over an operator algebra $\mathcal{B}$ is a consequence of the cocycle equation (3.7), according to which $j(0) k(y) = 0$, where

$$k(y) = j(y)\varsigma - \varsigma,. \quad \varsigma = -k(0). \tag{4.1}$$

Denoting by $\varsigma^\dagger$ the linear functional $\xi^\circ \mapsto (\varsigma|\xi^\circ)$ on $\mathcal{K}$ corresponding to the $\varsigma \in \mathcal{K}$, the condition (3.8) yields

$$l(y) = \frac{1}{2}\left(\varsigma^\dagger k(y) + k^*(y)\varsigma\right) = \varsigma^\dagger j(y)\varsigma - \varsigma^\dagger\varsigma. \tag{4.2}$$

Hence, in addition to $\lambda_n^m(y) = L_m^{\circ\dagger} j(y) L_n^\circ$ one can obtain the structure (3.3) with

$$\varphi(y) = \varsigma^\dagger j(y)\varsigma, \quad \varphi_n(y) = \varsigma^\dagger j(y) L_n^\circ, \quad \varphi^m(y) = L_m^{\circ\dagger} j(y)\varsigma, \tag{4.3}$$

and $\kappa = \varsigma^\dagger\varsigma - \delta$, $\kappa_n = \varsigma^\dagger L_n^\circ - L_n^-$. Thus, $\lambda(y) = \varphi(y) - \kappa$, where φ is a completely positive nonlinear map of B into the space $\mathcal{M}\left(\mathbb{C} \oplus \ell_\mathbb{N}^2\right)$ of complex matrices $x = (x_\nu^\mu)$. Moreover, φ is uniquely defined as the birth-map by the condition $\varphi(0) = 0$ with $\kappa = -\lambda(0) = (\kappa_\nu^\mu)$, where $\kappa_+^- = \kappa, \kappa_n^- = \kappa_n, \kappa_+^m = \overline{\kappa}_m$, and $\kappa_n^m = -\lambda_\nu^\mu(0)$, constituting a negative-definite matrix $\kappa_\bullet^\bullet = [\kappa_n^m]$. Any germ-matrix λ whose components are decomposed into the sums of the components φ_ν^μ of a PD map φ and $\lambda(0)$, are obviously CPD with respect to the orthoprojector p_0 in (2.4). As follows from the dilation theorem, there exists a family $\varsigma_- = \varsigma = \varsigma_+, \quad \varsigma_n = L_n^\circ - j(0) L_n^\circ, \quad n \in \mathbb{N}$ of vectors $\varsigma_\nu \in \mathcal{K}$ with $j(0)\varsigma_\nu = 0$ such that $\varphi_\nu^\mu(y) = \varsigma_\mu^\dagger j(y)\varsigma_\nu$ for all $\mu \in \{-, 1, 2, ...\}$, $\nu \in \{+, 1, 2, ...\}$. Thus the equation (2.1) for a completely positive exponential cocycle with bounded stochastic derivatives has the following general form

$$\mathrm{d}\phi_t(y) + \left(\gamma - \varsigma^\dagger j(y)\varsigma\right)\phi_t(y)\,\mathrm{d}t = \sum_{m,n=1}^{\infty}\left(\varsigma_m^\dagger j(y)\varsigma_n - \gamma_n^m\right)\phi_t(y)\,\mathrm{d}A_m^n$$

$$+\sum_{m=1}^{\infty}\left(\varsigma_m^\dagger j(y)\varsigma - \gamma_m^\dagger\right)\phi_t(y)\,\mathrm{d}A_m^+ + \sum_{n=1}^{\infty}\left(\varsigma^\dagger j(y)\varsigma_n - \gamma_n\right)\phi_t(y)\,\mathrm{d}A_-^n, \tag{4.4}$$

where $\gamma_\nu^\mu = -\alpha_\nu^\mu(0)$. If $M_t = \phi_t(1)$ is a martingale, the normalization condition $\sum_{k=1}^\infty \varsigma^{k\dagger}\varsigma^k = \kappa$ ($\leq \kappa$ if submartingale).

In the particular case $\mathcal{K} = \mathbb{C}\oplus\mathfrak{h}$, $j(y) = 1 \oplus y$, where $\mathfrak{h}$ is a Hilbert space of a representation $\mathcal{B} \subseteq \mathcal{B}(\mathfrak{h})$ of the C*-algebra $\mathcal{B}$ in the operator algebra $\mathcal{B}(\mathfrak{h})$, this gives a quantum stochastic generalization of the Poissonian birth semigroups [1] with the affine generators $\alpha^{\mu}_{\nu}(y) = \varsigma^{\dagger}_{\mu}X\varsigma_{\nu} - \gamma^{\mu}_{\nu}$. In the more general case when the space $\mathcal{K}$ is embedded into the Hilbert sum of all tensor powers of the space $\mathfrak{h}$ such that $j(y) = \oplus_{k=0}^{\infty} y^{\otimes k}$, the birth function φ is described by the components

$$\varphi^m_n(y) = \sum_{k=0}^{\infty} \varsigma^{k\dagger}_m y^{\otimes k}\varsigma^k_n, \qquad \varphi(y) = \sum_{k=1}^{\infty} \varsigma^{k\dagger} y^{\otimes k}\varsigma^k \tag{4.5}$$

$$\varphi^m(y) = \sum_{k=1}^{\infty} \varsigma^{k\dagger}_m y^{\otimes k}\varsigma^k, \qquad \varphi_n(y) = \sum_{k=1}^{\infty} \varsigma^{k\dagger} y^{\otimes k}\varsigma^k_n$$

with $\varsigma^k, \varsigma^k_n \in \mathfrak{h}^{\otimes k}$.

Note, if $\mathcal{B}$ is a W*-algebra and the germ map λ is w*-analytic, the completely positive function φ is also analytic, being defined by a w*-analytical representation $j = \oplus_{k=0}^{\infty} i^{\otimes k}$ in a full Fock space $\mathcal{K} = \oplus_{k=0}^{\infty}\mathcal{H}^{\otimes k}$, where i is a (linear) w*-representation of $\mathcal{B}$ on a Hilbert space $\mathcal{H}$. This gives the general form for the w*-analytical quantum stochastic quasi-Poisson birth process over the algebra $\mathcal{B}$.

The next theorem proves that these structural conditions which are necessary for complete positivity of the stochastic exponents, given by the equation (2.1), are also sufficient. In particular it proves the existence of the quantum birth cocycle ϕ for a given generating stochastic birth matrix-function φ.

Theorem 3. *Let the structural maps λ of the quantum stochastic PD exponent ϕ over the unit ball of an operator algebra $\mathcal{B}$. Then they are bounded in the unit ball of $\mathcal{B}$,*

$$\|\lambda\| < \infty, \qquad \|\lambda_\bullet\| = \left(\sum_{n=1}^{\infty}\|\lambda_n\|^2\right)^{\frac{1}{2}} = \|\lambda^\bullet\| < \infty, \qquad \|\lambda^\bullet_\bullet\| = \|\lambda^\bullet_\bullet(1)\| < \infty,$$

where $\|\lambda\| = \sup\{\|\lambda(y)\| : \|y\| < 1\}, \|\lambda^\bullet_\bullet(1)\| = \sup\{\langle\zeta^\bullet|\lambda^\bullet_\bullet(1)\zeta^\bullet\rangle \,|\, \|\zeta^\bullet\| < 1\}$, *and have the form (4.3) written as*

$$\lambda(y) = \varphi(y) - \kappa$$

with $\varphi = \varphi^-_+, \quad \varphi^m = \varphi^m_+, \quad \varphi_n = \varphi^-_n$ *and* $\varphi^m_n = \lambda^m_n$, *composing a bounded PD map*

$$\boldsymbol{\varphi} = \begin{bmatrix} \varphi & \varphi_\bullet \\ \varphi^\bullet & \varphi^\bullet_\bullet \end{bmatrix}, \quad \textit{and} \quad \boldsymbol{\kappa} = \begin{bmatrix} \kappa & \kappa_\bullet \\ \kappa^\bullet_\bullet & 0 \end{bmatrix} \tag{4.6}$$

with arbitrary κ and $\kappa_\bullet = (\kappa_1, \kappa_{2,\ldots})$. The equation (4.4) has the unique PD solution

$$\phi_t(y) = V_t^\dagger \exp\left[A^+_\bullet(t)\varphi^\bullet(y)\right]\varphi^\bullet_\bullet(y)^{A^\bullet_\bullet(t)}\exp\left[\varphi_\bullet(y)A^\bullet_-(t)\right]V_t\exp\left[t\varphi(y)\right], \tag{4.7}$$

where $V_t = \exp\left[-\kappa_\bullet A^\bullet_-(t) - \frac{1}{2}\kappa t I\right]$.

Proof. (Sketch) The PD solution to the quantum stochastic equation (4.4) can be obtained by the iteration of the equivalent quantum stochastic integral equation

$$\phi_t(y) = V_t^\dagger V_t + \int_0^t V_s^\dagger \phi^s_{t-s}(y) V_s \beta^\mu_\nu(y)\, \mathrm{d}A^\nu_\mu(s)$$

where $\beta_{\nu}^{\mu}(y) = \varphi_{\nu}^{\mu}(y) - \delta_{\nu}^{\mu}$ .Here V_t is the exponential vector cocycle $V_r^s V_s = V_{r+s}$, resolving the quantum stochastic differential equation

$$\mathrm{d}V_t + \kappa V_t \mathrm{d}t + \sum_{n=1}^{\infty} \kappa_n V_t \mathrm{d}A_{-}^{n} = 0$$

with the initial condition $V_0 = I$ in $\mathcal{D}$ and with $V_r^s = T_r^{\dagger} V_r T_s$, shifted by the time-shift co-isometry T_s in $\mathcal{D}$.

References

[1] Belavkin, V.P., Multiquantum Systems and Point Processes: Generating Functionals and Non-linear Semigroups. Reports on Mathematical Physics, **28** (1), pp57-90, 1989.

[2] Belavkin, V.P., Kernel Representations of $\star$ -semigroups Associated with Infinitely Divisible States. Quantum Probability and Related Topics, **8**, pp31-50, World Scientific, 1992.

[3] Schürmann, M., A class of Representations of Involutive Bialgebras. Math. Proc. Camb. Phil. Soc., **107**, pp149-175, 1990.

[4] Belavkin, V.P., Chaotic states and Stochastic Integration in Quantum Systems. Russian Math. Survey, **47** (1), pp47-106, 1992.

[5] Belavkin, V.P., A Pseudo-Euclidean Representation of Conditionally Positive Maps. Math. Notes, **49**, No.6, pp135-137, 1991.

[6] Hudson, R.S., and Parthasarathy, K.R., Quantum Itô's formula and Stochastic Evolution. Comm. Math. Phys., **93**, pp301-323, 1984.

[7] Araki, H., Factorizable Representations of Current Algebra, Publ. Res. Inst. Math. Sci., **5** pp361-422, 1970.

[8] Parthasarathy and K. Schmidt, Positive Definite Kernels, Continuous Tensor Products, and Central Limit Theorems of Probability Theory. Lecture notes in Mathematics, Schpringer-Verlag, **272** , 1972.

[9] Streater, R.K. Current Commutation Relations,Continuous Tensor Products and Infinitely Divisible Group Representations. In : Local Quantum Theory, pp247-263, Academic Press, 1969.

Mathematics Department, University of Nottingham,, NG7 2RD, UK.

E-mail address: vpb@maths.nott.ac.uk

Lévy Processes on Quantum Hypergroups

Uwe Franz [1]
Michael Schürmann [1]

Abstract. Quantum hypergroups are non-commutative versions of hypergroups and were introduced by Yu.A. Chapovsky and L.I. Vainerman. Assuming that the quantum hypergroup satisfies a certain positivity condition (Schoenberg's correspondence), we show that Lévy processes, like in the quantum group case, are given by solutions of quantum stochastic differential equations in the sense of R.L. Hudson and K.R. Parthasarathy. We prove that quantum hypergroups of double coset type satisfy Schoenberg's correspondence. As an example we discuss the quantum hypergroup $U\langle 2\rangle//U\langle 1\rangle$ with $U\langle n\rangle$ the non-commutative analogue of the coefficient algebra of the unitary group.

1. Intoduction

Let K be a hypergroup; see [2]. This means, among other conditions, that

- K is a (locally compact) topological space with a distinguished point $e \in K$.
- There is a binary operation, denoted by $\star$ and called *convolution*, on the space $\mathbf{M}_b$ of finite signed measures on K which turns $\mathbf{M}_b$ into an algebra.
- For probability measures μ and ν the convolution product $\mu \star \nu$ is again a probability measure.
- $\mu \star \delta_e = \delta_e \star \mu = \mu$ for all $\mu \in \mathbf{M}_b$

[1] Institut für Mathematik und Informatik, Universität Greifswald, Friedrich-Ludwig-Jahn-Str. 15a, 17487 Greifswald, Germany

where δ_x is the Dirac measure at x for $x \in K$.

For an appropriate complex-valued function f on K (for example, $f \in L^\infty(K)$) we define the function Δf on $K \times K$ by

$$\Delta f(x,y) = \int_K f \, d(\delta_x \star \delta_y).$$

If $f \in L^\infty(K)$ then $\Delta f \in L^\infty(K \times K)$. In many cases $L^\infty(K \times K)$ will be (the closure of) the tensor product $L^\infty(K) \otimes L^\infty(K)$ and we will have the following situation. There is a $*$-algebra $F(K)$ of functions on K such that Δ maps $F(K)$ to the tensor product $F(K) \otimes F(K)$. The hypergroup can then be described by a triplet (F, Δ, δ) with the properties

- F is a complex $*$-algebra
- $\Delta : F \to F \otimes F$ is a *positive* linear mapping satisfying $\Delta \mathbf{1} = \mathbf{1} \otimes \mathbf{1}$ and the coassociativity condition

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$$

- $\delta : F \to \mathbb{C}$ is a $*$-algebra homomorphism satisfying the counit condition

$$(\delta \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \delta) \circ \Delta$$

If we allow not only commutative $*$-algebras and if we replace the positivity of Δ by *complete positivity* we arrive at the notion of a *quantum hypergroup* (see [3]) or, more generally, of what we call a *hyper-bialgebra*.

Important examples of hypergroups are given by a double coset structure. Let G be a semi-group with unit element e. In order to stay in a purely algebraic framework, we consider the $*$-algebra $R(G)$ of functions which come from a finite-dimensional representation of G (i.e. $f \in R(G)$ if $f(x) = \langle \xi, \gamma(x)\zeta \rangle$ for $\xi, \zeta \in \mathbb{C}^n$ and γ a $*$-representation of the elements of G as $n \times n$-matrices). $R(G)$ becomes a $*$-bialgebra if we define the comultiplication by $\Delta_1(x,y) = f(xy)$ and the counit by $\delta_1(f) = f(e)$. Now let H be sub-semi-group of G equipped with a Haar measure λ. Since H is a semi-group, we can define a comultiplication Δ_2 and a counit δ_2 on $R(H)$ in the same manner as for G. Denote by $\pi : R(G) \to R(H)$ the restriction to H. Then π is a $*$-bialgebra homomorphism. Denote by $\mathrm{R}(G)//\mathrm{R}(H)$ the space of functions in $\mathrm{R}(G)$ satisfying

$$f(xzy) = f(z) \text{ for all } x, y \in G,\ z \in H,$$

that is $\mathrm{R}(G)//\mathrm{R}(H)$ consists of functions on $G//H$, the space of double cosets of G with respect to H. We have

$$\mathrm{R}(G)//\mathrm{R}(H) = \{f \in \mathrm{R}(G) \mid (\pi \otimes \mathrm{id}) \circ \Delta_1 f = \mathbf{1} \otimes f \text{ and } (\mathrm{id} \otimes \pi) \circ \Delta_1 f = f \otimes \mathbf{1}\}.$$

It can be shown that the $*$-algebra $\mathrm{R}(G)//\mathrm{R}(H)$ is turned into a hyper-bialgebra if we set

$$\Delta f(x, y) = \int f(xzy)\, \mathrm{d}\lambda(z)$$

and

$$\delta f = \delta_1 f = f(e),$$

$f \in \mathrm{R}(G)//\mathrm{R}(H)$. Then

$$\Delta = (\mathrm{id} \otimes (\lambda \circ \pi) \otimes \mathrm{id}) \circ (\Delta_1 \otimes \mathrm{id}) \circ \Delta_1 \lceil \mathrm{R}(G)//\mathrm{R}(H)$$

and $\delta = \delta_1 \lceil \mathrm{R}(G)//\mathrm{R}(H)$. Examples of this construction are given by double coset hypergroups. Moreover, this constuction can be turned over to the non-commutative setting; see [3] and Section 3 of this paper.

We will be concerned with quantum stochastic processes on hyper-bialgebras, in particular, with quantum Lévy processes. These are defined in analogy to Lévy processes on $*$-bialgebras: $*$-homomorphisms are replaced by completely positive mappings; cf. also [12]. We prove that Lévy processes on hyper-bialgebras can be realized as solutions of quantum stochastic differential equations on Bose-Fock space, thus generalizing the result for bialgebras, *under the condition* that the hyper-bialgebra fulfills the principle of *Schoenberg's correspondence* (Section 2). We were not able to prove Schoenberg's correspondence in the general case of a hyper-bialgebra but only for hyper-bialgebras of double coset type with the additional assumption that the Haar measure is faithful (Section 3). In Section 4 we introduce the example of the double coset hyper-bialgebra $\mathrm{U}\langle 2\rangle//\mathrm{U}\langle 1\rangle$ with $\mathrm{U}\langle n\rangle$ denoting the non-commutative analogue of the coefficient algebra of the unitary group U_n. In Section 5 we consider a class of Brownian motions on $\mathrm{U}\langle 2\rangle//\mathrm{U}\langle 1\rangle$ for which we analyze the corresponding quantum stochastic differential equations in Section 6.

Vector spaces will be over the field of complex numbers. Algebras are always assumed to be associative, complex and unital. For a vector space $\mathcal{V}$ we denote by $\mathcal{V}'$ the vector space of linear functionals on $\mathcal{V}$. For a coalgebra $(\mathcal{C}, \Delta, \delta)$ we define the n-times comultiplication $\Delta^{(n)} : \mathcal{C} \to \mathcal{C}^{\otimes n}$, $n = 0, 1, 2, \dots$, inductively by $\Delta^{(0)} = \delta$ and $\Delta^{(n+1)} = (\mathrm{id} \otimes \Delta^{(n)}) \circ \Delta$. Note that $\Delta^{(1)} = \mathrm{id}$ and $\Delta^{(2)} = \Delta$.

A $*$-algebra is an algebra $\mathcal{A}$ equipped with an involution, i.e. an antilinear mapping $a \mapsto a^*$ satisfying $(ab)^* = b^* a^*$ and $(a^*)^* = a$. An element of a $*$-algebra is called positive if it is a finite sum of elements of the form $a^* a$. A linear mapping Φ from a $*$-algebra $\mathcal{A}$ to a $*$-algebra $\mathcal{B}$ is called positive if $\Phi(a^* a)$ is a positive element in $\mathcal{B}$ for all $a \in \mathcal{A}$, i.e. if Φ maps positive elements to positive elements. We call Φ completely positive (c.p.) if $\Phi \mathbf{1} = \mathbf{1}$ and if $\Phi \otimes \mathrm{id} : \mathcal{A} \otimes \mathcal{M}_n(\mathbb{C}) \to \mathcal{B} \otimes \mathcal{M}_n(\mathbb{C})$ is positive for all $n \in \mathbb{N}$ where $\mathcal{M}_n(\mathbb{C})$ denotes the $*$-algebra of $n \times n$-matrices. The tensor product of two c.p. mappings is again c.p.

2. Lévy processes on hyper-bialgebras

A *quantum probability space* is a pair $(\mathcal{A}, \Phi)$ consisting of a $*$-algebra $\mathcal{A}$ and a state Φ on $\mathcal{A}$, see [1] and also [9, 8, 4, 13]. For a complex vector space $\mathcal{V}$ a linear mapping $j : \mathcal{V} \to \mathcal{A}$ is called a quantum random variable (q.r.v.). The unital sub-$*$-algebras $\mathcal{A}_1, \dots, \mathcal{A}_n$ of $\mathcal{A}$ are called (tensor) *independent* if $[\mathcal{A}_k, \mathcal{A}_l] = 0$ for $k \neq l$, and if $\Phi(a_1 \dots a_n) = \Phi(a_1) \dots \Phi(a_n)$ for all $a_k \in \mathcal{A}_k$, $k = 1, \dots, n$. The q.r.v. $j_1, \dots, j_n$, $j_k : \mathcal{V}_k \to \mathcal{A}$, are said to be independent if the $*$-algebras $*\text{-}\underline{\mathrm{alg}}\,(j_k(\mathcal{V}_k))$, $k = 1, \dots, n$, are independent where $*\text{-}\underline{\mathrm{alg}}$ means 'unital $*$-algebra generated by'.

The $*$-tensor algebra $\mathcal{T}(\mathcal{V})$ over a vector space $\mathcal{V}$ is defined to be the free $*$-algebra generated by $\mathcal{V}$. This space can be realized as the vector space

$$\bigoplus_{n=0}^{\infty} (\mathcal{V} \oplus \overline{\mathcal{V}})^{\otimes n}$$

with $\overline{\mathcal{V}}$ a complex conjugate copy of $\mathcal{V}$ and the $*$-algebra structure given by

$$(v_1 \otimes \dots v_n)\, v = v_1 \otimes \dots v_n \otimes v; \;\; v^* = \bar{v}.$$

For a q.r.v. j we denote by $\mathcal{T}(j)$ the unique extension of j to $\mathcal{T}(\mathcal{V})$ as a $*$-algebra homomorphism. The *distribution* of j is the state $\Phi \circ \mathcal{T}(j)$ on $\mathcal{T}(\mathcal{V})$.

A *Lévy process* on a coalgebra $\mathcal{C}$ is a family of q.r.v. (j_{st}) over the same quantum probability space, indexed by pairs (s,t) of real numbers with $0 \leq s \leq t$, and satisfying

- $j_{rs} \star j_{st} = j_{rt}, \; 0 \leq r \leq s \leq t$
- $j_{tt} = \delta \,\mathrm{id}$
- $j_{t_1 t_2}, \ldots, j_{t_n t_{n+1}}$ independent for $0 \leq t_1 \leq t_2 \leq \cdots \leq t_{n+1}$
- the distribution of j_{st} only depends on $t - s$ (we write Φ_t for the distribution of j_{0t})
- $\lim_{t \to 0+} \Phi_t(c_1 \otimes \ldots \otimes c_n) = \delta(c_1) \ldots \delta(c_n)$ (where we put $\delta(\bar{c}) = \overline{\delta(c)}$)

Notice that $(\mathcal{T}(\mathcal{C}), \mathcal{T}(\Delta), \mathcal{T}(\delta))$ is a $*$-bialgebra. The above definition of a Lévy processes says that $\mathcal{T}(j_{st})$ is a Lévy process on the $*$-bialgebra $\mathcal{T}(\mathcal{C})$ in the sense of [11]. Therefore, the theory of Lévy processes developed in [11] applies and we obtain a realization of our Lévy process on a Bose-Fock-space as the solution to a quantum stochastic differential equation in the sense of Hudson and Parthasarathy [7].

We describe the situation more precisely. Let D be a pre-Hilbert space. We denote by $\mathrm{L}(D)$ the $*$-algebra formed by all linear operators $R : D \to D$ which possess an adjoint R^* on D (i.e. there exists a linear operator $R^* : D \to D$ such that $\langle \xi, R\zeta \rangle = \langle R^*\xi, \zeta \rangle$ for all $\xi, \zeta \in D$.) Suppose that we are given

- a linear mapping $\mathrm{r} : \mathcal{C} \to \mathrm{L}(D)$
- a linear mapping $\mathrm{e} : \mathcal{C} \oplus \overline{\mathcal{C}} \to D$
- a linear mapping $\psi : \mathcal{C} \to \mathbb{C}$.

We put $\mathrm{r}(\bar{c}) = \mathrm{r}(c)^*$ and $\psi(\bar{c}) = \overline{\psi(c)}$ and we will always assume that the set $\{\mathrm{r}(b_1) \ldots \mathrm{r}(b_n)\mathrm{e}(b) \,|\, b, b_1, \ldots, b_n \in \mathcal{C} \oplus \overline{\mathcal{C}}\}$ is total in D.

Consider the quantum stochastic differential equation

$$\mathrm{d}j_{st} = j_{st} \star \mathrm{d}I_t \;; \quad j_{ss} = \delta$$

with

$$I_t(c) = A_t^*(\mathrm{e}(c)) + \Lambda_t(\mathrm{r}(c) - \delta(c)\mathrm{id}) + A_t(\mathrm{e}(\bar{c})) + \psi(c)t; \; c \in \mathcal{C} \oplus \overline{\mathcal{C}}$$

in the sense of [11], Theorem 2.5.1. Then the solution to these equations is a Lévy process on $\mathcal{C}$ whose generator $\Psi : \mathrm{T}(\mathcal{C}) \to \mathbb{C}$ is given by

$$\begin{aligned} \Psi(c) &= \psi(c) \text{ for } c \in \mathcal{C} \oplus \overline{\mathcal{C}} \\ \Psi(c_1 \otimes c_2) &= \langle \mathrm{e}(\bar{c}_1), \mathrm{e}(c_2) \rangle \text{ for } c_1, c_2 \in \mathcal{C} \oplus \overline{\mathcal{C}} \\ \Psi(c_1 \otimes \ldots \otimes c_n) &= \langle \mathrm{e}(\bar{c}_1), \mathrm{r}(c_2) \ldots \mathrm{r}(c_{n-1}) \mathrm{e}(c_n) \rangle \text{ for } c_1, \ldots, c_n \in \mathcal{C} \oplus \overline{\mathcal{C}}, \; n \geq 3 \end{aligned}$$

Conversely, starting from a Lévy process, by applying the GNS construction to its generator, one obtains D, e, r, ψ such that the above quantum stochastic differential equation yields a version of the process. The quantum probability space underlying our Fock-representation of the Lévy process is given by the $*$-algebra $\mathrm{L}(\mathcal{E}_D)$ and the vacuum state. Here

$$\mathcal{E}_D = \bigcap_{\alpha \geq 0} \mathrm{dom}\, \alpha^{\mathrm{N}} \cap \bigcup_E \Gamma(E)$$

with N the number operator, $\Gamma(E)$ the Bose-Fock-space over $\mathrm{L}^2(\mathbb{R}+) \otimes E$, and where the union is taken over all finite dimensional subspaces E of D.

We pose the following question. Let the coalgebra $\mathcal{B}$ also carry the structure of a $*$-algebra such that the following are satisfied

- the comultiplication $\Delta : \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}$ is completely positive (c.p.)
- the counit $\delta : \mathcal{B} \to \mathbb{C}$ is a $*$-algebra homomorphism

We call such an object a *hyper-bialgebra*; see [3] where the concept of a quantum hypergroup was introduced. What are the conditions on the coefficients e, r and ψ such that the corresponding Lévy process consists of c.p. mappings?

We equip $\mathcal{T}(\mathcal{B})$ with an other multiplication, denoted by $\cdot$, by setting

$$(b_1 \otimes \ldots \otimes b_2) \cdot (c_1 \otimes \ldots \otimes c_m) = b_1 \otimes \ldots \otimes b_{n-1} \otimes (b_n c_1) \otimes c_2 \otimes \ldots \otimes c_m.$$

which turns $\mathcal{T}(\mathcal{B})$ into a hyper-bialgebra. This new hyper-bialgebra has another interpretation. Consider the free product

$$\mathcal{B} \sqcup_1 \mathbb{C}(p)$$

of unital hyper-bialgebras $\mathcal{B}$ and $\mathbb{C}(p)$ (cf. [12]) where $\mathbb{C}(p)$ denotes the $*$-bialgebra generated by a single projection p (i.e. an indeterminate satisfying $p^2 = p = p^*$). Then

$$\mathcal{B} \sqcup_1 \mathbb{C}(p) = (\operatorname{kern} \mathcal{B}) \sqcup \operatorname{kern} \mathbb{C}(p) \oplus \mathbb{C}\mathbf{1}$$

(here $\sqcup$ is the free product of algebras) and the $*$-bialgebra $\mathcal{T}(\mathcal{B})$ can be recovered in $\mathcal{B} \sqcup_1 \mathbb{C}(p)$ if we identify $b_1 \otimes \ldots \otimes b_n$ with $p^\perp b_1 p^\perp \ldots p^\perp b_n p^\perp$. Moreover, the hyper-bialgebra $\mathcal{T}(\mathcal{B})$ is also a sub-hyper-bialgebra of $\mathcal{B} \sqcup_1 \mathbb{C}(p)$ if we send $b_1 \otimes \ldots \otimes b_n$ to $b_1 p^\perp \ldots p^\perp b_n$.

We say that a hyper-bialgebra $\mathcal{B}$ satisfies *Schoenberg's correspondence* if for a linear functional Ψ on $\mathcal{T}(\mathcal{B})$ the following are equivalent:

(i) $\Psi(\mathbf{1}) = 0$, $\Psi(B^*) = \overline{\Psi(B)}$ for all $B \in \mathcal{T}(\mathcal{B})$ and $\Psi(B^* \cdot B) \geq 0$ for all $B \in \operatorname{kern} \mathcal{T}(\delta)$

(ii) $\exp_*(t\Psi)(\mathbf{1}) = 1$ and $\exp_*(t\Psi)(B^* \cdot B) \geq 0$ for all $B \in \mathcal{T}(\mathcal{B})$

where the convolution in (ii) is with respect to the comultiplication $\mathcal{T}(\Delta)$.

A $*$-representation of an algebra $\mathcal{A}$ on a pre-Hilbert space D is a $*$-algebra homomorphism from $\mathcal{A}$ to $\mathrm{L}(D)$. For a $*$-representation ρ of the $*$-algebra $\mathcal{A}$ on a pre-Hilbert space D and for a $*$-homomorphism $\delta : \mathcal{A} \to \mathbb{C}$ the pre-Hilbert space D becomes a two-sided $\mathcal{A}$-module if we put

$$a.\xi.b = \rho(a)\xi\delta(b) \text{ for } a, b \in \mathcal{A} \text{ and } \xi \in D.$$

We speak of (ρ, δ)-cocycles and -coboundaries of the Hochschildt cohomology associated with this bimodule structure of D.

Theorem 2.1 *Let $\mathcal{B}$ be a hyper-bialgebra which we suppose to satisfy Schoenberg's correspondence. Let j_{st} be a Lévy process on $\mathcal{B}$ with coefficients D,* e, r *and ψ. Then the q.r.v. j_{st} are c.p. if and only if there exist*

- *a pre-Hilbert space E and an isometry $V : D \to E$*
- *a $*$-representation ρ of $\mathcal{B}$ on E*
- *a (ρ, δ)-1-cocycle $\eta : \mathcal{B} \to E$*

such that

- $e = V^* \circ \eta$
- $r(b) = V^* \circ \rho(b) \circ V$
- $-\langle \eta(b^*), \eta(c) \rangle$ is the (δ, δ)-coboundary of ψ .

Proof: Using Schoenberg's correspondence, it is not difficult to see that a Lévy process on $\mathcal{B}$ is c.p. if and only if its generator satisfies the condition (i) above. However, then we can apply Corollary 2.5 of [12].◇

Let j_{st} be a Lévy process with the extra property that the isometry V appearing in the canonical construction of j_{st} is unitary. We call such a process *basic*. In this case $\eta = e$, $r = \rho$ and $\Psi(b_1 \otimes \ldots \otimes b_n) = \psi(b_1 \ldots b_n)$. Therefore, a basic Lévy process is given by a conditionally positive, hermitian linear functional ψ with $\psi(\mathbf{1}) = 0$ on $\mathcal{B}$. In fact, there is a 1-1-correspondence between such functionals and basic Lévy processes.

3. Double coset hyper-bialgebras

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be $*$-bialgebras. Suppose that we are given a *Haar measure* λ on $\mathcal{B}_2$ that is $\lambda : \mathcal{B}_2 \to \mathbb{C}$ is a state satisfying

$$(\mathrm{id} \otimes \lambda) \circ \Delta_2 = \lambda \mathbf{1} = (\lambda \otimes \mathrm{id}) \circ \Delta_2.$$

We will also assume that λ is *faithful*, a condition needed for the proof of Theorem 3.1 below. Let $\pi : \mathcal{B}_1 \to \mathcal{B}_2$ be a $*$-bialgebra epimorphism. We put

$$\begin{aligned} \mathcal{B}_1/\mathcal{B}_2 &= \{ b \in \mathcal{B}_1 \mid (\mathrm{id} \otimes \pi) \circ \Delta_1(b) = b \otimes \mathbf{1} \} \\ \mathcal{B}_2 \backslash \mathcal{B}_1 &= \{ b \in \mathcal{B}_1 \mid (\pi \otimes \mathrm{id}) \circ \Delta_1(b) = \mathbf{1} \otimes b \} \\ \mathcal{B} &= \mathcal{B}_1/\mathcal{B}_2 \cap \mathcal{B}_2 \backslash \mathcal{B}_1 \end{aligned}$$

Next we define

$$\tilde{\Delta} : \mathcal{B}_1 \to \mathcal{B}_1 \otimes \mathcal{B}_1$$

by

$$\tilde{\Delta} b = (\mathrm{id} \otimes (\lambda \circ \pi) \otimes \mathrm{id}) \circ \Delta_1^{(3)}.$$

It is not difficult to check that $(\mathcal{B}, \Delta, \delta)$ with $\Delta = \tilde{\Delta} \lceil \mathcal{B}$ and $\delta = \delta_1 \lceil \mathcal{B}$ is an example of a hyper-bialgebra; see [3]. We sometimes write $\mathcal{B} = \mathcal{B}_1 // \mathcal{B}_2$ and call $\mathcal{B}$ a *double coset hyper-bialgebra*.

Theorem 3.1 *Double coset hyper-bialgebras satisfy Schoenberg's correspondence.*

The proof will be given at the end of this section.

To analyse the situation consider first a convolution semi-group φ_t on $\mathcal{B} = \mathcal{B}_1//\mathcal{B}_2$. We know that φ_t is the convolution exponential of $\psi = \frac{\mathrm{d}}{\mathrm{d}t}\varphi_t|_{t=0}$, the pointwise derivative at 0 of φ_t, i.e.

$$\varphi_t = \exp_\star(t\psi)$$

which is defined pointwise as the series

$$\sum_{n=0}^{\infty} \frac{\psi^{\star n}}{n!} t^n = \delta + \psi t + \frac{\psi^{\star 2}}{2!} t^2 + \dots,$$

see [10]. Now a linear functional β on $\mathcal{B} \subset \mathcal{B}_1$ can be extended to $\mathcal{B}_1$ by setting

$$\tilde{\beta} = \beta \circ \big((\lambda \circ \pi) \otimes \mathrm{id} \otimes (\lambda \circ \pi)\big) \circ \Delta^{(3)}$$

because $\big((\lambda\circ\pi)\otimes\mathrm{id}\otimes(\lambda\circ\pi)\big)\circ\Delta^{(3)}$ maps $\mathcal{B}_1$ to $\mathcal{B}$. Moreover, the restriction of $\tilde{\beta}$ to $\mathcal{B}$ gives back β. We may write

$$\tilde{\beta}\lceil\mathcal{B} = \beta \text{ and } \tilde{\beta} = (\lambda \circ \pi) \star \beta \star (\lambda \circ \pi).$$

The convolution semi-group φ_t is mapped to $\tilde{\varphi}_t$ with the properties

$$\begin{aligned} \tilde{\varphi}_{s+t} &= \tilde{\varphi}_s \star \tilde{\varphi}_t \text{ (with respect to } \Delta_1) \\ \tilde{\varphi}_t &\to \lambda \circ \pi = \tilde{\varphi}_0 \text{ for } t \to 0+ \end{aligned}$$

Thus $\tilde{\varphi}_t$ is a continuous convolution semi-group on $\mathcal{B}_1$ which does not start at the counit δ_1 but at $\lambda \circ \pi$!

This leads to the following general consideration. Let $\mathcal{B}$ be a $*$-bialgebra and suppose that we are given linear functionals φ_t satisfying

$$\begin{aligned} \varphi_{s+t} &= \varphi_s \star \varphi_t \\ \varphi_t &\to \varphi_0 \end{aligned}$$

Can we differentiate φ_t at 0? Let us look at matrices first. Let $A_t \in \mathcal{M}_d(\mathbb{C})$ with $A_{s+t} = A_s A_t$ and $A_t \to A_0$. Since $A_0^2 = A_0$ we can find a basis of $\mathbb{C}^d$ such that A_0 is of the form

$$\begin{pmatrix} \mathrm{I} & 0 \\ 0 & 0 \end{pmatrix}$$

with I the $n \times n$-unit matrix, $n \leq d$. We have $A_0 A_t = A_t = A_t A_0$ which means that A_t has the form

$$\begin{pmatrix} B_t & 0 \\ 0 & 0 \end{pmatrix}$$

with $B_t \in \mathcal{M}_n(\mathbb{C})$ and

$$B_{s+t} = B_s B_t, \quad B_t \to \mathrm{I}.$$

We know that $B_t = \mathrm{e}^{tG}$ with $G = \frac{\mathrm{d}}{\mathrm{d}t} B_t|_{t=0}$ and therefore

$$A_t = \begin{pmatrix} \mathrm{e}^{tG} & 0 \\ 0 & 0 \end{pmatrix} = A_0 \, \mathrm{e}^{t\tilde{G}} \tag{1}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} A_t|_{t=0} = \tilde{G}$$

where we put $\tilde{G} = \begin{pmatrix} G & 0 \\ 0 & 0 \end{pmatrix}$. In the case of a general coalgebra $\mathcal{C}$ and $\varphi_t \in \mathcal{C}'$, $\varphi_{s+t} = \varphi_s \star \varphi_t$, $\varphi_t \to \varphi_0$, we use for a given element b in $\mathcal{C}$ the fundamental theorem on coalgebras (see [14]) to find a finite-dimensional sub-coalgebra $\mathcal{C}_b$ of $\mathcal{C}$ containing b. For $\mathrm{T}_t : \mathcal{C}_b \to \mathcal{C}_b$, $\mathrm{T}_t(c) = (\mathrm{id} \otimes \varphi_t) \circ \Delta(c)$, $c \in \mathcal{C}_b$, we have $\mathrm{T}_{s+t} = \mathrm{T}_s \mathrm{T}_t$, $\mathrm{T}_t \to \mathrm{T}_0$. By what we saw for matrices it follows $\mathrm{T}_t = \mathrm{T}_0 \, \mathrm{e}^{t\tilde{G}}$ and

$$\varphi_t(c) = \delta \circ \mathrm{T}_t(c) = \varphi_0 \star \mathrm{e}_\star^{t\psi}(c) \text{ for } c \in \mathcal{C}_b$$

with $\psi = \delta \circ \tilde{G}$. We also have

$$\frac{\mathrm{d}}{\mathrm{d}t} \varphi_t(c)|_{t=0} = (\varphi_0 \star \psi)(c) = (\psi \star \varphi_0)(c) = \psi(c)$$

for $c \in \mathcal{C}_b$. Since the intersection of two sub-coalgebras is a sub-coalgebra, ψ can be defined on the whole of $\mathcal{B}$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} \varphi_t|_{t=0} = \psi; \; \varphi_t = \varphi_0 \star \mathrm{e}_\star^{t\psi}.$$

A sesqui-linear form L on a vector space $\mathcal{V}$ is called *positive* if $L(v, v) \geq 0$ for all $v \in \mathcal{V}$. In order to prove a Schoenberg type result for convolution semi-groups (on $*$-bialgebras) which do not start at the counit, we proceed like in [10] by showing

Lemma 3.2 *Let C be a coalgebra. We form the tensor product $(\bar{C} \otimes C, \Lambda, \bar{\delta} \otimes \delta)$ of the coalgebras $(\bar{C}, \bar{\Delta}, \bar{\delta})$ and (C, Δ, δ) where $\bar{C}$ denotes the complex conjugate coalgebra of C. Let L_t be linear functionals on $\bar{C} \otimes C$ (that is the L_t are sesqui-linear forms on C) satisfying*

- $L_{s+t} = L_s \star L_t$ *(with respect to Λ)*
- $L_t \to L_0$ *pointwise for $t \to 0+$*

Then for

$$K = \frac{\mathrm{d}}{\mathrm{d}t} L_t|_{t=0}$$

the following conditions are equivalent:

(i) *L_0 is positive and*

$K(c, c) \geq 0$ for all $c \in C$ with $L_0(c, c) = 0$, and $K(c, d) = \overline{K(d, c)}$ for all $c, d \in C$

(ii) *L_t are positive for all $t \in \mathbb{R}_+$*

Proof: The proof is similar to the counit case. We give it here in a version adapted to our situation.— (ii) $\Longrightarrow$ (i) is proved by differentiating. For the proof of (i) $\Longrightarrow$ (ii) it suffices to show that $L_0 \star \mathrm{e}_\star^K$ is positive. Thanks to the fundamental theorem on coalgebras we may restrict ourselves to a finite-dimensional C.

We choose a scalar product S in C. We begin by showing that to each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$L_0(c, c) \leq \delta \text{ and } \| c \| = 1 \implies K(c, c) > -\epsilon.$$

(Notice that by assumtion $K(c, c)$ is real.) To see this we form the sets

$$A_{n,\epsilon} = \{c \in C \mid \| c \| = 1 \text{ and } L_0(c, c) \leq \frac{1}{n} \text{ but } K(c, c) \leq -\epsilon\}.$$

The $A_{n,\epsilon}$ are closed with $\bigcap_n A_{n,\epsilon} = \emptyset$. The latter follows from the fact that $K(c,c) \geq 0$ if $L_0(c,c) = 0$. By compactness there is n_0 such that $A_{n_0,\epsilon} = \emptyset$. Put $\delta = \frac{1}{n_0}$.

Next we show that to each $\epsilon > 0$ there exists n_ϵ such that

$$L_0 + \frac{K + \epsilon S}{n}$$

is positive for all $n \geq n_\epsilon$. By the first part there is a $\delta > 0$ such that for $\| c \| = 1$

$$K(c,c) + \epsilon \geq 0 \text{ if } L_0(c,c) \leq \delta.$$

This means

$$(L_0 + \frac{K + \epsilon S}{n})(c,c) \geq 0$$

for all $c \in \mathcal{C}$ with $\| c \| = 1$ and $L_0(c,c) \leq \delta$. For $c \in \mathcal{C}$ with $\| c \| = 1$ and $L_0(c,c) > \delta$ we find n_ϵ such that

$$\left| \frac{K(c,c) + \epsilon}{n} \right| \leq \frac{\| K \| + \epsilon}{n} \leq \delta$$

for all $n \geq n_\epsilon$. Then

$$\left(L_0 + \frac{K + \epsilon S}{n} \right) (c,c) \geq 0$$

for all $c \in \mathcal{C}$, $\| c \| = 1$, $L_0(c,c) > \delta$, $n \geq n_\epsilon$. Thus

$$L_0 + \frac{K + \epsilon S}{n}$$

is positive for all $n \geq n_\epsilon$. Since the convolution product of two positive forms on $\mathcal{C}$ is positive we have that

$$L_0 \star \left(L_0 + \frac{K + \epsilon S}{n}\right) \star L_0 = L_0 \star \left(\bar{\delta} \otimes \delta + \frac{K + \epsilon\, L_0 \star S \star L_0}{n}\right) \geq 0$$

for all $n \geq n_\epsilon$ and

$$0 \leq L_0 \star \left(L_0 + \frac{K + \epsilon\, L_0 \star S \star L_0}{n} \right)^{\star n} = L_0 \star \left(\bar{\delta} \otimes \delta + \frac{K + \epsilon\, L_0 \star S \star L_0}{n} \right)^{\star n}$$

converges pointwise to the form $L_0 \star \mathrm{e}_\star^{K + \epsilon\, L_0 \star S \star L_0}$ which, therefore, must be positive. By lettting ϵ tend to 0, we arrive at the desired result.◇

As a direct consequence we have

Theorem 3.3 *Let $\mathcal{B}$ be a $*$-bialgebra and let $\varphi_t \in \mathcal{B}'$, $t \in \mathbb{R}_+$, satisfy*

- $\varphi_{s+t} = \varphi_s \star \varphi_t$
- $\varphi_t \to \varphi_0$

Then for $\psi = \frac{\mathrm{d}}{\mathrm{d}t}\varphi_t|_{t=0}$ *the following conditions are equivalent:*

(i) φ_0 *is positive and*

$\psi(b^*b) \geq 0$ *for all* $b \in \mathcal{B}$ *with* $\varphi_0(b^*b) = 0$, *and* $\varphi(b^*) = \overline{\varphi(b)}$ *for all* $b \in \mathcal{B}$

(ii) φ_t *is positive for all* $t \in \mathbb{R}_+$

Proof: We observe that, by applying the mapping

$$\mathcal{F} : \mathcal{B}' \to (\bar{\mathcal{B}} \otimes \mathcal{B})'$$

given by

$$\mathcal{F}(\varphi)(c, d) = \varphi(c^*d),$$

we can reduce everything to the situation of the preceeding lemma.◇

Proof of Theorem 3.1: Let $\mathcal{B} = \mathcal{B}_1//\mathcal{B}_2$ be a double coset hyper-bialgebra. Then we define the homomorphism $\tilde{\pi}$ from $(\mathcal{T}(\mathcal{B}_1), \cdot)$ to $\mathcal{B}_2$ by

$$\tilde{\pi}(b_1 \otimes \ldots \otimes b_n) = b_1 \ldots b_n.$$

It is straightforward to check that $\mathcal{T}(\mathcal{B})$ equals $\mathcal{T}(\mathcal{B}_1)//\mathcal{B}_2$, so that $\mathcal{T}(\mathcal{B})$ is again a double coset hyper-bialgebra. Thus it is sufficient to prove that for a linear functional ψ on a given double coset hyper-bialgebra we have

$$\psi \text{ conditionally positive and hermitian} \implies \varphi_t = \mathrm{e}_\star^{t\psi} \text{ positive}$$

However, $\tilde{\varphi}_t$ and $\tilde{\psi}$ satisfy the conditions of Theorem 3.3 with $\tilde{\varphi}_0 = \lambda \circ \pi$. To see that $\tilde{\psi}$ satisfies (i) of Theorem 3.3 we remark first that $(\lambda \circ \pi)(b^*b) = 0$ if and only if $b \in \ker \pi$ since λ is faithful. Then, using the fact that $\ker \pi$ is a bi-ideal, one shows that, for $b \in \ker \pi$, $\big((\lambda \circ \pi) \otimes \mathrm{id} \otimes (\lambda \circ \pi)\big) \circ \Delta_1^{(3)}\, b^*b$ is of the form $\sum c_i^* c_i$ with $c_i \in \ker \delta$. An application of Theorem 3.3 yields the positivity of $\tilde{\varphi}_t$ and of φ_t. ◇

4. The hyper-bialgebra U⟨2⟩//U⟨1⟩

For $d \in \mathbb{N}$ we denote by $\mathrm{U}\langle d\rangle$ the free (non-commutative!) $*$-algebra generated by indeterminates x_{kl}, $k, l = 1, \ldots, d$, with the unitarity relations

$$\sum_{n=1}^{d} x_{kn} x_{ln}^* = \delta_{kl} \qquad (2)$$

$$\sum_{n=1}^{d} x_{kn}^* x_{nl} = \delta_{kl} \qquad (3)$$

The $*$-algebra $\mathrm{U}\langle d\rangle$ is turned into a $*$-bialgebra if we put

$$\begin{aligned} \Delta_1 x_{kl} &= \sum_{n=1}^{d} x_{kn} \otimes x_{nl} \\ \delta_1 x_{kl} &= \delta_{kl}. \end{aligned}$$

This $*$-bialgebra has been investigated by P. Glockner und W. von Waldenfels [6]. If we assume that the generators x_{kl}, x_{kl}^* commute we obtain the coefficient algebra of the unitary group U_d. This is why $\mathrm{U}\langle d\rangle$ was sometimes called the *non-commutative analogue of the coefficient algebra of the unitary group*. It is equal to the $*$-algebra generated by mappings

$$\xi_{kl} : \mathcal{U}(\mathbb{C}^d \otimes \mathcal{H}) \to \mathcal{B}(\mathcal{H})$$

with

$$\xi_{kl}(U) = U_{kl}, \; U \in \mathcal{U}(\mathbb{C}^d \otimes \mathcal{H}) \subset \mathrm{M}_d(\mathcal{B}(\mathcal{H}))$$

where $\mathcal{H}$ is an infinite-dimensional Hilbert space and $\mathcal{U}(\mathbb{C}^d \otimes \mathcal{H})$ denotes the group of unitary operators on $\mathbb{C}^d \otimes \mathcal{H}$. Moreover, $\mathcal{B}(\mathcal{H})$ is the $*$-algebra of bounded operators on $\mathcal{H}$ and $\mathrm{M}_d(\mathcal{B}(\mathcal{H}))$ denotes the $*$-algebra of $d \times d$-matrices with elements from $\mathcal{B}(\mathcal{H})$.

Proposition 4.1
(a) *On* $\mathrm{U}\langle 1\rangle$ *a faithful Haar measure is given by* $\lambda(x^n) = \delta_{0,n}$, $n \in \mathbb{Z}$.
(b) *On* $\mathrm{U}\langle 1\rangle$ *an antipode is given by setting* $Sx = x^*$ *and extending* S *as a* $*$*-algebra homomorphism.*
(c) *For* $d > 1$ *the* $*$*-bialgebra* $\mathrm{U}\langle d\rangle$ *does not posses an antipode.*

Proof: Only (c) requires a proof. Let us suppose that we are given an antipode S on $\mathrm{U}\langle d\rangle$, $d > 1$. Then

$$\begin{aligned}
\sum_{m=1}^{d}\sum_{n=1}^{d} S(x_{kn})x_{nl}x_{lm}^{*} &= x_{lk}^{*} \\
&= \sum_{n=1}^{d} S(x_{kl}) \sum_{m=1}^{d} x_{nl}x_{lm}^{*} \\
&= \sum_{n=1}^{d} S(x_{kn})\delta_{nl}. \\
&= S(x_{kl})
\end{aligned}$$

Similarly, one proves that $S(x_{kl}^{*}) = x_{lk}$. Since S is an antipode it has to be an algebra anti-homomorphism. Therefore,

$$\begin{aligned}
S(\sum_{n=1}^{d} x_{kn}x_{ln}^{*}) &= \sum_{n=1}^{d} S(x_{ln}^{*})S(x_{kn}) \\
&= \sum_{n=1}^{d} x_{nl}x_{nk}^{*}
\end{aligned}$$

which is not equal to δ_{kl} if $d > 1$.◇

Using the result of Glockner and von Waldenfels, we can describe the coalgebra structure of $\mathrm{U}\langle d\rangle$ as follows. Define a mapping

$$\tilde{\Delta}_1 : \mathrm{U}\langle d\rangle \to \underline{\mathrm{Map}}\left(\mathcal{U}(\mathbb{C}^d \otimes \mathcal{H}) \times \mathcal{U}(\mathbb{C}^d \otimes \mathcal{H}), \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H})\right)$$

by setting

$$\tilde{\Delta}_1\, \xi_{kl}\,(U, V) = \sum_{n=1}^{d} U_{kn} \otimes V_{nl}.$$

An emdedding ι of $\mathrm{U}\langle d\rangle \otimes \mathrm{U}\langle d\rangle$ into $\underline{\mathrm{Map}}\left(\mathcal{U}(\mathbb{C}^d \otimes \mathcal{H}) \times \mathcal{U}(\mathbb{C}^d \otimes \mathcal{H}), \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H})\right)$ is given by

$$\iota(b \otimes c)(U, V) = a(U) \otimes b(V)$$

and we have

$$\tilde{\Delta}_1 \mathrm{U}\langle d\rangle \subset \iota(\mathrm{U}\langle d\rangle \otimes \mathrm{U}\langle d\rangle) \text{ with } \Delta_1 = \iota^{-1} \circ \tilde{\Delta}_1.$$

Let us now apply the construction in the beginning of this paragraph to the situation

$$\mathcal{B}_1 = \mathcal{U}\langle 2\rangle;\ \mathcal{B}_2 = \mathcal{U}\langle 1\rangle = \mathbb{C}\langle x, x^*\rangle / xx^* = \mathbf{1} = x^*x$$

and

$$\begin{pmatrix} \pi(x_{11}) & \pi(x_{12}) \\ \pi(x_{21}) & \pi(x_{22}) \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & \mathbf{1} \end{pmatrix}$$

In order to describe $\mathcal{B}$ in this case, we introduce two gradings l and g on $\mathrm{U}\langle d\rangle$ by setting

$$\begin{aligned} \mathrm{l}\,(x_{kl}^{(\epsilon)}) &= \begin{cases} 1 & \text{if} \quad k = 1 \text{ and } \epsilon = 0 \\ -1 & \text{if} \quad k = 1 \text{ and } \epsilon = 1 \\ 0 & \text{if} \quad k = 2 \text{ and } \epsilon = 1 \end{cases} \\ \mathrm{g}\,(x_{kl}^{(\epsilon)}) &= \mathrm{l}\,(x_{lk}^{(\epsilon)}) \end{aligned}$$

where we use the notation $x_{kl}^{(0)} = x_{kl}$ and $x_{kl}^{(1)} = x_{kl}^*$. Since (2) and (3) are homogeneous elements of the free $*$-algebra generated by x_{kl}, the gradings l and g are well-defined. Denote by $\mathcal{B}_1^{(0)}$ and $\mathcal{B}_{1,(0)}$ the space of homogeneous elements of degree 0 in $\mathrm{U}\langle d\rangle$ in the l- and g-grading repectively.

Proposition 4.2

$$\begin{aligned} \mathcal{B}_1^{(0)} &= \{b \in \mathrm{U}\langle 2\rangle \mid (\pi \otimes \mathrm{id}) \circ \Delta_1 = \mathbf{1} \otimes b\} \\ \mathcal{B}_{1,(0)} &= \{b \in \mathrm{U}\langle 2\rangle \mid (\mathbf{1} \otimes \pi) \circ \Delta_1 = b \otimes \mathbf{1}\} \\ \mathcal{B} &= \mathcal{B}_1^{(0)} \cap \mathcal{B}_{1,(0)} \end{aligned}$$

Proof: We prove the first identity. If we consider $(\pi \otimes \mathrm{id}) \circ \Delta_1 b$ as an element of $\underline{\mathrm{Map}}\,\big(\mathcal{U}(\mathcal{H}) \times \mathcal{U}(\mathbb{C}^d \otimes \mathcal{H}), \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H})\big)$ we have for a monomial $b = \xi_{k_1 l_1}^{(\epsilon_1)} \cdots \xi_{k_n l_n}^{(\epsilon_n)}$

$$\begin{aligned} (\pi \otimes \mathrm{id}) \circ \Delta_1 b\Big(u, \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}\Big) &= \xi_{k_1 l_1}^{(\epsilon_1)} \cdots \xi_{k_n l_n}^{(\epsilon_n)} \begin{pmatrix} u \otimes U_{11} & U \otimes U_{12} \\ \mathbf{1} \otimes U_{21} & \mathbf{1} \otimes U_{22} \end{pmatrix} \\ &= u^{\mathrm{l}(b)} \otimes \xi_{k_1 l_1}^{(\epsilon_1)} \cdots \xi_{k_n l_n}^{(\epsilon_n)} \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}. \end{aligned}$$

For an arbitrary element

$$b = \sum_{n \in \mathbb{Z}} b^{(n)},\ b^{(n)} \in \mathcal{B}_1^{(n)}$$

in $U\langle d\rangle$ we have

$$b\begin{pmatrix} u\otimes U_{11} & U\otimes U_{12} \\ \mathbf{1}\otimes U_{21} & \mathbf{1}\otimes U_{22}\end{pmatrix} = \sum_{n\in\mathbf{Z}} u^n \otimes b^{(n)}\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22}\end{pmatrix}$$

which is equal to

$$\sum_{n\in\mathbf{Z}} \mathbf{1}\otimes b^{(n)}\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22}\end{pmatrix}$$

for all $u \in \mathcal{U}(\mathcal{H})$ and all $U \in \mathcal{U}(\mathbb{C}^d \otimes \mathcal{H})$ if and only if $b^{(n)} = 0$ for $n \neq 0$.◇

$\mathcal{B}$ is not a $*$-bialgebra. We have $\Delta x_{22} = x_{22}\otimes x_{22}$ but

$$\Delta x_{22}x_{22}^* = x_{22}x_{22}^* \otimes x_{22}x_{22}^* + (\mathbf{1} - x_{22}x_{22}^*) \otimes (1 - x_{11}x_{11}^*).$$

Notice that $U_2//U_1$ is the unit sphere S^1, so, in this sense, $U\langle 2\rangle//U\langle 1\rangle$ might be regarded as a non-commutative version of S^1.

Following [11], Section 5, a basic Brownian motion on $\mathcal{B}$ is a basic Lévy process on $\mathcal{B}$ whose generator ψ satisfies

$$\psi(bc) = \psi(b)\delta(c) + \delta(b)\psi(c) + \overline{\mathrm{d}(c^*)}\mathrm{d}(b)$$

where d is a derivation on $\mathcal{B}$, i.e. a linear functional on $\mathcal{B}$ with

$$\mathrm{d}(bc) = \mathrm{d}(b)\delta(c) + \delta(b)\mathrm{d}(c), \; b, c \in \mathcal{B}.$$

5. Examples of generators on $U\langle 2\rangle//U\langle 1\rangle$

We will now consider a class of basic Lévy processes on $\mathcal{B} = U\langle 2\rangle//U\langle 1\rangle$. Let $B = (b_{ij})$ a hermitian 2×2-matrix and let A_{ij}, $1 \le i, j \le 2$, be four complex matrices. Define ρ, η, and ψ on the generators of $U\langle 2\rangle$ by

$$\begin{aligned}
\rho(x_{ij}) &= \delta_1(x_{ij})\mathrm{id}_{\mathcal{M}_d}, & 1 \le i, j \le 2, \\
\rho(x_{ij}^*) &= \delta_1(x_{ij})\mathrm{id}_{\mathcal{M}_d}, & 1 \le i, j \le 2, \\
\eta(x_{ij}) &= A_{ij}, & 1 \le i, j \le 2, \\
\eta(x_{ij}^*) &= -A_{ji}, & 1 \le i, j \le 2, \\
\psi(x_{ij}) &= ib_{ij} - \frac{1}{2}\sum_{k=1}^{2}\langle A_{kj}, A_{ki}\rangle, & 1 \le i, j \le 2, \\
\psi(x_{ij}^*) &= -ib_{ji} - \frac{1}{2}\sum_{k=1}^{2}\langle A_{ki}, A_{kj}\rangle, & 1 \le i, j \le 2,
\end{aligned}$$

where $\langle A, A' \rangle = \sum \overline{a_{ij}} a'_{ij}$ for $A = (a_{ij})$, $A' = (a'_{ij}) \in \mathcal{M}_d$ is a scalar product on $\mathcal{M}_d$. These maps extend to a unique triple on $\mathrm{U}\langle 2 \rangle$ in the sense of Definition 2.3 of [5]. Actually, this is the form of a general Gaussian triple on $\mathrm{U}\langle 2 \rangle$, cf. [11], Section 5 and [5]. The restrictions of ρ, η and ψ to $\mathcal{B}$ define a triple on $\mathcal{B}$ and therefore the quantum differential equation

$$\mathrm{d}j_{st} = j_{st} \star \mathrm{d}I_t \; ; \quad j_{ss} = \delta$$

yields a basic Lévy process on $\mathcal{B}$.
It is instructive to compare this process with the process $\tilde{\jmath}$ on $\mathrm{U}\langle 2 \rangle$ obtained by solving the quantum stochastic equation

$$\mathrm{d}\tilde{\jmath}_{st} = \tilde{\jmath}_{st} \star_1 \mathrm{d}I_t \; ; \quad \tilde{\jmath}_{ss} = \delta_1$$

where $\star_1$ denotes the convolution w.r.t. the coproduct Δ_1 of $\mathrm{U}\langle 2 \rangle$. Even though the differentials appearing in these two quantum differential equations coincide, in general they are different because they come from different coproducts. Therefore one expects the processes to be different, too. This is the case. It can be checked by computing the expectation values or by verifying that j_{st} is not a $*$-homomorphism (wheras $\tilde{\jmath}$ is a Lévy process and therefore always a $*$-homomorphism).

Let us study the first few moments of $\tilde{\jmath}_{0t}$: We have

$$\begin{aligned} \psi(x_{ij}x_{kl}) &= -\langle A_{ji}, A_{kl} \rangle + \delta_{ij}\psi(x_{kl}) + \delta_{kl}\psi(x_{ij}), \\ \psi(x^*_{ij}x_{kl}) &= \langle A_{ij}, A_{kl} \rangle + \delta_{ij}\psi(x_{kl}) + \delta_{kl}\overline{\psi(x_{ij})}, \\ \psi(x_{ij}x^*_{kl}) &= \langle A_{ji}, A_{lk} \rangle + \delta_{ij}\overline{\psi(x_{kl})} + \delta_{kl}\psi(x_{ij}), \\ \psi(x^*_{ij}x^*_{kl}) &= -\langle A_{ij}, A_{lk} \rangle + \delta_{ij}\overline{\psi(x_{kl})} + \delta_{kl}\overline{\psi(x_{ij})} \end{aligned}$$

for the values of ψ on products of the generators. In particular, we have $\psi(x_{11}x^*_{11}) = ||A_{11}||^2 + ib_{11} - \frac{||A_{11}||^2 - ||A_{21}||^2}{2} - ib_{11} - \frac{||A_{11}||^2 - ||A_{21}||^2}{2} = -||A_{21}||^2$, $\psi(x_{22}x^*_{22}) = -||A_{12}||^2$.

Due to the form of the coproduct Δ_1 on $\mathrm{U}\langle 2 \rangle$, we get

$$\begin{aligned} E(\tilde{\jmath}_{0t}(x_{ij})) &= \left(e^{t(\psi(x_{kl}))_{1 \le k,l \le 2}} \right)_{ij}, \\ E(\tilde{\jmath}_{0t}(x^*_{ij})) &= \left(e^{t(\overline{\psi(x_{kl})})_{1 \le k,l \le 2}} \right)_{ij}, \end{aligned}$$

and similar formulae for the second-order elements, i.e. write $\psi(x_{ij}^{(\epsilon)}x_{kl}^{(\epsilon)})$ as a matrix, with the elements ordered in the following way,

$$\psi\begin{pmatrix} x_{11}^{(\epsilon)}x_{11}^{(\epsilon)} & x_{11}^{(\epsilon)}x_{12}^{(\epsilon)} & x_{12}^{(\epsilon)}x_{11}^{(\epsilon)} & x_{12}^{(\epsilon)}x_{12}^{(\epsilon)} \\ x_{11}^{(\epsilon)}x_{21}^{(\epsilon)} & x_{11}^{(\epsilon)}x_{22}^{(\epsilon)} & x_{12}^{(\epsilon)}x_{21}^{(\epsilon)} & x_{12}^{(\epsilon)}x_{22}^{(\epsilon)} \\ x_{21}^{(\epsilon)}x_{11}^{(\epsilon)} & x_{21}^{(\epsilon)}x_{12}^{(\epsilon)} & x_{22}^{(\epsilon)}x_{11}^{(\epsilon)} & x_{22}^{(\epsilon)}x_{12}^{(\epsilon)} \\ x_{21}^{(\epsilon)}x_{21}^{(\epsilon)} & x_{21}^{(\epsilon)}x_{22}^{(\epsilon)} & x_{22}^{(\epsilon)}x_{21}^{(\epsilon)} & x_{22}^{(\epsilon)}x_{22}^{(\epsilon)} \end{pmatrix}$$

and then exponentiate this matrix.
For the moments of j_{0t} we get

$$\begin{aligned} E(j_{0t}(x_{22})) &= e^{t\psi(x_{22})} = \exp\left(itb_{22} - t\frac{||A_{12}||^2 + ||A_{22}||^2}{2}\right), \\ E(j_{0t}(x_{22}^*)) &= e^{t\overline{\psi(x_{22})}} = \exp\left(-itb_{22} - t\frac{||A_{12}||^2 + ||A_{22}||^2}{2}\right), \end{aligned}$$

and

$$\begin{aligned} E(j_{0t}(x_{22}x_{22})) &= e^{t\psi(x_{22}x_{22})} = \exp\left(2itb_{22} - t||A_{12}||^2 - 2t||A_{22}||^2\right), \\ E(j_{0t}(x_{22}^*x_{22}^*)) &= e^{t\overline{\psi(x_{22}x_{22}^*)}} = \exp\left(-2itb_{22} - t||A_{12}||^2 - 2t||A_{22}||^2\right). \end{aligned}$$

We have

$$\begin{aligned} \Delta x_{11}x_{11}^* &= x_{11}x_{11}^* \otimes x_{11}x_{11}^* + (1 - x_{11}x_{11}^*) \otimes (1 - x_{22}x_{22}^*), \\ \Delta x_{22}x_{22}^* &= x_{22}x_{22}^* \otimes x_{22}x_{22}^* + (1 - x_{22}x_{22}^*) \otimes (1 - x_{11}x_{11}^*), \end{aligned}$$

and therefore

$$\begin{aligned} \dot{\varphi}_1 &= \varphi_1\psi(x_{11}x_{11}^*) - (1-\varphi_1)\psi(x_{22}x_{22}^*), \\ \dot{\varphi}_2 &= \varphi_2\psi(x_{22}x_{22}^*) - (1-\varphi_2)\psi(x_{11}x_{11}^*), \end{aligned}$$

for $\varphi_i(t) = E(j_{0t}(x_{ii}x_{ii}^*))$, $i = 1,2$. Since $\varphi_i(0) = \delta(x_{ii}x_{ii}^*) = 1$, we get

$$\begin{aligned} E(j_{0t}(x_{11}x_{11}^*)) &= \varphi_1(t) = \frac{\psi(x_{11}x_{11}^*)e^{t(\psi(x_{11}x_{11}^*)+\psi(x_{22}x_{22}^*))} + \psi(x_{22}x_{22}^*)}{\psi(x_{11}x_{11}^*) + \psi(x_{22}x_{22}^*)}, \\ E(j_{0t}(x_{22}x_{22}^*)) &= \varphi_2(t) = \frac{\psi(x_{22}x_{22}^*)e^{t(\psi(x_{11}x_{11}^*)+\psi(x_{22}x_{22}^*))} + \psi(x_{11}x_{11}^*)}{\psi(x_{11}x_{11}^*) + \psi(x_{22}x_{22}^*)}, \end{aligned}$$

6. Quantum stochastic differential equations

On U⟨2⟩ we have

$$\mathrm{d}\tilde{j}_{st}(x_{ij}) = \sum_{k=1}^{2} \tilde{j}_{st}(x_{ik})\mathrm{d}I_t(x_{kj}),$$

where

$$I_t(x_{kj}) = A_t^*(A_{kj}) - A_t(A_{jk}) + \psi(x_{kj})t$$

On $\mathcal{B}$ we have, e.g.,

$$\mathrm{d}j_{st}(x_{22}) = j_{st}(x_{22})\mathrm{d}I_t(x_{22}), \qquad \mathrm{d}j_{st}(x_{22}^*) = j_{st}(x_{22}^*)\mathrm{d}I_t(x_{22}^*),$$

and

$$\begin{aligned} \mathrm{d}j_{st}(x_{22}x_{22}^*) &= j_{st}(x_{22}x_{22}^*)\mathrm{d}I_t(x_{22}x_{22}^*) + (j_{st}(x_{22}x_{22}^*) - \mathrm{id})\mathrm{d}I_t(x_{11}x_{11}^*) \\ &= j_{st}(x_{22}x_{22}^*)(\mathrm{d}I_t(x_{22}x_{22}^*) + \mathrm{d}I_t(x_{11}x_{11}^*)) - \mathrm{d}I_t(x_{11}x_{11}^*). \quad (4) \end{aligned}$$

For $j_{st}(x_{22})j_{st}(x_{22}^*)$, on the other hand, we get

$$\begin{aligned} \mathrm{d}(j_{st}(x_{22})j_{st}(x_{22}^*)) &= j_{st}(x_{22})\mathrm{d}j_{st}(x_{22}^*) + \mathrm{d}j_{st}(x_{22})j_{st}(x_{22}^*) + \mathrm{d}j_{st}(x_{22}) \bullet \mathrm{d}j_{st}(x_{22}^*) \\ &= j_{st}(x_{22})j_{st}(x_{22}^*)\mathrm{d}I_t(x_{22}^*) + j_{st}(x_{22})j_{st}(x_{22}^*)\mathrm{d}I_t(x_{22}) \\ &\quad + j_{st}(x_{22})j_{st}(x_{22}^*)\mathrm{d}I_t(x_{22}) \bullet \mathrm{d}I_t(x_{22}^*) \\ &= j_{st}(x_{22})j_{st}(x_{22}^*)(\mathrm{d}I_t(x_{22}^*) + \mathrm{d}I_t(x_{22}) + \mathrm{d}I_t(x_{22}) \bullet \mathrm{d}I_t(x_{22}^*)) \end{aligned}$$

But since $\mathrm{d}I_t$ is a $*$-homomorphism on ker δ_1 and $\mathrm{d}I_t(1) = 0$, we get

$$\begin{aligned} & \mathrm{d}I_t(x_{22}^*) + \mathrm{d}I_t(x_{22}) + \mathrm{d}I_t(x_{22}) \bullet \mathrm{d}I_t(x_{22}^*) \\ = \;& \mathrm{d}I_t(x_{22}^* - 1) + \mathrm{d}I_t(x_{22} - 1) + \mathrm{d}I_t(x_{22} - 1) \bullet \mathrm{d}I_t(x_{22}^* - 1) \\ = \;& \mathrm{d}I_t(x_{22} - 1 + x_{22}^* - 1 + (x_{22} - 1)(x_{22}^* - 1)) \\ = \;& \mathrm{d}I_t(x_{22}x_{22}^* - 1) = \mathrm{d}I_t(x_{22}x_{22}^*), \end{aligned}$$

and therefore

$$\mathrm{d}(j_{st}(x_{22})j_{st}(x_{22}^*)) = j_{st}(x_{22})j_{st}(x_{22}^*)\mathrm{d}I_t(x_{22}x_{22}^*) \qquad (5)$$

We see that the quantum stochastic differential equations (4) and (5) differ if $\mathrm{d}I_t(x_{11}x_{11}^*) \neq 0$, and therefore we get

$$j_{st}(x_{22})j_{st}(x_{22}^*) \neq j_{st}(x_{22}x_{22}^*)$$

in that case, i.e., j_{st} is not a homomorphism. Note that $I_t(x_{11}x_{11}^*)$ and $I_t(x_{22}x_{22}^*)$ are of the form

$$\begin{aligned} I_t(x_{11}x_{11}^*) &= -2A_t^*(A_{11}) - 2A_t(A_{11}) - ||A_{21}||^2 t, \\ I_t(x_{22}x_{22}^*) &= -2A_t^*(A_{22}) - 2A_t(A_{22}) - ||A_{12}||^2 t \end{aligned}$$

since $\psi(x_{11}x_{11}^*) = -||A_{21}||^2$, $\psi(x_{22}x_{22}^*) = -||A_{12}||^2$ and

$$\eta(x_{ii}x_{ii}^*) = \rho(x_{ii})\eta(x_{ii}^*) - \eta(x_{ii})\delta_1(x_{ii}^*) = -2A_{ii},$$

$i = 1, 2$, so that it is not difficult to give the explicite solutions of (4) and (5).

References

[1] Accardi, L., Frigerio, A., Lewis, J.T.: Quantum stochastic processes. Publ. RIMS Kyoto Univ. **18**, 97-133 (1982)

[2] Bloom, W.R., Heyer, H.: Harmonic analysis of probability measures on hypergroups. Berlin New York: de Gruyter 1995

[3] Chapovsky, Yu.A., Vainerman. L.I.: Compact quantum hypergroups. J. Operator Theory **41**, 1-29 (1999)

[4] Franz, U., Schott, R.: Stochastic processes and operator calculus on quantum groups. Dordrecht Boston London: Kluwer 1999

[5] Franz, U.: Lévy processes on quantum groups. To appear in: "Probability on Algebraic Structures" (G. Budzban, Ph. Feinsilver and A. Mukherjea, Eds.), Contemporary Mathematics, American Mathematical Society, 2000. EMAU Greifswald Preprint-Reihe Mathematik 7/99, 1999.

[6] Glockner, P., Waldenfels, W.v.: The relations of the non-commutative coefficient algebra of the unitary group. SFB-Preprint Nr. 460 Heidelberg 1988

[7] Hudson, R.L., Parthasarathy, K.R.: Quantum Ito's formula and stochastic evolutions. Commun. Math. Phys. **93**, 301-323 (1984)

[8] Meyer, P.-A.: Quantum probability for probabilists. (Lect. Notes Math., vol. 1538). Berlin Heidelberg new York: Springer 1993

[9] Parthasarathy, K.R.: An introduction to quantum stochastic calculus. Basel Boston Berlin: Birkhäuser 1992

[10] Schürmann, M.: Positive and conditionally positive linear functionals on coalgebras. In: Accardi, L., Waldenfels, W.v. (eds.) Quantum probability and applications II. proceedings Oberwolfach 1984. (Lect. Notes Math., vol. 1136). Berlin Heidelberg New York: Springer 1985

[11] Schürmann, M.: White noise on bialgebras. (Lect. Notes math., vol. 1544). Berlin Heidelberg New York: Springer 1993

[12] Schürmann, M.: Operator processes majorizing their quadratic variation. To appear in: Inf. Dim. Analysis Quantum Prob.

[13] Streater, R.F.: Classical and quantum probability. Preprint London 1999

[14] Sweedler, M.E.: Hopf algebras. New York: Benjamin 1969

Samples of algebraic central limit theorems based on Z/2Z

Yukihiro Hashimoto †

1. Introduction

Random walks associated with subgroups of an infinitly many free product G of $\mathbf{Z}/2\mathbf{Z}$ bring us various samples of algebraic central limit theorems. Let $\mathbf{F}_i, \sigma_i$ be a copy of $\mathbf{Z}/2\mathbf{Z}$ and its generator σ. Taking the left regular representation of G on $l^2(G)$, a pair $(\mathcal{A}, \phi)$ of a group $*$-algebra $\mathcal{A}$ of G and a tracial state $\phi(\cdot) := \langle \cdot\delta_e, \delta_e\rangle$ is considered an algebraic probability space, where δ_e is a characteristic function of the unit e of G.

It is well-known fact that the limit distribution under ϕ associated with a discrete Laplacian

$$\frac{\sigma_1 + \sigma_2 + \cdots + \sigma_N}{\sqrt{N}}$$

converges to the Wigner semi-circle law $\frac{1}{2\pi}\chi_{[-2,2]}\sqrt{4-x^2}dx$, of which limit process has a free Fock representation

$$\lim_{N\to\infty} \phi\left((\frac{\sigma_1 + \sigma_2 + \cdots + \sigma_N}{\sqrt{N}})^m \right) = \langle (A^\dagger + A)^m \mathbf{1}, \mathbf{1}\rangle,$$

where $A^\dagger$ and A are canonical creation and annihilation operators actiong on an 1-mode free Fock space $\Gamma(\mathbf{C})$ with a cyclic element $\mathbf{1}$.

Let us take a sequence $\{w_{ij} := \sigma_i\sigma_j \mid i \neq j\}$. The assymptotic behavior of a Laplacian

$$\frac{1}{\sqrt{N(N-1)}} \sum_{1\le i\ne j\le N} w_{ij}$$

under ϕ is grasped as a special case $\lambda = 1$ of a Fock representation

$$\langle (A^\dagger + A + \lambda P)^m \mathbf{1}, \mathbf{1}\rangle,$$

where P is a projection orthogonal to the vacuum $\mathbf{1}$, that coinsides with a representation obtained in the studies of Haagerup state [14] and [2], [3] where the concept of the singleton independence was investigated. Starting with a partial sum

$$S_2(\gamma, N) := \frac{1}{\sqrt{v}} \sum_{\substack{1\le i<j\le N \\ i\le \max\{\gamma N, 1\}}} (w_{ij} + w_{ji})$$

where v is a constant so that $\phi(S_2(\gamma, N)) = 1$, the limit process has a representation, for instance, if γ equals to a constant $0 \le \alpha \le 1$,

$$\lim_{N\to\infty} \phi\left(S_2(\gamma,N)^m\right) = \left\langle \left(\sqrt{\frac{\alpha}{2-\alpha}}(A^\dagger + A + P) + \sqrt{\frac{1-\alpha}{2-\alpha}}(X^\dagger + X + Y^\dagger + Y + Q + R) \right)^m \mathbf{1}, \mathbf{1} \right\rangle$$

on a 4-mode Fock space Γ, a free product of four 1-mode Fock spaces, where $A, A^\dagger, X, X^\dagger, Y, Y^\dagger$ are canonical creations and annihilations and P, Q, R are projections orthogonal to $\mathbf{1}$ with certain mutual relations (section 4).

Considering sequences such as $\{w_{ijk} = \sigma_i\sigma_j\sigma_k \mid i,j,k : \text{diffrent each other}\}$ drives us into another generalization. The asymptotic behavior of

$$\frac{1}{\sqrt{N(N-1)(N-2)}} \sum_{\substack{1\le i,j,k\le N \\ i,j,k\,:\,\text{different each other}}} w_{ijk}$$

has a representation

$$\langle ((A^\dagger)^3 + B^\dagger + B + A^3)^m \mathbf{1}, \mathbf{1}\rangle$$

on a 1-mode Fock space, where $A^\dagger$ and A are canonical creation and annihilation operators, $B^\dagger$ and B are 'conditional' creation and annihilation ones, which kill the vacuum $\mathbf{1}$, acting on the subspace orthogonal to $\mathbf{1}$ where $A^\dagger = B^\dagger$ and $A = B$ hold. (The term 'conditional' is borrowed from the significant paper [7].)

†This work is supported by JSPS Research Fellow.

Throughout this study, the lattice path counting works effectively, which gives exact solutions to moment problems associated with some of these limit processes, with the help of the reflection method (e.g. [16]) and residue calculi. In the case of the last sample, a residue

$$f(t) := Res_{z=0}\frac{1-z^6}{(1-t(z^3+z+\frac{1}{z}+\frac{1}{z^3}))z}$$

gives the moment generating function $F(t) = 1/(1-t^2f(t))$.

The aim of this study is to collect samples of algebraic central limit theorems for detecting new concepts of independences in the sense of the algebraic probability theory, in a category of 'non-free' algebra. Such researchs on relations between the independences and algebraic relations will bring us interpolative concepts from the classical independence to the free independence. It is an important work to interpret these samples in terms of interacting Fock spaces [1], giving us a united understanding of algebraic central limit theorems.

2. The Wigner semi-circle law on $*\mathbf{Z}/2\mathbf{Z}$

Let F_i and σ_i be a copy of $\mathbf{Z}/2\mathbf{Z}$ and its generator respectively. Taking the left regular representation π of $G = *F_i$, an infinitly many product of F_i's, a pair $(\mathcal{A},\phi)$ of a group $*$-algebra $\mathcal{A}$ of G and a tracial state $\phi(\cdot) := \langle\cdot\delta_e,\delta_e\rangle$ is considered an algebraic probability space, where δ_e is a characteristic function of the unit e of G.

To obtain the algebraic central limit theorem with respect to freely independent elements σ_i's,

$$S_1(N) := \frac{\sigma_1+\sigma_2+\cdots+\sigma_N}{\sqrt{N}},$$

let us observe the action of each terms $\pi(\sigma_{i_1})\pi(\sigma_{i_2})\cdots\pi(\sigma_{i_m})/(\sqrt{N})^m$ on δ_e, in an expansion of

$$\left(\frac{\pi(\sigma_1)+\pi(\sigma_2)+\cdots+\pi(\sigma_N)}{\sqrt{N}}\right)^m$$

(abbreviate π, the rest). Since σ_i's are algebraic free, only the terms with the subindices forming a non-crossing pair partition survive in the limit $N\to\infty$. For a term $\sigma_{i_1}\cdots\sigma_{i_m}$, the rule

$$\begin{array}{llll} \sigma_{i_m} & \longleftrightarrow & \nwarrow, & \\ \sigma_{i_k} & \longleftrightarrow & \nwarrow, & \text{if } |\sigma_{i_k}\sigma_{i_{k+1}}\cdots\sigma_{i_m}| > |\sigma_{i_{k+1}}\cdots\sigma_{i_m}| \quad \text{and} \\ \sigma_{i_k} & \longleftrightarrow & \swarrow, & \text{if } |\sigma_{i_k}\sigma_{i_{k+1}}\cdots\sigma_{i_m}| < |\sigma_{i_{k+1}}\cdots\sigma_{i_m}|, \end{array}$$

gives a correspondence of the terms $\sigma_{i_1}\cdots\sigma_{i_m}$ to sequences $\swarrow\cdots\nwarrow$ of up-down arrows, where $|\sigma_{i_1}\cdots\sigma_{i_m}|$ denotes the reduced length of the product. Such a sequence $\epsilon_1\cdots\epsilon_m$ of arrows $\epsilon_i = \nwarrow$ or $\swarrow$ satisfies

$$\#\{i \mid \epsilon_i = \nwarrow, k\le i\le m\} \ge \#\{i \mid \epsilon_i = \swarrow, k\le i\le m\}, \qquad \text{for } k>1 \qquad \text{and}$$
$$\#\{i \mid \epsilon_i = \nwarrow, 1\le i\le m\} = \#\{i \mid \epsilon_i = \swarrow, 1\le i\le m\},$$

which is called *a sequence of Catalan type* here. $\eta_1(\epsilon_1\cdots\epsilon_m)$ denotes *the height* of $\epsilon_1\cdots\epsilon_m$ defined as $\eta_1(\epsilon_1\cdots\epsilon_m)$ $= \eta_1(\epsilon_1)+\cdots+\eta_1(\epsilon_m)$ where $\eta_1(\nwarrow) = +1$ and $\eta_1(\swarrow) = -1$. Then, a sequence $\epsilon_1\cdots\epsilon_m$ is of Catalan type if and only if $\eta_1(\epsilon_k\cdots\epsilon_m)\ge 0$ $(k>1)$ and $\eta_1(\epsilon_1\cdots\epsilon_m) = 0$ hold. The number of terms of corresponding to a sequence $\epsilon_1\cdots\epsilon_m$ of Catalan type is

$$N(N-1)\cdots(N-\frac{m}{2}+1);$$

of order $O((\sqrt{N})^m)$, allowing an expression

$$M_m := \lim_{N\to\infty}\phi\left(\left(\frac{\sigma_1+\sigma_2+\cdots+\sigma_N}{\sqrt{N}}\right)^m\right) = \#\{\text{sequence } \epsilon_1\cdots\epsilon_m \text{ of up-down arrows of Catalan type}\}.$$

Taking $\nwarrow$ for a creation and $\swarrow$ for an annihilation, the right hand side coinsides with a Fock representation

$$\langle (A^\dagger + A)^m\mathbf{1},\mathbf{1}\rangle,$$

where $A^\dagger$ and A are canonical creation and annihilation operators respectively actiong on an 1-mode free Fock space $\Gamma(\mathbf{C})$ with a cyclic element $\mathbf{1}$.

A sequence $\epsilon_1\cdots\epsilon_{2m}$ of up-down arrows of Catalan type corresponds to *a Catalan path*: a minimal path on a lattice $\mathbf{Z}^2$ from $(0,0)$ to (m,m) laying under the diagonal line $y = x+1$. The reflection method (*cf.* [16][22]) shows that the number of Catalan paths with length $2m$ equals to

$$\#\{\text{ minimal path from } (0,0) \text{ to } (m,m)\ \} - \#\{\text{ minimal path from } (-1,1) \text{ to } (m,m)\ \},$$

which is eqivalent to

$$\begin{aligned}
&[z^0]\left(z+\frac{1}{z}\right)^m - [z^2]\left(z+\frac{1}{z}\right)^m \\
&= [z^0]\left(z+\frac{1}{z}\right)^m - [z^{-2}]\left(z+\frac{1}{z}\right)^m \\
&= \text{ constant term in } (1-z^2)\left(z+\frac{1}{z}\right)^m \\
&= Res_{z=0}\left\{\left(\frac{1-z^2}{z}\right)\left(z+\frac{1}{z}\right)^m\right\},
\end{aligned}$$

where $[z^k]f(z)$ denotes a coefficient of z^k in a Laurent series $f(z)$. Then a residue calculus gives the moment generating function

$$\begin{aligned}
f(t) &= \sum_{m=0}^{\infty} M_m t^m \\
&= Res_{z=0}\frac{1-z^2}{\left(1-t(z+\frac{1}{z})\right)z} \\
&= \frac{1-\sqrt{1-4t^2}}{2t^2}.
\end{aligned}$$

As the Cauchy transform of the limit distribution μ associated with $S_1(N)$ equals to

$$\frac{1}{t}f\left(\frac{1}{t}\right) = \frac{t-\sqrt{t^2-4}}{2},$$

the Stieltjes inversion formula (*cf.*[5]) yields the Wigner law

$$d\mu = \frac{1}{2\pi}\chi_{[-2,2]}\sqrt{4-x^2}dx.$$

3. Folding of free elements I

Let us consider elements $w_{ij} := \sigma_i\sigma_j$ $(i \neq j)$, which are not free each other. A noticeable difference from the previous section is that, in some cases, a muliplication by w_{ij} fixes the reduced length of a product, e.g., $|w_{12}w_{23}| = |\sigma_1\sigma_3| = 2 = |w_{23}|$. Thus, an observation of the action of a product $w_{i_1j_1}\cdots w_{i_mj_m}$ on δ_e allows a correspondens of such a product to a sequence of symbols $\nwarrow$, $\swarrow$ and $\smile$ by way of the rule

$$\begin{array}{llll}
w_{i_mj_m} & \longleftrightarrow & \nwarrow, & \\
w_{i_kj_k} & \longleftrightarrow & \nwarrow, & \text{if } |w_{i_kj_k}w_{i_{k+1}j_{k+1}}\cdots w_{i_mj_m}| > |w_{i_{k+1}j_{k+1}}\cdots w_{i_mj_m}|, \\
w_{i_kj_k} & \longleftrightarrow & \smile, & \text{if } |w_{i_kj_k}w_{i_{k+1}j_{k+1}}\cdots w_{i_mj_m}| = |w_{i_{k+1}j_{k+1}}\cdots w_{i_mj_m}|, \quad \text{and} \\
w_{i_kj_k} & \longleftrightarrow & \swarrow, & \text{if } |w_{i_kj_k}w_{i_{k+1}j_{k+1}}\cdots w_{i_mj_m}| < |w_{i_{k+1}j_{k+1}}\cdots w_{i_mj_m}|.
\end{array}$$

By definition, for a product $w_{i_1j_1}\cdots w_{i_mj_m}$,

$$\phi(w_{i_1j_1}\cdots w_{i_mj_m}) = 1$$

holds provided that the sequence $i_1j_1\cdots i_mj_m$ of subindices forms a non-crossing pair partition with $i_k \neq j_k$ $(k = 1,\ldots,m)$, and as seen in the previous section, only such products survive in the limit $N \to \infty$. Those products correspond to sequences $\epsilon_1\cdots\epsilon_m$ of symbols $\nwarrow$, $\swarrow$ and $\smile$ of *Catalan type with inner singletons* [2]:

Definition 3.1. A sequence $\epsilon_1\cdots\epsilon_m$ of symbols $\nwarrow$, $\swarrow$ and $\smile$ is called *Catalan type with inner singletons* provided that

(i) the rest sequence $\epsilon_{i_1}\cdots\epsilon_{i_k}$ removed all $\smile$'s from $\epsilon_1\cdots\epsilon_m$ is of Catalan type.

(ii) $\eta_2(\epsilon_{k+1}\cdots\epsilon_m) > 0$ holds if $\epsilon_k = \smile$, where $\eta_2(\epsilon_1\cdots\epsilon_m)$ denotes *the height* of $\epsilon_1\cdots\epsilon_m$ defined as $\eta_2(\epsilon_1\cdots\epsilon_m) = \eta_2(\epsilon_1)+\cdots+\eta_2(\epsilon_m)$, $\eta_2(\nwarrow) = +2$, $\eta_2(\swarrow) = -2$ and $\eta_2(\smile) = 0$. $\smile$ is called *an inner singleton* here.

Since the number of terms in an expansion of

$$S_2(N)^m := \left(\frac{1}{\sqrt{N(N-1)}}\sum_{1\leq i\neq j\leq N} w_{ij}\right)^m$$

corresponding to the same sequence $\epsilon_1 \cdots \epsilon_m$ of Catalan type with inner singletons, which is equivalent to nothing but the number of sequences $i_1 j_1 \cdots i_m j_m$ of subindices forming non-crossing pair partitions with $i_k \neq j_k$ $(k = 1, \dots, m)$, equals to

$$m! \binom{N}{m} = O(N^m),$$

the m-th moment has an expression

$$\lim_{N \to \infty} \phi(S_2(N)^m) = \#\{\text{sequence } \epsilon_1 \cdots \epsilon_m \text{ of Catalan type with inner singletons}\}.$$

$A^\dagger$, A and P denote a creation, an annihilation and a projection othogonal to the vacuum $\mathbf{1}$ respecticely, acting on an 1-mode free Fock space $\Gamma(\mathbf{C})$. Then, taking $\nwarrow$, $\swarrow$ and $\smile$ for $A^\dagger$, A and P respectively yields a Fock representation for assymptotic behavior of $S_2(N)$:

Theorem 3.2.

$$\lim_{N \to \infty} \phi \left(\left(\frac{1}{\sqrt{N(N-1)}} \sum_{1 \leq i \neq j \leq N} w_{ij} \right)^m \right) = \left\langle (A^\dagger + A + P)^m \mathbf{1}, \mathbf{1} \right\rangle.$$

In the investigation of the Haagerup state [2], a general representation

$$\left\langle (A^\dagger + A + \lambda P)^m \mathbf{1}, \mathbf{1} \right\rangle$$

with a parameter λ. A description

$$\left\langle (A^\dagger + A + \lambda P)^m \mathbf{1}, \mathbf{1} \right\rangle = \sum_{k=0}^{m-2} \#\{\epsilon_1 \cdots \epsilon_m \ : \text{ of Catalan type with } k \text{ inner singletons } \} \cdot \lambda^k$$

is connected with a lattce path counting on $\mathbf{Z}^2$ by way of the rule

$$\begin{array}{lll} \nwarrow & \longleftrightarrow & \Omega_+ : \ (x,y) \to (x+1,y) \to (x+2,y), \\ \swarrow & \longleftrightarrow & \Omega_- : \ (x,y) \to (x,y+1) \to (x,y+2) \qquad \text{and} \\ \smile & \longleftrightarrow & \Omega_0 : \ (x,y) \to (x,y+1) \to (x+1,y+1). \end{array}$$

A sequence $\epsilon_1 \cdots \epsilon_m$ of Catalan type with inner singletons corresponds to a lattice path $\omega_1 \cdots \omega_m$ from $(0,0)$ to (m,m) which consist of moves Ω_+, Ω_- and Ω_0, walking under the line $y = x + 1$ without accrossing the diagonal $y = x$. Let l be the largest number that $\eta_2(\epsilon_l \cdots \epsilon_m) = 0$ holds, then by definition, $\epsilon_m = \nwarrow$, $\epsilon_l = \swarrow$ and $2 \leq l \leq m$. In the part $\epsilon_{l+1} \cdots \epsilon_{m-1}$, $\smile$'s occur with no restrictions: only **Definition 3.1** (i) holds, named *of Catalan type with singletons*. The corresponding path $\omega_{l+1} \cdots \omega_{m-1}$ lays under the line $y = x$ without accrossing the line $y = x - 1$, connecting $(2,0)$ with $(m - l + 1, m - l - 1)$. Putting

$$\begin{aligned} F_m &:= \sum_{k=0}^{m-2} \#\{\epsilon_1 \cdots \epsilon_m \ : \text{ of Catalan type with } k \text{ inner singletons } \} \cdot \lambda^k \qquad \text{and} \\ f_m &:= \sum_{k=0}^{m} \#\{\epsilon_1 \cdots \epsilon_m \ : \text{ of Catalan type with } k \text{ singletons } \} \cdot \lambda^k, \end{aligned}$$

the decomposition

$$\epsilon_1 \cdots \epsilon_m = \epsilon_1 \cdots \epsilon_{l-1} \cdot \swarrow \epsilon_{l+1} \cdots \epsilon_{m-1} \nwarrow$$

implies a recurrence formula

$$F_m = \sum_{l=0}^{m-2} F_{l-1} f_{m-l-1}, \tag{3.1}$$

which is nothing but a conditional moment-cumulant formula [7] with a cumulant $R_2(\swarrow, \nwarrow) = 1$. Since $\smile$'s have no restrictions in the sequence $\epsilon_1 \cdots \epsilon_m$ of Catalan type **with singletons**, it follows that

$$\begin{aligned} &\#\{\epsilon_1 \cdots \epsilon_m \ : \text{ of Catalan type with } k \text{ singletons } \} \\ &= \binom{m}{k} \#\{\epsilon_1 \cdots \epsilon_{m-k} \ : \text{ of Catalan type}\} \\ &= \binom{m}{k} \cdot \text{ constant term in } (1 - z^2) \left(z + \frac{1}{z} \right)^{m-k}. \end{aligned}$$

Hence

$$\begin{aligned} f_m &= \sum_{k=0}^{m} \binom{m}{k} \lambda^k \cdot \text{ constant term in } (1-z^2)\left(z+\frac{1}{z}\right)^{m-k} \\ &= \text{constant term in } (1-z^2)\left(z+\frac{1}{z}+\lambda\right)^m \\ &= Res_{z=0}\left\{\left(\frac{1-z^2}{z}\right)\left(z+\frac{1}{z}+\lambda\right)^m\right\}. \end{aligned}$$

Then the generating function

$$f(t) := \sum_{m=0}^{\infty} f_m t^m$$

is given by

$$\begin{aligned} f(t) &= \sum_{m=0}^{\infty} Res_{z=0}\left\{\left(\frac{1-z^2}{z}\right)\left(z+\frac{1}{z}+\lambda\right)^m\right\} t^m \\ &= Res_{z=0}\left\{\frac{1-z^2}{\left(1-t\left(z+\frac{1}{z}+\lambda\right)\right)z}\right\} \\ &= \frac{1-\lambda t-\sqrt{((\lambda+2)t-1)((\lambda-2)t-1)}}{2t^2}. \end{aligned}$$

In view of (3.1), the generating function

$$F(t) := \sum_{m=0}^{\infty} F_m t^m$$

has a functional equation

$$F(t)-1 = t^2 f(t) F(t),$$

and hence

$$F(t) = \frac{1+\lambda t-\sqrt{((\lambda+2)t-1)((\lambda-2)t-1)}}{2(\lambda+t)t}.$$

The Cauchy transform $G(t)$ of the distribution μ_λ associated with the operator $A^\dagger + A + \lambda P$ under the tracial state $\langle \cdot \mathbf{1}, \mathbf{1}\rangle$ is given by

$$\begin{aligned} G(t) &= \frac{1}{t} F\left(\frac{1}{t}\right) \\ &= \frac{t+\lambda-\sqrt{(\lambda+2-t)(\lambda-2-t)}}{2(1+\lambda t)}. \end{aligned} \tag{3.2}$$

Again, the Stieltjes inversion formula yields a non-symmetric deformation of the semi-circle law:

Theorem 3.3. The distribution μ_λ associated with the operator $A^\dagger + A + \lambda P$ under the tracial state $\langle \cdot \mathbf{1}, \mathbf{1}\rangle$ is given by

$$\mu_\lambda = \begin{cases} \tilde{\mu}_\lambda, & \lambda^2 \leq 1, \\ \left(1-\frac{1}{\lambda^2}\right)\delta_{-1/\lambda} + \tilde{\mu}_\lambda, & \lambda^2 \geq 1, \end{cases}$$

where

$$d\tilde{\mu}_\lambda = \frac{1}{2\pi}\chi_{[\lambda-2,\lambda+2]}(x)\frac{\sqrt{(\lambda+2-x)(x-\lambda+2)}}{1+\lambda x}dx \tag{3.3}$$

for any $\lambda \in \mathbf{R}$.

Remark. In the study of Haagerup state [15], the same distribution (3.3) is obtained only for $-1 \le \lambda \le 0$. Moreover, a coordinate exchange

$$t = 1 + \lambda x \qquad \text{and} \qquad \beta = \lambda^2$$

give a connection with the free Poisson distribution (*cf.* [7])

$$\pi_{\beta,\beta} = \begin{cases} (1-\beta)\delta_0 + \tilde{\pi}_{\beta,\beta}, & 0 \le \beta \le 1, \\ \tilde{\pi}_{\beta,\beta}, & 1 \le \beta, \end{cases}$$

where

$$\begin{aligned} d\tilde{\pi}_{\beta,\beta} &= \frac{1}{2\pi}\chi_{[(1-\sqrt{\beta})^2,(1+\sqrt{\beta})^2]}(t)\frac{\sqrt{4\beta-(t-1-\beta)^2}}{t}dt \\ &= \lambda^2 d\tilde{\mu}_\lambda. \end{aligned}$$

According to a relation between the Cauchy transform of a distribution and its orthogonal polynomials (*cf.*[32]), a continued fractional expression

$$g(t) = \cfrac{1}{t - b_1 - \cfrac{c_2}{t - b_2 - \cfrac{c_3}{t - b_3 - \cdots}}}$$

of the Cauchy transform of a measure induces recurrence relations among its *monic* orthogonal polynomials $\{p_n(t)\}$,

$$\begin{aligned} &p_0(t) = 1, \qquad p_1(t) = t - b_1, \\ &p_n(t) = (t - b_n)p_{n-1}(t) - c_n p_{n-2}(t) \qquad (n \ge 2). \end{aligned}$$

In the case of $G(t)$ in (3.2), a direct calculation gives an unfavorable expression (*cf.*[7])

$$G(t) = \cfrac{1}{t + \lambda - \cfrac{1+\lambda t}{t + \lambda - \cfrac{1 + \lambda t}{t + \lambda - \cdots}}},$$

however, a small trick removes the difficulty. Note that $G(t)$ is a solution of a quadratic equation in G,

$$(t + \lambda - (1 + \lambda t)G)G = 1. \tag{3.4}$$

Put $(1 + \lambda t)G(t) = \alpha g(t) + \beta$ where α and β are constants, and suppose that $g(t)$ is a solution of

$$(t - b - cg)g = 1 \tag{3.5}$$

which implies $g(t)$ has a suitable continued fractional expression

$$g(t) = \cfrac{1}{t - b - \cfrac{c}{t - b - \cfrac{c}{t - b - \cdots}}}.$$

Substitution of g into (3.4) and comparison with (3.5) give the solution

$$\alpha = 1, \quad \beta = \lambda, \quad b = \lambda \ \text{ and } \ c = 1,$$

hence

$$g(t) = \frac{1}{t - \lambda - g(t)} = \cfrac{1}{t - \lambda - \cfrac{1}{t - \lambda - \cfrac{1}{t - \lambda - \cdots}}},$$

$$G(t) = \frac{1}{t - g(t)} = \cfrac{1}{t - \cfrac{1}{t - \lambda - \cfrac{1}{t - \lambda - \cdots}}}.$$

Thus, the monic orthogonal polynomials associated with $d\mu_\lambda$ are determined by

$$\begin{aligned} &p_0(t) = 1, \qquad p_1(t) = t, \\ &p_n(t) = (t - \lambda)p_{n-1}(t) - p_{n-2}(t) \qquad (n \ge 2), \end{aligned}$$

with the Jacobi parameters [1]

$$\alpha_1 = 0, \qquad \alpha_n = \lambda \qquad (n \geq 2),$$
$$\omega_n = 1 \qquad (n \geq 1), \tag{3.6}$$

which declares that **Theorem 3.2** gives nothing but an interacting Fock representation with the Jacobi parameters (3.6).

4. Folding of free elements II

Let us start with a partial sum of $S_2(N)$,

$$S_2(\gamma, N) := \frac{1}{\sqrt{v}} \sum_{\substack{1 \leq i < j \leq N \\ i \leq \max\{\gamma N, 1\}}} (w_{ij} + w_{ji})$$

where v denotes the variance $v = \gamma N((2-\gamma)N - 1)$ so that $\phi(S_2(\gamma, N)^2) = 1$. Contrast to the previous section, the asymmetricity on the subindices causes more rich phenomena, depending on the growth rate of γ to N. We observe the three cases:

(A) $\gamma N \equiv 1$,
(B) $\gamma N \to \infty$ and $\gamma \to 0$ as $N \to \infty$,
(C) γ equals to a constant $0 \leq \alpha \leq 1$.

A product $w_{i_1 j_1} \cdots w_{i_m j_m}$ is connected with a sequence $\epsilon_1 \cdots \epsilon_m$ of symbols ${}_{\circ}\!\nwarrow^{\bullet}$, ${}_{\bullet}\!\nwarrow^{\circ}$, ${}_{\bullet}\!\nwarrow^{\bullet}$, ${}^{\circ}\!\smile^{\bullet}$, ${}^{\bullet}\!\smile^{\circ}$, ${}^{\bullet}\!\smile^{\bullet}$, ${}^{\circ}\!\swarrow_{\bullet}$, ${}^{\bullet}\!\swarrow_{\circ}$ and ${}^{\bullet}\!\swarrow_{\bullet}$ by way of the following rule:

$$\begin{aligned} w_{i_m j_m} &\longleftrightarrow {}_{\bullet}\!\nwarrow^{\circ}, \text{ if } i_m \leq \gamma N < j_m, \\ w_{i_m j_m} &\longleftrightarrow {}_{\circ}\!\nwarrow^{\bullet}, \text{ if } j_m \leq \gamma N < i_m, \\ w_{i_m j_m} &\longleftrightarrow {}_{\bullet}\!\nwarrow^{\bullet}, \text{ if } i_m, j_m \leq \gamma N, \end{aligned}$$

in the case of $i_k \leq \gamma N < j_k$,

$$\begin{aligned} w_{i_k j_k} &\longleftrightarrow {}_{\bullet}\!\nwarrow^{\circ}, && \text{if } |w_{i_k j_k} w_{i_{k+1} j_{k+1}} \cdots w_{i_m j_m}| > |w_{i_{k+1} j_{k+1}} \cdots w_{i_m j_m}|, \\ w_{i_k j_k} &\longleftrightarrow {}^{\bullet}\!\smile^{\circ}, && \text{if } |w_{i_k j_k} w_{i_{k+1} j_{k+1}} \cdots w_{i_m j_m}| = |w_{i_{k+1} j_{k+1}} \cdots w_{i_m j_m}|, \qquad \text{and} \\ w_{i_k j_k} &\longleftrightarrow {}^{\bullet}\!\swarrow_{\circ}, && \text{if } |w_{i_k j_k} w_{i_{k+1} j_{k+1}} \cdots w_{i_m j_m}| < |w_{i_{k+1} j_{k+1}} \cdots w_{i_m j_m}|, \end{aligned}$$

in the case of $j_k \leq \gamma N < i_k$,

$$\begin{aligned} w_{i_k j_k} &\longleftrightarrow {}_{\circ}\!\nwarrow^{\bullet}, && \text{if } |w_{i_k j_k} w_{i_{k+1} j_{k+1}} \cdots w_{i_m j_m}| > |w_{i_{k+1} j_{k+1}} \cdots w_{i_m j_m}|, \\ w_{i_k j_k} &\longleftrightarrow {}^{\circ}\!\smile^{\bullet}, && \text{if } |w_{i_k j_k} w_{i_{k+1} j_{k+1}} \cdots w_{i_m j_m}| = |w_{i_{k+1} j_{k+1}} \cdots w_{i_m j_m}|, \qquad \text{and} \\ w_{i_k j_k} &\longleftrightarrow {}^{\circ}\!\swarrow_{\bullet}, && \text{if } |w_{i_k j_k} w_{i_{k+1} j_{k+1}} \cdots w_{i_m j_m}| < |w_{i_{k+1} j_{k+1}} \cdots w_{i_m j_m}|, \end{aligned}$$

in the case of $i_k, j_k \leq \gamma N$,

$$\begin{aligned} w_{i_k j_k} &\longleftrightarrow {}_{\bullet}\!\nwarrow^{\bullet}, && \text{if } |w_{i_k j_k} w_{i_{k+1} j_{k+1}} \cdots w_{i_m j_m}| > |w_{i_{k+1} j_{k+1}} \cdots w_{i_m j_m}|, \\ w_{i_k j_k} &\longleftrightarrow {}^{\bullet}\!\smile^{\bullet}, && \text{if } |w_{i_k j_k} w_{i_{k+1} j_{k+1}} \cdots w_{i_m j_m}| = |w_{i_{k+1} j_{k+1}} \cdots w_{i_m j_m}|, \qquad \text{and} \\ w_{i_k j_k} &\longleftrightarrow {}^{\bullet}\!\swarrow_{\bullet}, && \text{if } |w_{i_k j_k} w_{i_{k+1} j_{k+1}} \cdots w_{i_m j_m}| < |w_{i_{k+1} j_{k+1}} \cdots w_{i_m j_m}|. \end{aligned}$$

For instance, the product $w_{1a} w_{a2} w_{2b} w_{b1}$ $(a, b > \gamma N)$ corresponds to ${}^{\bullet}\!\swarrow_{\circ}\ {}^{\circ}\!\smile^{\bullet}\ {}^{\bullet}\!\smile^{\circ}\ {}_{\circ}\!\nwarrow^{\bullet}$.

Consider an action of the symbols $\nwarrow^{\bullet}$, $\nwarrow^{\circ}$, $\nwarrow^{\bullet}$, $\smile^{\bullet}$, $\smile$, $\smile$, $\swarrow_{\bullet}$, $\swarrow_{\circ}$ and $\swarrow_{\bullet}$ on a sequence $\kappa = \kappa_1 \cdots \kappa_m$ of o's and •'s given by

$$\nwarrow^{\circ}\kappa = \bullet\circ\kappa,\quad {}_{\circ}\nwarrow^{\bullet}\kappa = \circ\bullet\kappa,\quad {}_{\bullet}\nwarrow^{\bullet}\kappa = \bullet\bullet\kappa,$$

$$\nwarrow^{\circ}\mathbf{1} = \bullet\circ,\quad {}_{\circ}\nwarrow^{\bullet}\mathbf{1} = \circ\bullet,\quad {}_{\bullet}\nwarrow^{\bullet}\mathbf{1} = \bullet\bullet,\quad \nwarrow^{\circ}\mathbf{0} = {}_{\circ}\nwarrow^{\bullet}\mathbf{0} = {}_{\bullet}\nwarrow^{\bullet}\mathbf{0} = \mathbf{0},$$

$$\smile^{\bullet}\kappa = \begin{cases} \bullet\kappa_2\cdots\kappa_m, & \text{if } \kappa_1 = \circ, \\ \mathbf{0}, & \text{otherwise}, \end{cases} \qquad \swarrow_{\circ}\kappa = \begin{cases} \kappa_3\cdots\kappa_m, & \text{if } \kappa_1\kappa_2 = \circ\bullet, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

$$\smile^{\circ}\kappa = \begin{cases} \circ\kappa_2\cdots\kappa_m, & \text{if } \kappa_1 = \bullet, \\ \mathbf{0}, & \text{otherwise}, \end{cases} \qquad {}^{\circ}\swarrow_{\bullet}\kappa = \begin{cases} \kappa_3\cdots\kappa_m, & \text{if } \kappa_1\kappa_2 = \bullet\circ, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

$$\smile\kappa = \begin{cases} \kappa, & \text{if } \kappa_1 = \bullet, \\ \mathbf{0}, & \text{otherwise}, \end{cases} \qquad {}^{\bullet}\swarrow_{\bullet}\kappa = \begin{cases} \kappa_3\cdots\kappa_m, & \text{if } \kappa_1\kappa_2 = \bullet\bullet, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

where $\mathbf{0}$ is a fixed point of all symbols and $\mathbf{1}$ an initial point. The reduction rule among w_{ij}'s, such as $w_{1a}w_{a2} = \sigma_1\sigma_2$, is reflected faithfully in the above rule. The equation $w_{i_1j_1}\cdots w_{i_mj_m} = e$ corresponds to $\epsilon_1\cdots\epsilon_m\mathbf{1} = \mathbf{1}$ particularly. $\eta_2(\epsilon_1\cdots\epsilon_m)$ denotes the height of $\epsilon_1\cdots\epsilon_m$ given as the length of the sequence $\epsilon_1\cdots\epsilon_m\mathbf{1}$ of o's and •'s, putting the length of $\mathbf{1} = 0$ and that of $\mathbf{0} = -\infty$.

The action of the symbols produces a direct combinatorial expression on a free Fock space. Let $\Gamma = \Gamma(a,b,x,y)$ be a unital algebra over C freely generated by a,b,x,y with the unit $\mathbf{1}$, taken for a free product of four 1-mode Fock spaces, $\Gamma = \Gamma(\mathbf{C}a) * \Gamma(\mathbf{C}b) * \Gamma(\mathbf{C}x) * \Gamma(\mathbf{C}y)$, equipped with a canonical inner product. An interpretation

$$\bullet\bullet \leftrightarrow a,\qquad \circ\circ \leftrightarrow b,\qquad \bullet\circ \leftrightarrow x,\qquad \circ\bullet \leftrightarrow y,$$

induces operators $A^{\dagger}$,A,P,$X^{\dagger}$,X,$Y^{\dagger}$,Y,Q,R corresponding to ${}_{\bullet}\nwarrow^{\bullet}$, ${}^{\bullet}\swarrow_{\bullet}$, ${}^{\bullet}\smile$, ${}_{\bullet}\nwarrow^{\circ}$, ${}^{\circ}\swarrow_{\bullet}$, ${}_{\circ}\nwarrow^{\bullet}$, ${}^{\bullet}\swarrow_{\circ}$, ${}^{\circ}\smile$ and ${}^{\bullet}\smile$ respectively, acting on Γ, under the rule defined below: for $u \in \Gamma$,

$$A^{\dagger}u = au,\qquad Au = \begin{cases} u', & \text{if } u = au',\quad u' \in \Gamma, \\ 0, & \text{otherwise}, \end{cases}$$

$$X^{\dagger}u = xu,\qquad Xu = \begin{cases} u', & \text{if } u = xu',\quad u' \in \Gamma, \\ 0, & \text{otherwise}, \end{cases}$$

$$Y^{\dagger}u = yu,\qquad Yu = \begin{cases} u', & \text{if } u = yu',\quad u' \in \Gamma, \\ 0, & \text{otherwise}, \end{cases}$$

$$Pau = au,\quad Pbu = 0,\quad Pxu = xu,\quad Pyu = 0,\quad P\mathbf{1} = 0,$$

$$Qxu = bu,\quad Qyu = 0,\quad Qau = yu,\quad Qbu = 0,\quad Q\mathbf{1} = 0,$$

$$Rxu = 0,\quad Ryu = au,\quad Rau = 0,\quad Rbu = xu,\quad R\mathbf{1} = 0.$$

4.1. The case of (A): $\gamma N \equiv 1$.

Since a morphism $\omega_{1i} \to g_i$ (and then, $\omega_{i1} \to g_i^{-1}$) yields an isomorhism from the subgroup of $G = *\mathbf{Z}/2\mathbf{Z}$ generated by $\{w_{1i}\}$ to a group freely generated by $\{g_i\}$, $S_2(1/N, N)$ induces the free central limit theorem. A 1-mode Fock representation is given by

$$\lim_{N\to\infty} \phi\Big(S_2\left(\frac{1}{N}, N\right)^m\Big) = \big\langle (A^{\dagger} + A)^m\mathbf{1}, \mathbf{1}\big\rangle,$$

4.2. The case of (B): $\gamma N > 1$ and $\gamma \to 0$ as $N \to \infty$.

An effect of folding free elements appears, however, the asymmetricity on the subindices causes a difference from the previous section. Consider a product $w_{xa}w_{ab}w_{bx} = e$ with $a, b \leq \gamma N$ and $x \leq N$. This type of products have no contribution to the limit distribution, as the number of such indices (a,b,x) has smaller order than $\sqrt{v}^3$. This observation shows that a product $w_{i_1j_1}\cdots w_{i_mj_m}$ containing a factor $w_{i_kj_k}$ with $i_k, j_k \leq \gamma N$ has no contribution in the limit $N \to \infty$, exactly,

Lemma 4.1. For a equation $\epsilon_1\cdots\epsilon_m\mathbf{1} = \mathbf{1}$, let T_N be the number of products $w_{i_1j_1}\cdots w_{i_mj_m} = e$ of w_{ij}'s $(1 \leq i \neq j \leq N)$ corresponding to $\epsilon_1\cdots\epsilon_m$. Then,

$$\lim_{N\to\infty} \frac{T_N}{(\sqrt{v})^m} = \begin{cases} 0, & \text{if } k > 0, \\ \left(\frac{1}{\sqrt{2}}\right)^m, & \text{if } k = 0, \end{cases}$$

where k denotes the total number of ⟍•'s, ◡'s and ⟋•'s appear in $\epsilon_1\cdots\epsilon_m$.

Proof. By definitions, the number of choice of subindices $i_s j_s$'s assymptotically equals to

$$(\gamma N)^{\frac{m}{2}}((1-\gamma)N)^{\frac{m-k}{2}}(\gamma N)^{\frac{k}{2}},$$

hence the assertion. □

As a result, a Fock representation on $\Gamma(a,b,x,y)$ is obtained.

Theorem 4.2. The assymptotic behavior of $S_2(\gamma,N)$ with $\gamma N>1$ and $\gamma\to 0$ as $N\to\infty$ has a representation on the Fock space $\Gamma(a,b,x,y)$,

$$\lim_{N\to\infty}\phi\Big(S_2(\gamma,N)^m\Big)=\left\langle\left(\frac{1}{\sqrt{2}}(X^\dagger+X+Y^\dagger+Y+Q+R)\right)^m 1,1\right\rangle.$$

Suppose that $\epsilon_1\cdots\epsilon_m 1=1$ holds. Like the innner singletons, ◡'s and ◡'s occur only at the height >0, however, by definition, ◡ and ◡ should appear pairwise at the same height, which brings us another combinatorial description. Let us consider the Fock space $\Gamma(a,b,x,y)$ defined above. Putting $z=(x+y)/\sqrt{2}$ and $c=(a+b)/\sqrt{2}$, the action of $Z^\dagger=X^\dagger+Y^\dagger$, $Z=X+Y$ and $O=Q+R$ is given by

$$Z^\dagger u=\sqrt{2}zu,\ Zzu=\sqrt{2}u,\ Ozu=cu,\ Ocu=zu \qquad (u\in\Gamma(a,b,x,y)).$$

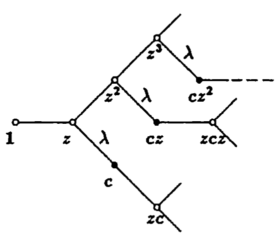
Hence we have

$$\left\langle\left(\frac{1}{\sqrt{2}}(X^\dagger+X+Y^\dagger+Y+Q+R)\right)^m 1,1\right\rangle=\left\langle(Z^\dagger+Z+\frac{1}{\sqrt{2}}O)^m 1.1\right\rangle$$

Let us consider more general situation

$$\langle(Z^\dagger+Z+\lambda O)^m 1,1\rangle$$

with a parameter λ, which is connected with the weighted walks, starting the origin 1 and returning there after m-step, on an induced subgraph of the binary tree. (The weights are given in the figuer below.)



Let F_m be the number of m-step walks leaving and returning to 1, allowed reaching 1 several times in the middle of the walks. Samely let f_m be the number of m-step walks leaving and returning to z without reaching 1, allowed reaching z several times in the middle of the walks. By the self-similarity of the graph, one has for $m\geq 2$,

$$\begin{aligned} f_m &= \sum_{k=0}^{m-2}(f_k+\lambda F_k)f_{m-k-2},\\ F_m &= \sum_{k=0}^{m-2} f_k F_{m-k-2}, \end{aligned}$$

where $f_0=F_0=1$. Putting the moment functions, $F(t)=\sum_m F_m t^m$ and $f(t)=\sum_m f_m t^m$, one has

$$\begin{aligned} f(t)-1 &= t^2(f(t)+\lambda F(t))f(t),\\ F(t)-1 &= t^2F(t)^2. \end{aligned}$$

Hence

$$\lambda^2t^2F(t)^3+(1-\lambda^2)t^2F(t)^2-F(t)+1=0,$$

and the Cauchy transform $G(t)$ of the distribution $d\mu_\lambda$ associated with **Theorem 4.2** is given as a solution of

$$\lambda^2 tG(t)^3 + (1-\lambda^2)G(t)^2 - tG(t) + 1 = 0.$$

Remark. Putting $\lambda^2 = 1/2$, $d\mu_\lambda$ coinside with the distribution in *Examples 1.5* (1.16) and (1.17) of [23], up to the variance, where the anti-commutation $ab + ba$ of semi-circle elements a, b which are free each other is observed. Indeed what we have done in the case of (B) is a calculation of the anti-commutation of semi-circle elements. Intuitively, this is because, in the limit we have

$$S_2(\gamma, N) \sim \left(\frac{\sigma_1 + \cdots + \sigma_{\gamma N}}{\sqrt{\gamma N}}\right)\left(\frac{\sigma_{\gamma N+1} + \cdots + \sigma_N}{\sqrt{N}}\right) + \left(\frac{\sigma_{\gamma N+1} + \cdots + \sigma_N}{\sqrt{N}}\right)\left(\frac{\sigma_1 + \cdots + \sigma_{\gamma N}}{\sqrt{\gamma N}}\right),$$

which is noting but the anti-commutation of semi-circle elements that are free each other.

4.3. The case of (C): γ equals to a constant $0 \le \alpha \le 1$.

In this case, such a product $w_{xa}w_{ab}w_{by}$ with $a, b \le \gamma N$ and $\gamma N < x, y \le N$ contributes to the limit distribution; the symbols ⤡•, ◡• and ⤢• appear.

Lemma 4.3. For a equation $\epsilon_1 \cdots \epsilon_m 1 = 1$, let T_N be the number of products $w_{i_1 j_1} \cdots w_{i_m j_m} = e$ of w_{ij}'s $(1 \le i \ne j \le N)$ corresponding to $\epsilon_1 \cdots \epsilon_m$. Then,

$$\lim_{N\to\infty} \frac{T_N}{(\sqrt{v})^m} = \left(\frac{\alpha}{2-\alpha}\right)^{\frac{k}{2}} \left(\frac{1-\alpha}{2-\alpha}\right)^{\frac{m-k}{2}}$$

where k denotes the total number of ⤡•'s, ◡•'s and ⤢•'s appear in $\epsilon_1 \cdots \epsilon_m$.

Proof. Just repeat the proof of **Lemma 4.1** in the case of (C). □

Then, again a Fock representation on $\Gamma(a, b, x, y)$ is in hand, which interpolates the distributions in **Theorem 3.2** and **Theorem 4.2**.

Theorem 4.4. The assymptotic behavior of $S_2(\gamma, N)$ with γ =constant α $(0 \le \alpha \le 1)$ has a representation on the Fock space $\Gamma(a, b, x, y)$,

$$\lim_{N\to\infty} \phi\Big((S_2(\gamma, N))^m\Big) = \left\langle \left(\sqrt{\frac{\alpha}{2-\alpha}}(A^\dagger + A + P) + \sqrt{\frac{1-\alpha}{2-\alpha}}(X^\dagger + X + Y^\dagger + Y + Q + R)\right)^m 1, 1 \right\rangle.$$

5. Multi-folding of free elements

In the previous sections, we saw that the double folding of free elements gives samples for conditionally free central limit theorems. However multi-folding of free elements suggests more general concept of independence. For instance, let us consider elements $w_{ijk} := \sigma_i\sigma_j\sigma_k$ $(i \ne j \ne k \ne i)$. Note that the difference of reduced length of $w_{i_1j_1k_1}w_{i_2j_2k_2}\cdots w_{i_mj_mk_m}$ and $w_{i_2j_2k_2}\cdots w_{i_mj_mk_m}$ equals to ± 3 or ± 1. Then, for a product $w_{i_1j_1k_1}\cdots w_{i_mj_mk_m}$, one associate a sequence of symbols $A^\dagger, A, B^\dagger, B$'s by way of the rule

$$\begin{aligned}
w_{i_mj_mk_m} &\longleftrightarrow A^\dagger,\\
w_{i_sj_sk_s} &\longleftrightarrow A^\dagger, \text{ if } |w_{i_sj_sk_s}w_{i_{s+1}j_{s+1}k_{s+1}}\cdots w_{i_mj_mk_m}| - |w_{i_{s+1}j_{s+1}k_{s+1}}\cdots w_{i_mj_mk_m}| = +3,\\
w_{i_sj_sk_s} &\longleftrightarrow B^\dagger, \text{ if } |w_{i_sj_sk_s}w_{i_{s+1}j_{s+1}k_{s+1}}\cdots w_{i_mj_mk_m}| - |w_{i_{s+1}j_{s+1}k_{s+1}}\cdots w_{i_mj_mk_m}| = +1,\\
w_{i_sj_sk_s} &\longleftrightarrow B, \text{ if } |w_{i_sj_sk_s}w_{i_{s+1}j_{s+1}k_{s+1}}\cdots w_{i_mj_mk_m}| - |w_{i_{s+1}j_{s+1}k_{s+1}}\cdots w_{i_mj_mk_m}| = -1, \text{ and}\\
w_{i_sj_sk_s} &\longleftrightarrow A, \text{ if } |w_{i_sj_sk_s}w_{i_{s+1}j_{s+1}k_{s+1}}\cdots w_{i_mj_mk_m}| - |w_{i_{s+1}j_{s+1}k_{s+1}}\cdots w_{i_mj_mk_m}| = -3.
\end{aligned}$$

Suppose that $w_{i_1j_1k_1}\cdots w_{i_mj_mk_m} = e$, that is the sequence of sub indices $i_1j_1k_1\cdots i_mj_mk_m$ forms a non crossing pair partition, which implies m is to be an even number. Let $\epsilon_1\cdots\epsilon_m$ be the corresponding sequence of $A^\dagger, A, B^\dagger, B$ defined above. By definitions, such a sequence $\epsilon_1\cdots\epsilon_m$ corresponds to a restricted Catalan path on $\mathbf{Z}^2$ from $(0,0)$ to $(3\lfloor m/2\rfloor, 3\lfloor m/2\rfloor)$ in the following way: each symbol ϵ_s is taken for a three step walk,

$$\begin{aligned}
A^\dagger &\longleftrightarrow \Omega_{+3} : (x,y) \to (x+1,y) \to (x+2,y) \to (x+3,y),\\
B^\dagger &\longleftrightarrow \Omega_{+1} : (x,y) \to (x,y+1) \to (x+1,y+1) \to (x+2,y+1),\\
B &\longleftrightarrow \Omega_{-1} : (x,y) \to (x,y+1) \to (x,y+2) \to (x+1,y+2) \text{ and}\\
A &\longleftrightarrow \Omega_{-3} : (x,y) \to (x,y+1) \to (x,y+2) \to (x,y+3),
\end{aligned}$$

and the corresponding lattice path consists of the walks $\Omega_{\pm 3}$ and $\Omega_{\pm 1}$, walking under the line $y = x + 1$ with out accrossing the diagonal $y = x$. Note that the walks Ω_{+1} and Ω_{-1} may start only from the trianguler areas under the line $y = x - 1$ and $y = x - 2$ respectively.

Let us observe the assymptotic behavior of

$$S_3(N) := \frac{1}{\sqrt{N(N-1)(N-2)}} \sum_{1\leq i\neq j\neq k\neq i\leq N} w_{ijk}.$$

From the argument above, it is easily seen that all odd moments vanish and the $2m$-th moment has an expression

$$\lim_{N\to\infty} \phi(S_3(N)^{2m}) = \#\{\text{Catalan path on } \mathbf{Z}^2 \text{ from } (0,0) \text{ to } (3m,3m) \text{ consisting of } \Omega_{\pm 3}, \Omega_{\pm 1}\}.$$

Summing up, we have an combinatorial description.

Theorem 5.1. Let $A^\dagger$ and A be canonical creation and annihilation operators on a 1-mode Fock space $\Gamma(\mathbf{C})$, and $B^\dagger$ and B be operators killing the vacuum $\mathbf{1}$, acting on the subspace orthogonal to $\mathbf{1}$ where $A^\dagger = B^\dagger$ and $A = B$ holds. Then the assymptotic behavior of $S_3(N)$ has a combinatorial description

$$\lim_{N\to\infty} \phi(S_3(N)^m) = \left\langle ((A^\dagger)^3 + B^\dagger + B + A^3)^m \mathbf{1}, \mathbf{1} \right\rangle_{\Gamma(\mathbf{C})}.$$

Remark. According to [7], Jacobi parameters associated with conditionally free central limit distributions are of the form

$$\omega_1 = p,\ \omega_n = q\ (n\geq 2),\ \alpha_n = 0\ (n \geq 0).$$

Contrast to the conditionally free case, above example has aperiodic Jacobi parameters,

$$\omega_1 = 1,\quad \omega_2 = 3,\ \omega_3 = 6,\ \omega_4 = 8/3,\ \omega_5 = 217/48,\ \ldots,$$
$$\alpha_n = 0 \quad (n \geq 0).$$

Acknowledgement. The auther expresses his gratitude to Professor R. Speicher for pointing out mistakes in the case of (B) of section 4, and for many valuable instructions.

References

[1] L. Accardi, M. Bożejko: *Interacting Fock spaces and Gaussianization of probability measures*, Infinite Dimensional Analysis and Quantum Probability **1** (1998), 663-670.

[2] L. Accardi, Y. Hashimoto and N. Obata: *Notions of independence related to the free group*, Infinite Dimensional Analysis and Quantum Probability **1** No.2 (1998), 201-220.

[3] ———: *Singleton independence*, Banach Center Publications **43** (1998), 9-24.

[4] ———: *A Role of Singletons in Quantum Central Limit Theorems*, J. Korean Math. Soc. **35** (1998), 675-690.

[5] N. Akhiezer: *Lectures on Integral Transforms*, Translations of Mathematical Monographs Vol. 70, AMS (1988).

[6] M. Akiyama and H. Yosida: *The distributions for linear combinations of a free family of projections and their orthogonal polynomials*, preprint (1998).

[7] M. Bożejko, M. Leinert and R. Speicher: *Convolution and limit theorems for conditionally free random variables*, Pacific J. Math. **175** (1996), 357-338.

[8] M. Bożejko and R. Speicher: *An example of a generalized Brownian motion*, Commun. Math. Phys. **137** (1991), 519-531.

[9] P. Deift: *Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach*, Courant Lecture Notes in Mathematics **3**, CIMS (1999).

[10] R. Durrett: *Probability: theory and examples*, Duxbury Press, Belmont, California (1991).

[11] A. Giacometti: *Exact closed form of the return probability on the Bethe lattice*, J. Phys. A: MAth. Gen. **28** (1995), L13-L17.

[12] V. Guba and M. Sapir: *Diagram Groups*, Memoirs of the Amer. Math. Soc. Vol. 130 Num. 620 (1997).

[13] N. Giri and W. von Waldenfels: *An algebraic version of the central limit theorem*, ZW **42** (1978), 129-134.

[14] Y. Hashimoto: *Deformations of the semi-circle law derived from random walks on free groups*, Prob. Math. Stat. **18** (1998), 399-410.

[15] ———: *A combinatorial approach to limit distributions of random walks: on discrete groups*, preprint (1996).

[16] P. Hilton and J. Pederson: *Catalan Numbers, Their Generalization, and Their Uses*, Math. Intelligencer **13** (1991), 64-75.

[17] A. Hora: *Central limit theorems and asymptotic spectral analysis on large graphs*, Infinite Dimensional Analysis and Quantum Probability **1** (1998), 221-246.

[18] H. Kesten: *Symmetric random walks on groups*, Trans. Amer. Math. Soc. **92** (1959), 336-359.

[19] Y. Lu: *On the interacting free Fock space and the deformed Wigner law*, Nagoya Math. J. **145** (1997), 1-28.

[20] ———: *Interacting Fock spaces related to the Anderson model*, Infinite Dimensional Analysis and Quantum Probability **1** No.2 (1998), 247-283.

[21] R. Lyndon and P. Schupp: *Combinatorial Group Theory*, Springer-Verlag (1977).

[22] G. Mohanty: *Lattice Path Counting and Applications*, Accademic Press (1979).

[23] A. Nica and R. Speicher: *Commutators of free random variables*, Duke Math. J. **92** No.3 (1998), 553-592.

[24] J. Serre: *Trees*, Springer-Verlag (1980).

[25] R. Simion and D. Ullman: *On the structure of the lattice of noncrossing partitions*, Discrete Math. **98** (1991), 193-206.

[26] R. Speicher: *Multiplicative functions on the lattice of non-crossing partitions and free convolution*, Math. Ann. **298** (1994), 611-628.

[27] ______: *On Universal Products*, in *Free Probability Theory*, Fields Institute Communications **12** [D. Voiculescu ed.] (1997), 257-279.

[28] ______: *Combinatorics of free probability theory*, Lecture note at the IHP in Paris on the special semester on 'Free probability and operator spaces' Oct. - Dec. (1999).

[29] R. Stanley: *Enumerative Combinatorics*, Vol. 1, 2nd ed., Cambridge (1986).

[30] ______: *Enumerative Combinatorics*, Vol. 2, Cambridge (1999).

[31] J. Stillwell: *Classical Topology and Combinatorial Group Theory*, Springer-Verlag (1993).

[32] G. Szegö: *Orthogonal Polynomials*, 4th ed., Amer. Math. Soc. Coll. Publ. Vol. 23 (1975).

[33] M. Voit: *Limit Theorems for Compact Two-Point Homogeneous Spaces of Large Dimensions*, J. Theor. Prob. **9**. No.2 (1996), 353-370.

[34] W. von Waldenfels: *Interval partitions and pair interactions*, in "Séminaire de Probabilités IX (P. A. Meyer, ed.)," p.565–588, Lect. Notes in Math. Vol. 465, Springer-Verlag (1975).

[35] W. Woess: *Random walks on infinite graphs and groups—A survey on selected topics*, Bull. London Math. Soc. **26** (1994), 1-60.

Chikusa-ku, Nagoya 464-8602, Japan
E-mail address: y-hashi@math.nagoya-u.ac.jp

Limit laws and semistability on infinite-dimensional locally compact groups

W. Hazod, Dortmund

The limit behaviour of automorphism-normalized products of independent random variables was investigated in the past and the possible limit laws, in particular stable and semistable laws are nowadays quite well understood, as long as the underlying group is a real or p-adic Lie group.
In fact, if the normalizing operators are localized on a *continuous* one-parameter group T then — without further restriction on the underlying group — the possible limit laws are concentrated on the contractible subgroup $C(T)$, in this case a closed Lie subgroup. But if the underlying group $\mathbb{G}$ is infinite-dimensional and if the normalizing automorphisms are not embedded into a continuous group then new (and unexpected) phenomena appear. There is still no general theory available but the stucture of possible limit laws can be investigated by a series of illustrative examples. As in the finite-dimensional setup the contractible subgroups $C(a)$ play an important role as the possible limit laws are concentrated on these subgroups.
The paper is organized as follows: It starts describing the role of contractible subgroups $C(a)$ showing that on metrizable locally compact groups semistable continuous convolution semigroups with trivial idempotent are representable as continuous injective homomorphic images of semistable continuous convolution semigroups on contractible completely metrizable topological groups. The investigation is continued with semistability on totally discontinuous groups including the p-adics as a detailed example.
Then, as particular examples of infinite-dimensional groups investigations of semistability on infinite products $K^{\mathbf{Z}}$ follow, including the shape of $C(a)$, marginal distributions and finally for Lie groups K, a comparision of Gaussian semistable limit laws on $K^{\mathbf{Z}}$ and on the corresponding (infinite-dimensional) Lie algebra. In fact, infinite products $\mathbb{G} = K^{\mathbf{Z}}$ of compact groups turn out to be of particular interest: The shift a defines an automorphism, a permutation of infinite order acting on the coordinates, and the existence of such automorphisms causes significant differences to the situation of finite products. We mention new features appearing in the situation $\mathbb{G} = K^{\mathbf{Z}}$:

- The intersection of the contractible parts $C(a) \cap C(a^{-1})$ is a dense subgroup.
- There exist (a, α)-semistable laws (for $\alpha \in (0,1)$) such that any projection to a finite product K^n is not semistable.

To simplify notations we shall troughout assume the underlying group $\mathbb{G}$ to be second-countable. We recall some well-known definitions. (See also [3], [14], [6], [7], [2]):
0.1. Definition. A continuous convolution semigroup $(\mu_t : t \geq 0)$ — in short $\mu_\bullet$ — is called (a, α)-*semistable* for $(a, \alpha) \in \mathrm{Aut}(\mathbb{G}) \times (0,1)$ if $a(\mu_t) = \mu_{\alpha t}, t \geq 0$.
$\mu_\bullet$ is *stable* w.r.t. a one-parameter group T iff $a_t(\mu_s) = \mu_{st}$ for $s, t > 0$, where $T = (a_t : t > 0) \subseteq \mathrm{Aut}(\mathbb{G})$ with multiplicative parametrization $a_t a_s = a_{t \cdot s}, t, s > 0$.
Note that in this definition of (semi-)stability local compactness of the underlying group is not necessary.
Continuous convolution semigroups in $\mathcal{M}^1(\mathbb{G})$ with idempotent $\mu_0 = \varepsilon_e$ are represented by *generating functionals* (cf. e.g. [9], [12]) defined on the test functions $\mathcal{D}(\mathbb{G})$

resp. on the regular functions $\mathcal{E}(\mathbb{G})$. Let $\mathcal{GF}(\mathbb{G})$ denote the cone of generating functionals.

Since (semi-)stability is closely related to the limit behaviour of automorphism-normalized convolution products we have to define domains of attraction:
0.2. Definition. $\mathrm{FDPA}(\mu_\bullet) := \{\nu \in \mathcal{M}^1(\mathbb{G}) : \exists (a_n) \subseteq \mathrm{Aut}(\mathbb{G}), k(n) \nearrow \infty,$ such that $a_n \nu^{[k(n)t]} \to \mu_t, t \geq 0\}$ (*domain of partial attraction*)
$\mathrm{FDSA}(\mu_\bullet) := \{\nu \in \mathrm{FDPA}(\mu_\bullet) : k(n)/k(n+1) \to \alpha \in (0,1)\}$ (*semi attraction*)
$\mathrm{FDA}(\mu_\bullet) := \{\nu \in \mathrm{FDPA}(\mu) : k(n) = n\}$ (*domain of attraction*).
If $a_n \in \{a^l : l \in \mathbb{Z}\}$ for some $a \in \mathrm{Aut}(\mathbb{G})$ (*normal attraction*) we use the notations $\mathrm{FDNPA}(\mu_\bullet)$ (if $a_n = a^{l(n)}, l(n) \nearrow \infty$), and $\mathrm{FDNSA}(\mu_\bullet)$ resp. $\mathrm{FDNA}(\mu_\bullet)$ (if $a_n = a^n$). (Cf. e.g. [5].)

The role of contractible subgroups

The investigations of the structure of the contractible subgroups $C(a), C_K(a)$ defined below play an important role in the theory of semistability on groups. We list some properties, pointing out in particular the additional features in case of *exponential Lie groups* (of course not to be expected in the infinite-dimensional situation).

We define (cf. [15], [6], [7], [2], [11]):
1.1. Definition. Let $a \in \mathrm{Aut}(\mathbb{G})$, let K denote a compact a-invariant subgroup. Then the contractible and K-contractible parts are defined as
$C(a) := \{x \in \mathbb{G} : a^n(x) \overset{n\to\infty}{\longrightarrow} e\}$ and $C_K(a) := \{x \in \mathbb{G} : a^n(x) \cdot K \to K\}$ respectively.
For a one-parameter group $T = (a_t : t > 0)$ we define analogously
$C(T) := \{x : a_t(x) \overset{t\to 0}{\longrightarrow} e\}$ and $C_K(T) := \{x : a_t \cdot K \overset{t\to 0}{\longrightarrow} K\}$.
More generally we define for a sequence $(a_n)_{n\in\mathbb{N}} \subseteq \mathrm{Aut}(\mathbb{G})$
$C((a_n)_{n\in\mathbb{N}}) := \{x \in \mathbb{G} : a_n(x) \to e\}$, and analogously $C_K((a_n))$ is defined.
Obviously, these contractible parts $C(a), C_K(a)$, etc. are subgroups of $\mathbb{G}$.

1.2. Remarks. The following observations are frequently used:
a) We have the following characterization: $C((a_n)_{n\geq 1}) =: C = \{x :$ for any subsequence $(n') \subseteq \mathbb{N}$ there exists a subsequence $(n'') \subseteq (n')$ with $a_n x \overset{(n'')}{\longrightarrow} e\}$.
We fix a sequence $(a_n)_{n\in\mathbb{N}}$. For a subsequence $(n') \subseteq \mathbb{N}$ put $C_{(n')} := C((a_n)_{n\in(n')})$.
b) Let d be a metric on the (second countable) group $\mathbb{G}$. Put for $\varepsilon > 0$ $C^{(\varepsilon)} := \{x : \limsup d(a_n(x), e) < \varepsilon\}$. Obviously, $C^{(\varepsilon)}$ is Borel measurable. Hence $C = \bigcap_{n\geq 1} C^{(1/n)}$ is Borel measurable, and analogously we obtain measurability of $C_{(n')}$.
c) Let $\mathbb{G}$ be an *exponential Lie group* with Lie algebra $\mathbb{V}$. For $a \in \mathrm{Aut}(\mathbb{G})$ let $\overset{o}{a}$ denote the differential, defined by $\exp(\overset{o}{a}(X)) = a(\exp(X))$, $X \in \mathbb{V}$. Let (a_n) and C as above. Define $\overset{o}{C} := \{X \in \mathbb{V} : \overset{o}{a}_n(X) \to 0\}$. Then $\overset{o}{C}$ is a subalgebra and we have $\exp(\overset{o}{C}) = C$. In particular, C and the subgroups $C_{(n')}$ defined in a) are closed connected subgroups.

For *exponential Lie groups* we observe with the notations introduced above:
1.3. Proposition. a) Assume that there exists a sequence $(a_n) \subseteq \mathrm{Aut}(\mathbb{G})$ which is contracting on $\mathbb{G}$, i.e. $a_n(x) \to e$ for all $x \in \mathbb{G}$. Then $\mathbb{G}$ is a contractible Lie group, hence nilpotent and simply connected.

b) More generally, for any sequence (a_n) the contractible part $C = C((a_n)_{n\geq 1})$ is a closed connected subgroup. If C is a_n-invariant for sufficiently large n then C is contractible, hence nilpotent. In particular if $a_n = a^n$ for some $a \in \mathrm{Aut}(\mathbb{G})$ then $C(a)$ is a contractible, nilpotent and a-invariant subgroup.

[We have $C = C((a_n)_{n\geq 1}) = \mathbb{G}$ by assumption. Hence $\mathbf{V} = \overset{o}{C}$ (cf. 1.2.c), i.e. $\overset{o}{a}_n(X) \to 0$ for all $X \in \mathbf{V}$. Therefore we obtain $\|\overset{o}{a}_n\| \to 0$, hence $\overset{o}{a}_n$ — and therefore also a_n — is contractive for sufficiently large n, i.e. $(a_n)^m x \overset{m\to\infty}{\longrightarrow} e$, $x \in \mathbb{G}$. See also [17]. The rest assertions follow immediately.]

The connections between semistability and contractibility are illuminated by the following observations. (See e.g. [15], [6], [7], see also 1.6 below):

1.4. Proposition. a) Let $\mathbb{G}$ be a locally compact group and let $\mu_\bullet$ be an (a,α)-semistable continuous convolution semigroup with trivial idempotent $\mu_0 = \varepsilon_e$ and Lévy measure η. Then $\mu_\bullet$ is concentrated on $C(a)$, i.e.

$\mu_t(\mathbf{Cp}C(a)) = 0$ for all t, and furthermore $\eta(\mathbf{Cp}C(a)) = 0$.

b) And with the same proof we obtain for non-trivial idempotents: If $\mu_0 = \omega_K$ then all the measures μ_t are concentrated on the K-contraction group $C_K(a)$ of a.

Analogously, for stable continuous convolution semigroups we have:

Let $T = (a_t)_{t>0} \subseteq \mathrm{Aut}(\mathbb{G})$ be a subgroup (with $a_{t\cdot s} = a_t a_s$). Let $\mu_\bullet$ be T-stable. Then $\mu_\bullet$ is (a_t, t)-semistable for all $t \in (0,1)$. Hence 1.4 applies. For stable laws with *continuous* group T we obtain a stronger result ([6]):

1.5. Proposition. Let $(\mu_t)_{t\geq 0}$ be a T-stable continuous convolution semigroup on a locally compact group $\mathbb{G}$ such that $\mu_0 = \omega_K$. Then all μ_t are concentrated on the K-contraction group $C_K(T)$ of T.

(Note that in this situation we need not assume $\mathbb{G}$ to be second countable since according to [6] the subgroups $C_K(T)$ and $C(T)$ are closed in $\mathbb{G}$ and hence measurable.)

Proposition 1.5 applies in particular for $K = \{e\}$. We obtain:

If $\mu_\bullet$ is a T-stable continuous convolution semigroup with trivial idempotent and if T is continuous then $\mu_\bullet$ is concentrated on the *closed* subgroup $C(T)$, isomorphic to a contractible simply connected nilpotent Lie group on which T acts contractively.

Hence for *continuous* groups T the investigation of T-stable laws with trivial idempotents is completely reduced to contractible simply connected nilpotent Lie groups.

Not only limit laws, also the attracted laws are concentrated on contractible parts. Generalizing the proof of 1.4 we obtain:

1.6. Proposition. Assume $\mu_\bullet$ to be a continuous convolution semigroup with trivial idempotent $\mu_0 = \varepsilon_e$. Let ν FDPA$(\mu_\bullet)$, i.e. $k(n) \nearrow \infty$, $a_n \in \mathrm{Aut}(\mathbb{G})$ such that $a_n(\nu)^{[k(n)t]} \to \mu_t, t \geq 0$ and assume moreover $\limsup k(n)/k(n+1) < 1$.

Then $\nu(C((a_n)_{n\geq 1}) = 1$.

[W.l.o.g. we assume $k(n)/k(n+1) \leq \kappa < 1$ for $n \geq 1$. Let $U \in \mathfrak{U}(e)$ be relatively compact Borel neighbourhoods. Let A denote the generating functional and η the Lévy measure of $\mu_\bullet$. According to a theorem of E. Siebert (cf. [13], [5], [4])

$$a_n(\nu)^{[k(n)t]} \to \mu_t,\ t \geq 0 \ \text{ iff } \ k(n)\cdot(a_n(\nu) - \varepsilon_e) \to A.$$

Hence $\sup_{n\geq 1} k(n)\cdot a_n(\nu)(\mathbf{Cp}U) \leq K(U) < \infty$. Therefore

$$\int \sum_{n\geq 1} 1_{\mathbf{Cp}U} \circ a_n d\nu = \sum_n a_n(\nu)(\mathbf{Cp}U)$$

$$= \frac{1}{k(1)} \sum_n \frac{k(1)}{k(2)} \cdots \frac{k(n-1)}{k(n)} \cdot k(n) \cdot a_n(\nu)(\mathbf{Cp}U) \leq \frac{K(U)}{k(1)} \cdot \sum \kappa^n < \infty .$$

Whence $1_{\mathbf{Cp}U} \circ a_n \to 0$ ν-a.e. In other words, $\{a_n(x)\}$ is relatively compact with $\mathrm{LIM}(a_n(x)) \subseteq U$ for ν-almost all x. (LIM denoting the set of accumulation points). Let $U_k \in \mathfrak{U}(e)$ with $U_k \downarrow \{e\}$. Repeating the above arguments we obtain $\nu(\bigcap_k \{x : \mathrm{LIM}(a_n x) \subseteq U_k\}) = \nu(C((a_n)_{n\in\mathbb{N}})) = 1$ as asserted.]

1.7. Corollary. a) Assume (as in the case of stable $\mu_\bullet$) that $k(n)/k(n+1) \to 1$. Then for any $\alpha \in (0,1)$ there exists a subsequence (n') with $k(n)/k(n+1) \overset{(n')}{\longrightarrow} \alpha$. And according to 1.6 we conclude $\nu(C((a_n)_{n\in(n')})) = 1$.
b) (*Domains of normal (semi-)attraction*). Let $a_n = a^n$ for some $a \in \mathrm{Aut}(\mathbb{G})$, $k(n)/k(n+1) \to \alpha \in (0,1)$ and $a^n \nu^{[k(n)t]} \to \mu_t, t \geq 0$. Then $\nu(C(a)) = 1$.

Let $a_n = a^{l(n)}$ with $l(n) \nearrow \infty$. Let $C := C((a_n))$. Then $\nu(C) = 1$, but in general $C \neq C(a)$ is possible. However, for *exponential Lie groups* we observe
1.8. Proposition. Let $\mathbb{G}$ be an exponential Lie group, let $a \in \mathrm{Aut}(\mathbb{G}), l(n) \nearrow \infty$. Then $C := C((a^{l(n)})_{n\in\mathbb{N}}) = C(a)$.
[Let $\mathbf{V}$ denote the Lie algebra of $\mathbb{G}$, let as above $\overset{o}{a} \in \mathrm{GL}(\mathbf{V})$ denote the differential of a defined by $\exp(\overset{o}{a} X) = a(\exp X), X \in \mathbf{V}$. For $x \in \mathbb{G}$ let $X = \exp^{-1}(x) \in \mathbf{V}$. $\mathbb{G}$ being exponential, $a^{l(n)}x \to e$ iff $\overset{o}{a}{}^{l(n)}X \to 0$. As easily seen, this is the case iff X belongs to the contractible $\overset{o}{a}$-invariant subspace $\bigcup_{|z|<1}\{Y : (\overset{o}{a} - zI)^k Y = 0$ for some $k \in \mathbb{N}\} = C(\overset{o}{a})$. Therefore $\overset{o}{a}{}^n X \overset{n\to\infty}{\longrightarrow} 0$; whence $a^n x \to e$ follows.]

The relevance of the description of $C((a_n)_{n\geq 1})$ in 1.1.a) is shown by the following
1.9. Proposition. Let $\mathbb{G}$ be a group in which the subgroups $C((a_n))$ are closed, e.g. an exponential Lie group (1.2.c)). Let (a_n) be a sequence in $\mathrm{Aut}(\mathbb{G})$ and let $\nu \in \mathcal{M}^1(\mathbb{G})$, such that $a_n(\nu) \to \varepsilon_e$ (infinitesimality). Then $\mathrm{supp}(\nu) \subseteq C((a_n))$.
[Let $\nu \in \mathcal{M}^1(\mathbb{G})$ and assume $a_n \nu \to \varepsilon_e$, for some sequence $(a_n) \subseteq \mathrm{Aut}(\mathbb{G})$. Consider the probability space $(\mathbb{G}, \mathcal{B}, \nu), \mathcal{B}$ denoting the Borel sets. Consider $(a_n = a_n(\cdot))_{n\in\mathbb{N}}$ as a sequence of $\mathbb{G}$-valued random variables on the probability space $(\mathbb{G}, \mathcal{B}, \nu)$. By assumption, $a_n(\nu) \to \varepsilon_e$, hence $a_n(\cdot)$ converge to e in distribution, equivalently in probability. Therefore for any subsequence $(n') \subseteq \mathbb{N}$ there exists a subsequence $(n'') \subseteq (n')$ with $a_n(\cdot) \overset{(n'')}{\longrightarrow} e$ ν-a.e. I.e., we have $\nu(C_{(n'')}) = 1$, with the notations from above.
$C_{(n'')}$ being closed, $\mathrm{supp}(\nu) \subseteq C_{(n'')}$ follows. Therefore, $\mathrm{supp}(\nu) \subseteq \bigcap_{(n')} C_{(n'')} = C$.]

Retopologisation of $C(a)$: Intrinsic topologies

We recall the following results from E. Siebert's investigations ([16]):
Let $a \in \mathrm{Aut}(\mathbb{G})$. Then there exists a unique topology $\mathcal{O}_r$ turning $C(a)$ into a topological Hausdorff group $\tilde{C}(a)$ (not necessarily locally compact), furthermore there exist $\tilde{a} \in \mathrm{Aut}(\tilde{C}(a))$ and a continuous injective homomorphism $\varphi : \tilde{C}(a) \to \mathbb{G}$ such that $\varphi \circ \tilde{a} = a \circ \varphi$ (hence $\varphi(\tilde{C}(a) = C(a))$.

2.1. Properties. a) If $\mathbb{G}$ is complete and metrizable and if $a \in \mathrm{Aut}(\mathbb{G})$ is contractive then we have $\tilde{C}(a) = C(a) = \mathbb{G}$.

b) $\mathcal{O}_\tau$ is stronger than the relative topology of $C(a)$ (as a subspace of $\mathbb{G}$).
c) If $\mathbb{G}$ is metrizable then $\tilde{C}(a)$ is metrizable too.
d) If $\mathbb{G}$ has a countable basis then $\tilde{C}(a)$ has a countable basis too.
e) If $\mathbb{G}$ is complete then $\tilde{C}(a)$ is complete too.
f) If $\mathbb{G}$ is totally disconnected then $\tilde{C}(a)$ is totaly disconnected too.

Let $\mu_\bullet$ be a continuous (a,α)-semistable convolution semigroup with $\mu_0 = \varepsilon_e$. According to 2.1 there exists a contractible completely metrizable group $\mathbb{H} := \tilde{C}(a)$ with contractive automorphism $\tilde{a} \in \mathrm{Aut}(\mathbb{H})$ and an injective continuous homomorphism $\varphi : \mathbb{H} \hookrightarrow \mathbb{G}$ such that $\varphi(\mathbb{H}) = C(a)$ and $\varphi \circ \tilde{a} = a \circ \varphi$.
Then φ^{-1} is a Borel isomorphism $C(a) \to \mathbb{H}$. Hence φ induces a bijection $\mathcal{M}^1(\mathbb{H}) \longleftrightarrow \{\nu \in \mathcal{M}^1(\mathbb{G}) : \nu(C(a)) = 1\}, \nu \mapsto \varphi(\nu) =: \mu$. In fact, a continuous affine bijective convolution homomorphism. But φ^{-1} need not be continuous.

Nevertheless any continuous convolution semigroup $\mu_\bullet$ concentrated on $C(a)$ generates a continuous convolution semigroup $\varphi^{-1}(\mu_\bullet) =: \nu_\bullet$ on $\mathbb{H}$:
2.2. Proposition. Let $\mathbb{G}$ and $\mathbb{H}$ be completely metrizable topological groups and $\varphi : \mathbb{H} \hookrightarrow \mathbb{G}$ be an injective continuous homomorphism. Put $\mathbb{L} := \varphi(\mathbb{H})$. If $\mathbb{H}$ is σ-compact then $\mathbb{L}$ is measurable.
a) If $\nu_\bullet \subseteq \mathcal{M}^1(\mathbb{H})$ is a continuous convolution semigroup then $\bigcup_{t>0} \mathrm{supp}(\nu_t)$ generates a (closed) σ-compact subgroup $\mathbb{H}_1$. Hence $\varphi(\nu_\bullet) = \mu_\bullet$ defines a continuous convolution semigroup in $\mathcal{M}^1(\mathbb{G})$ concentrated on the measurable subgroup $\mathbb{L}_1 := \varphi(\mathbb{H}_1) \subseteq \mathbb{G}$.
b) Conversely, assume $\mathbb{L} = \varphi(\mathbb{H})$ to be a measurable subgroup $\subseteq \mathbb{G}$. Let $\mu_\bullet$ be a continuous convolution semigroup in $\mathcal{M}^1(\mathbb{G})$ with $\mu_t(\mathbb{L}) = 1$ for $t \geq 0$. Then $\nu_\bullet = \varphi^{-1}(\mu_\bullet) \subseteq \mathcal{M}^1(\mathbb{H})$ is a continuous convolution semigroup (with $\varphi(\nu_\bullet) = \mu_\bullet$).
Proof: a) is obvious by continuity of φ.
To prove b) note first that $\nu_\bullet$ is uniquely defined by $\mu_\bullet$ and $\nu_\bullet$ is a convolution semigroup. We have to show that $t \mapsto \nu_t$ is continuous.
$\mathbb{H}$ is completely metrizable. Hence ν_t is tight for any $t \geq 0$, therefore the support is σ-compact. Hence w.l.o.g. we assume $\mathbb{H} = \bigcup K^{(m)}$ with an increasing sequence of compact sets $K^{(m)} \subseteq \mathbb{H}$. Hence in order to prove continuity it suffices to show that for any $t \geq 0$, for any sequence $t_n \to t$ and for any $K^{(m)}$ the restrictions $\nu_{t_n}|_{K^{(m)}} =: \kappa_n^{(m)}$ converge weakly to $\nu_t|_{K^{(m)}} =: \kappa^{(m)}$: [Indeed, we have the representations $\nu_{t_n} = \lim_{m\geq 1} \kappa_n^{(m)}$ and $\nu_t = \lim_{m\geq 1} \kappa^{(m)}$ with non-negative measures (convergence in norm). And therefore, if we can prove $\langle \kappa_n^{(m)}, f\rangle \overset{n\to\infty}{\longrightarrow} \langle \kappa^{(m)}, f\rangle$ for $f \in C^b(\mathbb{H})$, for all $m \in \mathbb{N}$, we easily conclude $\langle \nu_{t_n}, f\rangle \to \langle \nu_t, f\rangle$]
For any compact set $K \subseteq \mathbb{H}$ the restriction $\varphi|_K$ defines a topological isomorphism $K \to \varphi(K) =: K^\# \subseteq \mathbb{G}$. Hence for compact sets $K \subseteq \mathbb{H}$ we observe according to the portemanteau theorem applied to the continuous function $s \mapsto \varphi(\nu_s) = \mu_s$ that

$$\limsup \nu_{t_n}(K) = \limsup (\mu_{t_n})(K^\#) \leq \mu_t(K^\#) = \varphi(\nu_t)(\varphi(K)) = \nu_t(K).$$

Therefore, again by the portemanteau theorem applied to the restrictions $\nu_s|_K$ we conclude continuity of $s \mapsto \nu_s|_K$ for all compact $K \subseteq \mathbb{H}$.
In particular, $\kappa_n^{(m)} \overset{n\to\infty}{\longrightarrow} \kappa^{(m)}, m \in \mathbb{N}$, as asserted. ∎

Now we are ready to prove the following

2.3. Theorem. Suppose $\mathbb{G}$ to be a locally compact group and let $\mu_\bullet$ be a continuous convolution semigroup with trivial idempotent.

a) Let $\mathbb{G}$ be second countable and let $\mu_\bullet$ be (a,α)-semistable. Then there exist a completely metrizable topological contractible group $\mathbb{H}$ with contractive automorphism $\tilde{a}$, and a continuous injection $\varphi : \mathbb{H} \hookrightarrow \mathbb{G}$ such that $\varphi(\mathbb{H}) = C(a)$ and $\varphi \circ \tilde{a} = a \circ \varphi$. Furthermore there exists an $(\tilde{a},\alpha)$-semistable continuous convolution semigroup $\nu_\bullet \subseteq \mathcal{M}^1(\mathbb{H})$ with $\varphi(\nu_t) = \mu_t, t \geq 0$.

b) In particular, if $\mathbb{G}$ is a Lie group then $\mathbb{H}$ is a homogeneous (Lie) group.

c) Analogously, if $\mathbb{G}$ is totally disconnected then $\mathbb{H}$ is totally disconnected too.

d) Let $T = (a_t)_{t>0}$ be a continuous group in $\mathrm{Aut}(\mathbb{G})$ and let $\mu_\bullet$ be T- stable. Then $\mathbb{H} = C(T)$ is a closed subgroup, (isomorphic to) a homogeneous group, φ is the canonical injection and $\nu_\bullet$ is the restriction $\mu_\bullet|_{\mathbb{H}}$.

e) Let $\mathbb{G}$ be a p-adic Lie group. Then $\mathbb{H} = C(a)$ is a closed subgroup hence again φ is the canonical injection and $\nu_\bullet$ is the restriction $\mu_\bullet|_{\mathbb{H}}$.

Note again that in case b) (and d)) the investigations of (semi-)stable laws are completely reduced to simply connected nilpotent Lie groups.

[a) is an immediate consequence of Proposition 2.2 above. According to 1.5 $\mu_\bullet$ is concentrated on $C(a)$. Now b) and c) are immediate consequences of a), for d) see [6]. e) follows by [18], cf. [2].]

Note that within the category of complete and metrizable groups our knowledge of the structure of contractible groups is considerably poor. However, for special cases — if the group $\mathbb{H} = \tilde{C}(a)$ with the natural topology is locally compact — we obtain a reduction of the problems and a complete overview of possible semistable laws. We describe the situation for totally disconnected groups:

Semistable convolution semigroups on contractible totally disconnected groups

A locally compact totally disconnected group $\mathbb{G}$ is contractible with contractive $a \in \mathrm{Aut}(\mathbb{G})$ iff $\mathbb{G}$ admits a *filtration* $(G_n)_{n\in\mathbb{Z}}$ adapted to a, i.e. if there exist compact open subgroups $G_n \subseteq G_{n-1}$ with $\bigcap G_n = \{e\}$ aund $\bigcup G_n = \mathbb{G}$, such that $a(G_n) = G_{n+1}, n \in \mathbb{Z}$. The filtration $(G_n)_{n\in\mathbb{Z}}$ is said to be *normal* if G_n are compact open normal subgroups in $\mathbb{G}$. (See [15].)

3.1. Remark. Let $\mathbb{G}$ be a contractible totally disconnected locally compact group with contractive $a \in \mathrm{Aut}(\mathbb{G})$ and filtration $(G_n)_{n\in\mathbb{Z}}$. Assume the filtration to be *normal.* Then $\mathbb{G} = \lim_{n\in\mathbb{N}} \mathbb{G}/G_n$ is a projective limit of discrete groups. Therefore any continuous convolution semigroup $(\mu_t)_{t\geq 0}$ on $\mathbb{G}$ is a limit of Poisson semigroups $\mu_\bullet^{(n)}$ on $\mathbb{G}/G_n$. (Convolution semigroups on discrete groups are Poisson.) Let $\mu_\bullet$ be (a,α)-semistable. Then the automorphism a is not representable as limit of automorphisms of the factor groups $\mathbb{G}/G_n$ and $\mu_\bullet^{(n)}$ can not be semistable. [Semistable Lévy measures are infinite or trivial, hence semistable laws on discrete groups are trivial.]

For totally disconnected locally compact groups admitting a contractive automorphism a we obtain a complete description of all possible semistable laws. Let $(G_n)_{n\in\mathbb{Z}}$ be a filtration of $\mathbb{G}$ adapted to a. Then $Z := G_0\backslash G_1$ is a cross-section for the orbits $\{a^n(x) : n \in \mathbb{Z}\}$, $x \in \mathbb{G}\backslash\{e\}$. Let us remark that Z is locally compact.

First we note (Cf. [15]):

3.2. Proposition. Let η be a positive measure on $\mathcal{B}(\mathbb{G})$ with $\eta(\{e\}) = 0$, let $\alpha \in]0,1[$ and $a \in \mathrm{Aut}(\mathbb{G})$. Then the following assertions are equivalent:

(i) $\eta(\mathbf{C}pU) < \infty$ for all $U \in \mathfrak{U}(e)$; and $a(\eta) = \alpha \cdot \eta$;

(ii) there exists some finite positive measure κ on $\mathcal{B}(\mathbb{G})$ such that $\kappa(\mathbb{G}\backslash Z) = 0$ and such that $\eta = \sum_{k\in\mathbb{Z}} \alpha^{-k} \cdot a^k(\kappa)$. In fact, we have $\kappa = \eta|_Z$.

Since in the case of totally disconnected groups convolution semigroups are uniquely determined by their Lévy measures, proposition 3.2 provides a complete description of the possible semistable laws:

3.3. Corollary. Fix $\alpha \in (0,1)$. By $\kappa \mapsto \eta := \sum_{k\in\mathbb{Z}} \alpha^{-k} \cdot a^k(\kappa)$ there is given a bijection between the finite measures κ on $\mathcal{B}(\mathbb{G})$ concentrated on Z and the Lévy measures η on $\mathbb{G}$ such that $a(\eta) = \alpha \cdot \eta$ and $\kappa = \eta|_Z$, hence between $\kappa \in \mathcal{M}^1(\mathbb{G})$ with $\kappa(\mathbf{C}pZ) = 0$ and (a, α)-semistable continuous convolution semigroups $\mu_\bullet$.

We consider two examples of contractible totally disconnected groups:

3.4. Example. (*Semistable laws on the p-adics*) For some prime power p let $\mathbb{Q}_p$ denote the locally compact field of p-adic numbers. For any $t \in \mathbb{Q}_p$ we define the "homothetic" transformation $H_t : x \mapsto t \cdot x$. Via the mapping $t \mapsto H_t$ we obtain $\mathrm{Aut}(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times$, cf. [8], (26.18 d).) Let $|\cdot|_p$ denote the p-adic valuation of $\mathbb{Q}_p$. In view of $|H_t(x)|_p = |t|_p \cdot |x|_p$, the automorphism H_t is contractive iff $|t|_p < 1$. $\mathbb{Q}_p$ is totally disconnected. Moreover the subset $\Delta = \mathbb{Z}_p = \{x : |x|_p \leq 1\}$ of p-adic integers is a compact open subgroup of $\mathbb{Q}_p$; (see [8], § 10).

$\mathbb{Q}_p$ may be considered as the subset of the direct product $\otimes_{k\in\mathbb{Z}}\{0, \ldots, p-1\}$ consisting of sequences $\widehat{x} = (x(k))_{k\in\mathbb{Z}}$ such that $x(k) = 0, k \leq K$ for some $K = K(x) \in \mathbb{Z}$. (It is sometimes convenient to represent x equivalently as formal power series $\sum_{k\in\mathbb{Z}} x(k) \cdot p^k$ with $x(k) = 0$ for $k \leq K$.) Let $n := n(x) := \min\{k \in \mathbb{Z} : x(k) \neq 0\}$ if $x \neq 0$. The p-adic valuation is given as $|x|_p := p^{-n(x)}$, if $x \neq 0$, and $|0|_p := 0$.

The field of rational numbers $\mathbb{Q}$ is canonically densely embedded in $\mathbb{Q}_p$ and hence — endowed with the continuously extended algebraic operations of $\mathbb{Q}$ — the $|\cdot|_p$-closure $\mathbb{Q}_p$ is a locally compact totally disconnected topological field, and $\mathbb{Z}$ is dense in $\mathbb{Z}_p$.

Put $\Delta_n := \{x : |x|_p \leq p^{-n}\}, n \in \mathbb{Z}$, then $(\Delta_n)_{n\in\mathbb{Z}}$ is a nested sequence of compact open subgroups with $\bigcap \Delta_n = \{0\}, \bigcup \Delta_n = \mathbb{Q}_p$. And any compact subgroup is of the form Δ_n for some $n \in \mathbb{Z}$ ([8], 10.6).

Obviously, $H_{p^n}(\Delta_0) = \Delta_n, n \in \mathbb{Z}$, more generally, $H_t\Delta_0 = \Delta_n$ if $|t|_p = p^{-n}$. Hence in particular $(\Delta_n)_{n\in\mathbb{Z}}$ is a (normal) filtration adapted to $a := H_p$. For any $t \in \mathbb{Q}_p^\times$ with $|t|_p < 1$ the automorphism H_t is contractive. In particular, H_p is contractive.

If $|t|_p = p^{-d}, d \in \mathbb{N}$, then $(G_{(n)} := \Delta_{nd})_{n\in\mathbb{Z}}$ is a filtration adapted to H_t. We observe $G_{(n)}/G_{(n+1)} = \mathbb{Z}/(p^d \cdot \mathbb{Z})$.

The Haar measure ω_{Δ_n} is absolutely continuous to the Haar measure $\omega_{\mathbb{Q}_p}$:

Normalize $\omega_{\mathbb{Q}_p}$ such that $\omega_{\mathbb{Q}_p}(\Delta_0) = 1$. Then ω_{Δ_0} is the restriction $\omega_{\mathbb{Q}_p}|_{\Delta_0}$. And $\omega_{\Delta_n} = H_{p^n}(\omega_{\Delta_0}) = H_{p^n}(\omega_{\mathbb{Q}_p}|_{\Delta_0}) = \Delta(H_{p^n}) \cdot \omega_{\mathbb{Q}_p}|_{\Delta_n} = p^n \cdot \omega_{\mathbb{Q}_p}|_{\Delta_n}$.

In other words, $\int_{\mathbb{Q}_p} f d\omega_{\Delta_n} = p^n \cdot \int_{|x|_p \leq p^{-n}} f d\omega_{\mathbb{Q}_p}$ for $f \in L^1(\mathbb{Q}_p, \omega_{\mathbb{Q}_p})$.

Next we investigate in some details the following example of a semistable continuous convolution semigroup on the additive group $\mathbb{G} = (\mathbb{Q}_p, +)$.

Let $d \in \mathbb{N}$ and $t \in \mathbb{Q}_p$ with $|t|_p = p^{-d}$, put $a := H_t \in \mathrm{Aut}(\mathbb{Q}_p)$, and let $(G_{(n)} := \Delta_{nd})_{n\in\mathbb{Z}}$ be the corresponding filtration.

An (a,α)-semistable continuous convolution semigroup $\mu_\bullet$ is defined by the Lèvy measure $\eta = c \cdot \sum_{k\in\mathbb{Z}} \alpha^{-k} \cdot a^k(\nu)$, $0 < \alpha < 1$, $c \geq 0$, $\nu \in \mathcal{M}^1(G_{(0)} \setminus G_{(1)})$. (Cf. 3.3.)
We call $\lambda \in \mathcal{M}^1(\mathbb{Q}_p)$ *rotation invariant* if $H_x(\lambda) = \lambda$ for all $x \in \mathbb{U}$, where $\mathbb{U} = \{t : |t|_p = 1\}$ is the group of units in $\mathbb{Q}_p^\times \cong \mathrm{Aut}(\mathbb{Q}_p, +)$. (Cf. also [1], [19].)
The orbits $\mathbb{U} \cdot x$ are given by $\{y \in \mathbb{Q}_p : |y|_p = |x|_p\}$, hence a function $f : \mathbb{Q}_p \to \mathbb{C}$ is $\mathbb{U}$-invariant iff $f(\cdot) = \varphi(|\cdot|_p)$ for some function $\varphi : \mathbb{R}_+ \to \mathbb{C}$. Since for any $n \in \mathbb{Z}$ we have $u \cdot \Delta_n = \Delta_n$, $u \in \mathbb{U}$, we easily conclude that ω_{Δ_n} is rotation invariant.
Obviously, μ_t is rotation invariant if ν has this property, ν as above. We consider the special rotation invariant measure $\nu := \frac{p^d}{p^d-1} \cdot (\omega_{G_{(0)}} - \frac{1}{p^d}\omega_{G_{(1)}}) \in \mathcal{M}^1(G_{(0)})$. As easily seen, since $G_{(0)}/G_{(1)} \simeq \mathbb{Z}/p^d \cdot \mathbb{Z} \simeq \{0, \ldots, p^d - 1\}$, we have
$\omega_{G_{(0)}} = \sum_{k=0}^{p^d-1} \frac{1}{p^d} \cdot \varepsilon_{x_j} * \omega_{G_{(1)}}$. Hence $\nu = \frac{1}{p^d-1} \cdot \sum_{k=1}^{p^d-1} \varepsilon_{x_j} * \omega_{G_{(1)}}$.

$(\mathbb{Q}_p, +)$ is a locally compact *Abelian* group, hence $\mu_\bullet$ may be represented in terms of the Fourier transform: Following the representation in [8], § 25, we obtain the following description of $\widehat{\mathbb{Q}}_p$:
Fix a nontrivial continuous character $\varphi_1 : \mathbb{Q}_p \to \mathbb{T}$ with kernel $\ker\varphi_1 = \Delta_0$. ($\mathbb{T}$ denoting the torus $\{z \in \mathbb{C} : |z| = 1\}$.)
For $y \in \mathbb{Q}_p$ define $\varphi_y : x \mapsto \varphi_1(H_y(x)) = \varphi_1(y \cdot x)$. Any continuous character is obtained in that way and by $y \mapsto \varphi_y$ we obtain an isomorphism, hence $\widehat{\mathbb{Q}}_p \cong \mathbb{Q}_p$. Let $a = H_t \in \mathrm{Aut}(\mathbb{Q}_p)$ then we observe $\varphi_y(ax) = \varphi_{a(y)}(x) =: a^*(\varphi_y)(x)$. Hence $(\mathbb{Q}_p^\times, \cdot) \cong \mathrm{Aut}(\mathbb{Q}_p)$ acts in a natural way on $\widehat{\mathbb{Q}}_p$.
Now we have the means to compute explicitely the Fourier transform $\widehat{\nu}$ since
$\widehat{\omega}_{G_{(n)}}(\varphi_y) = 1$ iff $y \in G_{(-n)}$, and $= 0$ else. Hence
$\widehat{\nu}(\varphi_y) = \frac{p^d}{p^d-1}\widehat{\omega}_{G_{(0)}}(\varphi_y) - \frac{1}{p^d-1}\widehat{\omega}_{G_{(1)}}(\varphi_y)$. Therefore
$\widehat{\mu}_t(\varphi_y) = \exp t \cdot \int(\varphi_y - 1)d\eta$
$\qquad = \exp(tc \cdot \sum_{k\in\mathbb{Z}} \alpha^{-k}(\frac{p^d}{p^d-1}(\widehat{\omega}_{G_{(k)}}(\varphi_y) - 1) - \frac{1}{p^d-1}(\widehat{\omega}_{G_{(k+1)}}(\varphi_y) - 1))$
For simplification we assume now $d = 1, |t|_p = p^{-1}$, hence $G_{(n)} = \Delta_n, n \in \mathbb{Z}$. In this case, $\widehat{\omega}_{\Delta_k}(\varphi_y) - 1 = 0$ if $y \in \Delta_{-k}$ and = -1 else. And the representation yields: There exists some constant $C = C(\alpha, p) > 0$ such that $\widehat{\mu}_t(\varphi_y) = \exp(-t \cdot C \cdot \alpha^{-M})$ for $y \in \Delta_{-M} \setminus \Delta_{-M+1}$, i.e. for $|y|_p = p^M$. Define $\gamma := -\ln\alpha/\ln p > 0$, hence $\alpha = p^{-\gamma}$, then we obtain

$$\widehat{\mu}_t(\varphi_y) = \exp(-t \cdot C \cdot |y|_p^\gamma),\ y \in \mathbb{Q}_p$$

And conversely, $\widehat{\mu}_t(\varphi_y) = \exp(-t \cdot C|y|_p^\gamma)$ defines a rotation invariant (H_p, α)-semistable continuous convolution semigroup on $\mathbb{Q}_p$ for any $0 < \alpha < 1$ (and $\gamma = \gamma(\alpha)$ as above) and any $C > 0$.
At the first glance this representation is similar to the Fourier transform of (elliplically) symmetric stable laws on $\mathbb{R}$ or on real vector spaces $\mathbb{V}$. But note that there is an essential difference: In the real or vector space case we have $0 < \gamma \leq 2$, in the p-adic situation there is no restriction on $\gamma > 0$. Hence the similarity is only formal.

Some further remarks: The Lévy measure $\eta = c \cdot \sum_{k\in\mathbb{Z}} \alpha^{-k} a^k(\nu)$, with $a = H_p, \nu = \frac{p}{p-1} \cdot (\omega_{\Delta_0} - \frac{1}{p} \cdot \omega_{\Delta_1})$ as above, is absolutely continuous with respect to the Haar measure on $\mathbb{Q}_p$ and the density is given by $c \cdot \frac{p}{p-1} \cdot \sum_{k\in\mathbb{Z}} (\alpha/p)^{-k} \cdot 1_{\Delta_k \setminus \Delta_{k+1}}$, as easily seen inserting $d\omega_{\Delta_n}/d\omega_{\mathbb{Q}_p} = p^n \cdot 1_{\Delta_n}$ in the definition of η.

In fact, the Lévy measure η is absolutely continuous and unbounded, whence $\mu_t \ll \omega_{\mathbb{Q}_p}$ follows, cf. A. Janssen [10] resp. E. Siebert [14]. For more details see the investigations Albeverio et al. [1] and K. Yasuda [19] where Lévy processes with rotation invariant semistable continuous convolution semigroups are considered; these laws are called "*stable*" in [19] .

The following example points out once more the typical structure of totally disconnected contractible locally compact groups :

3.5. Example. (Cf. [15]). Let F be a finite group of order $r > 1$. By Λ we denote the set of all sequences $\widehat{x} = (x(k))_{k\in\mathbb{Z}} \in F^{\mathbb{Z}}$ such that $x(k) = e$ for all $k < k_0$ and for some $k_0 \in \mathbb{Z} \cup \{+\infty\}$. Defining the product of two such sequences componentwise, Λ becomes a group. Every subset $\Lambda_{(n)} := \{\widehat{x} = x(k) = e$ for all $k < n\}, n \in \mathbb{Z}$, is a normal subgroup of Λ. If n tends to $+\infty$ then the groups $\Lambda_{(n)}$ decrease to the identity e of Λ; if n tends to $-\infty$ then the groups $\Lambda_{(n)}$ increase to Λ.
We furnish Λ with the (unique) topology that turns Λ into a topological T_0 - group and has $(\Lambda_{(n)})_{n\in\mathbb{Z}}$ as a basis of the identity $\widehat{e}$ (cf. [8], (4.5) and (4.21)). Then Λ is a totally disconnected topological group.
Every factor group $\Lambda_{(n)}/\Lambda_{(n+1)}$ is finite (it is isomorphic with F); hence $\Lambda_{(0)}$ is totally bounded. Moreover Λ is complete with respect to its left uniform structure.
Thus $\Lambda_{(0)}$ is compact, and therefore Λ is locally compact.
Now let $\rho((x(k))_{k\in\mathbb{Z}}) := (x(k-1))_{k\in\mathbb{Z}}$ for all $\widehat{x} = (x(k))_{k\in\mathbb{Z}}$ in Λ (the shift restricted to Λ). It is easy to see that ρ is an automorphism of Λ such that $\rho(\Lambda_{(n)}) = \Lambda_{(n+1)}$ for all $n \in \mathbb{Z}$. Consequently, ρ is bicontinuous and contractive; and $(\Lambda_{(n)})_{n\in\mathbb{Z}}$ is a normal filtration of Λ adapted to ρ. In fact, it is easily verified that $\Lambda = \tilde{C}(a)$, and $\rho = \tilde{a}$ (cf. 2.1) where a denotes the shift on the direct product $F^{\mathbb{Z}}$. (See 4.1 below).

For later use we mention the following simple lemma generalizing 3.2, which enables us to construct semistable laws on general locally compact groups in concrete situations. Let $\mathcal{S}(a,\alpha) = \mathcal{S}(a,\alpha)(\mathbb{G}) := \{A \in \mathcal{GF}(\mathbb{G}) : a(A) = \alpha \cdot A\}$ denote the set of (a,α)-semistable generating functionals.

3.6. Lemma. Assume that $\mathbb{G}$ is a locally compact group, $a \in \mathrm{Aut}(\mathbb{G}), B \in \mathcal{GF}(\mathbb{G}), \alpha \in (0,1)$. Assume that for $f \in \mathcal{D}(\mathbb{G})$ the series $\sum_{k=-\infty}^{\infty} \alpha^{-k} \cdot \langle\, a^k(B), f\rangle$ is absolutely convergent.
Then $A : f \mapsto \langle\, A, f\rangle := \sum_{-\infty}^{\infty} \alpha^{-k} \cdot \langle\, B, f \circ a^k\rangle$ belongs to $\mathcal{S}(a,\alpha)$.
[As easily seen, A is almost positive and normalized (cf. [12], [9]). Hence $A \in \mathcal{GF}(\mathbb{G})$. And $a(A) = \alpha \cdot A$ obviously follows.]

(Semi-)stability on solenoidal groups

There exist compact connected finite-dimensional groups and stable semigroups of probabilities $\mu_\bullet$ with $\mathrm{supp}(\mu_t) = \mathbb{G}, t > 0$. ($\mathbb{G}$ cannot be a Lie group.) The corresponding group of automorphims $T = (a_t)_{t>0}$ is contractive on a dense subgroup (the range of the exponential map), but not contractive on $\mathbb{G}$. $t \mapsto a_t$ is not continuous in this example, and $\mathbb{G}$ is not second countable.

3.7. Example. Choose $\mathbb{R}_d$, the real line with the discrete topology, and let $\mathbb{G}$ be the solenoidal group $\mathbb{G} = (\mathbb{R}_d)^\wedge$ ($= \beta(\mathbb{R})$, the Bohr compactification of $\mathbb{R}$). Then $\psi : \mathbb{R}_d \to \mathbb{R}, \psi(x) := x$, is a continuous injective homomorphism, therefore the dual

homomorphism $\varphi : \widehat{\mathbb{R}}\ (\cong \mathbb{R}) \to (\mathbb{R}_d)^\wedge = \mathbb{G}$ is continuous, injective and has dense range. (Indeed $\mathbb{G}$ is one-dimensional and $\varphi : \mathbb{R} \to \mathbb{G}$ is just the exponential map.).
Now let $(\nu_t)_{t\geq 0}$ be strictly stable on $\mathbb{R}$, i.e. let $b_t = H_{t^\alpha} : x \mapsto t^\alpha \cdot x, t > 0, x \in \mathbb{R}$ and assume $b_t(\nu_s) = \nu_{st}$. b_t can be regarded as automorphism of $\mathbb{R}_d$, therefore the dual map $\widehat{b_t} =: a_t : \mathbb{G} \to \mathbb{G}$ is an automorphism of $\mathbb{G}$.
$y \in \mathbb{R}_d$ is identified with a character γ_y of $\mathbb{G}$, defined on the dense range $\varphi(\mathbb{R}_d)$ by $\langle \varphi(x), \gamma_y \rangle = e^{ixy}, x \in \mathbb{R}_d$. Therefore $\langle a_t(g), \gamma_y \rangle = \langle g, \gamma_{H_{t^\alpha}(y)} \rangle$ for all $t > 0$, $y \in \mathbb{R}_d$, $g \in \mathbb{G}$.
Define $\mu_t := \varphi(\nu_t)_{t\geq 0}$. Obviously $a_t(\mu_s) = a_t(\varphi(\nu_s)) = \varphi(b_t(\nu_s)) = \varphi(\nu_{ts}) = \mu_{ts}$ for $t, s > 0$. So $(\mu_s)_{s\geq 0}$ is stable w.r.t. $T = (a_t)_{t>0}$.
The group T is not contractive on (the compact group) $\mathbb{G}$, but T acts contractively on the range $\varphi(\mathbb{R})$: for $x \in \mathbb{R}$ we observe $a_t(\varphi(x)) = \varphi(t^\alpha \cdot x) \overset{t\to 0}{\longrightarrow} \varphi(0) = e$.
On the other hand $\mu_t = \varphi(\nu_t)$ is concentrated on $\varphi(\mathbb{R})$. ($\varphi(\mathbb{R})$ is σ-compact and hence measurable.) According to 2.3 any (semi-)stable law on $\mathbb{G}$ arises in that way. We note that $t \mapsto a_t$ is not continuous: There exist elements $g \in \mathbb{G}$ which are non-continuous characters on $\mathbb{R}$. But the set of continuity points $S(T)$ is dense.

Semistability on infinite products of compact groups

If $\mathbb{G}$ is a (real or p-adic) Lie group (not necessarily contractible) we have a more or less complete survey over semistable laws supported by $\mathbb{G}$. (See e.g. [3], [6], [7], [14], [2]). Beyond this class of groups there exist semistable laws, but the properties may differ in a characteristic manner. To point out those differences we investigate as a particular example infinite products $\mathbb{G} = K^{\mathbb{Z}}$ where $K \neq \{e\}$ is a compact group. Let a denote the shift, $a(\widehat{x})(k) := \widehat{x}(k+1)$ for $\widehat{x} \in \mathbb{G}$, $\widehat{x} : \mathbb{Z} \to K$.

4.1. Proposition. a) There exist non-trivial (a, α)-semistable laws on any group representable as infinite product $\mathbb{G} = K^{\mathbb{Z}}$, in particular on the infinite-dimensional torus $\mathbb{T}^{\mathbb{Z}}$, where a denotes the shift and $\alpha \in (0, 1)$.
b) Analogously, there exist non-trivial stable laws on any group $\mathbb{G} = K^{\mathbb{R}}$, for a nontrivial compact group K; in particular on the infinite-dimensional torus $\mathbb{T}^{\mathbb{R}}$. In this case the automorphism group T is the (non-continuous) group of shifts.
[a) Let $K \neq \{e\}$ be a compact group (e.g. $K = \mathbb{T}$). Define $\mathbb{G} := K^{\mathbb{Z}}$ and let $a : \mathbb{G} \to \mathbb{G}$ be the shift $a(\widehat{x})(k) := \widehat{x}(k+1)$, $k \in \mathbb{Z}$, for $\widehat{x} \in \mathbb{G}$, $\widehat{x} : \mathbb{Z} \to K$. For any $n_1 < n_2 \in \mathbb{Z}$, let $J := \{n_1, \ldots n_2\}$ and $\mathcal{K}_J := K^J$. Obviously, $\mathcal{D}(\mathbb{G}) = \mathcal{E}(\mathbb{G}) = \{f = f' \circ \pi_J$ for some $J \subseteq \mathbb{Z}$ and $f' \in \mathcal{D}(\mathcal{K}_J)\}$ ($\pi_J : K^{\mathbb{Z}} \to K^J$ denotes the canonical projection). Consequently, for any generating functional $B^\# \in \mathcal{GF}(\mathcal{K}_J)$ we define $B \in \mathcal{GF}(\mathbb{G})$ via $\langle B, f \rangle := \langle B^\#, f' \rangle$ where $f = f' \circ \pi_J$.
For any $f \in \mathcal{D}(\mathbb{G})$ obviously $\sum_{k\in\mathbb{Z}} \alpha^{-k} \cdot \langle B, f \circ a^k \rangle$ converges (indeed the entries are zero, except a finite number.) Therefore $A = \Sigma \alpha^{-k} \cdot a^k(B)$ is a semistable generating functional on $\mathbb{G}$ (cf. 3.6).
b) Let $\mathbb{G} = K^{\mathbb{R}}$ be represented as $\mathbb{G} = \{\widehat{x} : \mathbb{R}_+^\times \to K\}$ and define $T = (a_t)_{t>0}$ to be the group of shifts $a_t(\widehat{x})(s) := \widehat{x}(ts), \widehat{x} \in \mathbb{G}, t, s > 0$ (with multiplicative parametrization). T is a non-continuous group in $\mathrm{Aut}(\mathbb{G})$ fulfilling $a_t a_s = a_{ts}$. (If K is finite or $K = \mathbb{T}^m$ then there exist only trivial *continuous* groups in $\mathrm{Aut}(\mathbb{G})$).

Let $\lambda_\bullet^{(r)}$ be a continuous convolution semigroup on K $(\cong K^{(r)})$ and define $\mu_s := \otimes_{r>0}\lambda_s^{(r)}$ for any coordinate $r > 0$. Then, as immediately seen, $a_t(\mu_s) = \otimes_{r>0}\lambda_s^{(r/t)}$, $t > 0$. Hence for fixed $\gamma > 0$, we have

$$a_{t^\gamma}(\mu_s) = \mu_{s\cdot t}, s \geq 0, t > 0 \quad \text{iff} \quad \lambda_{s\cdot t}^{(r)} = \lambda_s^{(r/t^\gamma)} \ (r > 0).$$

In analogy to a), let $\nu_\bullet$ be an arbitrary continuous convolution semigroup on K with generating functional B. Then $\mu_s^{(\gamma)} := \otimes_{r>0}a_r(\nu_{s/r^{1/\gamma}})$ fulfil the relations $a_{t^\gamma}(\mu_s^{(\gamma)}) = \mu_{s\cdot t}^{(\gamma)}$.

If we identify K with $K^1 \subseteq \mathbb{G}$ and consider $B \in \mathcal{GF}(K)$ as generating functional $B \in \mathcal{GF}(\mathbb{G})$ then (e.g. for $\gamma = 1$) the generating functional of $\mu_\bullet^{(1)}$ is given by $A = \sum_{r>0} r^{-1} \cdot a_r(B)$. In fact, $f \in \mathcal{D}(\mathbb{G})$ depends only on finitely many coordinates, $\{r_1, \ldots, r_r\}$ say. Hence $\langle A, f\rangle$ is well defined and we have $a_t(A) = t \cdot A$ for all $t > 0$.]

4.2. Remark. In a) assume in particular $J = \{0\}$, consider $K = K^{(0)}$ as subgroup of $\mathbb{G}$. Let $B = B^{(0)} \in \mathcal{GF}(K)$ denote the generating functional of a continuous convolution semigroup $\mu_\bullet = \mu_\bullet^{(0)} \subseteq \mathcal{M}^1(K)$. Then the continuous convolution semigroup generated by A has product form $\mu_t = \otimes_{k\in\mathbb{Z}}\mu_t^{(k)}$, with $\mu_t^{(k)} = \mu_{\alpha^{-k}t}$.

$C(a) \cap C(a^{-1})$ on infinite products $K^{\mathbb{Z}}$

We consider the subgroups $\mathcal{F}_l := \mathcal{F}_l(a) := \{\widehat{x} \in \mathbb{G} : \lim_{k\to\infty}\widehat{x}(k) = e\}$, $\mathcal{F}_r := \{\widehat{x} \in \mathbb{G} : \lim_{k\to-\infty}\widehat{x}(k) = e\}$, $\mathcal{F}_0 := \{\widehat{x} \in \mathbb{G} : \lim_{|k|\to\infty}\widehat{x}(k) = e\} = \mathcal{F}_l \cap \mathcal{F}_r$ and $\mathcal{F} := \{\widehat{x} \in \mathbb{G} : \widehat{x}(k) \neq e \text{ finitely often}\}$.

Obviously, $C(a) = \mathcal{F}_l$, $C(a^{-1}) = \mathcal{F}_r$, and we observe $\mathcal{F} = \mathcal{F}_0$ iff K is finite.

If $\mathbb{G}$ is a Lie group then $C(\tau) \cap C(\tau^{-1}) = \{e\}$ for all $\tau \in \mathrm{Aut}(\mathbb{G})$. [This is easily proved e.g. repeating the arguments in [16], example 1.] Hence (a,α)- and (a^{-1},β)-semistable laws are concentrated on subgroups with trivial intersection.

In contrast, for $\mathbb{G} = K^{\mathbb{Z}}$ and if a denotes the shift as above then $\mathcal{F}$ and hence $\mathcal{F}_0 = C(a) \cap C(a^{-1})$ are dense in $\mathbb{G}$. However, for semistable laws in productform we obtain:

4.3. Proposition. Let $\rho_\bullet$ and $\sigma_\bullet$ be non-degenerate (a,α)- and (a^{-1},β)-semistable continuous convolution semigroups of product form considered in 4.2. Then, for $s,t > 0$, ρ_t and σ_s are concentrated on the disjoint measurable subsets $C(a)\backslash\mathcal{F}_0$ and $C(a^{-1})\backslash\mathcal{F}_0$ respectively.

Proof: In fact, if K is finite, the assertion follows since by construction semistable laws have infinite Lévy measures and are thus diffuse measures ([10], [14]). On the other hand, in this case $\mathcal{F}_0 = \mathcal{F}$ is countable. Whence $\rho_t(\mathcal{F}) = \sigma_s(\mathcal{F}) = 0, t, s > 0$.

If K is infinite, assume according to 4.2 $\rho_t = \otimes_{k\in\mathbb{Z}}\mu_t^{(k)}$ with $\mu_t^{(k)} = \mu_{\alpha^{-k}t}$ (where $\mu_\bullet$ is a continuous convolution semigroup in $\mathcal{M}^1(K) \cong \mathcal{M}^1(K^{(k)})$). And assume an analogous representation for $\sigma_\bullet$. We have to show $\rho_t(\mathcal{F}_0) = \sigma_s(\mathcal{F}_0) = 0$ for $s,t > 0$. Since μ_t is non-degenerate the limit set $\mathrm{LIM}\{\mu_t : t \to \infty\}$ is contained in $\{\varepsilon_x * \omega_H\}$ for some non-trivial subgroup $H \subset K$. Therefore, as easily seen, for a neighbourhood $U \in \mathfrak{U}(e)$ in K we have $\limsup \mu_t\{U\} < 1$. I.e. $\mu_t\{U\} \leq \kappa < 1$ for sufficiently large t, hence $\mu_t^{(k)}\{U\} = \mu_{\alpha^{-k}t}\{U\} \leq \kappa$ for sufficiently large k. For any $L \in \mathbb{N}$ we conclude

$$\rho_t\{\prod_{|j|\leq L} K \times \prod_{|j|>L} U\} = (\otimes_{j\in\mathbb{Z}}\mu_t^{(j)})\{\prod_{|j|\leq L} K \times \prod_{|j|>L} U\} = 0$$

since $\prod_{|j|>L} \mu_{t\cdot\alpha^{-j}}(U) = 0$.
Whence $\mu_t\{\mathcal{F}_0\} = 0$ for $t > 0$ as asserted since $\mathcal{F}_0 \subseteq \bigcup_{L\in\mathbb{N}} \prod_{|j|\leq L} K \times \prod_{|j|>L} V$ for any $V \in \mathfrak{U}\{e\}$. ∎

Marginals of semistable laws on ifinite products

4.4. Remarks. a) If K is a finite group, then K^n is finite for $n \in \mathbb{N}$, hence $\mathcal{S}(a,\alpha)(K^n)$ is trivial but $K^{\mathbb{Z}} = \mathbb{G}$ possesses non-trivial semistable laws. But according to 3.1 no finite-dimensional marginal distribution is semistable.
b) Finite-dimensional tori $\mathbb{T}^d, d \geq 2$, admit automorphisms with dense contractible subgroups and semistable laws on $\mathbb{T}^d$ are homomorphic images of operator semistable laws on subspaces of $\mathbf{V} = \mathbb{R}^d$.
Let a denote the shift on the *infinite-dimensional torus* $\mathbb{G} = \mathbb{T}^{\mathbb{Z}}$ acting contractively on the dense subgroup $\mathcal{F}_l$. Again, also in this case finite-dimensional marginals of (a,α)-semistable laws need not be semistable: Let $B \in \mathcal{GF}(\mathbb{G})$ be a generating functional such that the generated continuous convolution semigroup is concentrated on a finite-dimensional torus $\mathbb{H} := \mathbb{T}^I$, I finite $\subseteq \mathbb{Z}$, e.g. on $\mathbb{T}^{\{0\}}$. Assume $\alpha \in (0,1)$ and put $A := \sum_{k\in\mathbb{Z}} \alpha^{-k} \cdot a^k(B)$. According to 3.6 resp. 4.2 the continuous convolution semigroup generated by A is (a,α)-semistable. If B is a Poissongenerator then for any finite $I \subseteq \mathbb{Z}$ the projection onto $\mathbb{T}^I$ is Poisson and hence not semistable.

Limit laws on infinite-dimensional tori $\mathbb{T}^{\mathbb{Z}}$ *and on* $\mathbb{R}^{\mathbb{Z}}$

5.1. Example. $\mathbb{G} = \mathbb{T}^{\mathbb{Z}}$ is arcwise connected, with (infinite-dimensional Abelian) Lie algebra $\mathbb{R}^{\mathbb{Z}}$. In this case, the exponential map $\pi = \exp : \mathbf{V} := \mathbb{R}^{\mathbb{Z}} \to \mathbb{T}^{\mathbb{Z}}, \widehat{\phi} := (\phi(k) : k \in \mathbb{Z}) \mapsto (e^{i\cdot\phi(k)} : k \in \mathbb{Z})$, is surjective. There exists a linear subspace $\overset{o}{\mathcal{F}_l} := \{\widehat{\phi} \in \mathbb{R}^{\mathbb{Z}} : \lim_{k\to-\infty} \phi(k) = 0\}$ of $\mathbb{R}^{\mathbb{Z}}$ and an automorphism $\overset{o}{a}$, the shift on $\mathbb{R}^{\mathbb{Z}} = \mathbf{V}$, which acts contractively on $\overset{o}{\mathcal{F}_l}$, such that $a \circ \exp = \exp \circ \overset{o}{a}$ and such that $\exp(\overset{o}{\mathcal{F}_l}) = \mathcal{F}_l$. The restriction of the exponential map to $\overset{o}{\mathcal{F}_l}, \exp : \overset{o}{\mathcal{F}_l} \to \mathcal{F}_l$ is surjective but not injective. Moreover, it is not possible to describe $\overset{o}{a}$ by its action on finite dimensional subspaces. Also on $\mathbf{V} = \mathbb{R}^{\mathbb{Z}}$, finite-dimensional marginal distributions of $(\overset{o}{a},\alpha)$-semistable laws need not be semistable as shown analogously to the situation $\mathbb{T}^{\mathbb{Z}}$ in 4.4.b)

We avoided to develop a theory of generating functionals for the (non locally compact) group $\mathbb{R}^{\mathbb{Z}}$. Indeed, $\mathbf{V} = \mathbb{R}^{\mathbb{Z}}$ is a nuclear vector space and $\mathbb{G} = \mathbb{T}^{\mathbb{Z}}$ is a compact Abelian group. Hence Fourier transforms are available, and Fourier transforms in both cases are determined by finite-dimensional projections.
Let $\xi \in \widehat{\mathbf{V}}$, i.e. let $\phi \in \mathbf{V}'$ be a continuous linear functional, and $\langle \xi, X \rangle := e^{i\cdot\langle \phi, X\rangle}$, and let $\pi = \pi_I$ be a finite-dimensional projection. If ϕ is constant on cosets of ker π then $\mathbb{G} \ni x = \pi(X) \mapsto e^{i\cdot\langle \phi, X\rangle} =: \langle x, \widehat{\pi}(\xi)\rangle$ defines a character $\bar{\xi} = \widehat{\pi}(\xi)$ of $\mathbb{G}$; and any continuous character arises in this way.
Hence, with the notations introduced above the Fourier transforms fulfil
$\widehat{\overset{o}{\lambda}}(\xi) = \widehat{\lambda}(\pi(\xi))$ for $\overset{o}{\lambda} \in \mathcal{M}^1(\mathbf{V})$ resp. $\lambda = \exp(\overset{o}{\lambda}) \in \mathcal{M}^1(\mathbb{G})$.
Analogously, let $\mu_\bullet$ denote the Poisson semigroup on $\mathbb{G}$ with Fourier transform

$\widehat{\mu}_t = \exp(t(\widehat{\lambda} - 1))$, then $\widehat{\mu}_t^{o}(\xi) = \exp(t(\widehat{\lambda}^{o}(\pi(\xi)) - 1))$ defines the Poisson semigroup $\overset{o}{\mu}_{\bullet} = \exp\bullet(\overset{o}{\lambda} - \varepsilon_0)$ on $\mathbb{V}$.
Assume λ to be concentrated on $\mathbb{T}^{\{0\}}$, put $B := \lambda - \varepsilon_e$ and define $A \in \mathcal{GF}(\mathbb{G})$ as in 4.1.a). Assume further $\mathrm{supp}(\overset{o}{\lambda}) \subseteq [0, 2\pi]$. Then for $\bar{\xi} := \widehat{\pi}(\xi) \in \widehat{\mathbb{G}}, \bar{\xi} = (\bar{\xi}(k))_{k \in \mathbb{Z}} \in (\mathbb{T}^{\mathbb{Z}})^{\wedge} \cong \mathbb{Z}^{*\mathbb{Z}}$ (weak product), we obtain
$\widehat{A}(\bar{\xi}) = \sum \alpha^{-k} \cdot (a^k(\lambda - \varepsilon_e))^{\wedge}(\bar{\xi}) = \sum \alpha^{-k} \cdot (\widehat{\lambda}(\bar{\xi}(k)) - 1)$, and analogously,
$\widehat{A}^{o}(\xi) = \sum \alpha^{-k} \cdot (\overset{o}{a}{}^k(\overset{o}{\lambda} - \varepsilon_0))^{\wedge}(\xi) = \sum \alpha^{-k} \cdot (\widehat{\lambda}^{o}(\xi(k)) - 1)$.

Let $\nu_{\bullet}$ denote the semigroup on $\mathbb{G}$ defined by $\widehat{\nu}_t = e^{t\widehat{A}}$. Then by $\overset{o}{\nu}_t{}^{\wedge} := \exp(t\widehat{A}^{o})$ there is defined a continuous convolution semigroup $\overset{o}{\nu}_{\bullet} \subseteq \mathcal{M}^1(\mathbb{V})$ with $\pi(\overset{o}{\nu}_t) = \nu_t, t \geq 0$. (Fourier transforms $\widehat{A}$ resp. $\widehat{A}^{o}$ of generating functionals are logarithms of Fourier transforms of the generated probability measures, defined by $\widehat{\nu}_t = \exp(t \cdot \widehat{A})$ resp. $\widehat{\nu}_t^{o} = \exp(t \cdot \widehat{A}^{o})$. Hence $\widehat{A}^{o}$ is well defined, even if we avoided here to define generating functionals $\overset{o}{A}$ on $\mathbb{V}$.)
As immediately seen, $\nu_{\bullet}$ and $\overset{o}{\nu}_{\bullet}$ are (a, α)- resp. $(\overset{o}{a}, \alpha)$-semistable. But for any finite-dimensional projection $p : \mathbb{V} \to \mathbb{R}^I$ the Lévy measure of $p(\overset{o}{A})$ is concentrated on the compact subset $[0, 2\pi]^I \subseteq \mathbb{R}^I$, hence $p(\nu_{\bullet})$ can not be semistable.

Central limit laws and rescaled canonical random walks on Lie groups

6.1. Let $\mathbb{H}$ be a Lie group with Lie algebra $\mathbb{V}$. Let U and V be neighbourhoods of e and 0 in $\mathbb{H}$ and $\mathbb{V}$ respectively such that $\exp : V \to U$ is bijective. Let $\overset{o}{\gamma}_t$ be a Gaussian convolution semigroup on $\mathbb{V}$ with covariance I w.r.t. a basis $\{X_1, \ldots X_d\}$. Consider $\Delta = \frac{1}{2}\Sigma X_i^2$ as Laplacian on $\mathbb{H}$ and on $\mathbb{V}$ simultaneously. Hence Δ generates symmetric Gaussian semigroups $(\mu_{\bullet})$ in $\mathcal{M}^1(\mathbb{H})$ and $(\overset{o}{\gamma}_{\bullet})$ in $\mathcal{M}^1(\mathbb{V})$.
According to the usual central limit theorem (on vector spaces) $\overset{o}{\gamma}_t$ is representable as limit distribution of a canonical sequence of rescaled random walks:
Consider $\{\pm X_i : i = 1, \ldots d\}$, the nearest neighbours of 0 in $\mathbb{V}$ $(= \mathbb{R}^d)$. Let $(Y_j)_{j \geq 1}$ be a sequence of i.i.d. r.v. with distribution $\overset{o}{\nu}_0 = \frac{1}{2d} \sum \varepsilon_{\pm X_i}$. Then for all n $\{Y_j^{(n)} := n^{-1/2} Y_j\}_{n \geq 1}$ is an i.i.d. sequence on the (rescaled) lattice $n^{-1/2}\mathbb{Z}^d$ (w.r.t. the fixed basis $X_i, 1 \leq i \leq d$) with distribution $\overset{o}{\nu}_n = \frac{1}{2d} \sum \varepsilon_{\pm n^{-1/2} \cdot X_i}$.
Define $\xi_i(\cdot)$ to be the curves $\xi_i(t) := \exp(tX_i)_{t \in \mathbb{R}}$ in $\mathbb{H}$, put $\Psi_i^{(n)} := \exp(Y_i^{(n)})$, then $(\prod_{1 \leq i \leq m} \Psi_i^{(n)})_{m \geq 1}$ is a sequence of random walks on $\mathbb{H}$ with distribution $\nu_n = \frac{1}{2d} \sum \varepsilon_{\xi_i(\pm n^{-1/2})}$. (In some sense rescaled nearest neighbour random walks, but not necessarily concentrated on sublattices of $\mathbb{H}$).
In $\mathcal{M}^1(\mathbb{V})$ the CLT yields convergence of distributions of the rescaled random walks $n^{-1/2} \sum_0^{[nt]} Y_j = \sum_0^{[nt]} Y_j^{(n)}$, $\overset{o}{\nu}_n{}^{[nt]} \to \overset{o}{\gamma}_t, t \geq 0$.
According to E. Siebert's characterization of limit laws (cf. e.g. [13], [5]) this is equivalent to $n \cdot (\overset{o}{\nu}_n - \varepsilon_0) \to \Delta$ (for C_b^{∞}-functions on $\mathbb{V}$ with support in V).
Since exp is (locally) bijective, again by Siebert's theorem this is equivalent to $n \cdot (\nu_n - \varepsilon_e) \to \Delta$ (for C_b^{∞}-functions on $\mathbb{H}$ with support in U).
And again we obtain equivalence to $\nu_n^{[nt]} \to \mu_t, t \geq 0$.

Hence *Gaussian distributions* μ_t on a Lie group $\mathbb{H}$ are representable as limits of distributions of the rescaled random walks $\prod_{1\le i\le[nt]} \Psi_i^{(n)}$, and vice versa.

6.2. Remark. If Δ is a sub-Laplacian then the corresponding Gaussian semigroup $\overset{o}{\gamma_t}$ and the random walks $\overset{o}{\nu_n}{}^m$ are concentrated on a subspace of $\mathbb{V}$. But μ_t may have full support on $\mathbb{H}$.

6.3. Let $\mathbb{G}$ be a connected compact group with Lie algebra $\mathbb{V}$. $\mathbb{G}$ and $\mathbb{V}$ are projective limits $\mathbb{G} = \lim_{\leftarrow} \mathbb{G}^\alpha, \mathbb{G}^\alpha = \mathbb{G}/K_\alpha$, resp. $\mathbb{V} = \lim_{\leftarrow} \mathbb{V}^\alpha$. For fixed α let $\{X_1^\alpha, \dots X_{d_\alpha}^\alpha\}$ be a basis of $\mathbb{V}^\alpha$, let $\exp_\alpha : \mathbb{V}^\alpha \to \mathbb{G}^\alpha$ be the exponential mapping.
Let $(\mu_t)_{t\ge 0}$ be a Gaussian convolution semigroup on $\mathbb{G}$ and let for fixed α μ_t^α be the projected measures on $\mathbb{G}^\alpha$ with Laplacian $\Delta^\alpha = \sum (X_i^\alpha)^2$. And let $\overset{o}{\gamma_t}{}^\alpha$ be defined analogously. (W.l.o.g. we assume the basis of $\mathbb{V}^\alpha$ to be suitably chosen.) According to step 6.1 there exist random walks $(\nu_n^\alpha)^{[nt]}$ on $\mathbb{G}^\alpha$ and $(\overset{o}{\nu_n}{}^\alpha)^{[nt]}$ on $\mathbb{V}^\alpha$ converging to μ_t^α resp. to $\overset{o}{\gamma_t}{}^\alpha, t \ge 0$.

In particular we are interested in the following
6.4. Example. a) If $\mathbb{G} = \prod G_n$ is a product of compact connected Lie groups $G_n, n \in \mathbb{N}$, then we obtain a projective basis $\{X_i : i \ge 1\}$ of $\mathbb{V}$, such that $\{X_i : d_n + 1 \le i \le d_{n+1}\}$ is a basis of G_n, hence $\{X_i : 1 \le i \le d_{i+1}\}$ is a basis of $\prod_{1\le j\le n} G_j =: \mathbb{G}^n$.
If $\mu_t = \otimes_{k\in\mathbb{Z}} \mu_t^{(n)}$ is a product of Gaussian semigroups $\mu_t^{(n)} \in \mathcal{M}^1(G_n)$ with (Laplacian) generating functional Δ then the basis $\{X_i\}$ can be chosen in such a way that the Laplacians Δ_n corresponding to the projection $\mu_t^{(n)}$ to $\mathbb{G}^n$ have the form $\sum_1^{d_n} X_i^2$. In this case the approximating random walks admit a construction without making explicit use of the particular Lie groups: Elements of $\mathbb{V}$ may be represented as sequences $(c_j) \in \mathbb{R}^{\mathbb{Z}}$, formally as $\sum c_j X_j$. The random walks defined on $\mathbb{G}^n$ according to 6.3 form a projective family $(\overset{o}{\pi_m}(\overset{o}{\nu_n})^{[nt]})_{m=1,2,\dots}$, where $\overset{o}{\pi_m} : \mathbb{V} \to \mathbb{V}^m$ denote the canonical projections and $\overset{o}{\nu_n} \in \mathcal{M}^1(\mathbb{V})$ are of product form $\otimes_{k\in\mathbb{N}} \overset{o}{\nu_n}{}^{(k)}$.
Note that $\mathbb{V} = \lim_{\leftarrow} \mathbb{V}^n$ is a nuclear vector space, hence the projective families define probabilities $\overset{o}{\nu_n}$ on $\mathbb{V}$. And analogously, $(\pi_m(\nu_n)^{[nt]})_{m=1,2,\dots}$ form a projective spectrum on $\mathbb{G}$ with $\nu_n = \otimes_{k\in\mathbb{N}} \nu_n^{(k)}$.
Furthermore, $\overset{o}{\pi_m}(\overset{o}{\nu_n})^{[nt]} \to \overset{o}{\pi_m}(\gamma_t)$, $t \ge 0$, iff $\pi_m(\nu_n)^{[nt]} \to \pi_m(\mu_t)$, $t \ge 0$, for all $m \in \mathbb{N}$. But this is equivalent to the convergence $\nu_n^{[nt]} \to \mu_t, t \ge 0$.

Putting things together, for *Gaussian laws* we obtain equivalence of convergence of the random walks on $\mathbb{G}$ and $\mathbb{V}$ respectively, in other words,

$$\overset{o}{\nu_n}{}^{[nt]} \to \overset{o}{\gamma_t}, t \ge 0 \quad \text{iff} \quad \nu_n^{[nt]} \to \mu_t, t \ge 0 \qquad (*)$$

b) If we are in a situation analogous to 6.1, i.e. if $\mathbb{G} = K^{\mathbb{Z}}$, $\Delta = \Delta_0$ is a Laplacian on $K = K^{(0)}$, and $\Delta_n := \alpha^{-n} \cdot a^n(\Delta), n \in \mathbb{Z}$, ($a$ denoting again the shift), then the limits $\mu_\bullet$ and $\overset{o}{\gamma_\bullet}$ are Gaussian and (a, α)- resp. $(\overset{o}{a}, \alpha)$-semistable on $\mathbb{G}$ and $\mathbb{V}$ respectively.
In this situation, as easily seen, $\overset{o}{\nu_n}{}^{(k)}$ and $\nu_n^{(k)}$ in a) are representable as $(2d)^{-1} \cdot \sum_1^d \varepsilon_{\pm\alpha^{-k/2} \cdot n^{-1/2} \cdot X_i}$ and $(2d)^{-1} \cdot \sum_1^d \varepsilon_{\xi_i(\pm\alpha^{-k/2} \cdot n^{-1/2})}$, shifted by $\overset{o}{a}{}^k$ and a^k respectively. And we obtain $(*)$ with $\nu_k = \otimes_{k\in\mathbb{Z}} \nu_n^{(k)}$ and $\overset{o}{\nu_k} = \otimes_{k\in\mathbb{Z}} \overset{o}{\nu_n}{}^{(k)}$.

6.5. Remark. Note that the equivalence ($*$) can only be proved for Gaussian limits: The construction makes heavy use of the fact that for finite-dimensional projections (at least for large n) $\mathrm{supp}(\nu_n)$ and $\mathrm{supp}(\overset{o}{\nu}_n)$ are contained in neighbourhoods U and V on which exp is bijective. Hence considering finite-dimensional projections we conclude that the limits have to be Gaussian:
If $\mathbb{H}$ is a compact connected Lie group with Lie algebra $\mathbb{V}$ and $\exp : V \to U$ bijective then (U and) V must be bounded. Hence in particular, $\overset{o}{\nu}_n$ being concentrated on V has finite second moments. And therefore, if $\overset{o}{\nu}_n^{[nt]} \to \overset{o}{\gamma}_t$, $t \geq 0$, for some convolution semigroup $\overset{o}{\gamma}_\bullet$, then $\overset{o}{\nu}_n$ belongs to the domain of attraction of $\overset{o}{\gamma}_t$ and has finite second moments, hence the limit must be Gaussian.

References

[1] **Albeverio, S., Kaworski, W.:** A random walk on p-adics — the generator and its spectrum. Stochastic Proc. Applications 53, 1–22 (1994).

[2] **Dani, S. G., Shah, R.:** Contraction subgroups and semistable measures on p-adic Lie groups. Math. Proc. Camb. Phil. Soc. 110, 299–306 (1991).

[3] **Hazod, W.** Stable probabilities on locally compact groups. In: Probability measures on groups. Proceedings Oberwolfach 1981, 183–208. Lecture Notes Math. 928 (1982).

[4] **Hazod, W.** A generalization of E. Siebert's theorem on convergence of convolution semigroups and accompanying laws. Theory Prob. Appl. 40, 929–934 (1995).

[5] **Hazod, W. Scheffler, H-P.:** The domains of partial attraction of probabilities on groups and on vectorspaces. J. Theoretical Probability 6, 175–186 (1993).

[6] **Hazod, W., Siebert, E.:** Continuous automorphism groups on a locally compact group contracting modulo a compact subgroup and applications to stable convolution semigroups. Semigroup Forum 33, 111–143 (1986).

[7] **Hazod, W., Siebert, E.:** Automorphisms on a Lie group contracting modulo a compact subgroup and applications to semistable convolution semigroups. J. Theoretical Probability 1, 211–226 (1988).

[8] **Hewitt, E., Ross, K. A.:** Abstract Harmonic Analysis I. Berlin–Göttingen–Heidelberg: Springer (1963).

[9] **Heyer, H.:** Probability Measures on Locally Compact Groups. Berlin–Heidelberg–New York. Springer (1977).

[10] **Janssen, A.:** Continuous convolution semigroups with unbounded Lévy measures on locally compact groups. Arch. Math. 38, 565–576 (1982).

[11] **Jaworski, W.:** Contractive automorphisms of locally compact groups and the concentration function problem. J. Theoret. Probab. 10, 967–989 (1997).

[12] **Siebert, E.:** Über die Erzeugung von Faltungshalbgruppen auf beliebigen lokalkompakten Gruppen. Math. Z. 131, 313–333 (1973).

[13] **Siebert, E.:** Fourier Analysis and limit theorems for convolution semigroups on a locally compact group. Adv. Math. 39, 111–154 (1981).

[14] **Siebert, E.:** Semistable convolution semigroups on measurable and topological groups. Ann. IHP. 20, 147–164 (1984).

[15] **Siebert, E.:** Contractive automorphisms on locally compact groups. Math. Z. 191, 73–90 (1986).

[15] **Siebert, E.:** Semistable convolution semigroups and the topology of contraction groups. In: Probability measures on groups IX, Proceedings Oberwolfach 1988, 325–343. Lecture Notes Math. 1379 (1989).
[17] **Shah, R.:** Selfdecomposable measures on simply connected nilpotent Lie groups. J. Theor. Probab. 13, 65–83 (2000).
[18] **Wang, S. P.:** The Mautner phenomen for p-adic Lie groups. Math. Z. 185, 403–411 (1984).
[19] **Yasuda, K.:** Additive processes on local fields. J. Math. Sci. Tokyo 3, 629–654 (1996).

W. Hazod, Department Mathematics, University of Dortmund, D-44221 Dortmund, Germany

e-mail: hazod@math.uni-dortmund.de

Functional central limit theorems for locally compact groups: the use of infinite dimensional Fourier analysis

by Herbert Heyer

In the theory of functional central limit theorems one considers scaled sums of infinitesimal arrays of d-dimensional random vectors of the form

$$X_n(t) := \sum_{\ell=1}^{k_n(t)} X_{n\ell}$$

on a probability space $(\Omega, \mathfrak{A}, \mathbf{P})$ and studies the corresponding sequences $\{X_n : n \in \mathbf{N}\}$ of stochastic processes $X_n = \{X_n(t) : t \in \mathbf{R}_+\}$ as functions in the Skorokhod space $D(\mathbf{R}_+, \mathbf{R}^d)$. One of the most profound contributions to the theory was to establish necessary and sufficient conditions for a sequence $\{X_n : n \in \mathbf{N}\}$ of process X_n to converge in distribution on $D(\mathbf{R}_+, \mathbf{R}^d)$ towards an increment process $X := \{X(t) : t \in \mathbf{R}_+\}$. A classical tool used in solving the convergence problem is the Lévy-Khintchine bijection

$$\mathbf{P}_X \leftrightarrow (a, B, \eta) \tag{1}$$

between the set $\mathcal{IP}(\mathbf{R}^d)$ of distributions of increment processes X in $\mathbf{R}^d$ and the set $P(\mathbf{R}_+, \mathbf{R}^d)$ of characteristic triplets (a, B, η) consisting of shift mappings a, diffusion mappings B and Lévy measures η. The solution to the problem given for example in [12] consists in characterizing the convergence

$$X_n \to X \tag{2}$$

of an increment process in terms of convergence conditions on the scaled sums of moments towards the characteristic objects in the triplet (a, B, η).

Functional central limit theorems of the described type can also be looked at within the framework of general locally compact groups G provided a Lévy-Khintchine bijection similar to (1) is available. For Lie projective groups G this work was carried out in [8] and [13]. On the other hand the Lévy-Khintchine bijection for Moore groups G described in [14] and [6] suggests the search for at least sufficient conditions for the convergence (2) in terms of generalized characteristic functions of G-valued random variables or synonymously, in terms of the Fourier transforms of their distributions on the dual of G. The definition of the Fourier transform of a probability measure on G therefore involves infinite dimensional unitary representations of G. The method of infinite dimensional Fourier transforms has been efficiently applied to commutative arrays and stationary increment processes in [15]. In their papers [9] and [10] G. Pap and the author make use of infinite dimensional Fourier transforms in order to propose sufficient conditions in terms of integrating families related to the given infinitesimal array.

The present article aims at surveying the methodical tools and some of the results achieved on the way to a solution of the problem in (2). In particular the author will elaborate on an axiomatic approach to the Lévy continuity property which plays an important role in arriving at the desired functional central limits. The subsequent discussion can be viewed as a supplement actualizing the very useful survey [13].

1. The case of a Lie projective group

For the general setting we suppose that G is a second countable locally compact group with neutral element e. Given an array $\{X_{n\ell} : n, \ell \in \mathbf{N}\}$ of rowwise independent G-(valued) random variables and a scaling sequence $\{k_n : n \in \mathbf{N}\}$ consisting of increasing càd functions $k_n : \mathbf{R}_+ \to \mathbf{Z}_+$ with $k_n(o) = o$ and $k_n(\mathbf{R}_+) = \mathbf{Z}_+$, such that the family $\{X_{n\ell} : n \in \mathbf{N}, 1 \leq \ell \leq k_n(t)\}$ is infinitesimal in the sense that

$$\lim_{n\to\infty} \max_{1\leq \ell \leq k_n(t)} \mathbf{P}([X_{n\ell} \in V^c]) = o$$

for all Borel neighborhoods V of e and all $t \in \mathbf{R}_+$, we look at the sequence $\{X_n : n \in \mathbf{N}\}$ of functional processes

$$X_n := \prod_{\ell=1}^{k_n(\cdot)} X_{n\ell}$$

(with G as their state space). For any increment process $X = \{X(t) : t \in \mathbf{R}_+\}$ in G (normalized by $X(o) = e$ and càdlàg) the family $\{\mu(s,t) : (s,t) \in \mathbf{S}\}$ of distributions $\mu(s,t) := \mathbf{P}_{X(s)^{-1}X(t)}$ forms a convolution hemigroup in the set $M^1(G)$ of all probability measures on G, i.e. $\mu(s,r) * \mu(r,t) = \mu(s,t)$ for all $s \leq r \leq t, \mu(t,t) = \varepsilon_e$, and the mapping $(s,t) \mapsto \mu(s,t)$ from $\mathbf{S} = \{(u,v) \in \mathbf{R}_+^2 : u \leq v\}$ into $M^1(G)$ (together with the weak topology $\mathcal{T}_w$) is càdlàg in each variable. X is stochastically continuous if and only if $(s,t) \mapsto \mu(s,t)$ is continuous. Returning to the initial array and to the sequence $\{X_n : n \in \mathbf{N}\}$ of functional processes in G we have finite dimensional convergence

$$X_n \to X$$

if and only if

$$\prod_{\ell=k_n(s)+1}^{k_n(t)\ *} \mu_{n\ell} \to \mu(s,t)$$

for all $(s,t) \in \mathbf{S}$ in the sense of the topology $\mathcal{T}_w$ on $M^1(G)$.

Applying the fact that to any continuous convolution hemigroup $\{\mu(s,t) : (s,t) \in \mathbf{S}\}$ in $M^1(G)$ there corresponds the family $\{T_{s,t} : (s,t) \in \mathbf{S}\}$ of translation operators $T_{s,t} := T_{\mu(s,t)}$ defined in the space $\mathcal{L}(C^o(G), C^o(G))$ of all linear operators on the space $C^o(G)$ of all continuous functions on G vanishing at infinity, by

$$T_{s,t}f(x) := T_{\mu(s,t)}f(x) := \int_G f(xy)\mu(s,t)(dy)$$

whenever $f \in C^o(G), x \in G$, one obtains a bijection

$$\mathbf{H}(G) \leftrightarrow Evol(C^o(G))$$

between the sets $\mathbf{H}(G)$ of continuous convolution hemigroups in $M^1(G)$ and $Evol(C^o(G))$ of (strongly continuous, positive, left invariant) evolution families of contractions on $C^o(G)$. This bijection extends to a bijection

$$\mathbf{S}(G) \leftrightarrow Contr(C^o(G))$$

between continuous convolution semigroups and semigroups of contraction operators on $C^o(G)$.

For the following we assume to be known what it means that a mapping F from $\mathbf{S}$ or $\mathbf{R}_+$ into a Banach space E is of (continuous) finite (bounded) variation. A convolution hemigroup $\{\mu(s,t) : (s,t) \in \mathbf{S}\}$ is said to be of (continuous) *weak finite variation on a subspace* C of $C^o(G)$ if

$$(s,t) \mapsto (T_{\mu(s,t)} - I)f(e)$$

from $\mathbf{S}$ into $\mathbf{R}$ is of (continuous) bounded variation for every $f \in C$.

From now on let G be a Lie projective group with Lie algebra $L(G)$, projective basis $\{X_i : i \in I\}$ and projective (weak) coordinate system $\{x_i : i \in I\}$ (associated with $\{X_i : i \in I\}$). Examples of Lie projective groups are all locally compact abelian groups, all compact groups, in particular the torus group $\mathbf{T}^{\mathbf{N}}$ and the solenoidal group $\mathbf{Q}_d^\wedge$ (which both are not Lie groups), and all maximally almost periodic groups generated by a compact neighborhood of the identity. For Lie projective groups G the space $D(G)$ of (Bruhat) test functions is contained in the space $C_2(G)$ of twice left differentiable functions on G. The bijection

$$\mathbf{S}(G) \leftrightarrow P(G)$$
$$\{\mu(t) : t \in \mathbf{R}_+\} \leftrightarrow (a, B, \eta)$$

between $\mathbf{S}(G)$ and the set $P(G) := \mathbf{R}^I \times \mathbf{M}_{I,+} \times \mathbf{L}(G)$ of triplets (a, B, η) consisting of vectors a, symmetric positive semidefinite matrices B and Lévy measures η has been established in final form in [2], where also the tools for the general framework have been collected. The corresponding bijection

$$\mathbf{H}_{wfv}(G) \leftrightarrow P_{fv}(\mathbf{R}_+, G)$$
$$\{\mu(s,t) : (s,t) \in \mathbf{S}\} \leftrightarrow (a, B, \eta)$$

between the set $\mathbf{H}_{wfv}(G)$ of continuous hemigroups $\{\mu(s,t) : (s,t) \in \mathbf{S}\}$ of weakly finite variation on G and the set $P_{fv}(\mathbf{R}_+, G)$ of triplets (a, B, η), where a is a continuous mapping $\mathbf{R}_+ \to \mathbf{R}^I$ of finite variation with $a(o) = o, B$ an increasing continuous mapping $\mathbf{R}_+ \to \mathbf{M}_{I,+}$ with $B(o) = o$ and η a measure in $M^1(\mathbf{R}_+ \times G)$ such that $\eta(\mathbf{R}_+ \times \{e\}) = o, \eta([o,t] \times \cdot) \in \mathbf{L}(G)$ for all $t \in \mathbf{R}_+$, and

$$t \mapsto \int f(y)\eta([o,t] \times dy)$$

is continuous for all $f \in D(G)_+$ with $f(e) = o$. The set of all such measures η will be denoted by $\mathbf{L}(\mathbf{R}_+, G)$. While the first cited (Hunt) bijection is produced by a generating function, the letter one requires generating mappings and the notion of a weak backward equation.

The following functional convergence result has been proved in [8].

1.1 Theorem. Let $\{\mu_{n\ell} : n, \ell \in \mathbf{N}\}$ be an array of measures in $M^1(G), \{k_n : n \in \mathbf{N}\}$ a scaling sequence, and let D denote a dense subset of $\mathbf{R}_+$. It is assumed that

(i) there exists a continuous function $t \mapsto a(t) = (a_i(t))_{i \in I}$ on $\mathbf{R}_+$ such that for all $t \in D, i \in I$

$$\sum_{\ell=1}^{k_n(t)} \int x_i d\mu_{n\ell} \to a_i(t) \text{ as } n \to \infty,$$

(ii) there exists a continuous function $t \mapsto B(t) := (b_{ij}(t))_{i,j \in I}$ on $\mathbf{R}_+$ such that for all $t \in D, i, j \in I$

$$\sum_{\ell=1}^{k_n(t)} \int x_i x_j d\mu_{n\ell} \to b_{ij}(t) + \int_G x_i(y)x_j(y)\eta([o,t] \times dy) \text{ as } n \to \infty,$$

(iii) there exists a measure $\eta \in \mathbf{L}(\mathbf{R}_+, G)$ such that for all $t \in D$ and bounded continuous functions f on G vanishing in a neighborhood of e

$$\sum_{\ell=1}^{k_n(t)} \int f d\mu_{n\ell} \to \int_G f(y)\eta([o,t] \times dy),$$

(iv) for all $T > o, i \in I$

$$\limsup_{n\to\infty} \sup_{\substack{o \le s \le t \le T \\ t-s \le \delta}} \sum_{\ell=k_n(s)+1}^{k_n(t)} \left| \int x_i d\mu_{n\ell} \right| \to o \text{ as } \delta \to o.$$

Then $(a, B, \eta) \in P_{fv}(\mathbf{R}_+, G)$, and

$$\prod_{\ell=k_n(s)+1}^{k_n(t)} \mu_{n\ell} \to \mu(s,t)$$

for all $(s,t) \in \mathbf{S}$, where $\{\mu(s,t) : (s,t) \in \mathbf{S}\} \in \mathbf{H}_{wfv}$ and

$$\{\mu(s,t) : (s,t) \in \mathbf{S}\} \leftrightarrow (a, B, \eta).$$

The proof of the theorem is based on the corresponding result for a Lie group G established in [7].

2. Infinite dimensional Fourier transforms

In this section G is assumed to be an arbitrary locally compact group. By a representation of G we always mean a continuous homomorphism U from G into the group $\mathcal{U}(\mathcal{H}(U))$ of unitary operators on the complex representing Hilbert space $\mathcal{H}(U)$. The set of all representations of G will be denoted by $Rep(G)$. Of particular importance is the subset $Irr(G)$ of all irreducible representations U of G which by definition admit no nontrivial closed U-invariant subspace of $\mathcal{H}(U)$. The famous Gelfand-Raikov theorem states that $Irr(G)$ separates the points of G. We also introduce for any cardinal α the α-dimensional Hilbert space $\mathcal{H}(\alpha)$ and the sets $Rep_\alpha(G)$ and $Irr_\alpha(G)$ of all $U \in Rep(G)$ or $U \in Irr(G)$ respectively with $\mathcal{H}(U) = \mathcal{H}(\alpha)$. For the union of the sets $Rep_n(G)$ for $n \in \mathbf{N}$ we write $Rep_f(G)$. The prominent class of *Moore groups* G is defined by the inclusion $Irr(G) \subset Rep_f(G)$. It contains all compact and all abelian locally compact groups and has a well understood structure as is cited in [5].

Now we look at the set $\hat{G} := Irr(G)/\sim$ of unitary equivalence classes of irreducible representations. In the standard references [4] and [18] from which we pick most of the subsequent information, $\hat{G}$ is called the *dual* of G. For any $U \in \hat{G}$ we consider the space $\mathcal{H}_{(1)}(U)$ of all $u \in \mathcal{H}(U)$ with $\|u\| = 1$. We note that the symbol U will be used for the class in $\hat{G}$ as well as for any of its representations. For a given $U \in \hat{G}$ and $u, v \in \mathcal{H}(U)$ the corresponding coefficient of U is defined by $p_{u,v}(U) :=< U(\cdot)u, v >$. In the case that $u = v$ we write $p_u(U)$ instead of $p_{u,v}(U)$. The next definition concerns the *reduced dual* of G introduced as the set $\hat{G}_r$ of all $U \in \hat{G}$ such that there exists a $u \in \mathcal{H}_{(1)}(U)$ admitting the approximation (in the sense of the compact open topology $\mathcal{T}_{co}$)

$$p_u(U) = \lim_{n\to\infty} f_n * f_n^{\sim}$$

for some sequence $(f_n)_{n\geq 1}$ in $C^c(G)$.

Since $\hat{G}$ can be identified with the dual $C^*(G)^\wedge$ of the C^*-algebra $C^*(G)$ of G where $C^*(G)^\wedge$ carries the hull-kernel topology, we obtain the *Fell topology* on $\hat{G}$. A base of the Fell topology at the identity representation **1** of G is given by the family of finite intersections of sets of the form

$$V(C,\varepsilon) := \{U \in \hat{G} : \text{There exists } u \in \mathcal{H}_{(1)}(U) : |p_u(U)(x) - 1| < \varepsilon \text{ for all } x \in C\},$$

where C is a compact subset of G and $\varepsilon > o$. Furnished with the Fell topology $\hat{G}$ is a quasi-locally compact (Baire) space which is second countable if G is second countable. $\hat{G}_r$ is a closed subspace of $\hat{G}$. The equality $\hat{G}_r = \hat{G}$ can be characterized by either of the subsequent statements

(i) $\mathbf{1} \in \hat{G}_r$

(ii) Every continuous positive definite functions on G can be approximated (in the sense of $\mathcal{T}_{co}$) by functions of the form $f * f^{\sim}$ with $f \in C^c(G)$.

(iii) The constant function 1 on G can be approximated (in the sense of $\mathcal{T}_{co}$) by function of the form $f * f^{\sim}$ with $f \in C^c(G)$.

For any cardinal α the sets $Rep_\alpha(G)$ and $Rep_\alpha(C^*(G))$ are bijectively related to each other. Consequently the weak topology on $Rep_\alpha(C^*(G))$ induces a topology on $Rep_\alpha(G)$ which supplies an equivalent definition of the topology of $\hat{G}_\alpha$ as the subspace $\hat{G}$ consisting of all $U \in \hat{G}$ of dimension α.

We are now prepared to introduce the main tool of harmonic analysis on a locally compact group G: the *Fourier transform* $\hat{\mu}$ of a measure $\mu \in M^b(G)$ given for any $U \in Rep(G)$ as an element $\hat{\mu}(U)$ of the space $\mathcal{L}(\mathcal{H}(U))$ of all linear operators on $\mathcal{H}(U)$, by

$$< \hat{\mu}(U)u, v > := \int p_{u,v}(U) d\mu$$

whenever $u, v \in \mathcal{H}(U)$. Clearly, $\|\hat{\mu}\| \leq \|\mu\|$. Moreover, the application $\mu \mapsto \hat{\mu}$ from $M^b(G)$ into the set of mappings from $Rep(G)$ into $\bigcup\{\mathcal{L}(\mathcal{H}(U)) : U \in Rep(G)\}$ is linear, multiplicative, injective and bicontinuous in the sense of the following equivalences expressed for a sequence $(\mu_n)_{n \geq 1}$ and a measure μ both in $M^1(G)$:

(i) $\mu_n \to \mu$ (in the weak topology $\mathcal{T}_w$)

(ii) $\hat{\mu}_n(U)u \to \hat{\mu}(U)u$ for all $U \in Irr(G), u \in \mathcal{H}(U)$.

(iii) $< \hat{\mu}_n(U)u, v > \to < \hat{\mu}(U)u, v >$ for all $U \in Irr(G), u, v \in \mathcal{H}(U)$.

The implication (iii)⇒(i) can be considered as a narrow version of the Lévy continuity theorem for probability measures on a locally compact group. For the problem dealt with in [10] it turned out to be helpful to work with a wider version of Lévy's theorem which is axiomatized as follows.

2.1 Definition. G is said to *admit* the *Lévy continuity property* (LCP) *with respect to a subset* Γ of $Rep(G)$ if there exists a topology on Γ with the following property: Given a sequence $\{\mu_n : n \in \mathbb{N}\}$ in $M^1(G)$ and a mapping $h : \Gamma \to \bigcup\{\mathcal{L}(\mathcal{H}(U)) : U \in \Gamma\}$ which is continuous on $\Gamma \cap Rep_\alpha(G)$ for all cardinals α, satisfying

$$\hat{\mu}_n(U) \to h(U)$$

whenever $U \in \Gamma$ then there exists a measure $\mu \in M^1(G)$ such that

$$\mu_n \to \mu$$

and

$$\hat{\mu}(U) = h(U)$$

for all $U \in \Gamma$.

It is shown in [5] that any Moore group G admits (LCP) with respect to $\Gamma := Rep_f(G)$ the topology on Γ being $\mathcal{T}_{co}$ on $\bigcup\{Rep_n(G) : n \in \mathbf{N}\}$.

Following the note [3] we report on a different axiomatization of the Lévy continuity theorem.

Let G be a second countable locally compact group and Γ a subset of $\hat{G}$ such that $\mathbf{1} \in \Gamma$. A mapping $h : \Gamma \to \mathcal{L} := \bigcup\{\mathcal{L}(\mathcal{H})/\sim : \mathcal{H}$ is a Hilbert space$\}$ with $h(\mathbf{1})$ being a scalar (operator) is said to be continuous in $\mathbf{1}$ if for every $\varepsilon > o$ there exists a neighborhood V of $\mathbf{1}$ (with respect to the Fell topology in $\hat{G}$) satisfying the following property: If $U \in V \cap \Gamma$ then there is a representative $h(U)$ of the class $h(U) \in \mathcal{L}(\mathcal{H})/\sim$ for some Hilbert space $\mathcal{H}$, and a vector $u \in \mathcal{H}$ with $\|u\| = 1$ such that

$$| < h(U)u, u > -h(\mathbf{1})| < \varepsilon.$$

Obviously, the Fourier transform $\hat{\mu}$ of any measure $\mu \in M^b(G)$ considered as mapping $\Gamma \to \mathcal{L}$ is continuous at $\mathbf{1}$.

For subsets Γ of $\hat{G}$ (for groups G that are amenable and of type I) such that $\sigma(\Gamma^c) = o$, where σ denotes a representing measure (in the direct integral decomposition) of the left regular representation of G, the following modification of (LCP) holds.

2.2 Definition. Let G be a second countable locally compact group and $\Gamma \subset \hat{G}$ with $\mathbf{1} \in \Gamma$. G is said to *admit the modified Lévy continuity property* (MLCP) *with respect to* Γ if for any given sequence $\{\mu_n : n \in \mathbf{N}\}$ in $M^1(G)$ and any mapping $h : \Gamma \to \mathcal{L}$ which is continuous at $\mathbf{1}$ and satisfies

$$\hat{\mu}_n(U) \to h(U) \in \mathcal{L}$$

for all $U \in \Gamma$ there exists a measure $\mu \in M^1(G)$ such that

$$\mu_n \to \mu$$

and

$$\hat{\mu}(U) = h(U)$$

for all $U \in \Gamma$.

Following the exposition in [3] we note that if G is of type I (f.e. if G is nilpotent or solvable or a Moore group) then there exists a representing measure σ of the left regular representation of G such that $\sigma(\hat{G}_r^c) = o$. If, in addition, G is amenable (f.e. if G is an almost connected nilpotent or a Moore group) then $\mathbf{1} \in supp\sigma$ for every representing measure σ, and hence G admits (MLCP) with respect to any subset Γ of $\hat{G}$ with $\sigma(\Gamma^c) = o$.

On the other hand G admits (MLCP) with respect to $\hat{G}$ provided every neighborhood of $\mathbf{1}$ (in $\hat{G}$) contains a representation U such that for any $u \in \mathcal{H}(U)$ the coefficient $p_u(U)$ vanishes at infinity. Applying this fact it turns out that a noncompact, connected simple Lie group G with finite center admits (MLCP) with respect to $\hat{G}$ if and only if G violates the Kazdhan property which states that $\mathbf{1}$ is isolated in $\hat{G}$.

3. Convergence of scaled arrays of distributions

A (continuous) convolution hemigroup $\{\mu(s,t) : (s,t) \in \mathbf{S}\}$ of probability measures on a locally compact group G is characterized by the fact that the corresponding family $\{\hat{\mu}(s,t)(U) : (s,t) \in \mathbf{S}\}$ of operators in $\mathcal{L}(\mathcal{H}(U))$ is a (continuous) evolution family for each $U \in Irr(G)$. Given a subset Γ of $Rep(G)$ we define a convolution hemigroup $\{\mu(s,t) : (s,t) \in \mathbf{S}\}$ in $M^1(G)$ to be of (continuous) *$\mathcal{F}$-finite variation with respect to* Γ if for each $U \in \Gamma$ the mapping

$$(s,t) \mapsto \hat{\mu}(s,t)(U) - I$$

from $\mathbf{S}$ into $\mathcal{L}(\mathcal{H}(U))$ is of (continuous) finite variation.

3.1 Definition. Let $\{\mu(s,t) : (s,t) \in \mathbf{S}\}$ be a convolution hemigroup in $M^1(G)$ and let $\Gamma \subset Rep(G)$. A family $\{\varphi^U : U \in \Gamma\}$ of mappings $\varphi^U \in FV(\mathbf{R}_+, \mathcal{L}(\mathcal{H}(U)))$ is called an *integrating family related to* $\{\mu(s,t) : (s,t) \in \mathbf{S}\}$ if for all $U \in \Gamma, \varphi^U(o) = o$ and

$$\mu(s,t)^\wedge(U) = I + \int_{]s,t]} \hat{\mu}(s,\tau-)^\wedge(U)\varphi^U(d\tau)$$

whenever $(s,t) \in \mathbf{S}$.

If a convolution hemigroup $\{\mu(s,t) : (s,t) \in \mathbf{S}\}$ admits an integrating family for $\Gamma \subset Rep(G)$ then $\{\mu(s,t) : (s,t) \in \mathbf{S}\}$ is of $\mathcal{F}$-finite variation with respect to Γ. Conversely, if $\{\mu(s,t) : (s,t) \in \mathbf{S}\}$ is a convolution hemigroup of $\mathcal{F}$-finite variation with respect to Γ then it admits an integrating family for Γ. Moreover, let $\{\mu(s,t) : (s,t) \in \mathbf{S}\}$ be a convolution hemigroup of continuous $\mathcal{F}$-finite variation with respect to $\Gamma \subset Rep(G)$. Then the integrating family $\{\varphi^U : U \in \Gamma\}$ related to $\{\mu(s,t) : (s,t) \in \mathbf{S}\}$ is uniquely determined, and $\varphi^U \in C(\mathbf{R}_+, \mathcal{L}(\mathcal{H}(U)))$ for all $U \in \Gamma$.

In the classical situation of $G = \mathbf{R}^d$ (for $d \geq 1$), where $Irr(G) \cong \mathbf{R}^d$, any convolution hemigroup $\{\mu(s,t) : (s,t) \in \mathbf{S}\}$ in $M^1(G)$ can be characterized by a triplet (a, B, η) in $P(\mathbf{R}_+, G)$ such that

$$\mu(s,t)^\wedge(U) = exp\{i < U, a(t) - a(s) > -\frac{1}{2} < U, (B(t) - B(s))U >$$

$$+ \int (e^{i<U,y>} - 1 - i < U, h(y) >)\eta(]s,t] \times dy)\}$$

for all $U \in Irr(G)((s,t) \in \mathbf{S})$, where h denotes a truncation function on G. It turns out that $\{\mu(s,t) : (s,t) \in \mathbf{S}\}$ is of $\mathcal{F}$-finite variation if and only if a is of finite variation, and in this case the integrating family $\{\varphi^U : U \in Irr(G)\}$ related to $\{\mu(s,t) : (s,t) \in \mathbf{S}\}$ consists of functions $\varphi^U \in FV(\mathbf{R}_+, \mathcal{L}(\mathcal{H}(U)))$ given by

$$\varphi^U(\tau) = \log \ \mu(o,\tau)^\wedge(U)$$

whenever $\tau \in \mathbf{R}_+$. In terms of increment processes associated with hemigroups the above stated Lévy-Khintchine correspondence

$$\{\mu(s,t) : (s,t) \in \mathbf{S}\} \leftrightarrow (a,b,\eta)$$

between the sets $\mathbf{H}(G)$ and $P(\mathbf{R}_+, G)$ is proved in [12].

A similar description of the integrating family can be given in the case of Moore groups G which are known to be Lie projective. The necessary argument relies on Section 5 of [5] and the method developed in [14]. In the special case of abelian locally compact groups a comparison of the various versions of convolution hemigroups of finite variation has been carried out in [1].

Results for specified limits

3.2 Theorem. For every $n \in \mathbf{Z}_+$ let $\{\mu_n(s,t) : (s,t) \in \mathbf{S}\}$ be a convolution hemigroup admitting an integrating family $\{\varphi_n^U : U \in Irr(G)\}$. Suppose that for every $U \in Irr(G)$

(i) there exists a dense subset D of $\mathbf{R}_+$ such that for all $t \in D$

$$\varphi_n^U(t) \to \varphi_o^U(t),$$

(ii) for the sequence of moduli of continuity

$$\limsup_{n\to\infty} \omega_T(V_{\varphi_n^U}; \delta) \to o \text{ as } \delta \to o$$

whenever $T > o$.

Then

$$\mu_n(s,t) \to \mu_o(s,t)$$

for all $(s,t) \in \mathbf{S}$, and $\{\mu_o(s,t) : (s,t) \in \mathbf{S}\}$ is a convolution hemigroup of continuous $\mathcal{F}$-finite variation with respect to $Irr(G)$.

3.3 Theorem (Convergence). Let $\{\mu_{n\ell} : n, \ell \in \mathbf{N}\}$ be an array in $M^1(G)$ and $\{k_n : n \in \mathbf{N}\}$ a scaling sequence. Moreover, let $\{\mu(s,t) : (s,t) \in \mathbf{S}\}$ be a convolution hemigroup in $M^1(G)$ admitting an integrating family $\{\varphi^U : U \in Irr(G)\}$. Suppose that for every $U \in Irr(G)$

(i) there exists a dense subset D of $\mathbf{R}_+$ such that for all $t \in D$

$$\sum_{\ell=1}^{k_n(t)} (\hat{\mu}_{n\ell}(U) - I) \to \varphi^U(t),$$

(ii)

$$\limsup_{n\to\infty} \sup_{\substack{0\le s\le t\le T\\ t-s\le\delta}} \sum_{\ell=k_n(s)+1}^{k_n(t)} \|\hat{\mu}_{n\ell}(U) - I\| \to o \text{ as } \delta \to o$$

whenever $T > o$.

Then

$$\prod_{\ell=k_n(s)+1}^{k_n(t)} {}^{*}\ \mu_{n\ell} \to \mu(s,t)$$

for all $(s,t) \in \mathbf{S}$, and $\{\mu(s,t) : (s,t) \in \mathbf{S}\}$ is a convolution hemigroup of continuous $\mathcal{F}$-finite variation with respect to $Irr(G)$.

Results for unspecified limits

Here we assume that G is a locally compact group admitting (LCP) for some fixed $\Gamma \subset Rep(G)$.

3.4 Theorem. For every $n \in \mathbf{N}$ let $\{\mu_n(s,t) : (s,t) \in \mathbf{S}\}$ be a convolution hemigroup in $M^1(G)$ admitting an integrating family $\{\varphi_n^U : U \in \Gamma\}$. Suppose that for every $U \in \Gamma$

(i) there exists a dense subset D of $\mathbf{R}_+$ such that for all $t \in D$ the sequence $\{\varphi_n^U : n \in \mathbf{N}\}$ converges in $\mathcal{L}(\mathcal{H}(U))$,

(ii) $\limsup_{n\to\infty)} \omega_T(V_{\varphi_n^U};\delta) \to o$ as $\delta \to o$ whenever $T > o$.

Then there exists a family $\{\varphi^U \in FV(\mathbf{R}_+, \mathcal{L}(\mathcal{H}(U))) \cap C(\mathbf{R}_+, \mathcal{L}(\mathcal{H}(\mathcal{U}))) : \mathcal{U} \in \Gamma\}$ such that

$$\varphi_n^U \to \varphi^U$$

locally uniformly for all $U \in \Gamma$.

If, in addition,

(iii) the mapping $U \mapsto \varphi^U$ from $\Gamma \cap Rep_\alpha(G)$ into $C(\mathbf{R}_+, \mathcal{L}(\mathcal{H}(\alpha)))$ is continuous for each α,

(iv) the mapping $U \mapsto V_{\varphi^U}$ from $\Gamma \cap Rep_\alpha(G)$ into $C(\mathbf{R}_+, \mathbf{R}_+)$ is locally bounded for each α,

then there exists a convolution hemigroup $\{\mu(s,t) : (s,t) \in \mathbf{S}\}$ of continuous $\mathcal{F}$-finite variation with respect to Γ such that

$$\mu_n(s,t) \to \mu(s,t)$$

for all $(s,t) \in \mathbf{S}$, and $\{\varphi^U : U \in \Gamma\}$ is an integrating family related to $\{\mu(s,t) : (s,t) \in \mathbf{S}\}$.

3.5 Theorem (Convergence). Let $\{\mu_{n\ell} : n, \ell \in \mathbf{N}\}$ be an array in $M^1(G)$ and $\{k_n : n \in \mathbf{N}\}$ a scaling sequence. Suppose that for every $U \in \Gamma$

(i) there exists a dense subset D of $\mathbf{R}_+$ such that for all $t \in D$

$$\left\{ \sum_{\ell=1}^{k_n(t)} (\hat{\mu}_{n\ell}(U) - I) : n \in \mathbf{N} \right\} \text{ converges in } \mathcal{L}(\mathcal{H}(U)),$$

(ii)

$$\limsup_{n\to\infty} \sup_{\substack{o \le s \le t \le T \\ t-s \le \delta}} \sum_{\ell=k_n(s)+1}^{k_n(t)} \|\hat{\mu}_{n\ell}(U) - I\| \to o \text{ as } \delta \to o$$

whenever $T > o$.

Then there exists a family $\{\varphi^U \in FV(\mathbf{R}_+, \mathcal{L}(\mathcal{H}(U))) \cap C(\mathbf{R}_+ \mathcal{L}(\mathcal{H}(U))) : U \in \Gamma\}$ such that

$$\sup_{t\in[o,T]} \| \sum_{\ell=1}^{k_n(t)} (\hat{\mu}_{n\ell}(U) - I) - \varphi^U(t)\| \to o$$

for all $U \in \Gamma$ whenever $T > o$.

If, in addition, conditions (iii) and (iv) of Proposition 3.4 hold, then

$$\prod_{\ell=k_n(s)+1}^{k_n(t)\ \bullet} \mu_{n\ell} \to \mu(s,t)$$

for all $(s,t) \in \mathbf{S}$, and $\{\mu(s,t) : (s,t) \in \mathbf{S}\}$ is a convolution hemigroup of continuous $\mathcal{F}$-finite variation admitting $\{\varphi^U : U \in \Gamma\}$ as its related integrating family.

For the technical background and proofs of the results we refer the reader to [10]. The main idea is to reduce the study of convolution hemigroups on G via Fourier transform to the study of evolution families of operators and related operator-valued integrating functions which are chosen to be of finite variation. These integrating functions are applied in order to obtain integral representations of the given evolution families the integral involved being a (Bogdanowicz) generalization of the (bilinear) Lebesgue-Bochner-Stieltjes integral for operator-valued integrands and integrators.

4. Convergence of scaled arrays of random variables

In this section we wish to reformulate the previous results in terms of increment processes and scaled products of random variables taking their values in a second countable locally compact group G which is also a complete separable metric group. Let $X := \{X(t) : t \in \mathbf{R}_+\}$ be an increment process in second countable G and let $\{\mu(s,t) : (s,t) \in \mathbf{S}\}$ denote the associated convolution hemigroup of distributions $\mu(s,t)$ of increments $X(s)^{-1}X(t)$ of X. The process X is said to be of (continuous) finite $\mathcal{F}$-variation with respect to $\Gamma \subset Rep(G)$ if the convolution hemigroup $\{\mu(s,t) : (s,t) \in \mathbf{S}\}$ is of $\mathcal{F}$-finite variation with respect to Γ in the sense of Section 3, and to admit an integrating family for $\Gamma \subset Rep(G)$ if $\{\mu(s,t) : (s,t) \in \mathbf{S}\}$ does.

Results for specified limits

4.1 Theorem. For every $n \in \mathbf{N}$ let $X_n = \{X_n(t) : t \in \mathbf{R}_+\}$ be a càdlàg increment process in G which is of $\mathcal{F}$-finite variation with respect to $Irr(G)$ and admits an integrating family $\{\varphi_n^U : U \in Irr(G)\}$. Moreover, let $\{\mu(s,t) : (s,t) \in \mathbf{S}\}$ denote any convolution hemigroup of $\mathcal{F}$-finite variation with respect to $Irr(G)$ and let $\{\varphi^U : U \in Irr(G)\}$ be some integrating family related to $\{\mu(s,t) : (s,t) \in \mathbf{S}\}$. We assume the conditions (i) and (ii) of Theorem 3.2 to be satisfied.

Then there exists a G-valued stochastically continuous càdlàg increment process $X = \{X(t) : t \in \mathbf{R}_+\}$ of continuous $\mathcal{F}$-finite variation with respect to $Irr(G)$ such that

$$X_n \to X$$

in distribution on $D(\mathbf{R}_+, G)$, and $\mathbf{P}_{X(s)^{-1}X(t)} = \mu(s,t)$ whenever $(s,t) \in \mathbf{S}$.

4.2 Theorem. Let $\{X_{n\ell} : n, \ell \in \mathbf{N}\}$ be an array of rowwise independent random variables with values in G, and let $\{k_n : n \geq 1\}$ be a scaling sequence. Moreover, let $\{\mu(s,t) : (s,t) \in \mathbf{S}\}$ denote any convolution hemigroup in $M^1(G)$ admitting an integrating family $\{\varphi^U : U \in Irr(G)\}$. We assume that for every $U \in Irr(G)$

(i) there exists a dense subset D of $\mathbf{R}_+$ such that for all $t \in D$

$$\sum_{\ell=1}^{k_n(t)} (\mathbf{E}(U \circ X_{n\ell}) - I) \to \varphi^U(t),$$

(ii)

$$\limsup_{n\to\infty} \sup_{\substack{o \leq s \leq t \leq T \\ t-s \leq \delta}} \sum_{\ell=k_n(s)+1}^{k_n(t)} \|\mathbf{E}(U \circ X_{n\ell}) - I\| \to o \text{ as } \delta \to o$$

whenever $T > o$.

Then there exists a G-valued stochastically continuous càdlàg increment process $X = \{X(t) : t \in \mathbf{R}_+\}$ of $\mathcal{F}$-finite variation with respect to $Irr(G)$ such that

$$\prod_{\ell=1}^{k_n(\cdot)} X_{n\ell} \to X$$

in distribution on $D(\mathbf{R}_+, G)$, and $\mathbf{P}_{X(s)^{-1}X(T)} = \mu(s,t)$ whenever $(s,t) \in \mathbf{S}$.

Results for unspecified limits

Similar to Section 3 we need also here the additional hypothesis that G admits (LCP) for some fixed $\Gamma \subset Rep(G)$.

4.3 Theorem. For every $n \in \mathbf{N}$ let $X_n := \{X_n(t) : t \in \mathbf{R}_+\}$ be a càdlàg increment process in G which is of $\mathcal{F}$-finite variation with respect to Γ and admits an integrating family $\{\varphi_n^U : U \in \Gamma\}$. Suppose that for every $U \in \Gamma$ conditions (i) and (ii) of Theorem 3.4 are satisfied.

Then there exists a family $\{\varphi^U : U \in \Gamma\}$ of mappings $\varphi^U \in FV(\mathbf{R}_+, \mathcal{L}(\mathcal{H}(U))) \cap C(\mathbf{R}_+, \mathcal{L}(\mathcal{H}(U)))$ such that

$$\varphi_n^U \to \varphi^U$$

locally uniformly for all $U \in \Gamma$.

If, in addition, conditions (iii) and (iv) of Theorem 3.4 are fulfilled, then there exists a stochastically continuous càdlàg increment process $X = \{X(t) : t \in \mathbf{R}_+\}$ of continuous $\mathcal{F}$-finite variation with respect to Γ such that

$$X_n \to X$$

in distribution on $D(\mathbf{R}_+, G)$, and $\{\varphi^U : U \in \Gamma\}$ is an integrating family related to the convolution hemigroup of distributions of increments $X(s)^{-1}X(t)$ of X.

4.4 Theorem. Let $\{X_{n\ell} : n, \ell \in \mathbf{N}\}$ be an array of rowwise independent random variables with values in G, and let $\{k_n : n \geq 1\}$ be a scaling sequence. Suppose that for every $U \in \Gamma$

(i) there is a dense subset D of $\mathbf{R}_+$ such that for all $t \in D$ the sequence

$$\left\{ \sum_{\ell=1}^{k_n(t)} (\mathbf{E}(U \circ X_{n\ell}) - I) : n \in \mathbf{N} \right\}$$

converges in $\mathcal{L}(\mathcal{H}(U))$,

(ii)

$$\limsup_{n \to \infty} \sup_{\substack{o \leq s \leq t \leq T \\ t-s \leq \delta}} \sum_{\ell=k_n(s)+1}^{k_n(t)} \|\mathbf{E}(U \circ X_{n\ell}) - I\| \to o \text{ as } \delta \to o$$

whenever $T > o$.

Then there exists a family $\{\varphi^U : U \in \Gamma\}$ of mappings $\varphi^U \in FV(\mathbf{R}_+, \mathcal{L}(\mathcal{H}(U))) \cap C(\mathbf{R}_+, \mathcal{L}(\mathcal{H}(U)))$ such that

$$\varphi_n^U \to \varphi^U$$

locally uniformly for all $U \in \Gamma$.

If, in addition, conditions (i) and (ii) of Theorem 3.4 are fulfilled, then there exists a stochastically continuous càdlàg increment process $X = \{X(t) : t \in \mathbf{R}_+\}$ of continuous $\mathcal{F}$-finite variation with respect to Γ such that

$$\prod_{\ell=1}^{k_n(\cdot)} X_{n\ell} \to X$$

in distribution on $D(\mathbf{R}_+, G)$, and $\{\varphi^U : U \in \Gamma\}$ is an integrating family related to the convolution hemigroup of distributions of increments $X(s)^{-1}X(t)$ of X.

5. Suggestions for further research on the subject

An open problem in functional limit theory for locally compact groups is the specification of sufficient conditions enforcing the limiting process to be a diffusion. For Lie projective groups diffusion hemigroups and their corresponding increment processes have been characterized in [8] and [1]. We recall the following

5.1 Definition. A convolution hemigroup $\{\mu(s,t) : (s,t) \in \mathbf{S}\}$ on a locally comact group G is said to be a *diffusion hemigroup* if for all $T > o$ and for every neighborhood V of e

$$\lim_{\substack{t-s\to o \\ o\leq s<t\leq T}} \frac{1}{t-s}\mu(s,t)(V^c) = o.$$

Under Lipschitz conditions one shows that a convolution hemigroup on G is a diffusion hemigroup if and only if the corresponding increment process is a *diffusion process* in the sense that it has continuous paths.

For convolution semigroups $\{\mu(t) : t \in \mathbf{R}_+\}$ on G and their corresponding stationary increment processes the analoguous diffusion property

$$\lim_{t\to o} \frac{1}{t}\mu(t)(V^c) = o$$

valid for every neighborhood V of e defines *Gaussian semigroups* and *Gaussian processes* respectively.

In the sequel we shall sketch theorems on the convergence towards a Gaussian semigroup and on the martingale characterization of Gaussian semigroups, two results whose possible extensions to diffusion hemigroups by means of infinite dimensional Fourier transforms would be of great value for the development of functional central limit theory.

Let $\{\mu(t) : t \in \mathbf{R}_+\}$ be a convolution semigroup on G and $\{\mu(t)^\wedge(U) : t \in \mathbf{R}_+\}$ the associated semigroup of operators $\mu(t)^\wedge(U)$ in $\mathcal{L}(\mathcal{H}(U))$ whenever $U \in Rep(G)$. For any $U \in Rep(G)$ one introduces the infinitesimal generator$(N(U),\mathcal{N}(U))$ of the *representing semigroup* $\{\mu(t)^\wedge(U) : t \in \mathbf{R}_+\}$. It turns out that the domain $\mathcal{N}(U)$ of $N(U)$ contains the space $\mathcal{H}_o(U)$ of U-differentiable vectors of $\mathcal{H}(U)$, and $\mathcal{H}_o(U)$ contains the Gårding space $\mathcal{H}_1(U)$. If $U \in Rep_f(G)$ then $\mathcal{H}_1(U) = \mathcal{H}_o(U) = \mathcal{H}(U)$. For arbitrary $U \in Rep(G)$ the operator $N(U)$ admits a Lévy-Khintchine representation on $\mathcal{H}_o(U)$, and $\{\mu(t) : t \in \mathbf{R}_+\}$

is uniquely determined by the family $\{Res_{\mathcal{H}_1(U)}N(U) : U \in Irr(G)\}$. The author of [15] studies the convergence of sequences of convolution semigroups towards a limiting convolution semigroup on G. In particular he achieves the following central limit result.

5.2 Theorem. Let G be a Lie projective group, and let $\{\mu_{n\ell} : n, \ell \in \mathbf{N}\}$ be a commutative infinitesimal array in $M^1(G)$ satisfying the condition that

$$\lim_{n\to\infty} \sum_{\ell=1}^{k_n} \mu_{n,\ell}(V^c) = o$$

whenever V is a neighborhood of e. Suppose, moreover, that

$$\limsup_{n\to\infty} \sum_{\ell=1}^{k_n} | < \hat{\mu}_{n\ell}(U)u - u, u > | < \infty$$

for all $U \in Irr(G)$ and $u \in \mathcal{H}_o(U)$.

Then the sequence $\{\mu_n : n \in \mathbf{N}\}$ of row products

$$\mu_n := \prod_{\ell=1}^{k_n}{}^{*} \mu_{n\ell}$$

is uniformly tight, and for any of its nondegenerate limit points μ there exists a Gaussian semigroup $\{\mu(t) : t \in \mathbf{R}_+\}$ on G such that $\mu(1) = \mu$.

Next we describe a martingale characterization of a Gaussian semigroup or process in terms of its representing semigroup as it is shown in [16].

For any Hilbert space $\mathcal{H}$ we consider $\mathcal{L}(\mathcal{H})$-*martingales* $\{Z(t) : t \in \mathbf{R}_+\}$ (with respect to a filtration $\{\mathcal{F}(t) : t \in \mathbf{R}_+\}$) defined by the property that for all $u, v \in \mathcal{H}$ the $\mathbf{C}$-valued process $\{< Z(t)u, v >: t \in \mathbf{R}_+\}$ is a martingale with respect to $\{\mathcal{F}(t) : t \in \mathbf{R}_+\}$). Now, let $\{\mu(t) : t \in \mathbf{R}_+\}$ be a convolution semigroup with representing semigroup $\{\mu(t)^\wedge(U) : t \in \mathbf{R}_+\}$ for $U \in Rep(G)$. Let Γ be a subset of $Rep(G)$ such that for all $U \in \Gamma$ and all $t \in \mathbf{R}_+$ the operator $\mu(t)^\wedge(U)$ is invertible in $\mathcal{L}(\mathcal{H}(U))$, and that the Fourier mapping $\mu \mapsto \hat{\mu}$ from $M^b(G)$ into the set of mappings from Γ into $\bigcup\{\mathcal{L}(\mathcal{H}(U)) : U \in \Gamma\}$ is injective. Then a stochastic process $X = \{X(t) : t \in \mathbf{R}_+\}$ in G is a (stationary) increment process corresponding to $\{\mu(t) : t \in \mathbf{R}_+\}$ if and only if for each $U \in \Gamma$ the process $\{\mu(t)^\wedge(U)^{-1}U \circ X(t) : t \in \mathbf{R}_+\}$ is an $\mathcal{L}(\mathcal{H}(U))$-valued martingale with respect to the canonical filtration of X. One notes that this equivalence holds provided G is almost periodic in the sense that $Rep_f(G)$ separates the points of G, and $\Gamma := Irr(G) \cap Rep_f(G)$. If, moreover, G is a Moore group, it clearly holds for $S := Irr(G)$.

5.3 Theorem. Let G be a compact group for which a faithful representation $F \in Rep_f(G)$ exists. Given a convolution semigroup $\{\mu(t) : t \in \mathbf{R}_+\}$ on G and a stochastic process $X = \{X(t) : t \in \mathbf{R}_+\}$ in G with filtration $\{\mathcal{F}(t) : t \in \mathbf{R}_+\}$ which has continuous paths, the following statements are equivalent:

(i) X is a Gaussian process corresponding to $\{\mu(t) : t \in \mathbf{R}_+\}$.

(ii) For each $U \in \{F, F \otimes F\}$ the process $\{\mu(t)^\wedge(U)^{-1} U \circ X(t) : t \in \mathbf{R}_+\}$ is an $\mathcal{L}(\mathcal{H}(U))$-valued martingale with respect to the filtration of X.

As for the hypothesis on G in the theorem it should be noted that a compact group G admits a faithful finite dimensional representation if and only if G is isomorphic as a topological group to a (compact) group of orthogonal (or unitary) matrices, or equivalently to G being a Lie group. Further equivalences can be found in [11].

In the proof of the implication (ii)⇒(i) of the theorem the author of [16] applies the fact that for any convolution semigroup $\{\mu(t) : t \in \mathbf{R}_+\}$ on a locally compact group G and any càdlàg process $\{X(t) : t \in \mathbf{R}_+\}$ in G the process $\{\mu(t)^\wedge(U)^{-1} U \circ X(t) : t \in \mathbf{R}_+\}$ is an $\mathcal{L}(\mathcal{H}(U))$-valued local L^2-martingale (for $U \in Rep(G)$) if and only if the process $\{U \circ X(t) - N(U) \int_o^t U \circ X(s) ds : t \in \mathbf{R}_+\}$ has that property.

In the case of an arbitrary locally compact group G admitting a faithful real representation in $Rep_f(G)$ a result similar to Theorem 5.3 can be found in [17].

References

[1] M.S. Bingham, H. Heyer: On diffusion hemigroups of probability measures on an abelian locally compact group
To appear in Resultate der Mathematik 2000

[2] E. Born: An explicit Lévy-Hincin formula for convolution semigroups on locally compact groups
J.Theor. Probab. 2 (1989), 325-342

[3] Ph. Bougerol: Extension du théorème de continuité Paul Lévy aux groupes moyennables
In: Probability Measures on Groups VII, Oberwolfach, 1983, pp. 10-22. Lecture Notes in Mathematics 1064, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo 1984

[4] J. Dixmier: Les C*-algèbres et leurs représentations
Gauthier-Villars, Paris 1964

[5] H. Heyer: Probability Mesures on Locally Compact Groups
Springer-Verlag, Berlin-Heidelberg-New York 1977

[6] H. Heyer: Semigroupes de convolution sur un groupe localement compact et applications à la théorie des probabilités
In: Ecole d'Eté de Probabilités de Saint Flour VII, 1977, pp. 173-236. Lecture Notes in Mathematics 678, Springer-Verlag, Berlin-Heidelberg-New York 1978

[7] H. Heyer, G. Pap: Convergence of noncommutative triangular arrays of probability measures on a Lie group
J.Theor. Probab. 10 (1997), 1003-1052

[8] H. Heyer, G. Pap: Convolution hemigroups of bounded variation on a Lie projective group
J.London Math. Soc. (2) 59 (1999), 369-384

[9] H. Heyer, G. Pap: Convergence of convolution hemigroups on Moore groups
In: Analysis on infinite-dimensional Lie groups and algebras, Proceedings, Luminy 1997, pp. 122-144.
World Scientific, Singapore-New Jersey-London-Hong Kong 1998

[10] H. Heyer, G. Pap: Convergence of evolution operator families of finite variation and convergence of triangular systems of random variables in a locally compact group
Submitted to Publ.Math. Debrecen

[11] K.H. Hofmann, S.A. Morris: The Structure of Compact Groups
Walter de Gruyter, Berlin-New York 1998

[12] J. Jacod, A.N. Shiryayev: Limit Theorems for Stochastic Processes
Springer-Verlag, Berlin-Heidelberg-New York-London-Paris- Tokyo 1987

[13] G. Pap: Functional central limit theorems on Lie groups: a survey
In: 7th International Vilnius Conference on Probability Theory and Mathematical Statistics and 22nd European Meeting of Statisticians.
Proceedings, Vilnius, 1998. VSP, Utrecht

[14] E. Siebert: On the Lévy-Chintschin formula on locally compact maximally almost periodic groups
Math.Scand. 41 (1977), 331-346

[15] E. Siebert: Fourier analysis and limit theorems for convolution semigroups on a locally compact group
Advances in Mathematics 39,2 (1981), 111-154

[16] M. Voit: Martingale characterizations of stochastic processes on compact groups
To appear in Probab.Math.Statist.

[17] M. Voit: A Lévy characterization for Gaussian processes on matrix groups
To appear in Proc.Amer.Math.Soc.

[18] G. Warner: Harmonic Analysis on Semi-Simple Lie Groups II
Springer-Verlag, Berlin-Heidelberg-New York 1972

Harmonic Analysis on Complex Random Systems

TAKEYUKI HIDA
MEIJO UNIVERSITY
NAGOYA, JAPAN

Abstract White noise analysis has an aspect of harmonic analysis arising from the infinite dimensional rotation group $O(E)$ which is formed by all the linear isomorphisms of a basic nuclear space $E \subset L^2(R^d)$. In fact, the white noise measure μ is kept invariant under the action of the group $O^*(E^*)$ consisting of the adjoint transformations g^* of the members g in $O(E)$.

In this report, particular attentions will be paid to a subgroup generated by the so-called whiskers. A whisker, we mean, is a continuous one-parameter subgroup $\{g_t\}$ of $O(E)$, where each member g_t comes from a diffeomorphism of the time (or space-time) parameter space of the white noise. The most important whisker is the time shift. With this choice of a whisker, one can define a one-parameter unitary group $\{U_t\}$ acting on the Hilbert space $L^2(E^*, \mu)$ and speak of the spectral multiplicity. This notion enables us to consider a sort of degree of complexity of random evolutional phenomena that propagate as the time or space-time parameter moves.

Another interesting subgroup of $O(E)$ is the conformal group $C(d)$ generated by certain various whiskers involving the shift. The group structure of $C(d)$ is well known, since it is locally isomorphic to the Lie group $SO(d+1,1)$, so that it is ready to be applied to white noise theory. Indeed, this group $C(d)$ plays important roles, in particular, in the investigations of reversibility and of variations of a random field $X(C)$ when C is deformed by the action of the group $C(d)$.

Together with some other significant examples of whiskers, we can carry on an **essentially infinite dimensional harmonic analysis** in line with the white noise analysis.

§1. Introduction and background

The subject of *harmonic analysis on white noise space* has undergone a vast development: Laplacians, Fourier transform and operator theory in general. While, complexity or complex system is proposing interesting future directions in various fields in science. We shall, in this note, focus our attention to random phenomena, namely *random complex systems* and in fact, they can be discussed in line with white noise analysis. Note that the white noise analysis has an aspect of an infinite dimensional harmonic analysis that arises from the infinite dimensional rotation group. Thus, our present aim is to investigate complex random systems expressed in terms of white noise by appealing to the theory of infinite diemsional rotation group.

We shall briefly review the white noise space and the rotation group as background.

White noise is a measure space (E^*, μ), where E^* is a space of genralized functions on R^d and it is taken to be the dual space of some nuclear space E, and where μ is a measure on E^* determined by a characteristic functional

$$C(\xi) = \exp\left[-\frac{1}{2}\|\xi\|^2\right], \quad \xi \in E.$$

Set $(L^2) = L^2(E^*, \mu)$. Then, we have a Fock space:

$$(L^2) = \oplus_n H_n.$$

A Gel'fand triple

$$(S) \subset (L^2) \subset (S)^*$$

defines the space $(S)^*$ of *generalized white noise functionals.*

To have a visualized expression of $(S)^*$-functional φ is an S-transform (Kubo–Takenaka) defined by

$$(S\varphi)(\xi) = C(\xi) \int \exp[\langle x, \xi\rangle]\varphi(x)d\mu(x).$$

The S-transform is usuful to define operators, like annihilation operator ∂_t and creation operator ∂_t^*, that act on the space $(S)^*$. Indeed, S is a bijective mapping from $(S)^*$ to its range.

We then come to the rotation group $O(E)$ of E. Let g be a linear homeomorphism of E such that

$$\|g\xi\| = \|\xi\|, \quad \xi \in E.$$

Then, g is called a *rotation* of E. The collection $O(E)$ of all rotations of E forms a group under the usual product. Also, the compact-open topology is introduce to $O(E)$, so that it is a topological group.

Definition. The topological group $O(E)$ is called the *rotation group* of E. If E is not specified, it is called an *infinite dimensional rortation group* and is denoted by O_∞.

Let g^* be the adjoint operator of g. Necessarily g^* is a continuous linear operator acting on the space E^*.

Proposition. *The group $O^*(E^*)$ is isomorphic to $O(E)$ under the correspondence $g^* \leftrightarrow g^{-1}$.*

With the help of the characteristic functional we can prove

Theorem 1. *The white noise measure μ is invariant under the action of the group $O^*(E^*)$:*

$$g^* \cdot \mu = \mu.$$

Hence, the operator U_g given by

$$U_g\varphi(x) = \varphi(g^*x)$$

is unitary. We can therefore introduce the unitary representation of the group $O(E)$ on the Hilbert space (L^2).

§2. Subgroups of $O(E)$ and their roles

The group $O(E)$ is, in a sense, quite big; in fact, it is not even locally compact, and its structure is very complex. It would be a good idea to take subgroups separately and investigate their roles in white noise analysis.

B. Finite dimesional subgroups

Take a finite dimensional subspace, say E_n isomorphic to R^n. The collection of rotations g such that their restrictions to E_n are its rotations and identity on $E_n^{\perp}$ forms a subgroup, denoted by G_n. Obviously, G_n is isomorphic to the linear group $SO(n)$.

I. Hyperfinite dimensional subgrpoup

Set

$$G_\infty = \vee_n G_n.$$

Then, the infinite dimensional Laplace-Beltrami operator Δ_∞ is determined by the subgroup G_∞ and is expressed in the form

$$\Delta_\infty = \int \partial_t^* \partial_t dt.$$

Also, we can prove (see [2]) the unitary representation $\{U_g, g \in G_\infty\}$ on H_n, $n \geq 1$, is irreducible. As a result, Δ_∞ takes a constant value, in fact $-n$, on the subspace H_n.

II. Infinite dimensional subgroup: The Lévy group

As is well known the Lévy group $\mathcal{G}$ is essentially infinite dimensional. Its action can generally not be approximated by finite dimensional rotations. Contrary to the case I above, the Lévy Laplacian Δ_L acts effectively on the space $(S)^*$ and annihilates the basic space (L^2). There is a formal expression (due to H.-H. Kuo) of the Lévy Laplacian that helps to understand its actions.

$$\Delta_L = \int (\partial_t)^2 (dt)^2.$$

It is noted that the subgroups that have appeared so far depend on the choice of a complete orthonormal system for $L^2(R^d)$.

III. Ultra infinite dimensional subgroups: Whiskers

There are significant one-parameter subgroups that come from the diffeomorphisms of the parameter space R^d. They are called whiskers. The most important whisker is the shift. Define S_t^j by

$$S_t^j \xi(u) = \xi(u - te_j), \quad \xi \in E; \quad t \in R; \quad j = 1, 2, ..., d,$$

where e_j is the j-th coordinate vector of R^d. There are many other whiskers that have good relations (commutation relations) with shift. A significant class of whiskers is isomorphic to the conformal group $C(d)$.

As we shall discuss in what follows, the shift expresses the change of time or space-time and illustrates the propagation of random phenomena.

§3. Complex systems

What we shall be concerned with are random complex systems which are time-oriented or space-time-oriented. Assume further that the systems in question are functionals of white noise. This means that we tacitly assume that white noise input is provided behind the system. The observed data shall be expressed as a stochastic process $X(t)$ depending on the time t or a random field $X(C)$ indexed by a manifold C, say a contour, that runs through a Euclidean space. Mathematically they are functionals, maybe generalized functionals, of white noise.

There are various approaches to those random complex systems; among others we propose the *innovation approach*. The original idea came from P. Lévy's paper [4], where he has proposed a *stochastic infinitesimal equation* for a stochastic process $X(t)$. This can also be extended to the case of a random firld $X(C)$, although the existence of the proposed equation can not always be expected. With the help of the innovation we can measure thr complexity of random complex systems. In some cases we can form the innovation for our purpose, and they are now in order.

Starting from a Brownian motion or a white noise , which is a basic elementary stochastic process or generalized stochastic process, resp., we discuss functions of Brownian motion (or white noise) taking the time development (shift) into account.

1) Gaussian system

Let $X(t)$ be a Gaussian process with mean $E(X(t)) = 0$. Assume that $X(t)$ is separable and has *unit multipliocity* in the time domain. Then, there exists a white noise $\dot{B}(t)$ such that

$$X(t) = \int^t F(t,u)\dot{B}(u)du,$$

where $F(t, u)$ is a non random kernel function. In addition, $\{X(u), u \leq t\}$ has the same information as $\{\dot{B}(u), u \leq t\}$ for every t. A representation satisfying these conditions is called *canonical.*

The notion of multiplicity can be understood in such a way that associated with each t is a projection $E(t)$ corresponding to the space spanned by the variables $X(s)$, $s \leq t$, (if necessary $E(t)$ is modified so as to be right continuous) so that the spectrum as well as the (spectral) multiplicity can be defined by the Helliger–Hahn theorem.

The unit multiplicity means that the given Gaussian process represented by a single Brownian motion (white noise) which we could call an elemental stochastic process. There are many Gaussian processes with higher multiplicity and number of the multiplicity expresses the "degree of complexity."

2) Nonlinear functionals of white noise

There are a lot of significant stochastic processes that are expressed by nonlinear functionals of a white noise (Brownian motion). There is requested a calculus, called white noise analysis, where a white noise $\{\dot{B}(t)\}$ is taken to be the system of variables.

In order to establish the causal calculus of complex systems of the above form of a stochastic process, it is necessary to generalize the notion the multiplicity. Namely, a one-parameter unitary group $\{U(t), t \in R\}$, acting on the space of white noise functionals and representing the time propagation, is introduced. Actually, $U(t)$ is defined so as to hold the relation $U(t)\dot{B}(s) = \dot{B}(t+s)$.

Once the unitary group is introduced, one can see a cyclic subspace of the form

$$H(f) = \text{span}\{U(t)f, t \in R\}.$$

Again the Hellinger–Hahn theorem claims that there is a system $\{H(f_n); n = 1, 2....\}$ such that it is an orthogonal system and that the entire complex system in question is expressed as the direct sum of those cyclic subspaces. Those subspaces are arranged in the order of the spectral measures. The number of the cyclic subspaces is the *multiplicity* in the general sense. This multiplicity is different from the Gaussian case, but it also serves to the measurement of complexity.

Remark. A stochastic process formed by some nonlinear functional for which its innovation is actually obtained (see [3]) can be discussed directly for degree of complexity.

Example. The Wiener expansion. There is a famous application called the Wiener expansion. We want to identify an unknown system that permits white noise input as is illustrated below.

input $\longrightarrow$ nonlinear system $\longrightarrow$ output

Let the known nonlinear systems be provided in advance. If the same input as that to the nonlinear system is given, then their outputs can be compared to those of the unknown system. Thus, the Wiener expansion provides a tool to identify a random complex system that admits white noise input. Nonlinear system has usually infinite multiplicity which means we need, theoretically speaking, infinitely many known systems.

§4. Reversibility and irreversibility: Roles of whiskers

Reversibility and irreversibility of random evolutional phenomena may be expressed in terms of the $\dot{B}(t)$ instead of the time parameter t itself and both properties are defined with respect to the conformal transformations mapping a time interval onto another in a time reverse order.

We start our discussion with a simple example in Gaussian case where the time interval is taken to be $[0, 1]$ to fix the idea.

1) A Brownian motion $\{B(t), t \in [0, 1]\}$ is certainly irreversible, since it is an accumulated sum of independent variables $\dot{B}(t)$'s at every instant t, and both variance and entropy increase as t proceeds.

2) Let a Brownian motion $B(t)$ be pinned at $t = 1$ to a position c, namely let $B(1) = c$. Then, we are given a Gaussian process, denoted by $X_c(t)$. The reversibility maybe understood to be an invariant property of a process under the simple time reflection. If so, we have

Proposition. *The probability distributipon of $X_0(t)$, $t \in [0, 1]$, is invariant under the time refelection: $t \mapsto 1 - t$.*

Proof easily comes from the computation of the covariace function:

$$\Gamma(t,s) = (t \wedge s)\{(1-t) \wedge (1-s)\}.$$

There are observations.

1. It is easily seen that a Brownian motion $B(t)$, which is an irreversible process, is viewed as a superposition of reversible processes $X_c(t), c \in R^1$, with the weight of the standard Gaussian measure $g(1,c)dc$ to which $B(1)$ is subject.

2. The (forward) canonical representation of $X(t)$ is expressed in the form

$$X_1(t) = (1-t)\int_0^t \frac{1}{1-u}\dot{B}_1(u)du, \quad t \in [0,1].$$

   The above $B_1(t)$ is a new Brownian motion that has the same information as $X_1(t)$. While, the reversal canonical representation is given by

$$X_2(t) = t\int_t^1 \frac{1}{u}\dot{B}_2(u)du, \quad t \in [0,1].$$

   Two representations given above express the same Brownian bridge as a Gaussian process and they are linked by the projective transformation of the parameter t (see [2:Chapter 5]). There, a role of whiskers can be seen.

The reversibility of a Gaussian process $X(t)$ in white noise analysis is to be considered in terms of the innovation. Since the time domain is limited to a finite interval, the innovation should be formed locally in time. This implies that there is a differential operator L_t such that

$$L_t X(t) = \dot{B}(t).$$

Now the reversibility of a Gaussian process may be dealt with as follows.

a) We understand that a Brownian bridge is an *elemental reversible Gaussian process.* Thus, starting from a Brownian bridge we may consider general reversible Gaussian processes.

b) We generalize the reversible property in such a way that the canonical kernels of forward and reversal representations are linked by conformal transformations.

Thus, in the present situation we may assume that

c) the system of the fundamental solutions of the differential equation

$$L_t f = 0$$

   consists of polynomials in $(t-1)$.

Summing up we now have

Theorem 2. *Let a bridged Gaussian process $X(t)$ satisfy the conditions a), b) and assumption c). Assume that the order of the differential operator L_t is N uniformly in t. Then, the process $X(t)$ is reversible.*

PROOF. By assumption, we have the canonical representation of $X(t)$ (see [1]):

$$X(t) = \int_0^t R(t, u)\dot{B}(u)du,$$

where $R(t, u)$ is Riemann's function of the form

$$R(t, u) = \sum_{k=1}^{N} a_k \frac{(1-t)^k}{(1-u)^k},$$

where we may assume $a_1 = 1$ so that all the a_k's are uniquely determined. Then, as a generalization of the Proposition a conformal map of the interval $[0, 1]$ defines a new representation of $X(t)$ by using the forward and reversal canonical representations.

§5. Concluding remark

With a generalization explained at the end of the last section, we are suggested to think of reversibility of a random field $X(C)$. To fix the idea, C is taken to be a contour in the plane. To discuss reversibility, it is necessary to have an oriented family $\mathbf{C}$ of contours. Denote it by $\mathbf{C} = \{C_t, t_0 \leq t \leq t_1\}$ with the order $C_s < C_t$ for $s < t$ denoting C_s is inside of C_t. Most important requirement is that the C_t expands as t increases from C_0 to C_1 smoothly by the action of continuous family $\{g_t\}$ of conformal transformations. With this setup a reversibility of $X(C)$ can be discussed, where $X(C)$ is an integral of white noise over the domain (C) enclosed by a contour C (cf. causality).

It seems to be interesting to note that $X(C_t)$, $t_0 \leq t \leq t_1$, denotes a trajectory (path) of a Gaussian random field and on the set of the trajectories a Gaussian measure is naturally introduced. It is, therefore, our hope that we are ready to apply to the path integral. Actual computations have been given in the case where $\{C_T\}$ is a family of concentric circles.

References

[1] T. Hida, Canonical representation of Gaussian processes and their applications. Mem. College of Sci. Univ. of Kyoto, 33 (1960), 109–155.

[2] T. Hida, Brownian motion. Springer–Verlag. 1980

[3] T. Hida and Si Si, Innovations for random fields. Infinite Dimensional Analysis, Quantum Probability and Related Topics. 1 no.4 (1998), 499–509.

[4] P. Lévy. Random functions: General theory with special reference to Laplacian random functions. Univ. of California Publications in Statistics. 1. no.12 (1953), 331–388.

[5] P. Lévy, Problemes concret d'analyse fonctionnelle. Gauthier–Viller, 1951.

[6] Si Si, Random irreversible phenomena. Entropy in subordination. Proceedings Les Treilles Conf. to appear.

[7] N. Wiener, Nonlinear problems in random theory. MIT Press, 1958.

Hypergroup Actions and Wavelets

Juri Hinz, Mathematisches Institut
Universität Tübingen, 72076 Tübingen, Germany
email: juri.hinz@uni-tuebingen.de

May 8, 2000

Abstract

In analogy to wavelet transforms, we use group–like structures in order to introduce a class of integral transformations. We consider them in the context of Hilbert spaces and study their inversion.

0 Introduction

Wavelet analysis was introduced as a mathematical tool by A. Grossmann, J. Morlet, and T. Paul in [4] and was motivated by applications in signal processing. Many examples of important transformations can be recognized as wavelet transforms or are closely related to them (see [5], [6]). The mathematics of wavelet transform, as given in [5], is based on the theory of square integrable representations of locally compact groups and has a considerable range of generality.
In this paper we consider some integral transformations of wavelet type acting on the space of square integrable functions on a *commutative hypergroup*. They generalize the *classical wavelet transform* and the *windowed Fourier transform*. This work was motivated by a preprint of M. Rösler [10] and a series of papers by K. Trimèche (see [12], [13], [14], [15], [11]).

The first section recalls some results about commutative hypergroups. In the second section we define the left–transform. In the third section we discuss some special cases of the left–transform corresponding to transitive group actions.

1 Commutative hypergroups

Throughout this paper the following notation will be used: Let K be a locally compact space and denote by $C_b(K)$, $C_0(K)$, and $C_c(K)$ the spaces of continuous functions on K which are bounded, vanishing at infinity, and with compact support respectively. The symbol $M(K)$ denotes the space of Borel measures on K, $M_+(K)$, $M^b(K)$, and $M^b_+(K)$ are its subsets consisting of positive, bounded, and bounded positive measures, respectively. The σ–algebra of Borel measurable sets of K is denoted by $\mathcal{B}(K)$.

The notion of a hypergroup generalizes that of a locally compact group. (For additional reading on hypergroups we recommend [1] and [7].) A hypergroup K is a locally compact topological space with an axiomatically defined convolution $*$ on the Banach space $M^b(K)$ of bounded measures. With this operation, $M^b(K)$ forms a Banach algebra. The convolution $*$ satisfies several requirements which are natural for locally compact groups: For example, $*$ is weakly continuous, the convolution of probability measures is again a probability measure, there exists $e \in K$ such that the Dirac measure ε_e is the unit of the algebra $(M^b(K), *)$. Furthermore, there also exists a homeomorphism $^- : K \to K$ with $\int_K f(z^-)\varepsilon_x * \varepsilon_y(dz) = \int_K f(z)\varepsilon_{y^-} * \varepsilon_{x^-}(dz)$ for all $x, y \in K$, $f \in C_b(K)$. (In the case that K is a group, $^-$ is given by inversion.)

The hypergroup K is commutative if the algebra $(M^b(K), *)$ is commutative. If K is commutative then there exists (up to a constant) a uniquely determined measure $m \in M_+(K)$ satisfying $\varepsilon_x * m = m$ for all $x \in K$; m is called the *Haar measure.* As in the case of groups, family $(T_x)_{x \in K}$ of translation operators can be defined: For each $x \in K$ the corresponding T_x acts on suitable classes of functions by $f \mapsto T_x f$, $(T_x f)(y) = \int_K f d\varepsilon_x * \varepsilon_y$. Translation operators are contractions on $L^2(K, m)$ and $T_x^* = T_{x^-}$ holds for all $x \in K$. For commutative hypergroups, a Fourier transform and a Plancherel identity are available. A bounded measurable function $\chi : K \to \mathbb{C}$ is called *character*, if $\chi(e) = 1$, $\overline{\chi(x)} = \chi(x^-)$, and $T_x\chi = \chi(x)\chi$ are satisfied for all $x \in K$. The set $\widehat{K}$ of characters is endowed with the compact open topology. The Fourier transform $L^1(K, m) \to C_0(\widehat{K})$, $f \mapsto \hat{f}$ is defined by $\hat{f}(\chi) := \int_K \overline{\chi(x)} f(x) m(dx)$. There exists a unique measure $\pi \in M_+(\widehat{K})$ (*the Plancherel measure*), such that the Fourier transform maps $L^1(K, m) \cap L^2(K, m)$ into $L^2(\widehat{K}, \pi)$ L^2–isometrically; it can be extended to a unitary operator $\mathcal{F} : L^2(K, m) \mapsto L^2(\widehat{K}, \pi)$. Similarly, the inverse Fourier transform $L^1(\widehat{K}, \pi) \to C_0(K)$, $g \mapsto \check{g}$, $\check{g}(\chi) := \int \chi(x) g(\chi)\pi(d\chi)$ maps $L^1(\widehat{K}, \pi) \cap L^2(\widehat{K}, \pi)$ into $L^2(K, m)$ also L^2–isometrically. Its extension to $L^2(\widehat{K}, \pi)$ is the unitary operator $\mathcal{F}^{-1}$. We point out that in general the support S of the Plancherel measure is a *proper* subset of $\widehat{K}$. Translation operators are diagonalized by $\mathcal{F}$ in the following sense: For all $x \in K$ the operator $\mathcal{F}T_x\mathcal{F}^{-1}$ acts on $L^2(\widehat{K}, \pi)$ as the multiplication by the function $\widehat{K} \to \mathbb{C}$, $\chi \mapsto \chi(x)$.

We explain the basic idea of this paper by means of the examples of the classical wavelet transform and of the windowed Fourier transform on $\mathbb{R}$:

1. Given a function $0 \neq v \in L^2(\mathbb{R})$, we define $L_v : L^2(\mathbb{R}) \to C(\mathbb{R} \times \mathbb{R} \setminus \{0\})$, $h \mapsto L_v h$ as
$$(L_v h)(b, a) = \int \frac{1}{|a|} \overline{v(\frac{r}{a})} h(r + b) dr, \qquad \forall \qquad h \in L^2(\mathbb{R}),$$
$b \in \mathbb{R}$, and $a \in \mathbb{R} \setminus \{0\}$. The function $(b, a) \mapsto (L_v h)(b, a)$ is up to the factor $(b, a) \mapsto |a|^{\frac{1}{2}}$, the usual wavelet transform of h. Let us introduce on $L^2(\mathbb{R})$ the families $(T_b)_{b \in \mathbb{R}}$ and $(D_a)_{a \in \mathbb{R} \setminus \{0\}}$ of translation and dilation operators respectively as $(T_b f)(r) := f(b + r)$, $(D_a f)(r) := \frac{1}{|a|} f(\frac{r}{a})$ for all $f \in L^2(\mathbb{R})$, $r \in \mathbb{R}$. With these operators we may write $(L_v h)(b, a) = \langle D_a v, T_b h \rangle$ for all $h \in L^2(\mathbb{R})$, $b \in \mathbb{R}$, $a \in \mathbb{R} \setminus \{0\}$.

2. Given a function $0 \neq v \in L^2(\mathbb{R})$ we define the transform $W_v : L^2(\mathbb{R}) \to C(\mathbb{R} \times \mathbb{R})$,

as $(W_v h)(b,a) = \int_{\mathbb{R}} e^{iar}\overline{v(r)}h(r+b)dr$, which is up to a factor the windowed Fourier transform. Again, using dilation (in this case modulation) operators $(D'_a)_{a\in\mathbb{R}}$ given by $(D'_a f)(r) := e^{-iar}f(r)$ for all $f \in L^2(\mathbb{R})$, $r \in \mathbb{R}$, and $a \in \mathbb{R}$, the transform W_v may be written as $(W_v h)(b,a) = \langle D'_a v, T_b h\rangle$ for all $h \in L^2(\mathbb{R})$, $a, b \in \mathbb{R}$.

The following remarkable observation should be pointed out: If we define the actions β and β' of the groups $(\mathbb{R}\setminus\{0\},\cdot)$ and $(\mathbb{R},+)$ on the dual $\widehat{\mathbb{R}}$ of $\mathbb{R}$ as

$$\begin{array}{ll} \beta : \widehat{\mathbb{R}} \times \mathbb{R}\setminus\{0\} \mapsto \widehat{\mathbb{R}} & \beta(\chi,a) := \chi\cdot a, \\ \beta' : \widehat{\mathbb{R}} \times \mathbb{R} \mapsto \widehat{\mathbb{R}} & \beta'(\chi,a) := \chi + a, \end{array}$$

then for each $g \in L^2(\mathbb{R})$ we obtain all dilations $(D_a)_{a\in\mathbb{R}\setminus\{0\}}$ as $\mathcal{F}D_a\mathcal{F}^{-1}g = g(\beta(.,a))$ and dilations $(D'_a)_{a\in\mathbb{R}}$ as $\mathcal{F}D'_a\mathcal{F}^{-1}g = g(\beta'(.,a))$. In both cases the dilations are unitarily equivalent via $\mathcal{F}$ to operators on $L^2(\widehat{\mathbb{R}})$, induced by an action of a group on $\widehat{\mathbb{R}}$.

Motivated by this observation we start with a commutative hypergroup K, a function $v \in L^2(K,m)$, and an action β of a locally compact group G on $\widehat{K}$. We study the linear operator $L_v : L^2(K,m) \to \mathbb{C}^{K\times G}$, given by $(L_v h)(b,a) := \langle D_a v, T_b h\rangle$ for all $h \in L^2(K,m)$, $(b,a) \in K \times G$. Here $(D_a)_{a\in G} \subset B(L^2(K))$ are dilations defined by $\mathcal{F}D_a\mathcal{F}^{-1}g := g(\beta(.,a))$ for all $g \in L^2(\widehat{K},\pi)$, and $(T_b)_{b\in K}$ are the usual translations of the hypergroup K.

2 The left–transform

Let (K,m) be a commutative hypergroup K equipped with a fixed Haar measure m. We assume that a locally compact group G acts continuously on the support of the Plancherel measure $S = \operatorname{supp}\pi \subset \widehat{K}$: That means that there exists a continuous mapping $\beta : S \times G \to S$, $(\chi, a) \mapsto \chi^a$ satisfying $(\chi^{a_1})^{a_2} = \chi^{a_1 a_2}$ for all $\chi \in S$ and $a_1, a_2 \in G$.

Let μ be a fixed left Haar measure of G. We introduce the set $\{\mu^\chi : \chi \in S\}$ of image measures of μ induced by the mappings $G \to S$, $a \mapsto \chi^a$: For each $\chi \in S$ we obtain $\mu^\chi(B) = \mu(\{a \in G : \chi^a \in B\})$ for all $B \in \mathcal{B}(S)$. Let us also define the set $\{\pi^a : a \in G\}$ of image measures of $\pi|_S$ induced by the mappings $S \to S$, $\chi \mapsto \chi^a$. For each $a \in G$ we obtain $\pi^a(B) = \pi(\{\chi \in S : \chi^a \in B\})$ for all $B \in \mathcal{B}(S)$. We suppose the following assumption to be satisfied:

Assumption 1. *For all $a \in G$ the measure π^a is absolutely continuous with respect to $\pi|_S$ and the corresponding Radon–Nikodym derivative satisfies $\frac{d\pi^a}{d\pi|_S} \in L^\infty(S,\pi|_S)$.*

For each $a \in G$ and $f \in \mathbb{C}^S$ we define the function $f^a \in \mathbb{C}^S$ as $f^a(\chi) := f(\chi^a)$ for all $\chi \in S$. Due to the above assumption, the mapping $f \mapsto f^a$ defines a continuous linear operator $L^2(S,\pi|_S) \to L^2(S,\pi|_S)$ for each $a \in G$. Since the Hilbert spaces $L^2(S,\pi|_S)$ and $L^2(\widehat{K},\pi)$ are naturally isomorphic we may consider the mapping $f \mapsto f^a$ as a continuous linear operator on $L^2(\widehat{K},\pi)$.

Definition.
(i) The operators $(D_a)_{a\in G} \subset B(L^2(K,m))$, defined by $D_a : L^2(K,m) \to L^2(K,m)$, $h \mapsto \mathcal{F}^{-1}(\mathcal{F}h)^a$ for all $a \in G$, are called dilation operators.

(ii) For each $v \in L^2(K,m)$, the linear mapping $L_v : L^2(K,m) \to \mathbb{C}^{K\times G}$, $h \mapsto L_v h$, given by $(L_v h)(b,a) := \langle D_a v, T_b h\rangle$ for all $(b,a) \in K \times G$, is called the left-transform corresponding to v.
(iii) The elements of $\mathcal{A} := \{v \in L^2(K,m) : (\chi \mapsto \int_G |\mathcal{F}v(\chi^a)|^2\mu(da)) \in L^\infty(S, \pi|_S)\}$ are called admissible vectors. The elements of $\mathcal{A}\setminus\{0\}$ are called wavelets.

Given $v_1, v_2 \in \mathcal{A}$, we define $C_{v_2,v_1} : S \to \mathbb{C}$ as $C_{v_2,v_1}(\chi) := \int_G \overline{(\mathcal{F}v_2)(\chi^a)}(\mathcal{F}v_1)(\chi^a)\mu(da)$ for all $\chi \in S$. It follows from Cauchy–Schwarz inequality that $C_{v_2,v_1} \in L^\infty(S,\pi|_S)$. We remark that the function C_{v_2,v_1} is constant on each orbit:

Lemma 1. *For all $v_1, v_2 \in \mathcal{A}$, $\tilde\chi \in S$ and $\chi_0 \in \beta(\tilde\chi, G)$ we have $C_{v_2,v_1}(\chi_0) = C_{v_2,v_1}(\tilde\chi)$.*

Proof. For $\chi_0 \in \beta(\tilde\chi, G)$ there exists $a_0 \in G$ with $\chi_0 = \tilde\chi^{a_0}$, and it follows that

$$C_{v_2,v_1}(\chi_0) = \int_G \overline{(\mathcal{F}v_2)(\chi_0^a)}(\mathcal{F}v_1)(\chi_0^a)\mu(da) = \int_G \overline{(\mathcal{F}v_2)(\tilde\chi^{a_0a})}(\mathcal{F}v_1)(\tilde\chi^{a_0a})\mu(da).$$

We conclude that

$$\int_G \overline{(\mathcal{F}v_2)(\tilde\chi^{a_0a})}(\mathcal{F}v_1)(\tilde\chi^{a_0a})\mu(da) = \int_G \overline{(\mathcal{F}v_2)(\tilde\chi^a)}(\mathcal{F}v_1)(\tilde\chi^a)\mu(da) = C_{v_2,v_1}(\tilde\chi)$$

since μ is a left Haar measure on G. (The same argument implies that $\mu^\chi = \mu^{\tilde\chi}$ for $\chi \in \beta(\tilde\chi, G)$). ■

For an admissible vector v the left–transform can actually be discussed in the framework of Hilbert spaces:

Proposition 1:
(i) Given $v \in \mathcal{A}$ the mapping $h \mapsto L_v h$ defines a bounded linear operator from $L^2(K,m)$ into $L^2(K \times G, m \otimes \mu)$.
(ii) $\langle L_{v_1}h_1, L_{v_2}h_2\rangle = \int_S \overline{(\mathcal{F}h_1)(\chi)}(\mathcal{F}h_2)(\chi)C_{v_2,v_1}(\chi)\pi(d\chi)$ holds for all $v_1, v_2 \in \mathcal{A}$ and $h_1, h_2 \in L^2(K,m)$.

Proof. (i) Let $v \in \mathcal{A}$ and $h \in L^2(K,m)$. The function $L_v h$ is measurable since

$$\begin{aligned} L_v h(b,a) &= \langle D_a v, T_b h\rangle = \langle \mathcal{F}D_a v, \mathcal{F}T_b h\rangle = \int_{\widehat{K}} \overline{(\mathcal{F}v)^a(\chi)}\chi(b)(\mathcal{F}h)(\chi)\pi(d\chi) \\ &= \int_S \overline{(\mathcal{F}v)(\chi^a)}\chi(b)(\mathcal{F}h)(\chi)\pi(d\chi), \end{aligned}$$

and the integrand $K \times G \times S \to \mathbb{C}$, $(b,a,\chi) \mapsto \overline{(\mathcal{F}v)(\chi^a)}\chi(b)(\mathcal{F}h)(\chi)$ is measurable in view of continuity of $\beta : (\chi, a) \mapsto \chi^a$.

Now $L_v h \in L^2(K \times G, m \otimes \mu)$ is seen as follows:

$$\begin{aligned} \infty > \int_S |(\mathcal{F}h)(\chi)|^2 C_{v,v}(\chi)\pi(d\chi) &= \int_S |(\mathcal{F}h)(\chi)|^2 \int_G |((\mathcal{F}v)(\chi^a)|^2\mu(da)\pi(d\chi) \\ &= \int_G\int_S |\overline{(\mathcal{F}v)(\chi^a)} \cdot (\mathcal{F}h)(\chi)|^2\pi(d\chi)\mu(da) \\ &= \int_G\int_{\widehat{K}} |\overline{(\mathcal{F}v)^a} \cdot (\mathcal{F}h)|^2 d\pi\mu(da). \end{aligned}$$

showing that $\overline{(\mathcal{F}v)^a} \cdot (\mathcal{F}h) \in L^2(\widehat{K})$ for μ–almost all $a \in G$. The isometry of $\mathcal{F}$ ensures that

$$\begin{aligned}
\infty > \int_G \int_{\widehat{K}} |\overline{(\mathcal{F}v)^a} \cdot (\mathcal{F}h)|^2 d\pi \mu(da) &= \int_G \int_K |((\overline{\mathcal{F}v)^a}\mathcal{F}h)^\vee(b)|^2 m(db)\mu(da) \\
&= \int_G \int_K |\langle(\mathcal{F}v)^a, \mathcal{F}T_b h\rangle|^2 m(db)\mu(da) \\
&= \int_{K\times G} |(L_v h)(b,a)|^2 m \otimes \mu(d(b,a)).
\end{aligned}$$

The first equality holds since

$$\langle(\mathcal{F}v)^a, \mathcal{F}T_b h\rangle = \int_{\widehat{K}} \overline{(\mathcal{F}v)^a(\chi)}\chi(b)(\mathcal{F}h)(\chi)\pi(d\chi) = ((\overline{\mathcal{F}v)^a}\mathcal{F}h)^\vee(b).$$

(ii) Polarizing

$$\langle L_v h, L_v h\rangle = \int_S |(\mathcal{F}h)(\chi)|^2 C_{v,v}(\chi)\pi(d\chi) \quad \forall v \in \mathcal{A}, h \in L^2(K,m), \tag{1}$$

we obtain

$$\langle L_{v_1} h_1, L_{v_2} h_2\rangle = \int_S \overline{(\mathcal{F}h_1)}(\chi)(\mathcal{F}h_2)(\chi)C_{v_2,v_1}(\chi)\pi(d\chi) \qquad \forall v_1, v_2 \in \mathcal{A}, h_1, h_2 \in L^2(K,m).$$

■

Remark. (The inversion of the left–transform.) Let us suppose that for a given $v_2 \in \mathcal{A}$ there exists $v_1 \in \mathcal{A}$ satisfying $C_{v_2,v_1} = 1$. In this situation we obviously obtain $\langle L_{v_1} h_1, L_{v_2} h_2\rangle = \langle h_1, h_2\rangle$ for all $h_1, h_2 \in L^2(K,m)$, which means $L_{v_1}^* L_{v_2} = \mathbb{1}$.

3 Transitive group action

In this section a special group action is considered: We suppose that there is essentially only one orbit in S, which implies that the function C_{v_2,v_1} is constant $\pi|_S$–almost everywhere on S. This assumption is analogous to that of irreducibility for square integrable group representations.

Assumption 2. *The action β of G on S is assumed to be transitive, which means that there exists $\tilde{\chi} \in S$ with $\pi(\widehat{K} \setminus \beta(\tilde{\chi}, G)) = 0$. Furthermore, we assume the measures $\mu^{\tilde{\chi}} \in M_+(S)$ and $\pi|_S$ to be equivalent.*

Remark A similar condition is discussed in the case of groups in [2] Proposition 2.
We denote by R the function given as $R : \widehat{K} \to \mathbb{R}_+$, $R(\chi) := \frac{d\mu^{\tilde{\chi}}}{d\pi|_S}(\chi)$ for all $\chi \in S$, and $R(\chi) := 0$ for all $\chi \in \widehat{K} \setminus S$. Obviously $R > 0$ π–almost everywhere on $\widehat{K}$.

Lemma 2. *Assumption 2 implies:*
(i) $\mathcal{A} = \{v \in L^2(K,m) \,:\, R^{\frac{1}{2}}\mathcal{F}v \in L^2(\widehat{K},\pi)\}$. In particular $\mathcal{A}$ is a dense linear subspace of $L^2(K,m)$.
(ii) If v is a wavelet then L_v is, up to a positive factor, an isometric operator.

Proof. (i): Let us choose an arbitrary $\tilde{\chi} \in S$ satisfying $\pi(\widehat{K} \setminus \beta(\tilde{\chi}, G)) = 0$ and $v \in L^2(K, m)$. From

$$\int_G |(\mathcal{F}v)(\tilde{\chi}^a)|^2 \mu(da) = \int_S |\mathcal{F}v|^2 d\mu^{\tilde{\chi}} = \int_S |\mathcal{F}v|^2 \frac{\mu^{\tilde{\chi}}}{d\pi_S} d\pi|_S = \int_{\widehat{K}} |\mathcal{F}v|^2 R d\pi$$

it follows that $v \in \mathcal{A}$ if and only if the above integrals are finite.
(ii): Since $0 \neq v \in \mathcal{A}$ we obtain from the above arguments that $L^\infty(S, \pi|_S) \ni C_{v,v} = \int_{\widehat{K}} |v(\chi')|^2 \underbrace{R(\chi')}_{>0} \pi(d\chi') > 0$. It follows for all $h \in L^2(K, m)$ that

$$\langle L_v h, L_v h \rangle = \int_S |(\mathcal{F}h)(\chi)|^2 C_{v,v}(\chi) \pi(d\chi) = \|h\|^2 \underbrace{\int_{\widehat{K}} |(\mathcal{F}v)(\chi')|^2 R(\chi') \pi(d\chi')}_{>0}. \qquad \blacksquare$$

Polarizing the last equality, we are led to the following orthogonality relation:

$$\langle L_{v_1} h_1, L_{v_2} h_2 \rangle = \langle R^{\frac{1}{2}} \mathcal{F} v_2, R^{\frac{1}{2}} \mathcal{F} v_1 \rangle \langle h_1, h_2 \rangle \qquad \forall v_1, v_2 \in \mathcal{A}, \quad h_1, h_2 \in L^2(K \times G, m \otimes \mu).$$

For admissible vectors we may normalize the left–transform and obtain an isometric operator:

Definition. *Let Assumption 2 be satisfied and $v \in L^2(K, m)$ be a wavelet. The isometric operator $\mathcal{L}_v := \frac{1}{\|L_v v\|} L_v$ is called the wavelet transform corresponding to the wavelet v.*

Remark. As in the case of groups the wavelet transform $\mathcal{L}_v$ is inverted on its range by its adjoint $\mathcal{L}_v^*$, what means $\mathcal{L}_v^* \mathcal{L}_v = \mathbb{1}$; here

$$\mathcal{L}_v^* \xi = \frac{1}{\|L_v v\|} \int_{K \times G} \xi(b, a) T_{b^-} D_a v \;\; m \otimes \mu(d(b, a))$$

holds in the weak sense for all $\xi \in L^2(K \times G, m \otimes \mu)$. The range of $\mathcal{L}_v$ consists precisely of those $\xi \in L^2(K \times G, m \otimes \mu)$ satisfying $\mathcal{L}_v \mathcal{L}_v^* \xi = \xi$, where the last assertion is equivalent to

$$\xi(b, a) = \int_{K \times G} \frac{\langle T_{b^-} D_a v, T_{b'^-} D_{a'} v \rangle}{\|L_v v\|^2} \xi(b', a') m \otimes \mu(d(b', a')) \qquad \forall (b, a) \in K \times G.$$

3.1 A remark on discretization

The most important feature of the classical wavelet transform is the discretization technique, since multiresolution analysis based on orthogonal wavelets provide tools for the design of fast algorithms. The discretization of the classical wavelet transform is possible due to Poisson's summation formula on $\mathbb{R}$. Unfortunately, no corresponding result is available for commutative hypergroups. For this reason, no straightforward discretization technique can be done in the context of hypergroups and we can present only a discretization of the diation parameter. An alternative approach to discretization is based on a direct construction of the so–called wavelet frames. This construction is known in some special cases, see [10].

Let the assumptions 1 and 2 be satisfied and v be a wavelet. A *discretization* of $\mathcal{L}_v$ is given by a set $\mathcal{D} \subset K \times G$ such that $\mathcal{L}_v h|_{\mathcal{D}}$ determines $\mathcal{L}_v h$ uniquely. The most desirable case is that where $\mathcal{D}$ is discrete and $h \mapsto \mathcal{L}_v h|_{\mathcal{D}}$ is a bounded injective operator from $L^2(K,m)$ into $l^2(\mathcal{D})$. In our setting, we consider only the case $\mathcal{D} = K \times G_d$, where $G_d \subset G$ is a discrete subgroup of G. The group action β is restricted to the action β_d of the discrete subgroup G_d. The first assumption still holds for β_d, but the transitivity of β_d (second assumption) fails in general. However, for $v \in \mathcal{A}_d$ (admissible vector for β_d) the operator $h \mapsto \mathcal{L}_v h|_{\mathcal{D}}$ mapping from $L^2(K)$ into $L^2(K \times G_d)$ is still bounded. It is also injective, if $\inf_{\chi \in S} C_{v,v}(\chi) > 0$. This follows from (1):

$$\langle L_v h, L_v h\rangle_{L^2(K\times G_d)} = \int_S |(\mathcal{F}v)(\chi)|^2 C_{v,v}(\chi)\pi(d\chi) \geq \|h\|^2 \inf_{\chi\in S} C_{v,v}(\chi) \;\forall\, h \in L^2(K).$$

Note that here $C_{v,v}$ also corresponds to β_d and is given by:

$$C_{v,v}(\chi) = \int_{G_d} |\mathcal{F}v(\chi^a)|^2 \mu_{G_d}(da) \qquad \forall \chi \in S.$$

4 Examples

Example 1. (The wavelet transform on $\mathbb{R}$). The hypergroup, endowed with the Haar measure m, is given as $(K, m(dr)) := (\mathbb{R}, dr)$; this choice implies $(\widehat{K}, \pi(d\chi)) := (\mathbb{R}, \frac{1}{2\pi}d\chi)$ and $S = \widehat{K}$. The translations $(T_b)_{b\in K}$ are given as $(T_b h)(r) = h(b+r)$ for all $h \in L^2(K,m)$, $b \in K$. Let us define $(G, \mu(da)) := (\mathbb{R}\setminus\{0\}, \frac{1}{|a|}da)$. The group G acts on $\widehat{K}$ by multiplication: $\beta : (\chi, a) \mapsto \chi \cdot a$. Assumptions 1 and 2 are automatically satisfied. We obtain for all $a \in G$ $\pi^a(d\chi) := \frac{1}{2\pi|a|}d\chi$, and, putting $\tilde{\chi} := 1$, the image measure $\mu^{\tilde{\chi}}$ is given by $\mu^{\tilde{\chi}}(d\chi) = \frac{1}{|\chi|}d\chi$. The dilation (here modulation) operators are easily seen as acting as $(D_a h)(r) = (\mathcal{F}^{-1}(\mathcal{F}h)(.\cdot a))(r) = \frac{1}{|a|}h(\frac{r}{a})$ for all $a \in G$, $r \in K$, $h \in L^2(K,m)$. Given $v \in L^2(K,m)$, we obtain the left–transform of $h \in L^2(K,m)$ as

$$(L_v h)(b,a) = \langle D_a v, T_b h\rangle = \int_{\mathbf{R}} \frac{1}{|a|}\bar{v}(ra^{-1})h(r+b)dr = \int_{\mathbf{R}} \frac{1}{|a|}\bar{v}(\frac{u-b}{a})h(u)du$$

for all $(b,a) \in K \times G$. The function R is calculated by $R(\chi) := \frac{d\mu^{\tilde{\chi}}}{d\pi}(\chi) = \frac{2\pi}{|\chi|}$ for all $\chi \in \widehat{K}$. By definition, $0 \neq v \in L^2(K,m)$ is a wavelet if

$$\int_{\widehat{K}} R(\chi)|\mathcal{F}v(\chi)|^2\pi(d\chi) = \int_{\mathbf{R}} \frac{2\pi}{|\chi|}|\mathcal{F}v(\chi)|^2\frac{1}{2\pi}d\chi = \int_{\mathbf{R}} |\mathcal{F}v(\chi)|^2\frac{1}{|\chi|}d\chi < \infty.$$

Example 2. (The windowed Fourier transform on $\mathbb{R}$). We choose $(K, m(dr))$, $(\widehat{K}, \pi(d\chi))$ and $(T_b)_{b\in K}$ as in the previous example. Let us define the group as $(G, \mu(da)) := (\mathbb{R}, da)$. The group G acts on $\widehat{K}$ by addition: $\beta : (\chi, a) \mapsto \chi + a$. Assumptions 1 and 2 are then satisfied. We obtain $\pi^a(d\chi) := \frac{1}{2\pi}d\chi$ for all $a \in G$, and, putting $\tilde{\chi} := 0$, the image measure $\mu^{\tilde{\chi}}$ is found as $\mu^{\tilde{\chi}}(d\chi) = d\chi$. The dilation (here modulation) operators are easily seen as acting as $(D_a h)(r) = (\mathcal{F}^{-1}(\mathcal{F}h)(.+a))(r) = e^{-iar}h(r)$ for all $a \in G$, $r \in K$. For a given $v \in L^2(K,m)$, we obtain the left–transform of h as

$$(L_v h)(b,a) = \langle D_a v, T_b h\rangle = \int_{\mathbf{R}} \overline{e^{-iar}v(r)}h(r+b)dr$$

for all $(b,a) \in K \times G$. Since $R(\chi) := \frac{d\mu^{\tilde{\chi}}}{d\pi}(\chi) = 2\pi$ each $0 \neq v \in L^2(K,m)$ is a wavelet.

Example 3. (Radial wavelet transform, a special case of [10]). The Bessel–Kingman hypergroup K with parameter $\alpha > -\frac{1}{2}$ is given as $K := \mathbb{R}_+$, the Haar measure is just $m(dr) := r^{2\alpha+1}dr$, and the convolution $*$ of point measures satisfies

$$(\varepsilon_x * \varepsilon_y)(dr) = \frac{\Gamma(\alpha+1)}{\Gamma(\frac{1}{2})\Gamma(\alpha+\frac{1}{2})2^{\alpha-1}} \frac{[(r^2-(x-y)^2)((x+y)^2-r^2)]^{\alpha-\frac{1}{2}}}{(xyr)^{2\alpha}} 1_{[|x-y|,x+y]}dr.$$

The set of characters of K is just

$$\{r \mapsto j_\alpha(\chi \cdot r) \mid \chi \in \mathbb{R}_+, j_\alpha \text{ is the modified Bessel function of order } \alpha\},$$

$$j_\alpha(z) := \sum_{k \geq 0} \frac{(-1)^k\Gamma(\alpha+1)}{2^{2k}k!\Gamma(\alpha+k+1)} z^{2k} \qquad \forall z \in \mathbb{C}$$

and via this parameterization the dual $\widehat{K}$ can be identified topologically with $\mathbb{R}_+$. The Plancherel measure π, associated with (K,m), is given by $\pi(d\chi) = \frac{\chi^{2\alpha+1}}{(2^\alpha\Gamma(\alpha+1))^2}d\chi$, and its support S is equal to $\widehat{K}$. Let the group $G := \mathbb{R}_+ \setminus \{0\}$ act on $\widehat{K}$ by multiplication: $\beta : (\chi, a) \mapsto \chi \cdot a$. We fix the Haar measure μ on G as $\mu(da) := \frac{1}{a}da$. Assumption 1 is satisfied since $\pi^a(d\chi) := \frac{\chi^{2\alpha+1}}{(2^\alpha\Gamma(\alpha+1))^2a^{2\alpha+2}}d\chi$ for all $a \in G$. The dilation operators can be obtained explicitly: It follows from

$$\begin{aligned}
(D_a h)(r) &= (\mathcal{F}^{-1}(\mathcal{F}h)(\cdot\, a))(r) \\
&= \int_0^\infty j_\alpha(\chi r)(\mathcal{F}h)(\chi a)\frac{\chi^{2\alpha+1}}{(2^\alpha\Gamma(\alpha+1))^2}d\chi \\
&= \int_0^\infty j_\alpha(a\chi\frac{r}{a})(\mathcal{F}h)(\chi a)\frac{(\chi a)^{2\alpha+1}}{(2^\alpha\Gamma(\alpha+1))^2 a^{2\alpha+1}}d\chi \\
&= \frac{a^{-1}}{a^{2\alpha+1}} \underbrace{\int_0^\infty j_\alpha(\chi\frac{r}{a})(\mathcal{F}h)(\chi)\frac{\chi^{2\alpha+1}}{(2^\alpha\Gamma(\alpha+1))^2}d\chi}_{h(\frac{r}{a})} \qquad \forall h \in C_c(K)
\end{aligned}$$

that $(D_a h)(r) = \frac{1}{a^{2\alpha+2}}h(\frac{r}{a})$ for all $h \in L^2(K,m)$, $a \in G$, and $r \in K$. Finally to see Assumption 2 is satisfied, we set $\tilde{\chi} := 1$ and obtain $\mu^{\tilde{\chi}}(d\chi) = \frac{1}{\chi}d\chi$. This implies that $R(\chi) := \frac{d\mu^{\tilde{\chi}}}{d\pi}(\chi) = \frac{1}{\chi}\frac{(2^\alpha\Gamma(\alpha+1))^2}{\chi^{2\alpha+1}} > 0$ for all $\chi \in \widehat{K}$. The function $0 \neq v \in L^2(K,m)$ is a wavelet if

$$\infty > \int_{\widehat{K}} R|\mathcal{F}v|^2 d\pi = \int_0^\infty \frac{1}{\chi}\frac{(2^\alpha\Gamma(\alpha+1))^2}{\chi^{2\alpha+1}}|\mathcal{F}v(\chi)|^2\frac{\chi^{2\alpha+1}}{(2^\alpha\Gamma(\alpha+1))^2}d\chi = \int_0^\infty |\mathcal{F}v(\chi)|^2\frac{1}{\chi}d\chi.$$

Example 4. Here we consider the wavelet transform on Chébli–Trimèche hypergroups. This is a generalization of the previous example. A Chébli–Trimèche hypergroup K with Haar measure m is given by $(K, m(dr)) := (\mathbb{R}_+, A(r)dr)$. The mapping $A : \mathbb{R}_+ \to \mathbb{R}_+$, called the Chébli–Trimèche function, is assumed to satisfy several conditions. (For the exact definition of Chébli–Trimèche hypergroups we refer the reader to [1], p. 209). The set of characters $\widehat{K}$ is identified with $\mathbb{R}_+ \cup i[0,\rho]$, (the constant $\rho \in \mathbb{R}_+$ is called the index of the hypergroup). By this identification the support of the Plancherel measure is given by $S := \mathbb{R}_+$. Furthermore there exists a function $C : \mathbb{R}_+ \to \mathbb{C}$ with $\pi|_S(d\chi) = |C(\chi)|^{-2}d\chi$. By a result of Trimèche (see [12]):

$$\sup_{\chi>0} \frac{|C(\frac{\chi}{a})|^{-2}}{|C(\chi)|^{-2}} < \infty \qquad \forall a \in \mathbb{R}_+ \setminus \{0\}. \tag{2}$$

Let us define the action of the group $G := \mathbb{R}_+ \setminus \{0\}$ on S by multiplication: $\beta : (\chi, a) \mapsto \chi \cdot a$. It follows from

$$\begin{aligned} \int_S f(\chi)\pi^a(d\chi) &= \int_S f(\chi^a)\pi(d\chi) = \int_0^\infty f(\chi \cdot a)|C(\chi)|^{-2}d\chi \\ &= \int_0^\infty f(\chi \cdot a)|C(\frac{\chi \cdot a}{a})|^{-2}d\chi \\ &= \int_0^\infty f(\chi)|C(\frac{\chi}{a})|^{-2}\frac{1}{a}d\chi \qquad \forall f \in C_c(\widehat{K}) \end{aligned}$$

that $\pi^a(d\chi) = \frac{|C(\frac{\chi}{a})|^{-2}}{a}d\chi$ for all $a \in G$. We conclude that Assumption 1 is satisfied since in view of (2) $\frac{d\pi^a}{d\pi} \in L^\infty(S, \pi|_S)$ holds for all $a \in G$. As in the previous example, we endow the group G with the Haar measure $\mu(da) = \frac{1}{a}da$. Choosing $S \ni \tilde{\chi} := 1$ the action β is easily seen to be transitive. It follows from $\frac{d\mu^{\tilde{\chi}}}{d\pi|_S}(\chi) = \frac{1}{\chi}\frac{1}{|C(\chi)|^{-2}} > 0$ that Assumption 2 is satisfied. The function $0 \neq v \in L^2(K, m)$ is a wavelet if

$$\int_{\widehat{K}} R|\mathcal{F}v|^2 d\pi = \int_0^\infty \frac{1}{\chi}\frac{1}{|C(\chi)|^{-2}}|\mathcal{F}v(\chi)|^2|C(\chi)|^{-2}d\chi = \int_0^\infty \frac{1}{\chi}|\mathcal{F}v(\chi)|^2 d\chi < \infty.$$

References

[1] W. R. Bloom, H. Heyer, *Harmonic Analysis of Probability Measures on Hypergroups.* Walter de Gruyter, Berlin, New York (1995).

[2] D. Bernier, K. Taylor *Wavelets from square integrable representations.* SIAM J. of Math. Anal. **27**, 594-608.

[3] I. Daubechies. *Ten Lectures on Wavelets.* CBMS–NFS Regional Conference Series in Applied Mathematics, Philadelphia (1992).

[4] A. Grossmann, J. Morlet, T. Paul. *Transformations associated to square integrable group representations I: General results.* J. Math. Phys., **26**, 2473-9, (1985).

[5] A. Grossmann, J. Morlet, T. Paul. *Transformations associated to square integrable group representations II: Examples.* Ann. Inst. Henri Poincaré Physique Théoretique, **45**, 293-309, (1986).

[6] M. Holschneider. *Wavelets an Analytic Tool.* Oxford University Press, Oxford (1995).

[7] R. I. Jewett. *Spaces with an abstract convolution of measures.* Adv. in Math. **18**, 1-101, (1975).

[8] A. K. Louis, P. Maaß, A. Rieder. *Wavelets Theorie und Anwendungen.* Teubner, Stuttgart 1994.

[9] Y. Meyer. *Wavelets and Operators.* Cambridge Studies in Advanced Mathematics, 37, Cambridge University Press (1992).

[10] M. Rösler. *Radial wavelets on hypergroups.* Preprint.

[11] K. Trimèche. *Generalized Wavelets and Hypergroups.* Gordon and Breach, Amsterdam (1997).

[12] K. Trimèche. *Wavelets on hypergroups.* International Conference on Harmonic Analysis, Birkhäuser, Boston, Basel, Berlin (1997).

[13] K. Trimèche. *Inversion of the spherical mean operator and its dual using spherical wavelets.* Proceedings Oberwolfach 23–29 October 1994, World Scientific Publishing, Singapore (1995).

[14] K. Trimèche. *Continuous wavelet transform on semisimple Lie groups and on Cartan motion group.* C. R. Acad. Sci. Canada, Vol. XVI, No. **4**, 161–165, (1994).

[15] K. Trimèche. *Continuous wavelet transform on semisimple Lie groups and inversion of the Abel transform and its dual.* C. R. Math. Rep. Acad. Sci. Canada, Vol. XVII, No. **2**, 83–86, (1995).

On Inductive Limits of Topological Algebraic Structures in relation to the Product Topologies

By

Takeshi HIRAI, Hiroaki SHIMOMURA
Nobuhiko TATSUUMA, and Etsuko HIRAI

Abstract. In infinite-dimensional harmonic analysis, we encounter naturally inductive limits of certain topological algebraic objects, such as Lie groups, Banach algebras, topological semigroups and so on. In such cases, the inductive limit algebraic structures are not necessarily consistent with the inductive limit topologies, contrary to the affirmative statement in [Enc, Article 210]. This phenomenon is studied in [TSH] in the case of topological groups.

We study in this paper similar situations for other categories of topological algebraic structures. Further, in relation to this, we study certain properties of general topological spaces for the 'commutativity' of (1) taking direct products and (2) taking inductive limits.

This paper is a summarized version of [HSTH].

§1. Inductive limits and direct products

1.1. Preliminaries. Let us consider an inductive system in a certain category $\mathcal{C}$, of topological spaces, of topological groups, of topological vector spaces, or of topological algebras, etc., as

$$\{\, (X_\alpha, \tau_{X_\alpha}),\ \alpha \in A\,;\ \phi_{\beta,\alpha},\ \alpha \preceq \beta,\ \alpha, \beta \in A \,\},$$

where the index set A is a directed set, each X_α is an object in $\mathcal{C}$ with topology τ_{X_α}, and $\phi_{\beta,\alpha}$ is a (continuous) homomorphism $X_\alpha \to X_\beta$ in $\mathcal{C}$ satisfying the consistency condition: $\phi_{\gamma,\beta} \circ \phi_{\beta,\alpha} = \phi_{\gamma,\alpha}$ for any $\alpha \preceq \beta \preceq \gamma$.

Then, on an inductive limit space $X := \varinjlim X_\alpha$, we define the corresponding algebraic structure. On the other hand, we have also an inductive limit topology, denoted as $\varinjlim \tau_{X_\alpha}$ or simply as τ_{ind}^X, in which a subset D of X is open, by definition, if and only if $\phi_\alpha^{-1}(D) \subset X_\alpha$ is open in τ_{X_α} for each $\alpha \in A$. Here, ϕ_α denotes the canonical homomorphism from X_α to X.

In this paper, we study about the harmonicity of the limit topology τ_{ind}^X with the algebraic structure on X. Furthermore, we consider an appropriate variant of τ_{ind}^X in each category $\mathcal{C}$ (denote it by $\tau_{\mathcal{C}}^X$ provisionally here) and study various kinds of harmonicity, and propose several problems.

Meantime, we find that one of the important points of discussions is the problem of commutativity of (1) taking the inductive limit $\tau_{\mathcal{C}}^X$ and (2) taking direct products. This commutativity is expressed symbolically as $\tau_C^X \times \tau_C^Y \cong \tau_C^{X\times Y}$, for two inductive systems $\{(X_\alpha, \tau_{X_\alpha}), \alpha \in A\}$ and $\{(Y_\alpha, \tau_{Y_\alpha}), \alpha \in A\}$ with $Y = \varinjlim Y_\alpha$. In the case where this commutativity holds, we say that the condition (DPA) (= *Direct Product is Admitted*) holds for $\tau_C^{\{*\}}$.

More in detail, let us explain our problems in the following.

1.2. Inductive limits of topological groups.

Let $\{(G_\alpha, \tau_{G_\alpha}); \alpha \in A\}$ be an inductive system of topological groups with a directed set A as index set. Here τ_{G_α} denotes the group topology on G_α and we are given an inductive system of continuous group homomorphisms $\phi_{\alpha_2,\alpha_1}; G_{\alpha_1} \to G_{\alpha_2}$ $(\alpha_1, \alpha_2 \in A, \alpha_1 \preceq \alpha_2)$ satisfying $\phi_{\alpha_3,\alpha_2} \circ \phi_{\alpha_2,\alpha_1} = \phi_{\alpha_3,\alpha_1}$ for $\alpha_1 \preceq \alpha_2 \preceq \alpha_3$. Put $G := \varinjlim G_\alpha$ and $\tau_{ind}^G := \varinjlim \tau_{G_\alpha}$ the inductive limit of groups and that of topologies respectively. Then, as seen in [TSH], the multiplication $G \times G \ni (g,h) \mapsto gh \in G$ is not necessarily continuous with respect to the inductive limit topology τ_{ind}^G, or more exactly, with respect to $(\tau_{ind}^G \times \tau_{ind}^G, \tau_{ind}^G)$.

Inspired by this rather critical phenomenon, we start to study the inductive limit topologies in detail in more general setting.

1.3. A continuity criterion.

Let $\{(X_\alpha, \tau_{X_\alpha}); \alpha \in A\}$ be an inductive system of topological spaces. Take another inductive system $\{(Z_\alpha, \tau_{Z_\alpha}); \alpha \in A\}$ of topological spaces with the same index set A and with an inductive system of continuous maps $\phi'_{\alpha_2,\alpha_1} : Z_{\alpha_1} \to Z_{\alpha_2}$. Then, assume that we are given a system of maps F_α of X_α to Z_α for $\alpha \in A$ which is *consistent* in the sense that $F_{\alpha_2} \circ \phi_{\alpha_2,\alpha_1} = \phi'_{\alpha_2,\alpha_1} \circ F_{\alpha_1}$ for $\alpha_1, \alpha_2 \in A, \alpha_1 \preceq \alpha_2$. Then this system induces a map $F : X \to Z := \varinjlim Z_\alpha$ such that $F \circ \phi_\alpha = \phi'_\alpha \circ F_\alpha\,(\alpha \in A)$, where ϕ_α (resp. ϕ'_α) denotes the natural map from X_α to X (resp. Z_α to Z), continuous with respect to $(\tau_{X_\alpha}, \tau_{ind}^X)$ (resp. to $(\tau_{Z_\alpha}, \tau_{ind}^Z)$). Furthermore the following fact is easy to prove.

Lemma 1.1. *If every map $F_\alpha : X_\alpha \to Z_\alpha$ is continuous in $(\tau_{X_\alpha}, \tau_{Z_\alpha})$ for $\alpha \in A$, then the induced map $F : X \to Z$ is continuous in $(\tau_{ind}^X, \tau_{ind}^Z)$.*

Let us apply this lemma to the above case of inductive limits of topological groups, by setting

$$(X_\alpha, \tau_{X_\alpha}) = (G_\alpha \times G_\alpha, \tau_{G_\alpha} \times \tau_{G_\alpha}), \quad (Z_\alpha, \tau_{Z_\alpha}) = (G_\alpha, \tau_{G_\alpha}),$$

and $F_\alpha : X_\alpha \to Z_\alpha$ as $F_\alpha(g_\alpha, h_\alpha) = g_\alpha h_\alpha$. Then, since τ_{G_α} is a group topology on G_α, the map F_α is continuous for each $\alpha \in A$, and so, as their natural limit, the multiplication map $F(g,h) = gh$ of $X = G \times G$ to $Z = G$ is continuous, by Lemma 1.1, with respect to the topologies $\tau_{ind}^{G\times G} := \varinjlim (\tau_{G_\alpha} \times \tau_{G_\alpha})$ on $G \times G = X$

and $\tau_{ind}^G := \varinjlim \tau_{G_\alpha}$ on $G = Z$.

1.4. Direct products of inductive limits of topologies.

On the other hand, it is easy to see the following fact for the direct product of inductive limits of topologies. Take two inductive limits of topological spaces $(X, \tau_{ind}^X) = \left(\varinjlim X_\alpha, \varinjlim \tau_{X_\alpha}\right)$ and $(Y, \tau_{ind}^Y) = \left(\varinjlim Y_\alpha, \varinjlim \tau_{Y_\alpha}\right)$, and consider their direct products.

Proposition 1.2. *The product space $X \times Y$ is naturally identified with the inductive limit space $\varinjlim (X_\alpha \times Y_\alpha)$. On this space the direct product of inductive limit topologies $\tau_{ind}^X \times \tau_{ind}^Y = \left(\varinjlim \tau_{X_\alpha}\right) \times \left(\varinjlim \tau_{Y_\alpha}\right)$ is weaker than or equal to the inductive limit of product topologies $\tau_{ind}^{X\times Y} := \varinjlim (\tau_{X_\alpha} \times \tau_{Y_\alpha})$, or in a symbolic notation, $\tau_{ind}^X \times \tau_{ind}^Y \preceq \tau_{ind}^{X\times Y}$. In particular, for a subset of product type $D \times E \subset X \times Y$, it is open in the former topology if and only if so is in the latter.*

For an inductive limit of topological groups $G := \varinjlim G_\alpha$, taking into account the above result, we see from Lemma 1.1 that, in the case where the multiplication $G \times G \ni (g, h) \mapsto gh \in G$ is not continuous with respect to τ_{ind}^G, the product topology $\tau_{ind}^G \times \tau_{ind}^G$ should be strictly weaker than the inductive limit topology $\tau_{ind}^{G\times G} := \varinjlim (\tau_{G_\alpha} \times \tau_{G_\alpha})$. Thus we come naturally to the following problem.

Problem A. *Let the notations be as above. Then, give a necessary and sufficient condition for the equivalence of two topologies $\tau_{ind}^X \times \tau_{ind}^Y$ and $\tau_{ind}^{X\times Y} := \varinjlim (\tau_{X_\alpha} \times \tau_{Y_\alpha})$ on $X \times Y$, where $(X, \tau_{ind}^X) = \left(\varinjlim X_\alpha, \varinjlim \tau_{X_\alpha}\right)$ and $(Y, \tau_{ind}^Y) = \left(\varinjlim Y_\alpha, \varinjlim \tau_{Y_\alpha}\right)$.*

1.5. Examples and further problems.

Let us examine the simple example, Example 1.2 in [TSH], from the stand point of general topology.

Example 1.1. Let $G_n = F^n \times \mathbf{Q}, F = \mathbf{R}$, $\mathbf{Q}$ or $\mathbf{T}$ with the usual non-discrete topology τ_n for $n \in \mathrm{N}$. Then, $G = \varinjlim G_n = (\prod' F) \times \mathbf{Q}$, where $\prod' F$ denotes the restricted direct product of countable number of F's. The multiplication on G is not continuous with respect to $\tau_{ind}^G = \varinjlim \tau_{G_n}$. Hence, $\tau_{ind}^G \times \tau_{ind}^G \prec \tau_{ind}^{G\times G}$.

Furthermore, considering G_n as a topological space and express it as a direct product of two spaces as $X_n \times Y$, with $X_n = F^n, Y = \mathbf{Q}$. Then, $X := \varinjlim X_n = \varinjlim F^n = \prod' F$, and we see that the direct product topology $\tau_{ind}^X \times \tau_Y$ is strictly weaker than $\tau_{ind}^{X\times Y} = \varinjlim (\tau_{X_n} \times \tau_Y)$ at every point of $X \times Y$, by reexamining the proof in Example 1.2 in [TSH] for non-continuity of the multiplication on G.

In the above case, the topological space Y is fixed, and so the following problem is also important to study.

Problem B. *Let* $(X, \tau_{ind}^X) = \left(\varinjlim X_\alpha, \varinjlim \tau_{X_\alpha}\right)$ *be an inductive limit of topological spaces and* (Y, τ_Y) *a fixed topological space. Then, give a necessary and sufficient condition for the equivalence of two topologies* $\tau_{ind}^X \times \tau_Y$ *and* $\tau_{ind}^{X\times Y} := \varinjlim (\tau_{X_\alpha} \times \tau_Y)$ *on* $X \times Y$.

The former Problem A contains this Problem B, but it is worth to study Problem B by itself. We may expect that a solution to Problem B helps to solve Problem A. However the situation is not so simple that Problem A is reduced to Problem B, because, for instance, the topology τ_Y cannot be in general recovered from the system $\tau_{Y_n} = \tau_Y|_{Y_n}$. So we propose the following problem.

Problem C. *Let* (Y, τ_Y) *be a topological space and* $\{(Y_\alpha, \tau_{Y_\alpha}); \alpha \in A\}$ *be an inductive system of topological spaces such that* $Y_\alpha \subset Y$ *and* $Y = \varinjlim Y_\alpha$ *as sets. Assume that the restriction* $\tau_Y|_{Y_\alpha}$ *of the topology* τ_Y *onto* Y_α *is equal to* τ_{Y_α}. *Then,* $\tau_Y \preceq \tau_{ind}^Y := \varinjlim \tau_{Y_\alpha}$. *Look for a necessary and sufficient condition for the equivalence of these two topologies on* Y.

1.6. A characterization of the product topology $\tau_{ind}^X \times \tau_{ind}^Y$.

For the product $X \times Y$ of two inductive limits of topological spaces $(X, \tau_{ind}^X) = \left(\varinjlim X_\alpha, \varinjlim \tau_{X_\alpha}\right)$ and $(Y, \tau_{ind}^Y) = \left(\varinjlim Y_\alpha, \varinjlim \tau_{Y_\alpha}\right)$, we have by Proposition 1.2, the relation $\tau_{ind}^X \times \tau_{ind}^Y \preceq \tau_{ind}^{X\times Y} := \varinjlim (\tau_{X_\alpha} \times \tau_{Y_\alpha})$.

Further we can characterize the product topology as the strongest topology on $X \times Y$ among direct product topologies weaker than $\tau_{ind}^{X\times Y}$. More exactly, we have the following.

Theorem 1.3. *Let* τ'_X *and* τ'_Y *be topologies on* X *and* Y *respectively such that* $\tau'_X \times \tau'_Y \preceq \tau_{ind}^{X\times Y}$. *Then,* $\tau'_X \preceq \tau_{ind}^X$, $\tau'_Y \preceq \tau_{ind}^Y$, *and so* $\tau'_X \times \tau'_Y \preceq \tau_{ind}^X \times \tau_{ind}^Y$.

The above facts evoke studies on inductive limit topologies in various kinds of categories, such as the Bamboo-Shoot topology τ_{BS}^G in the category of topological groups in [TSH] and its generalization, the locally convex vector topology τ_{lcv}^X in the category of locally convex topological vector spaces, and so on.

§2. Inductive limit topologies in various categories

As mentioned in **1.2**, for an inductive limit $G = \varinjlim G_n$ of topological groups

$G_n, n \geq 1$, the multiplication map is not necessarily continuous with respect to the inductive limit topology $\tau_{ind}^G = \varinjlim \tau_{G_n}$. So we have introduced in [TSH] a so-called Bamboo-Shoot topology τ_{BS}^G on G as the strongest group topology $\preceq \tau_{ind}^G$, under the condition (PTA) on the inductive system $\{ G_n \}$.

In these respects, it is also natural to ask the similar question for other topological algebraic objects, such as topological vector spaces (= TVSs), topological semigroups, topological rings, and topological algebras etc.

2.1. Case of locally convex topological vector spaces.

A good category of TVSs is the category of locally convex topological vector spaces (= LCTVSs) over a field $F = \mathbf{R}$ or $\mathbf{C}$. In that category, we know well how to define an inductive limit of topologies.

Let $\{(X_\alpha, \tau_{X_\alpha}); \alpha \in A\}$ be an inductive system of LCTVSs with $\phi_{\alpha_2,\alpha_1} : X_{\alpha_1} \to X_{\alpha_2}$, $\alpha_1, \alpha_2 \in A, \alpha_1 \preceq \alpha_2$, a homomorphism in the category of LCTVSs, that is, a continuous linear map. On the vector space $X = \varinjlim X_\alpha$, we usually consider a locally convex vector topology as follows.

On the limit space $X = \varinjlim X_\alpha$ of an inductive system $\{ X_\alpha \}$ of LCTVSs, a locally convex vector topology, denoted by lcv-$\varinjlim \tau_{X_\alpha}$ or τ_{lcv}^X, is defined as the one for which a fundamental system of neighbourhood of the null element 0 is given as $\{U \subset X$; τ_{ind}^X-open, convex, balanced (i.e., $\lambda x \in U$ for $x \in U, \lambda \in F, |\lambda| \leq 1$), and absorbing $\}$ (cf. [Yo, I.1, Definition 6, p.27]). Further we have also a simple characterization of neighboufoods of $0 \in X$, as is given in [Tr, §13, p.126].

Now we propose the following problem.

Problem D. *Assume that every space X_α in an inductive system of LCTVSs has an additional structure or operation of the same kind, which induces as its inductive limit such a structure or an operation on the limit space $X := \varinjlim X_\alpha$. Is this structure or operation consistent with the lcv-limit topology τ_{lcv}^X ?*

2.2. Multiplication or product in an inductive system.

Let us first consider two concrete cases to show what kind of things we want to study.

Let M be a non-compact differentiable manifold, and $M_n \nearrow M, n \geq 1$, be an increasing sequence of relatively compact, open submanifolds such that the closure $\overline{M_n}$ is contained in M_{n+1}. The space of complex-valued test functions (C^∞-functions with compact supports) on M, denoted by $\mathcal{D}(M)$, is a LCTVS obtaind as an inductive limit of the inductive system $X_n = \mathcal{D}(\overline{M_n}) := \{\varphi \in C^\infty(M); \text{supp}(\varphi) \subset \overline{M_n}\}$, $n \in \mathbf{N}$. Here $\mathcal{D}(\overline{M_n})$ is topologized in a usual manner by means of a countable number of seminorms.

Let us consider two kinds of operations in $X = \mathcal{D}(M)$. First one is the pointwise multiplication $T : X \times X \to X$, given as $T(\varphi_1, \varphi_2)(p) = \varphi_1(p)\,\varphi_2(p)$ ($p \in M$), and the second one is the convolution $T(\varphi_1, \varphi_2) = \varphi_1 * \varphi_2$ in the case of $M = \mathbf{R}^k$. We ask if they are continuous or not in $(\tau^X_{lcv} \times \tau^X_{lcv}, \tau^X_{lcv})$.

Note that, for the first T, $\mathrm{supp}(\varphi_1\varphi_2) \subset \mathrm{supp}(\varphi_1) \cap \mathrm{supp}(\varphi_2)$, and so it maps $X_n \times X_n$ into X_n. On the other hand, for the second T, $\mathrm{supp}(\varphi_1 * \varphi_2)$ becomes bigger and is in general comparable to $\mathrm{supp}(\varphi_1) + \mathrm{supp}(\varphi_2)$, and so T maps $X_n \times X_n$ into $X_{\beta(n)}$ with a $\beta(n) > n$.

Proposition 2.1. *In the space of test functions $X = \mathcal{D}(M)$, the multiplication map $T(\varphi_1, \varphi_2) = \varphi_1\,\varphi_2$ is continuous in $(\tau^X_{lcv} \times \tau^X_{lcv}, \tau^X_{lcv})$.*

Proposition 2.2. *In the space of test functions $X = \mathcal{D}(\mathbf{R}^k)$, the convolution map $T(\varphi, \psi) = \varphi * \psi$ is continuous in $(\tau^X_{lcv} \times \tau^X_{lcv}, \tau^X_{lcv})$.*

In the above two cases, the proofs are not routine as may be expected. Here multiplications T are both commutative, but in our proofs the commutativity is not important but the special structure of the space $\mathcal{D}(M)$ is fully used. So, the proofs can not be generalized directly in the following general situation.

Problem E. *Assume that an inductive system $\{X_\alpha ; \alpha \in A\}$ of LCTVSs has multiplications, consistent in the sense that, for any α, there exists a $\beta(\alpha)$ such that $T_\alpha : X_\alpha \times X_\alpha \to X_{\beta(\alpha)}$ is a continuous bilinear map, and that, for any $\alpha_1, \alpha_2 \in A$, there exists a $\gamma \in A$ such that $\gamma \succeq \alpha_j$, $\beta(\gamma) \succeq \beta(\alpha_j)$, $j = 1, 2$, and T_{α_j}'s are naturally induced from T_γ. Then the system $\{T_\alpha\}$ induces as its inductive limit a multiplication T on $X = \varinjlim X_\alpha$.*

Is the limit map T continuous with respect to $\tau^X_{lcv} = \mathrm{lcv}\text{-}\varinjlim \tau_{X_\alpha}$?

2.3. Multiplication map between two spaces of test functions.

Let M and M' be two differentiable manifolds. We assume that at least one of them, say M', is non-compact.

The space of testing functions $X = \mathcal{D}(M)$ is equipped with a locally convex vector topology τ'_X, where $\tau'_X = \tau_X$ the usual C^∞-topology in the case M is compact, and $\tau'_X = \tau^X_{lcv} := \mathrm{lcv}\text{-}\varinjlim \tau_{X_n}$ with $X_n = \mathcal{D}(M_n)$ as above in the case M is non-compact. The space $Y = \mathcal{D}(M')$ is equipped with the lcv-limit topology $\tau^Y_{lcv} := \mathrm{lcv}\text{-}\varinjlim \tau_{Y_n}$ with $Y_n = \mathcal{D}(M'_n)$, where $\{M'_n ; n = 1, 2, \dots\}$ is a sequence of relatively compact open submanifolds such that $\overline{M'_n} \subset M'_{n+1}$ and $M' = \cup_{n \ge 1} M'_n$. We can give to the product space $X \times Y = \mathcal{D}(M) \times \mathcal{D}(M')$ the lcv-limit topology $\tau^{X\times Y}_{lcv}$ which is equal to $\mathrm{lcv}\text{-}\varinjlim\,(\tau_X \times \tau_{Y_n})$ if M is compact, and to $\mathrm{lcv}\text{-}\varinjlim\,(\tau_{X_n} \times \tau_{Y_n})$ if M is non-compact.

Now put $Z := \mathcal{D}(M \times M')$. Then, we ask if the multiplication (or product) map $T : X \times Y \to Z$, given as $T(\varphi, \psi)(p, p') = \varphi(p) \cdot \psi(p')$, $p \in M, p' \in M'$, for

$\varphi \in X, \psi \in Y$, is continuous with respect to $(\tau'_X \times \tau^Y_{lcv}, \tau^Z_{lcv})$.

Theorem 2.3. *Let M and M' be two differentiable manifolds. Assume that one of them, say M', is non-compact. Then, the multiplication map $T : \mathcal{D}(M) \times \mathcal{D}(M') \ni (\varphi, \psi) \mapsto \varphi \cdot \psi \in \mathcal{D}(M \times M')$ is not continuous in $(\tau'_X \times \tau^Y_{lcv}, \tau^Z_{lcv})$, where $X = \mathcal{D}(M), Y = \mathcal{D}(M'), Z = \mathcal{D}(M \times M')$, and $\tau'_X = \tau_X$ or $\tau'_X = \tau^X_{lcv}$ according as M is compact or not.*

The proof is interesting but we have no space to write it down here.

Taking into account Propositions 2.1, 2.2 and Theorem 2.3, we propose the following problem.

Problem F. *Take three inductive systems of LCTVSs $\{(X_\alpha, \tau_{X_\alpha}); \alpha \in A\}$, $\{(Y_\alpha, \tau_{Y_\alpha}); \alpha \in A\}$, and $\{(Z_\alpha, \tau_{Z_\alpha}); \alpha \in A\}$, and let their inductive limits be (X, τ^X_{lcv}), (Y, τ^Y_{lcv}) and (Z, τ^Z_{lcv}). Assume that, for every $\alpha \in A$, there exists a continuous multiplication (bilinear map) $T_\alpha : X_\alpha \times Y_\alpha \to Z_{\beta(\alpha)}$ with a $\beta(\alpha) \succeq \alpha$, which are consistent with these inductive systems so that there exists a multiplication $T : X \times Y \to Z$ as their inductive limit. Then, under what conditions, T is continuous in $(\tau^X_{lcv} \times \tau^Y_{lcv}, \tau^Z_{lcv})$?*

Remark 2.1. In comparison to the so-called kernel theorem for distributions (cf. [Tr, Th.51.7]), we give a remark. In the situation in Theorem 2.3 with M' non-compact, take a distribution S on $M \times M'$ or $S \in \mathcal{D}'(M \times M')$. Then the bilinear functional $\mathcal{D}(M) \times \mathcal{D}(M') \ni (\varphi, \psi) \mapsto S(T(\varphi, \psi))$ is not necessarily continuous in the product topology, because so is not the bilinear map $T : \mathcal{D}(M) \times \mathcal{D}(M') \to \mathcal{D}(M \times M')$.

2.4. Spaces of finitely many times differentiable functions.

Let r be a non-negative integer and M' is a non-compact $C^{(r)}$-class differentiable manifold. Let us consider the space $Y = C^{(r)}_c(M')$ of $C^{(r)}$-class functions with compact supports. For $r = 0$, Y is nothing but the space of continuous functions with compact supports. Further let $Z = C^{(\infty,r)}_c(M \times M')$ be the space of functions $f(x, y)$ in $(x, y) \in M \times M'$, which is simultaneously of class $C^{(\infty)}$ in $x \in M$ and of class $C^{(r)}$ in $y \in M'$, and compactly supported. We topologize Y and Z respectively as inductive limits of sequences of Banach spaces $Y_n = C^{(r)}(\overline{M'_n})$, and $Z_n = C^{(\infty,r)}(\overline{M_n} \times \overline{M'_n})$.

Theorem 2.4. *Let M be a differentiable manifold and M' be a non-compact $C^{(r)}$-class manifold for some r, $0 \le r < \infty$. Put $X = \mathcal{D}(M), Y = C^{(r)}_c(M')$ and $Z = C^{(\infty,r)}_c(M \times M')$. Then, the multiplication map $T : X \times Y \ni (\varphi, \psi) \mapsto \varphi \cdot \phi \in Z$ is not continuous in $(\tau'_X \times \tau^Y_{lcv}, \tau^Z_{lcv})$, where $\tau'_X = \tau_X$ if M is compact,* *and* $\tau'_X = \tau^X_{lcv}$ *if* M *is non-compact.*

§3. Bamboo-Shoot topology τ^G_{BS} and locally convex topology τ^X_{lcv}

3.1. Bamboo-Shoot topology for PTA-groups.

For an inductive system of topological groups $\{(G_\alpha, \tau_{G_\alpha}); \alpha \in A\}$, assume that the index set A is cofinal to a sub-directed-set isomorphic to $\mathbf{N}$. Then we introduced in [TSH, §2] a condition called (PTA), and under this condition, we defined the so-called Bamboo-Shoot topology τ^G_{BS} on $G = \varinjlim G_\alpha$, and proved that it is the strongest one among group topologies weaker than or equal to the inductive limit topology τ^G_{ind} on G.

3.2. Bamboo-Shoot topology and locally convex topology.

The group topology τ^G_{BS} has an intimate relation to the locally convex vector topology τ^X_{lcv} as in the following problem.

Problem G. *Let* $\{(X_n, \|\cdot\|_n); n \in \mathbf{N}\}$ *be an inductive system of Banach algebras. Then* $X = \varinjlim X_n$ *has naturally a structure of algebra. Take an inductive system of topological subgroups* G_n *of* $(X_n^\times, \tau_{X_n^\times})$ *the group of all invertible elements in* X_n*, with the restriction* $\tau_{X_n^\times}$ *of* $\|\cdot\|_n$*-topology on* $X_n^\times$*. In the case where the condition* (PTA) *holds, what is the relation between the Bamboo-Shoot topology* τ^G_{BS} *on* $G = \varinjlim G_n$ *and the restriction* $\tau^X_{lcv}|_G$ *onto* G *of the locally convex vector topology* τ^X_{lcv} *?*

A. Yamasaki[Ya] and T. Edamatsu[Ed] studied certain special cases of this problem.

Slitely generalizing the situation, we also propose the following proplem.

Problem H. *Assume that every* (X_n, τ_{X_n}) *is locally convex as a TVS. Then, with the locally convex limit topology* τ^X_{lcv}*, does the algebra* X *become a topological algebra ?*

Furthermore, let $G_n := X_n^\times$ *be the set of all invertible elements in* X_n*. Then,* G_n *is a topological group with the relative topology* $\tau_{G_n} := \tau_{X_n}|_{G_n}$*, and they form an inductive system of topological groups. Then, under the condition* (PTA)*, what is the relation between the Bamboo-Shoot topology* τ^G_{BS} *on* G *and the restriction* $\tau^X_{lcv}|_G$ *onto* G *of the locally convex limit topology* τ^X_{lcv} *on* X *?*

We also remark here that studies in different directions on inifinite dimensional Lie groups, containing the theory of their representations, are continued for example in [Boy] and in [NRW].

3.3. Extension of Bamboo-Shoot topologies and their products.

In the category of topological groups, we can extend in an abstract way the notion of Bamboo-Shoot topology on an inductive limit group $G = \varinjlim G_\alpha$ for any (not necessarily countable) inductive system $\{ (G_\alpha, \tau_{G_\alpha}), \alpha \in A; \phi_{\beta,\alpha}, \alpha \preceq \beta \}$.

In fact, we see easily from axioms of neighbouhood system of the unit element for a topological group (e.g., (GT1) $\sim$ (GT5) in [TSH, §1.3]) that there exists, on an inductive limit group $G = \varinjlim G_\alpha$, the strongest group topology under the condition that every canonical homomorphism $\phi_\alpha : G_\alpha \to G$ is continuous. We call it the *extended Bamboo-Shoot topology* and denote it again by τ_{BS}^G.

In the case where the inductive system is countable and the condition (PTA) holds for it, this topology coincides with the Bamboo-Shoot topology τ_{BS}^G constructed explicitly in [TSH].

In the category of topological groups, the problem similar to Problem A is affirmatively solved as follows. Let $\{ (G_\alpha, \tau_{G_\alpha}) ; \alpha \in A \}$ and $\{ (H_\alpha, \tau_{H_\alpha}) ; \alpha \in A \}$ be inductive systems of topological groups. Let $G = \varinjlim G_\alpha$ and $H = \varinjlim H_\alpha$ be their inductive limit groups, and the canonical homomorphisms be $\phi_\alpha : G_\alpha \to G$ and $\psi_\alpha : H_\alpha \to H$.

Then, we have the direct product of inductive systems as $\{ (G_\alpha \times H_\alpha, \tau_{G_\alpha \times H_\alpha}) ; \alpha \in A \}$ with $\tau_{G_\alpha \times H_\alpha} = \tau_{G_\alpha} \times \tau_{H_\alpha}$. Its inductive limit is canonically identified with the direct product $G \times H$.

Theorem 3.1. (i) *Let $G = \varinjlim G_\alpha$, $H = \varinjlim H_\alpha$, and $G \times H = \varinjlim (G_\alpha \times H_\alpha)$ be as above. Then the extended Bamboo-Shoot topologies τ_{BS}^G, τ_{BS}^H, and $\tau_{BS}^{G\times H}$ on G, H, and $G \times H$ respectively satisfy*

$$\tau_{BS}^G \times \tau_{BS}^H \cong \tau_{BS}^{G\times H} \qquad \text{on} \quad G \times H.$$

(ii) *In the case of countable inductive systems, if $\{ (G_n, \tau_{G_n}) ; n \in \mathrm{N} \}$ and $\{ (H_n, \tau_{H_n}) ; n \in \mathrm{N} \}$ satisfy the condition* (PTA), *then so does their direct product $\{ (G_n \times H_n, \tau_{G_n \times H_n}) ; n \in \mathrm{N} \}$.*

3.4. Direct product of locally convex vector topology.

Let $\{(X_\alpha, \tau_{X_\alpha}); \alpha \in A\}$ and $\{(Y_\alpha, \tau_{Y_\alpha}); \alpha \in A\}$ be inductive systems of LCTVSs, and put $X = \varinjlim X_\alpha$, $Y = \varinjlim Y_\alpha$. The direct product of these systems is defined as $\{(X_\alpha \times Y_\alpha, \tau_{X_\alpha \times Y_\alpha}); \alpha \in A\}$ with $\tau_{X_\alpha \times Y_\alpha} := \tau_{X_\alpha} \times \tau_{Y_\alpha}$. Then its inductive limit is isomorphic to the direct product $X \times Y$ as vector spaces. For topologies on this space, we already know that $\tau_{lcv}^X \times \tau_{lcv}^Y \preceq \tau_{lcv}^{X\times Y} := \text{lcv-}\varinjlim \tau_{X_\alpha \times Y_\alpha}$.

On the other hand, we can translate the proof of Theorem 3.1 appropriately in the category of LCTVSs, and see that the condition (DPA) holds in general for the 'lcv-limit functor' $\tau_{lcv}^{\{*\}}$ as follows.

Theorem 3.2. *Let $X = \varinjlim X_\alpha, Y = \varinjlim Y_\alpha$ be inductive limits in the category of LCTVSs. The direct product space $X \times Y$ is identified with the inductive limit of the direct product of inductive systems. Then, as locally convex vector topologies on $X \times Y$, there holds the equivalence*

$$\tau_{lcv}^X \times \tau_{lcv}^Y \cong \tau_{lcv}^{X\times Y} := \text{lcv-}\varinjlim \tau_{X_\alpha \times Y_\alpha}.$$

§4. Sufficient conditions for Problem A

For sufficient conditions for Problem A or B, the local compactness and the local sequential compactness play important roles. Here we study them for Problem A.

4.1. A sufficient condition for $\tau_{ind}^X \times \tau_{ind}^Y \cong \tau_{ind}^{X\times Y}$.

As in **1.4**, take two inductive systems of topological spaces and put $X = \varinjlim X_\alpha, Y = \varinjlim Y_\alpha$. First let us give a simple sufficient condition for the 'commutativity' of (1) taking inductive limits and (2) taking direct products, for inductive limits of topologies, that is, the condition (DPA) for $\tau_{ind}^{\{*\}}$.

Theorem 4.1. *Assume that A has a cofinal sub-directed-set isomrphic to* $\mathbf{N}$. *For two inductive systems of topological spaces, assume that every X_α and Y_α are locally compact Hausdorff spaces. Then, as topologies on $X \times Y$ with $X = \varinjlim X_\alpha, Y = \varinjlim Y_\alpha$, identified with $\varinjlim\,(X_\alpha \times Y_\alpha)$, the product topology $\tau_{ind}^X \times \tau_{ind}^Y$ and the inductive limit topology $\tau_{ind}^{X\times Y} := \varinjlim\,(\tau_{X_\alpha} \times \tau_{Y_\alpha})$ are mutually equivalent: $\tau_{ind}^X \times \tau_{ind}^Y \cong \tau_{ind}^{X\times Y}$, that is, the condition* (DPA) *holds.*

4.2. Other sufficient conditions.

We give other sufficient conditions assuming on X_n and Y_n a stronger condition (SC) than the local sequential compactness.

Definition 4.1. For a subset D of a topological space Z, its *sequential closure*, denoted by $\mathrm{scl}(D)$, is defined as

$\mathrm{scl}(D) := \{\, z \in Z\,;\, \exists z_n \in D \text{ such that } \lim_{n\to\infty} z_n = z \,\}$,

and D is called *sequentially compact* if every sequence in it has a subsequence converging to a point in D, and further Z is called *locally sequentially compact* if every point in it has an open neighbourhood U for which $\mathrm{scl}(U)$ is sequentially compact.

Our condition (SC) on Z is defined as follows.

(SC) *For every sequentially compact subset K and an open set O containing it, there exists an open set G such that $K \subset G \subset \mathrm{scl}(G) \subset O$ and that $\mathrm{scl}(G)$ is sequentially compact.*

Under this condition (SC), we can give two kinds of sufficient conditions for Problem A as follows. For an inductive system, assume that $A = \mathrm{N}$, and that $X_1 \subset \cdots \subset X_n \subset X_{n+1} \subset \cdots \subset X$ canonically by the identification through the canonical maps ϕ_n.

Theorem 4.2. *Let $A = \mathrm{N}$ for an inductive system of topological spaces, and assume that every (X_n, τ_{X_n}) and (Y_n, τ_{Y_n}) satisfies the condition* (SC). *Then, in the case where they all satisfy the first countability axiom, the condition* (DPA) *holds, i.e., for $X = \varinjlim X_n$ and $Y = \varinjlim Y_n$, there holds the equivalence $\tau_{ind}^X \times \tau_{ind}^Y \cong \tau_{ind}^{X \times Y} := \varinjlim (\tau_{X_n} \times \tau_{Y_n})$ on $X \times Y$.*

Theorem 4.3. *Let $A = \mathrm{N}$, and assume the condition* (SC) *for every (X_n, τ_{X_n}) and (Y_n, τ_{Y_n}). Then, in the case where the system satisfies $\tau_{X_{n+1}}|_{X_n} = \tau_{X_n}$, $\tau_{Y_{n+1}}|_{Y_n} = \tau_{Y_n}$ for $n \geq 1$, and the condition*

(Gδ) $\quad$ *X_n is a G_δ-set of X_{n+1}, and Y_n is a G_δ-set of Y_{n+1}, for $n \geq 1$,*

there holds for $X \times Y$ the equivalence $\tau_{ind}^X \times \tau_{ind}^Y \cong \tau_{ind}^{X \times Y} := \varinjlim (\tau_{X_n} \times \tau_{Y_n})$.

§5. The case of a fixed Y and Problem B

In the following, we study in detail Problems A and B, especially necessary conditions for converses of theorems in §4. In this section, we study the case where Y is fixed, or the case where $(Y_n, \tau_{Y_n}) = (Y, \tau_Y)$ for any $n \geq 1$. This is our Problem B.

5.1. Comments to converses of Theorems 4.1, 4.2 and 4.3.

Statements for direct converses of these theorems contain necessarily a global characterization such as "*X_n is a locally compact space*". However, this kind of global characterization of spaces X_n and Y_n are not possible in its nature of inductive sequences of topological spaces, and so, possible converses should be at first stated in languages of local characterizations of these spaces. This can be seen from the following examples.

Example 5.1. Let $X = \mathbf{R}$ and $X_n = (-n, n) \cup \mathbf{Q}$ with an open interval $(-n, n)$, where X is equipped with a usual topology $\tau_{\mathbf{R}}$ of $\mathbf{R}$, and X_n with its relative topology $\tau_{X_n} = \tau_{\mathbf{R}}|_{X_n}$. Then, no X_n is locally compact, whereas so is the inductive limit space X (cf. Theorems 5.2 and 5.3). Note that the space

$(\mathbf{Q}, \tau_{\mathbf{Q}} = \tau_{\mathbf{R}}|_{\mathbf{Q}})$ is totally disconnected and normal.

Example 5.2. Let $Y = \prod'_{k\geq 1} \mathbf{R}_k$ with $\mathbf{R}_k = \mathbf{R}$ be the restricted direct product of $\mathbf{R}$. Put $Y_n = \prod_{k=1}^{n} \mathbf{R}_k = \mathbf{R}^n$, $Y'_n = \left(\prod_{k=1}^{n-1} \mathbf{R}_k\right) \times \mathbf{Q} \subset Y_n$, and imbed Y_n into Y_{n+1} as $Y_n \ni y \mapsto (y, 0) \in Y_{n+1}$. The space Y_n is equipped with the usual Euclidean metric, and the space Y'_n with its relative topology. Then, Y_n is locally compact, whereas no point of Y'_n has a compact neighbourhood. However the topological space Y considered as the inductive limit of (Y_n, τ_{Y_n}), $n \geq 1$, is also equal to the inductive limit of $(Y'_n, \tau_{Y'_n})$, $n \geq 1$, since there is a mixed inductive system given by $Y''_{2n+1} := Y_n$, $Y''_{2n} := Y'_n$, $(n \geq 1)$, which converges to (Y, τ^Y_{ind}).

Now let $\{ X_n ; n \in \mathrm{N}\}$ be an inductive system of separable locally compact spaces and put $X = \varinjlim X_n$. Consider two inductive systems of direct product type as $\{ X_n \times Y_m ; (n, m) \in \mathrm{N} \times \mathrm{N}\}$, and $\{ X_n \times Y'_m ; (n, m) \in \mathrm{N} \times \mathrm{N}\}$, where $(n, m) \preceq (n', m')$ in $\mathrm{N} \times \mathrm{N}$ if and only if $n \leq n'$, $m \leq m'$. Then we get as their inductive limits the same space $X \times Y$. Denote by $\tau^{X\times Y}_{ind,1}$ and $\tau^{X\times Y}_{ind,2}$ the inductive limit topologies on $X \times Y$ corresponding to the first and the second system respectively. We assert that $\tau^{X\times Y}_{ind,1} \cong \tau^{X\times Y}_{ind,2} \cong \tau^X_{ind} \times \tau^Y_{ind}$.

In fact, the first equivalence is affirmed by considering a mixed inductive system (Z_n, τ_{Z_n}), $n > 1$, with $(Z_{2n+1}, \tau_{Z_{2n+1}}) := (X_n \times Y_n, \tau_{X_n} \times \tau_{Y_n})$, $(Z_{2n}, \tau_{Z_{2n}}) := (X_n \times Y'_n, \tau_{X_n} \times \tau_{Y'_n})$. Another equivalence $\tau^{X\times Y}_{ind,1} \cong \tau^X_{ind} \times \tau^Y_{ind}$ is guaranteed by Theorem 4.1 thanks to the local compactness of X_n's and Y_n's.

Furthermore, in the case the index m is fixed, as for the topologies on $\lim_{n\to\infty} (X_n \times Y_m) = X \times Y_m$ and on $\lim_{n\to\infty} (X_n \times Y'_m) = X \times Y'_m$, we get the equivalence $\tau^X_{ind} \times \tau_{Y_m} = \tau^{X\times Y_m}_{ind}$ by Theorem 4.1, but the inequivalence $\tau^X_{ind} \times \tau_{Y'_m} \prec \tau^{X\times Y'_m}_{ind}$ by Theorem 5.2 below.

5.2. A sufficient condition for $\tau^X_{ind} \times \tau_Y \cong \tau^{X\times Y}_{ind}$.

Let us now begin to treat Problem B. Fix a topological space (Y, τ_Y). Put $Z_n = X_n \times Y$, $\tau_{Z_n} = \tau_{X_n} \times \tau_Y$, and $Z = \varinjlim Z_n, \tau^Z_{ind} = \varinjlim \tau_{Z_n}$. We identify Z with $X \times Y$ and τ^Z_{ind} with $\tau^{X\times Y}_{ind}$. We know in general $\tau^X_{ind} \times \tau_Y \preceq \tau^{X\times Y}_{ind}$, and the problem is to guarantee the converse relation. A simple sufficient condition is given as follows.

Proposition 5.1. *Assume for the inductive system $\{ (X_n, \tau_{X_n}) \}$ that X_n is imbedded homeomprphically into X_{n+1} for $n \geq 1$, and for the counter part (Y, τ_Y) that Y is locally compact Hausdorff. Then there holds the equivalence $\tau^X_{ind} \times \tau_Y \cong \tau^{X\times Y}_{ind}$.*

5.3. Normalization of situations.

To simplify the situations we put some natural assumptions from the beginning.

First we assume for simplicity that the index set A contains a cofinal subset isomorphic to N as directed set, and so we take $A = \mathrm{N}$ later on except when the contrary is announced. It may be assumed without essential loss of generality that

(00-X) *each canonical map* $\phi_{n+1,n} : X_n \to X_{n+1}$ $(n \geq 1)$ *is injective,*

and so considering as $X_n \subset X_{n+1}$ and $X = \bigcup_{n \geq 1} X_n$, we can omit the notations $\phi_{m,n}$ and ϕ_n rather freely, and then,

(01-X) *each* $\phi_{n+1,n}$ *is a homeomorphism, or* $\tau_{X_{n+1}}|_{X_n} \cong \tau_{X_n}$.

For (01-X), we remark that the topologies τ_{X_n} can be replaced by $\tau_{ind}^X|_{X_n}$ to get the same inductive limit topology τ_{ind}^X, and then (01-X) holds for new topologies on X_n's. From now on, we assume (00-X) and (01-X) for $\{ X_n \}$.

Taking an appropriate cofinal sequence if necessary, we may put the following assumption for $\{ X_n \}$ from the beginning:

(1-X) *for any* n, X_n *as a subset of* X_{n+1} *has no* $\tau_{X_{n+1}}$*-inner point of* X_{n+1}.

5.4. Necessary conditions for $\tau_{ind}^X \times \tau_Y \cong \tau_{ind}^{X \times Y}$.

We follow the discussion of A. Yamasaki in [Ya] to get the following necessary condition.

Theorem 5.2. *Let* $A = \mathrm{N}$ *and* Y *be fixed. Assume the condition* (1-X) *and the following:*

(2-x_0) *for* $n \gg 1$, $x_0 \in X_n$ *has a countable fundamental system of* τ_{X_n}*-neighbouhoods;*

(3-y_0) $y_0 \in Y$ *has a countable fundamental system of neighbourhoods consisting of closed ones;*

(4-y_0) $y_0 \in Y$ *does not have a sequentially compact neighbourhood.*

Then, $\tau_{ind}^X \times \tau_Y \prec \tau_{ind}^{X \times Y} := \varinjlim (\tau_{X_n} \times \tau_Y)$ *at* $(x_0, y_0) \in X \times Y$.

Reformulating the above result in a global form, we get a kind of converse, in the case of a fixed Y, of affirmative assertions in theorems in §4 as follows.

Theorem 5.3. *Assume* (1-X) *and the following:*

(2-X) *each* (X_n, τ_{X_n}) *satisfies the first countability axiom;*

(3-Y) Y *is regular and satisfies the first countability axiom.*

Then, $\tau_{ind}^X \times \tau_Y \prec \tau_{ind}^{X \times Y}$ *at any point* $(x, y) \in X \times Y$ *for which* $y \in Y$ *has no sequentially compact neighbourhood.*

§6. Necessary conditions for $\tau_{ind}^X \times \tau_{ind}^Y \cong \tau_{ind}^{X\times Y}$ and Problem A

Let $A = \mathbf{N}$. Let us consider two inductive systems $\{X_n\}$ and $\{Y_n\}$, and put $Z_n = X_n \times Y_n$ and identify $Z = \varinjlim Z_n$ with $X \times Y$, then $\tau_{ind}^Z = \tau_{ind}^{X\times Y}$. Assume (00-X) and (01-X) for $\{X_n\}$ and similarly (00-Y) and (01-Y) for $\{Y_n\}$, for simplicity.

6.1. Conditions for $\tau_{ind}^X \times \tau_{ind}^Y \prec \tau_{ind}^{X\times Y}$ at a point.

We study when the above two inductive limit topologies on $Z = X \times Y$ are different from each other at a point $z_0 = (x_0, y_0) \in Z$.

Theorem 6.1. *Assume the following:*

(1-X) *X_n has no $\tau_{X_{n+1}}$-inner point of X_{n+1} for $n \geq 1$;*

(2-X) *X_n satisfies the first countability axiom for $n \geq 1$;*

(3-Y_{n_0}) *Y_{n_0} is regular and satisfies the first countability axiom;*

(4-Y_{n_0}-y_0) *$y_0 \in Y_{n_0}$ has no sequentially compact neighbourhood;*

(5-Y_{n_0}) *Y_{n_0} is τ_{Y_n}-closed in Y_n for all $n > n_0$.*

Then, $\tau_{ind}^X \times \tau_{ind}^Y \prec \tau_{ind}^{X\times Y}$ at $(x_0, y_0) \in X \times Y$ for any $x_0 \in X_{n_0}$.

Reformulating the above result in a global form, we get a converse of Theorem 4.1 as follows.

Theorem 6.2. *Assume* (1-X) *and* (2-X)*and further assume the following:*

(3′-Y) *each (Y_n, τ_{Y_n}) is regular and satisfies the first contability axiom;*

(5′-Y) *Y_n is closed in $(Y_{n+1}, \tau_{Y_{n+1}})$, for $n \geq 1$.*

Then, if $y_0 \in Y$ has no sequentially compact neighbourhood in any (Y_n, τ_{Y_n}), there holds $\tau_{ind}^X \times \tau_{ind}^Y \prec \tau_{ind}^Z$ at $(x_0, y_0) \in Z$ for any $x_0 \in X$.

To get much faithful converses to Theorems 4.1, 4.2 and 4.3, we should get rid of the first countability axiom.

Theorem 6.3. *Let X_n and Y_n be all regular Hausdorff spaces satisfying the first countability axiom. Assume the conditions* (1-X) *and* (5′-X) *for $\{X_n\}$ and similarly* (1-Y) *and* (5′-Y) *for $\{Y_n\}$. Then $\tau_{ind}^X \times \tau_{ind}^Y \cong \tau_{ind}^{X\times Y}$ if and only if X_n and Y_n are all locally sequentially-compact.*

6.2. Case of metrizable spaces.

In the case of metrizable spaces, they are automatically regular and satisfy the first countability axiom, and furthermore sequential compactness is equivalent to compactness. Therefore, in that case, we get from Theorems 4.1 and 6.2 the following simple necessary and sufficient condition for the commutativity of "inductive limit" and "direct product": $\tau_{ind}^X \times \tau_{ind}^Y \cong \tau_{ind}^{X\times Y} := \varinjlim (\tau_{X_n} \times \tau_{Y_n})$.

Theorem 6.4. *Assume the conditions* (00-X), (01-X), (1-X) *and* (5′-X) *for* $\{X_n\}$, *and similarly* (00-Y), (01-Y), (1-Y) *and* (5′-Y) *for* $\{Y_n\}$. *Let* X_n *and* Y_n *be all metrizable spaces. Then,* $\tau_{ind}^X \times \tau_{ind}^Y \cong \tau_{ind}^{X\times Y}$ *if and only if* X_n *and* Y_n *are locally compact.*

References

[Boy] R.P. Boyer, Representation theory of infinite dimensional unitary groups, Contemporary Math., **145**(1993).

[Ed] T. Edamatsu, On the bamboo-shoot topology of certain inductive limits of topological groups, to appear in J. Math. Kyoto Univ.

[Enc] "Encyclopedic Dictionary of Mathematics", Second Edition, MIT, 1987.

[Hi] T. Hirai, Group topologies and unitary representations of the group of diffeomorphisms, in *Analysis on inifinite-dimensional Lie groups and algebras*, International Colloquium Marseille 1997, pp.145-153, World Scientific.

[HSTH] T. Hirai, H. Shimomura, N. Tatsuuma and E. Hirai, Inductive limits of topologies, their direct products, and problems related to algebraic structures, Preprint Kyoto-Math 2000-01, Kyoto University, January, 2000.

[NRW] L. Natarajan, E. Rodoríguez-Carrington, and J.A. Wolf, New classes of infinite-dimensional Lie groups, Proc. of Symposia in Pure Math., **56**(1994), Part 2, 377-392.

[TSH] N. Tatsuuma, H. Shimomura and T. Hirai, On group topologies and unitary representations of inductive limits of topological groups and the case of the group of diffeomorphisms, J. Math. Kyoto Univ., **38** (1998), 551-578.

[Tr] F. Treves, Topological vector spaces, distributions and kernels, Academic Press, 1967.

[Ya] A. Yamasaki, Inductive limit of general linear groups, J. Math. Kyoto Univ., **38**(1998), 769-779.

[Yo] K. Yosida, Functional analysis, 6th edition, Springer-Verlag, 1995.

Department of Mathematics, Kyoto University

Department of Mathematics, Fukui University

Hakutaka Sake Brewing Company and Kôyô Gakuin

Department of Mathematics, Kyoto Sangyô University

Scaling Limit of the Spectral Distributions of the Laplacians on Large Graphs

Akihito HORA

Okayama University (Okayama, JAPAN)

Abstract

We examine several scaling limits of the spectral distributions of Laplacians (or equivalently adjacency operators) on regular graphs and their second quantization on Fock spaces as the graphs grow infinitely in certain manners.

1 Introduction

The present note reports our recent development in asymptotic spectral theory for Laplacians on certain graphs. Main references are [10], [11] and [12], while the material in §5 first appears in published form in this note.

Let us begin with an abstract setting. A regular graph $\Gamma = (V, E)$, V and E being its vertex set and edge set respectively, has by definition the same degree at every vertex x : $\kappa := |\{y \in V | x \sim y\}|$. Here $x \sim y$ denotes that x and y are adjacent vertices. The Laplacian operator Δ on Γ acts on $f : V \longrightarrow \mathbf{C}$ as

$$(\Delta f)(x) := \sum_{y \sim x} f(y) - \kappa f(x) \ ,$$

which is a formal expression when Γ is an infinite graph.

Taking a state ϕ on the algebra generated by Δ and I (the identity), one considers the spectral distribution of Δ for which the distribution function is determined by values of ϕ at the projectors in the spectral decomposition of Δ. In this note, we will deal with vacuum states and analogs of Gibbs states. We are interested in asymptotic behaviour of the spectral distribution along a growing family of graphs, especially in the case where $\kappa \to \infty$. We try to read a statistical property of the spectral distribution through a scaling limit. The scaling agrees with that of the central limit theorem (CLT, for short). Actually, our problem is closely related to the CLT in algebraic probability theory which was initiated by von Waldenfels et al. (e.g. [7], [15]).

It is convenient to refer to Cayley graphs to see the way CLT comes out. Let G be a group generated by $\Omega = \{\omega_1, \cdots, \omega_\kappa\} \not\ni e$, assuming that $\Omega^{-1} = \Omega$ as a set. Two vertices $x, y \in G$ are defined to be adjacent if $yx^{-1} \in \Omega$. The Laplacian on this Cayley graph is expressed as

$$\Delta = \sum_{j=1}^{\kappa} \pi_L(\omega_j) - \kappa I \tag{1}$$

where π_L denotes the left regular representation of G. Let us take vacuum state $\phi := \langle \delta_e, \cdot \delta_e \rangle_{\ell_2(G)}$. According to the formulation of CLT, our problem is to discuss weak convergence of the spectral distribution of

$$\frac{\Delta - \phi(\Delta)}{\sqrt{\phi((\Delta - \phi(\Delta))^2)}} = \frac{1}{\sqrt{\kappa}} \sum_{j=1}^{\kappa} \pi_L(\omega_j) \tag{2}$$

with respect to ϕ as G grows in a certain manner with $\kappa \to \infty$. Noncommuting summands $\pi_L(\omega_j)$ have a sort of (in)dependence reflecting the structure of G. It may reveal a new convolution structure of the limit distribution, yielding Gauss and Wigner as the extremal ones (see [8], [5]). Furthermore, replacing π_L and ϕ by other representations and states will be also interesting.

2 Preliminaries

2.1 Symmetric group and Young diagram

Let $\mathcal{S}_n$ denote the symmetric group of degree n and $\mathcal{S}_\infty := \bigcup_{n=1}^{\infty} \mathcal{S}_n$ their inductive limit. We follow the convention that a Young diagram is expressed as a finite array of left-aligned nonincreasing rows. Let $\mathcal{Y}$ denote the set of Young diagrams and $\mathcal{D}$ the subset of $\mathcal{Y}$ whose element has no rows consisting of a single box. If $\lambda \in \mathcal{Y}$ contains $k^{(j)}$ rows of length j, we use the notation $\lambda = (1^{k^{(1)}} 2^{k^{(2)}} \cdots j^{k^{(j)}} \cdots)$. The number of boxes contained in λ is $|\lambda| := \sum_j j k^{(j)}$. The conjugacy classes in $\mathcal{S}_\infty$ except the trivial one $\{e\}$ are parametrized by the diagrams in $\mathcal{D}$. Let C_λ be the conjugacy class in $\mathcal{S}_\infty$ corresponding to $\lambda \in \mathcal{D}$ and set $C_\lambda^{(n)} := \mathcal{S}_n \cap C_\lambda$ for $n \geq |\lambda|$. $C_\lambda^{(n)}$ is also a conjugacy class in $\mathcal{S}_n$. One sees

$$|C_\lambda^{(n)}| = n^{\underline{|\lambda|}} / \prod_{j \geq 2} j^{k^{(j)}} k^{(j)}!$$

for $\lambda = (2^{k^{(2)}} 3^{k^{(3)}} \cdots)$ with $n^{\underline{r}} := n(n-1) \cdots (n-r+1)$. π_L denoting the left regular representation of $\mathcal{S}_\infty$, we set

$$A_\lambda^{(n)} := \sum_{x \in C_\lambda^{(n)}} \pi_L(x) \quad \text{and formally} \quad A_\lambda := \sum_{x \in C_\lambda} \pi_L(x) \tag{3}$$

for $\lambda \in \mathcal{D}$. The representation matrix of $A_\lambda^{(n)}|_{\ell^2(\mathcal{S}_n)}$ with respect to the basis $\{\delta_x | x \in \mathcal{S}_n\}$ is an adjacency matrix of the group association scheme of $\mathcal{S}_n$. The complex linear hull of these adjacency matrices is closed under multiplication and hence becomes an algebra. (See [1].) We call A_λ also an adjacency operator on $\mathcal{S}_\infty$.

Regarding $\mathcal{Y}$ as a vertex set and joining two Young diagrams if one diagram is made by adding a box to the other, one obtains the Young graph (or Young lattice). Later in §5, we will mention the Young graph equipped with multiplicity (or colour) on each edge.

2.2 Distance-regular graph

Let S be a v-set (i.e. $|S| = v$) and set $V := \{x \subset S | |x| = d\}$ as a vertex set. (Assume $2d \leq v$ without loss of generality.) $x, y \in V$ are defined to be adjacent if $|x \cap y| = d - 1$. Obviously, $|V| = \binom{v}{d}$ and $\kappa = d(v - d)$ (degree). This graph $J(v, d)$ is called a Johnson graph. The Laplacian on $J(v, d)$ describes the classical Bernoulli-Laplace model imitating a kind of diffusion of sparse gases.

We give a quick review on distance-regular graphs (DRG, for short), among which $J(v, d)$ plays a central role in this note. See [1] for details. Let $\Gamma = (V, E)$ be a finite connected graph. $\partial(x, y)$ denotes the distance (i.e. minimal length) between $x, y \in V$ and $\mathrm{diam}\Gamma := \max_{x,y \in V} \partial(x, y)$ the diameter of Γ. Γ is called a DRG with diameter d if, for $\forall h, i, j \in \{0, 1, \cdots, d\}$, $|\{z \in V | \partial(x, z) = i, \partial(z, y) = j\}| =: p_{ij}^h$ does not depend on the choice of x, y whenever $\partial(x, y) = h$. In particular, $p_{11}^0 = \kappa$ (degree of Γ). Set $\kappa_i := p_{ii}^0$. The ith adjacency operator A_i ($i = 0, 1, \cdots, d$) is defined as

$$(A_i f)(x) := \sum_{\partial(x,y)=i} f(y) \qquad \text{for} \quad f : V \longrightarrow \mathbf{C} \,.$$

In particular, $A_0 = I$, $A_1 = A$ (adjacency operator) and $\Delta = A - \kappa I$. The condition of distance-regularity is translated into a linearizing formula for adjacency operators :

$$A_i A_j = \sum_{h=0}^{d} p_{ij}^h A_h \,.$$

The commutative algebra $\mathcal{A}(\Gamma)$ generated by A and I is called the adjacency algebra of Γ. Clearly, $\{A_0, A_1, \cdots, A_d\}$ is a linear basis of $\mathcal{A}(\Gamma)$. Then one sees that $\mathrm{diam}\Gamma + 1 = \dim\mathcal{A}(\Gamma) =$ the number of distinct eigenvalues of A. (For a general graph, the former '$=$' should be replaced by '$\leq$'. A DRG has high symmetry and its eigenvalues are thus degenerated.) Letting θ_0 $(= \kappa) > \theta_1 > \cdots > \theta_d$ be distinct eigenvalues of A and E_j the orthogonal projector on $\ell^2(V)$ corresponding to θ_j, one has

$$A = \sum_{j=0}^{d} \theta_j E_j \,, \qquad A_i = \sum_{j=0}^{d} v_i(\theta_j) E_j \quad (i = 0, 1, \cdots, d) \,.$$

Here v_i is shown to be a polynomial of degree i such that $v_i(A) = A_i$. $\{E_0, E_1, \cdots, E_d\}$ also forms a linear basis of $\mathcal{A}(\Gamma)$.

3 Central Limit Theorem for Adjacency Operators on $\mathcal{S}_\infty$

It is quite interesting to seek out statistical properties of large symmetric groups as is seen in [13], [2], [3] etc. In this section, we report the main result in [10] which extends the result in [13]. We follow the notations in §§2.1.

Let $\phi := \langle \delta_e, \cdot \delta_e \rangle_{\ell^2(\mathcal{S}_\infty)}$ be the vacuum state. For each $\lambda \in \mathcal{D}$, one sees

$$\phi(A_\lambda^{(n)}) = 0 \ , \quad \phi(A_\lambda^{(n)2}) = |C_\lambda^{(n)}|$$

as the mean and the variance of $A_\lambda^{(n)}$ with respect to ϕ respectively. Hence we consider an asymptotic spectral behaviour of $A_\lambda^{(n)}/\sqrt{|C_\lambda^{(n)}|}$ as $n \to \infty$ from the viewpoint of CLT. Let $H_r(x)$ denote the Hermite polynomial of degree r obeying the recurrence formula :

$$H_{r+1}(x) = xH_r(x) - rH_{r-1}(x) \ , \quad H_0(x) = 1 \ , \quad H_1(x) = x \ .$$

Theorem 1 ([10]) *For all $\lambda_1, \cdots, \lambda_m \in \mathcal{D}$ and for all $p_1, \cdots, p_m \in \mathbf{N}$, we have*

$$\lim_{n\to\infty} \phi\Big(\Big(\frac{A_{\lambda_1}^{(n)}}{\sqrt{|C_{\lambda_1}^{(n)}|}}\Big)^{p_1} \cdots \Big(\frac{A_{\lambda_m}^{(n)}}{\sqrt{|C_{\lambda_m}^{(n)}|}}\Big)^{p_m}\Big)$$
$$= \prod_{j\geq 2} \int_{-\infty}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}} \Big(\frac{H_{k_1^{(j)}}(x)}{\sqrt{k_1^{(j)}!}}\Big)^{p_1} \cdots \Big(\frac{H_{k_m^{(j)}}(x)}{\sqrt{k_m^{(j)}!}}\Big)^{p_m} dx \ , \tag{4}$$

where $\lambda_i = (2^{k_i^{(2)}} 3^{k_i^{(3)}} \cdots j^{k_i^{(j)}} \cdots)$ $(i = 1, \cdots, m)$.

From (4) we can read how adjacency operators $A_{\lambda_1}, \cdots, A_{\lambda_m}$ are correlated with respect to ϕ. The strucure of the right hand side of (4) tells that rows of different length among $\lambda_1, \cdots, \lambda_m$ essentially play independent roles while there remain some interfering effects among rows of the same length and in different diagrams. In our computation, this asymptotic independence along length j is attributed to disjoint union structure of a certain graph. The left hand side of (4) can be expressed in terms of the irreducible characters of $\mathcal{S}_n$ and the Plancherel measure on $\hat{\mathcal{S}}_n$. Under this formulation, Kerov showed in [13] the corresponding result to (4) for one-row Young diagrams.

4 Central Limit Theorems on Distance-Regular Graphs

Since the Laplacian Δ on a DRG does not yield such a canonical decomposition as (1) or (3), the original feature of CLT which describes a macroscopic effect of sums of small 'independent' fluctuations through appropriate scaling may seem to go somewhat backward. However it has a good meaning to consider

$$(\Delta - \Phi(\Delta))/\sqrt{\Phi((\Delta - \Phi(\Delta))^2)} \tag{5}$$

with respect to some state Φ on adjacency algebra $\mathcal{A}(\Gamma)$ in the situation that DRG Γ grows in some manner. Then the (in)dependence of summands should be transformed into topological structure of the graph. In this section, we survey our results concerning the Johnson graph as examples of such CLT on a DRG as (5). We follow the notations in §§2.2.

4.1 Vacuum state

For DRG Γ, we define vacuum state Φ_0 on $\mathcal{A}(\Gamma)$ as

$$\begin{aligned} \Phi_0(X) &:= \frac{1}{|V|}\,\mathrm{tr}X \qquad (X \in \mathcal{A}(\Gamma)) \\ &= \langle \delta_x, X\delta_x \rangle_{\ell^2(V)} \qquad (X \in \mathcal{A}(\Gamma)) \qquad \text{for all } x \in V\,. \end{aligned}$$

Theorem 2 ([11]) *Let $\Gamma = J(2d,d)$ (Johnson graph) and $\Phi = \Phi_0$ (vacuum state) in (5). Then the spectral distribution of (5) with respect to Φ_0 converges weakly to*

$$e^{-(\xi+1)} I_{[-1,\infty)}(\xi)d\xi$$

as $d \to \infty$. Here $I_.$ denotes an indicator function.

4.2 Gibbs state

We announce the main result in [12]. For DRG Γ with diameter d, we define linear functional Φ_q on $\mathcal{A}(\Gamma)$ by

$$\Phi_q(A_h) := \kappa_h q^h \qquad (h = 0, 1, \cdots, d)$$

where q is a parameter. It is shown that, for $\Gamma = J(v,d)$ and $0 \le q \le 1$, Φ_q is actually a state (namely, enjoys positivity) on $\mathcal{A}(J(v,d))$. Φ_q is regarded as analogue of the Gibbs state with inverse temperature parameter $\beta = -\log q$ ($q = 0 \iff$ vacuum state Φ_0).

Theorem 3 ([12]) *Let $\Gamma = J(2d,d)$ and $\Phi = \Phi_q$ in (5) where $0 \le q \le 1$. Then the spectral distribution of (5) with respect to Φ_q converges weakly to the following as $d \to \infty$:*

(Case 1) if $q = r/d^\alpha$ where $r \geq 0$ and $\alpha > 1$ are constants,

$$e^{-(\xi+1)} I_{[-1,\infty)}(\xi) d\xi \; ; \tag{6}$$

(Case 2) if $q = r/d$ where $r \geq 0$ is a constant,

$$\sqrt{2r+1} e^{-(\xi\sqrt{2r+1}+2r+1)} J_0(i2\sqrt{r(\xi\sqrt{2r+1}+r+1)}) I_{[-(r+1)/\sqrt{2r+1},\infty)}(\xi) d\xi \; . \tag{7}$$

Here

$$J_0(z) := \sum_{k=0}^{\infty} \frac{(-z^2/4)^k}{(k!)^2} \qquad (z \in \mathbf{C})$$

is the 0th Bessel function.

In both cases, $d \to \infty$ and $q \to 0$ hence "temperature of the graph" tends to 0.

Remark (communicated to the author by P.Biane) Checking the characteristic function of (7), one sees that (7) is expressed as

$$\delta_{-(r+1)/\sqrt{2r+1}} * \mu_r * \nu_r \quad \text{where} \quad \mu_r(d\xi) := \sqrt{2r+1} e^{-\xi\sqrt{2r+1}} I_{[0,\infty)} d\xi$$

and ν_r is the infinitely divisible distribution whose characteristic function is given by

$$\exp \int_0^\infty (e^{is\xi} - 1) r\sqrt{2r+1} e^{-\xi\sqrt{2r+1}} d\xi \; .$$

Note that

$$\delta_{-(r+1)/\sqrt{2r+1}} * \mu_r \longrightarrow (6) \quad \text{and} \quad \nu_r \longrightarrow \delta_0 \quad \text{as} \quad r \to 0 \; .$$

5 Second Quantization and Central Limit Theorem

In this section, we give some observations on CLT for the second quantizations of discrete Laplacians.

5.1 Second quantization

Let $\mathcal{F}(\mathcal{H})$ be the Boson Fock space over Hilbert space $\mathcal{H}$:

$$\mathcal{F}(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \mathcal{H}^{\circ n} \; , \qquad \mathcal{H}^{\circ 0} := \mathbf{C}\mathbf{1}$$

where $\circ$ denotes the symmetric tensor product and $\mathbf{1}$ the vacuum vector. The creator $a^*(\xi)$ and annihilator $a(\xi)$ on $\mathcal{F}(\mathcal{H})$ are defined by

$$a^*(\xi)\xi_1 \circ \cdots \circ \xi_n := \sqrt{n+1}\,\xi \circ \xi_1 \circ \cdots \circ \xi_n \; , \qquad a^*(\xi)\mathbf{1} := \xi$$

$$a(\xi)\xi_1 \circ \cdots \circ \xi_n := \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \langle \xi, \xi_j \rangle_{\mathcal{H}} \xi_1 \circ \cdots \circ \check{\xi_j} \circ \cdots \circ \xi_n \; , \quad a(\xi)\mathbf{1} := 0$$

$(\xi, \xi_1, \cdots, \xi_n \in \mathcal{H})$. Here $\check{}$ indicates the conventional notation for removal of a component. The exponential vector defined as

$$e(\xi) := \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \xi^{\circ n} \quad \text{satisfies} \quad \langle e(\xi), e(\eta) \rangle_{\mathcal{F}(\mathcal{H})} = e^{\langle \xi, \eta \rangle_{\mathcal{H}}}$$

$(\xi, \eta \in \mathcal{H})$. The (differential) second quantization of operator A on $\mathcal{H}$ is

$$d\Gamma(A) := \sum_{n=1}^{\infty} \sum_{j=1}^{n} I \otimes \cdots \otimes I \otimes A \otimes I \otimes \cdots \otimes I \ ,$$

where A sits on the jth component in the product of the right hand side.

Let us work on Cayley graph (G, Ω), i.e. Ω is a generator set of group G such that $\Omega^{-1} = \Omega \not\ni e$. Assume that Ω is an infinite set. For each $n \in \mathbb{N}$, take finite subset Ω_n of Ω such that $\Omega_n^{-1} = \Omega_n$ and $\Omega_n \nearrow \Omega$ (as a set) as $n \to \infty$. (Recall the discussion in §3 on the conjugacy classes in $\mathcal{S}_\infty$.) We consider adjacency operators on $\ell^2(G)$:

$$A := \sum_{\omega \in \Omega} \pi_L(\omega) \quad \text{(formally)} \quad \text{and} \quad A_n := \sum_{\omega \in \Omega_n} \pi_L(\omega) \ . \tag{8}$$

The second quantizations of them on $\mathcal{F}(\ell^2(G))$ are expressed in terms of creators and annihilators as

$$d\Gamma(A) = \sum_{\omega \in \Omega} \sum_{x \in G} a^*_{\omega x} a_x \quad \text{(formally) and} \quad d\Gamma(A_n) = \sum_{\omega \in \Omega_n} \sum_{x \in G} a^*_{\omega x} a_x \ ,$$

where we set $a_x := a(\delta_x)$ and $a^*_x := a^*(\delta_x)$ $(x \in G)$ for simplicity. These operators describe the (nearest neighbour) random walk on G from the viewpoint of quantum fields. Setting

$$\Phi := \langle e^{-1/2} e(\delta_e), \cdot \, e^{-1/2} e(\delta_e) \rangle_{\mathcal{F}(\mathcal{H})} \qquad \text{coherent state}$$

(sorry for confusing usage of several 'e's), we have

$$\Phi(d\Gamma(A_n)) = 0 \quad \text{and} \quad \Phi(d\Gamma(A_n)^2) = |\Omega_n| \ .$$

Hence our problem of CLT is to discuss weak convergence of the spectral distribution of the operator:

$$d\Gamma(A_n / \sqrt{|\Omega_n|}) = \frac{1}{\sqrt{|\Omega_n|}} \sum_{\omega \in \Omega_n} \sum_{x \in G} a^*_{\omega x} a_x$$

with respect to Φ as $n \to \infty$. This can be solved by relating the moments of an operator on $\mathcal{H}$ to those of its second quantization.

5.2 Moments with respect to coherent state

In general, let $\mathcal{H}$ be a Hilbert space, $\xi \in \mathcal{H}$ a unit vector, and A a self-adjoint operator on $\mathcal{H}$. Set

$$\phi := \langle \xi, \cdot \xi \rangle_{\mathcal{H}} \quad \text{and} \quad \Phi := \langle e^{-1/2} e(\xi), \cdot e^{-1/2} e(\xi) \rangle_{\mathcal{F}(\mathcal{H})} . \tag{9}$$

The relation between the moments of A and $d\Gamma(A)$ are as follows.

Proposition 1 *Set $m_r := \phi(A^r)$ and $M_r := \Phi(d\Gamma(A)^r)$ for $r \in \mathbf{N}$. Then we have*

$$M_r = \sum_{|\lambda|=r, \lambda \in \mathcal{Y}} d(\lambda) m_1^{k(1)} m_2^{k(2)} \cdots m_r^{k(r)} \tag{10}$$

where $\lambda = (1^{k(1)} 2^{k(2)} \cdots r^{k(r)})$ in each term and

$$d(\lambda) := \frac{r!}{1!^{k(1)} 2!^{k(2)} \cdots r!^{k(r)} k(1)! k(2)! \cdots k(r)!} . \tag{11}$$

(10) is the same relation as that between moments of a probability measure and its cumulants. Note that one has

$$\Phi(e^{-itd\Gamma(A)}) = \exp\{\phi(e^{-itA}) - 1\} \qquad (\forall t \in \mathbf{R}) .$$

Combined with the following elementary formula, this yields Proposition 1.

Lemma 1

$$\frac{d^r}{dt^r} e^{f(t)} = e^{f(t)} \sum_{|\lambda|=r, \lambda \in \mathcal{Y}} d(\lambda) f'(t)^{k(1)} f''(t)^{k(2)} \cdots f^{(r)}(t)^{k(r)}$$

where $\lambda = (1^{k(1)} 2^{k(2)} \cdots r^{k(r)})$ and $d(\lambda)$ is given by (11).

Lemma 1 is easily shown by induction on r.

Coming back to Cayley graph (G, Ω), we set $\xi = \delta_e$ in (9):

$$\phi = \langle \delta_e, \cdot \delta_e \rangle_{\ell^2(G)} , \quad \Phi = \langle e^{-1/2} e(\delta_e), \cdot e^{-1/2} e(\delta_e) \rangle_{\mathcal{F}(\ell^2(G))} ,$$

and consider A_n in (8). The limits of moments of $A_n / \sqrt{|\Omega_n|}$ with respect to ϕ are, if they exist, majorized by the Gaussian ones, i.e.

$$\lim_{n \to \infty} \phi((A_n / \sqrt{|\Omega_n|})^{2p}) \leq \frac{(2p)!}{2^p p!} \qquad (\forall p \in \mathbf{N})$$

(see [8]) where the right hand side is the $2p$th moment of the standard normal distribution. Applying Proposition 1 to the Gaussian case, in which $m_{2p} = (2p)!/(2^p p!)$ and the odd moments vanish, we have

$$M_{2p} = \frac{(2p)!}{2^p p!} B(p) \tag{12}$$

by using the pth Bell number $B(p)$ i.e. the number of classification of p objects. Taking into account the asymptotic of $B(p)$ as $p \to \infty$, we can majorize (12) and hence limiting moments of $d\Gamma(A_n)/\sqrt{|\Omega_n|}$ with respect to Φ.

Proposition 2 *If for* $\forall r \in \mathrm{N}$

$$\lim_{n\to\infty} \phi\Big(\Big(\frac{A_n}{\sqrt{|\Omega_n|}}\Big)^r\Big) =: m_r \quad \textit{exists, then} \quad \lim_{n\to\infty} \Phi\Big(\Big(\frac{d\Gamma(A_n)}{\sqrt{|\Omega_n|}}\Big)^r\Big) =: M_r$$

also exists for all $r \in \mathrm{N}$ *and satisfies*

$$\lim_{p\to\infty} M_{2p}^{1/2p}/\,2p < \infty\,. \tag{13}$$

(13) is a modification of Carleman's condition. It ensures the unique existence of a probability whose rth moment is M_r (see e.g. [6]).

5.3 Branching, q-deformation

We end the section with two remarks.

Let the Young graph be equipped with multiplicity function $\kappa(\lambda, \mu)$ on each edge with $\lambda, \mu \in \mathcal{Y}$ such that $|\mu| = |\lambda| + 1$. Then the Young graph is simply called a branching. We refer to [14] for terminology and examples of branchings. $\lambda_0 \in \mathcal{Y}$ denotes the diagram consisting of a single box. To each path $u = (\lambda_0, \lambda_1, \cdots, \lambda_n)$, in which $|\lambda_{i+1}| = |\lambda_i| + 1$, going from λ_0 to $\lambda = \lambda_n$, one assigns the weight $w_u := \prod_{i=0}^{n-1} \kappa(\lambda_i, \lambda_{i+1})$. Then

$$d(\lambda) := \sum_{u=(\lambda_0,\cdots,\lambda_n),\lambda_n=\lambda} w_u \tag{14}$$

is called the combinatorial dimension function on the branching. If the multiplicity function is trivial i.e. $\kappa(\lambda, \mu) \equiv 1$, $d(\lambda)$ agrees with the number of standard tableaux in λ and hence with the dimension of the irreducible representation of $\mathcal{S}_{|\lambda|}$ associated with λ. We see that $d(\lambda)$ in (11) is the combinatorial dimension function on the branching determined by the following multiplicity function. Let $\lambda, \mu \in \mathcal{Y}$ such that $|\mu| = |\lambda| + 1$.
(i) If μ is made by adding a box to a row (say, of length j) in λ and λ contains r rows of length j, then set $\kappa(\lambda, \mu) := r$.
(ii) If μ is made by adding a box to λ as the new bottom row, then set $\kappa(\lambda, \mu) := 1$.
This observation helps recurrent computation of $d(\lambda)$ in (11).

A parallel discussion to the preceding subsections can proceed if one considers the second quantization on a q-Fock space ($0 < q < 1$). See e.g. [4] for the structure of the inner product, the creators and the annihilators on a q-Fock space. An exponential vector and a coherent state in (9) are naturally q-deformed. Then it is shown that Proposition 1 and the branching in the last paragraph yield their 'q-analogue'. Namely, the combinatorial dimension function $d(\lambda)$ is given by (14), but the rule assigning the multiplicity function $\kappa(\lambda, \mu)$ should be slightly modified depending on q.

References

[1] Bannai,E., Ito,T.: *Algebraic combinatorics I, association schemes.* Menlo Park, California: Benjamin / Cummings, 1984

[2] Biane,P.: Permutation model for semi-circular systems and quantum random walks. *Pacific J. Math.* **171**, 373 - 387 (1995)

[3] Biane,P.: Representations of symmetric groups and free probability. *Adv. Math.* **138**, 126 - 181 (1998)

[4] Bożejko,M., Kümmerer,B., Speicher,R.: q-Gaussian processes: non-commutative and classical aspects. *Commun. Math. Phys.* **185**, 129 - 154 (1997)

[5] Bożejko,M., Wysoczański,J.: New examples of convolutions and non-commutative central limit theorems. *Quantum Probability (Gdańsk 1997)*, Banach Center Publ. **43**, 95 - 103, Polish Acad. Sci., Warsaw, 1998

[6] Durrett,R.: *Probability: theory and examples.* Duxbury Press, 1991

[7] Giri,N., von Waldenfels,W.: An algebraic version of the central limit theorem. *Z. Wahr. verw. Geb.* **42**, 129 - 134 (1978)

[8] Hashimoto,Y.: A combinatorial approach to limit distributions of random walks on discrete groups. Preprint 1996

[9] Hashimoto,Y.: Deformations of the semi-circle law derived from random walks on free groups. *Prob. Math. Stat.* **18**, 399 - 410 (1998)

[10] Hora,A.: Central limit theorem for the adjacency operators on the infinite symmetric group. *Commun. Math. Phys.* **195**, 405 - 416 (1998)

[11] Hora,A.: Central limit theorems and asymptotic spectral analysis on large graphs. *Inf. Dim. Anal., Quant. Probab., Rel. Topics* **1**, No.2, 221 - 246 (1998)

[12] Hora,A.: Gibbs state on a distance-regular graph and its application to a scaling limit of the spectral distributions of discrete Laplacians. To appear

[13] Kerov,S.V.: Gaussian limit for the Plancherel measure of the symmetric group. *C. R. Acad. Sci. Paris* **316**, 303 – 308 (1993)

[14] Kerov,S.V.: The boundary of Young lattice and random Young tableaux. *DIMACS Series in Discrete Mathematics and Theoretical Computer Science* **24**, 133 – 158 (1996)

[15] von Waldenfels,W.: An algebraic central limit theorem in the anticommuting case. *Z. Wahr. verw. Geb.* **42**, 135 – 140 (1978)

Initial Value Problem For White Noise Operators And Quantum Stochastic Processes

UN CIG JI
DEPARTMENT OF MATHEMATICS
SOGANG UNIVERSITY
SEOUL, 121-742 KOREA
AND
NOBUAKI OBATA
GRADUATE SCHOOL OF MATHEMATICS
NAGOYA UNIVERSITY
NAGOYA, 464-8602 JAPAN

Introduction

Over the past few years the interest has been increasing in white noise approach to both classical and quantum stochastic differential equations. It is the fundamental idea of white noise theory, or also called Hida calculus [7], [8], that randomness is reduced to its elemental components represented by deterministic vectors in an infinite dimensional space and that stochastic analysis is translated into an infinite dimensional calculus. This approach has been discussed along with classical stochastic calculus, see e.g., [9], [10], [14], and references cited therein, and has created a completely new idea of nonlinear extension of stochastic calculus via quantum domain [1], [2]. Namely, by means of white noise theory a traditional quantum stochastic differential equation introduced by Hudson and Parthasarathy [11] is brought into a normal-ordered white noise differential equation:

$$\frac{d\Xi}{dt} = L_t \diamond \Xi, \qquad \Xi|_{t=0} = I,$$

where $\{L_t\}$ is a quantum stochastic process involving lower powers (at most one) of quantum white noises. This observation led us naturally to construct a general scheme of normal-ordered white noise differential equations. In fact, in the series of papers [3], [4], [19], [20], we have established unique existence of a solution in the space of white noise operators and a method of examining its regularity properties in terms of weighted Fock spaces. However, the results were obtained only for linear equations as above though such equations are already far beyond the traditional Itô theory in the sense that the coefficients $\{L_t\}$ may involve very singular noises such as higher powers or higher order derivatives of quantum white noises.

This paper aims at a small step towards a systematic study of nonlinear white noise differential equations. We shall focus on an initial value problem of the form:

$$\frac{d\Xi}{dt} = F(t, \Xi), \qquad \Xi|_{t=0} = \Xi_0, \qquad 0 \le t \le T.$$

For technical reason it seems reasonable to start with the case that $F : [0,T] \times \mathcal{L}(\mathcal{W}, \mathcal{W}^*) \to \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ is a continuous function, where $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ stands for the space of white noise operators. A difficulty is caused by the fact that $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ is not a Banach space but a nuclear Fréchet space. For example, it seems very hard to obtain efficient norm estimates for a formal solution constructed by successive approximations. We shall surmount this obstacle by exploiting symbol calculus, which is a peculiar tool in white noise theory with a useful theorem of characterization [16], see also [2]. The main result is stated in Theorem 10 in Section 5.

1 White Noise Distributions

As usual, let us start with the Gaussian space (E^*, μ), that is, $E^* = \mathcal{S}'(\mathbf{R})$ is the space of tempered distributions and μ is the Gaussian measure on E^* defined by

$$e^{-|\xi|_0^2/2} = \int_{E^*} e^{i\langle x, \xi\rangle} \mu(dx), \qquad \xi \in E,$$

where $|\xi|_0$ stands for the norm of $\xi \in H = L^2(\mathbf{R})$ and $\langle \cdot, \cdot\rangle$ for the canonical bilinear form on $E^* \times E = \mathcal{S}'(\mathbf{R}) \times \mathcal{S}(\mathbf{R})$. The probability space (E^*, μ) is called the *Gaussian space* and plays a key role in white noise theory. For example, a Gaussian random variable

$$B_t(x) = \langle x, 1_{[0,t]}\rangle, \qquad x \in E^*, \quad t \geq 0, \tag{1}$$

is defined in the sense of $L^2(E^*, \mu)$ and $\{B_t\}$ becomes a realization of a Brownian motion. However, the time derivative of the Brownian motion, called the *white noise*, is not well-defined in $L^2(E^*, \mu)$. In fact, we obtain from (1) a rather formal representation:

$$W_t(x) = \langle x, \delta_t\rangle, \qquad x \in E^*, \quad t \geq 0.$$

The above ill-definedness will be easily conquered by introducing a particular Gelfand triple:

$$\mathcal{W} \subset L^2(E^*, \mu) \subset \mathcal{W}^*, \tag{2}$$

where the white noise process becomes a smooth map $t \mapsto W_t \in \mathcal{W}^*$, and moreover, nonlinear functions of $\{W_t\}$ are managed in $\mathcal{W}^*$.

As for the construction of (2), we adopt a general framework due to Cochran, Kuo and Sengupta [5]. We first take a sequence of positive numbers $\alpha = \{\alpha(n)\}_{n=0}^{\infty}$ satisfying the following five conditions:

(A1) $\alpha(0) = 1 \leq \alpha(1) \leq \alpha(2) \leq \cdots$;

(A2) the generating function $G_\alpha(t) = \sum_{n=0}^{\infty} \frac{\alpha(n)}{n!} t^n$ has an infinite radius of convergence;

(A3) the power series $\widetilde{G}_\alpha(t) = \sum_{n=0}^{\infty} \frac{n^{2n}}{n!\alpha(n)} \left\{ \inf_{s>0} \frac{G_\alpha(s)}{s^n} \right\} t^n$ has a positive radius of convergence;

(A4) there exists a constant $C_{1\alpha} > 0$ such that $\alpha(n)\alpha(m) \leq C_{1\alpha}^{n+m}\alpha(n+m)$ for any n, m;

(A5) there exists a constant $C_{2\alpha} > 0$ such that $\alpha(n+m) \leq C_{2\alpha}^{n+m}\alpha(n)\alpha(m)$ for any n, m.

Given such a positive sequence, we define a weighted Fock space:

$$\Gamma_\alpha(E_p) = \left\{ \phi \sim (f_n)_{n=0}^{\infty} ;\ f_n \in E_p^{\widehat{\otimes} n},\ \| \phi \|_{p,+}^2 \equiv \sum_{n=0}^{\infty} n!\, \alpha(n) \, | f_n |_p^2 < \infty \right\}, \tag{3}$$

where

$$E_p = \left\{ \xi \in H \,;\, | \xi |_p \equiv | A^p \xi |_0 < \infty \right\}, \qquad A = 1 + t^2 - \frac{d^2}{dt^2}.$$

We then define

$$\Gamma_\alpha(E) = \operatorname{proj\,lim}_{p\to\infty} \Gamma_\alpha(E_p), \tag{4}$$

which bears a resemblance to $\mathcal{S}(\mathbf{R}) \equiv E = \operatorname{proj\,lim}_{p\to\infty} E_p$. The constant numbers

$$\| A^{-1} \|_{\mathrm{OP}} = \frac{1}{2}, \qquad \| A^{-q} \|_{\mathrm{HS}}^2 = \sum_{j=0}^{\infty} \frac{1}{(2j+2)^{2q}}, \quad q > \frac{1}{2},$$

with the simple inequality:

$$| \xi |_p \le \| A^{-1} \|_{\mathrm{OP}}^q \, | \xi |_{p+q}, \qquad \xi \in E, \quad p \in \mathbf{R}, \quad q \ge 0, \tag{5}$$

will be used in various norm estimates below.

We denote by $\mathcal{W}_\alpha$ the complexification of $\Gamma_\alpha(E)$ defined in (4). It is easily proved that $\mathcal{W}_\alpha$ is a nuclear space whose topology is given by the family of norms $\{\| \cdot \|_{p,+} \,;\, p \in \mathbf{R}\}$ defined in (3). Taking the celebrated Wiener–Itô–Segal isomorphism $L^2(E^*, \mu) \cong \Gamma(H_{\mathbf{C}})$ into account, where $\Gamma(H_{\mathbf{C}})$ is the usual Fock space, i.e., the weighted Fock space with weight one, we obtain a Gelfand triple:

$$\mathcal{W}_\alpha \subset \Gamma(H_{\mathbf{C}}) \cong L^2(E^*, \mu) \subset \mathcal{W}_\alpha^*. \tag{6}$$

This is called the *Cochran-Kuo-Sengupta space* (or *CKS-space* shortly) associated with α. If there is no danger of confusion, we simply set $\mathcal{W} = \mathcal{W}_\alpha$. The canonical bilinear form on $\mathcal{W}^* \times \mathcal{W}$ is denoted by $\langle\langle \cdot, \cdot \rangle\rangle$. Then

$$\langle\langle \Phi, \phi \rangle\rangle = \sum_{n=0}^{\infty} n! \, \langle F_n, f_n \rangle, \qquad \Phi \sim (F_n) \in \mathcal{W}^*, \quad \phi \sim (f_n) \in \mathcal{W},$$

and it holds that

$$|\langle\langle \Phi, \phi \rangle\rangle| \le \| \Phi \|_{-p,-} \, \| \phi \|_{p,+}, \qquad \| \Phi \|_{-p,-}^2 = \sum_{n=0}^{\infty} \frac{n!}{\alpha(n)} \, | F_n |_{-p}^2 .$$

As is easily verified, the Brownian motion $t \mapsto B_t$ is differentiable in $\mathcal{W}^*$ and the white noise process $t \mapsto W_t \in \mathcal{W}^*$ is defined.

Here we mention some special cases. The *Hida-Kubo-Takenaka space* [13] is the CKS-space with $\alpha(n) \equiv 1$ and is denoted by $\mathcal{W} = (E)$. The *Kondratiev-Streit space* [12] is also the CKS-space with $\alpha(n) = (n!)^\beta$, $0 \le \beta < 1$, and is denoted by $\mathcal{W} = (E)_\beta$. Another interesting example is given by the k-th order Bell numbers $\{B_k(n)\}$ defined by

$$G_{\mathrm{Bell}(k)}(t) = \frac{\overbrace{\exp(\exp(\cdots(\exp}^{k-\text{times}} t)\cdots))}{\exp(\exp(\cdots(\exp 0)\cdots))} = \sum_{n=0}^{\infty} \frac{B_k(n)}{n!} t^n. \qquad (7)$$

We record some properties of the generating function $G_\alpha(t)$ defined in (A2), whose proofs are straightforward.

Lemma 1 *Let $\alpha = \{\alpha(n)\}$ be a positive sequence satisfying (A1) and (A2), and $G_\alpha(t)$ the generating function defined therein. Then,*

(1) $G_\alpha(0) = 1$ *and* $G_\alpha(s) \le G_\alpha(t)$ *for* $0 \le s \le t$;

(2) $e^s G_\alpha(t) \le G_\alpha(s+t)$ *and* $e^t \le G_\alpha(t)$ *for* $s, t \ge 0$;

(3) $c[G_\alpha(t) - 1] \le G_\alpha(ct) - 1$ *for any* $c \ge 1$ *and* $t \ge 0$.

Lemma 2 *Let $\alpha = \{\alpha(n)\}$ be a positive sequence and $G_\alpha(t)$ the generating function defined therein. If α satisfies conditions (A1), (A2) and (A4), then*

$$G_\alpha(s) G_\alpha(t) \le G_\alpha(C_{1\alpha}(s+t)), \qquad s, t \ge 0.$$

If conditions (A1), (A2) and (A5) are fulfilled, then

$$G_\alpha(s+t) \le G_\alpha(C_{2\alpha}s) G_\alpha(C_{2\alpha}t), \qquad s, t \ge 0.$$

2 White Noise Operators

A continuous linear operator from $\mathcal{W}$ into $\mathcal{W}^*$ is called a *white noise operator.* The space of such operators is denoted by $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ and is equipped with the topology of uniform convergence on every bounded subset. In other words, the topology of $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ is defined by the seminorms:

$$\| \Xi \|_{B,B'} = \sup \{ |\langle\langle \Xi\phi, \psi \rangle\rangle| \,;\, \phi \in B,\, \psi \in B' \},$$

where B, B' run over all bounded subsets of $\mathcal{W}$. Similarly, the topology of $\mathcal{L}(\mathcal{W}, \mathcal{W})$ is defined by

$$\| \Xi \|_{B,p} = \sup \{ \| \Xi\phi \|_p \,;\, \phi \in B \},$$

where B runs over all bounded subsets of $\mathcal{W}$ and $p \ge 0$. Note that the canonical inclusion $\mathcal{L}(\mathcal{W}, \mathcal{W}) \to \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ is continuous.

A useful tool for analyzing white noise operators is the operator symbol. With each $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ we associate a $\mathbf{C}$-valued function on $E_{\mathbf{C}} \times E_{\mathbf{C}}$ defined by

$$\widehat{\Xi}(\xi, \eta) = \langle\langle \Xi\phi_\xi, \phi_\eta \rangle\rangle, \qquad \xi, \eta \in E_{\mathbf{C}},$$

where ϕ_ξ is the exponential vector defined by

$$\phi_\xi(x) = e^{\langle x, \xi \rangle - \langle \xi, \xi \rangle/2} \quad \sim \quad \left(1, \xi, \frac{\xi^{\otimes 2}}{2!}, \cdots, \frac{\xi^{\otimes n}}{n!}, \cdots\right).$$

The above $\widehat{\Xi}$ is called the *symbol* of Ξ. Every operator in $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ is uniquely specified by its symbols since the exponential vectors $\{\phi_\xi\,;\, \xi \in E_{\mathbf{C}}\}$ span a dense subspace of $\mathcal{W} = \mathcal{W}_\alpha$ for any α. The following analytic characterization theorem for operator symbol is a peculiar consequence of white noise theory.

Theorem 3 [2] *A function $\Theta : E_{\mathbf{C}} \times E_{\mathbf{C}} \to \mathbf{C}$ is the symbol of a white noise operator $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$, i.e., $\Theta = \widehat{\Xi}$, if and only if the following two conditions are satisfied:*

(O1) *for any $\xi, \xi_1, \eta, \eta_1 \in E_{\mathbf{C}}$ the function $(z, w) \mapsto \Theta(z\xi + \xi_1, w\eta + \eta_1)$ is entire holomorphic on $\mathbf{C} \times \mathbf{C}$;*

(O2) *there exist constant numbers $C \geq 0$ and $p \geq 0$ such that*

$$|\Theta(\xi, \eta)|^2 \leq C G_\alpha(|\,\xi\,|_p^2) G_\alpha(|\,\eta\,|_p^2), \qquad \xi, \eta \in E_{\mathbf{C}}.$$

In that case

$$\|\,\Xi\phi\,\|_{-(p+q),-}^2 \leq C\widetilde{G}_\alpha^2(\|\,A^{-q}\,\|_{\mathrm{HS}}^2)\,\|\,\phi\,\|_{p+q,+}^2, \qquad \phi \in \mathcal{W},$$

where $q > 1/2$ is taken in such a way that $\widetilde{G}_\alpha(\|\,A^{-q}\,\|_{\mathrm{HS}}^2) < \infty$.

Among white noise operators the most fundamental are the annihilation and creation operators at a point $t \in \mathbf{R}$. Let us now recall the definitions. For any $\phi \in \mathcal{W}$ the limit

$$a_t\phi(x) = \lim_{\theta \to 0} \frac{\phi(x + \theta\delta_t) - \phi(x)}{\theta}, \qquad x \in E^*, \quad t \in \mathbf{R},$$

always exists and a_t becomes a continuous operator from $\mathcal{W}$ into itself, i.e., $a_t \in \mathcal{L}(\mathcal{W}, \mathcal{W})$. Hence by duality $a_t^* \in \mathcal{L}(\mathcal{W}^*, \mathcal{W}^*)$. These operators a_t and a_t^* are called the *annihilation operator* and the *creation operator* at a time point t, respectively.

3 Stochastic Processes as Continuous Flows

Following [17] we introduce some notions. A continuous map $t \mapsto \Phi_t \in \mathcal{W}^*$ defined on an interval is reasonably called a *classical stochastic process* (in the sense of white noise theory). Basic examples are the Brownian motion $\{B_t\}$ and the white noise process $\{W_t\}$. Similarly, a continuous map $t \mapsto \Xi_t \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ defined on an interval is called a *quantum stochastic process* (in the sense of white noise theory). The annihilation operators $\{a_t\}$ and the creation operators $\{a_t^*\}$ form quantum stochastic processes. In some literature the pair $\{a_t, a_t^*\}$ is called the *quantum white noise process.* Moreover, we have

Proposition 4 *Both maps $t \mapsto a_t \in \mathcal{L}(\mathcal{W}, \mathcal{W})$ and $t \mapsto a_t^* \in \mathcal{L}(\mathcal{W}^*, \mathcal{W}^*)$ are infinitely many times differentiable.*

The proof is easy with the help of the norm estimates of derivatives of the delta function, see [18, Appendix]. We next mention a criterion of the continuity of $t \mapsto \Xi_t$ in terms of operator symbols. The proof is a straightforward modification of the argument for the Kondratiev–Streit space [20, Theorem 1.8].

Lemma 5 *Let T be a locally compact space. Then a function $t \mapsto \Xi_t \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$, $t \in T$, is continuous if and only if for any $t_0 \in T$ there exist $K \geq 0$, $p \geq 0$ and an open neighborhood U_0 of t_0 such that*

$$|\widehat{\Xi}_t(\xi, \eta)|^2 \leq K G_\alpha(|\xi|_p^2) G_\alpha(|\eta|_p^2), \qquad \xi, \eta \in E_{\mathbf{C}}, \quad t \in U_0,$$

and

$$\lim_{t \to t_0} \widehat{\Xi}_t(\xi, \eta) = \widehat{\Xi}_{t_0}(\xi, \eta), \qquad \xi, \eta \in E_{\mathbf{C}}.$$

Although an immediate consequence from the above, the next result is also useful.

Lemma 6 *Let $\Xi_n, \Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$, $n = 1, 2, \cdots$. Then Ξ_n converges to Ξ in $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ if and only if there exist $K \geq 0$, $p \geq 0$ such that*

$$|\widehat{\Xi}_n(\xi, \eta)|^2 \leq K G_\alpha(|\xi|_p^2) G_\alpha(|\eta|_p^2), \qquad \xi, \eta \in E_{\mathbf{C}}, \quad n = 1, 2, \cdots,$$

and

$$\lim_{n \to \infty} \widehat{\Xi}_n(\xi, \eta) = \widehat{\Xi}(\xi, \eta), \qquad \xi, \eta \in E_{\mathbf{C}}.$$

We are now in a position to clarify the classical-quantum correspondence in white noise theory. It can be verified that the pointwise multiplication in $\mathcal{W}$ gives rise to a continuous bilinear map: $\mathcal{W} \times \mathcal{W} \to \mathcal{W}$. Hence by duality, for $\Phi \in \mathcal{W}^*$ and $\phi \in \mathcal{W}$ there exists a unique element denoted by $\Phi\phi \in \mathcal{W}^*$ such that

$$\langle\langle \Phi, \phi\psi \rangle\rangle = \langle\langle \Phi\phi, \psi \rangle\rangle, \qquad \psi \in \mathcal{W}.$$

Moreover, the map $\widetilde{\Phi} : \phi \mapsto \Phi\phi$ becomes a white noise operator, i.e., $\widetilde{\Phi} \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$. Thus, every $\Phi \in \mathcal{W}^*$ gives rise to a white noise operator by multiplication and we obtain a continuous inclusion $\mathcal{W}^* \hookrightarrow \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$. In this sense, every classical stochastic process $\{\Phi_t\}$ is identified with a quantum stochastic process. Conversely, given a quantum stochastic process $\{\Xi_t\}$ and a white noise function $\phi \in \mathcal{W}$, $\{\Phi_t = \Xi_t\phi\}$ becomes a classical stochastic process. In particular, a classical stochastic process $\{\Phi_t\}$ is recovered from the corresponding quantum stochastic process $\{\widetilde{\Phi}_t\}$ as $\Phi_t = \widetilde{\Phi}_t\phi_0$, where ϕ_0 is the vacuum vector. We often identify $\widetilde{\Phi}_t$ with Φ_t and denote them by the common symbol for simplicity.

4 Integration of Quantum Stochastic Processes

Let $L^1_{\mathrm{loc}}(\mathbf{R})$ be the space of all $\mathbf{C}$-valued locally integrable functions on $\mathbf{R}$. We begin with the following

Lemma 7 *Let $\{L_t\}$ be a quantum stochastic process defined on an interval $I \subset \mathbf{R}$. Then for any $a, t \in I$ and $f \in L^1_{\mathrm{loc}}(\mathbf{R})$ there exists a unique operator $\Xi_{a,t}(f) \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ such that*

$$\langle\langle \Xi_{a,t}(f)\phi, \psi \rangle\rangle = \int_a^t f(s) \langle\langle L_s\phi, \psi \rangle\rangle ds, \qquad \phi, \psi \in \mathcal{W}. \tag{8}$$

Moreover, $t \mapsto \Xi_{a,t}(f) \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ is continuous.

PROOF. Let $[a,b] \subset I$ be a closed finite interval. Since $s \mapsto L_s$ is continuous, the interval $[a,b]$ is mapped to a compact subset $K \subset \mathcal{L}(\mathcal{W},\mathcal{W}^*) \cong (\mathcal{W}\otimes\mathcal{W})^*$. Hence there exists some $p \geq 0$ such that

$$C \equiv \sup_{a\leq s\leq b} \| L_s \|_{-p} < \infty.$$

Then for any $s \in [a,b]$ we have

$$|\langle\langle L_s\phi,\ \psi\rangle\rangle| = |\langle\langle L_s,\ \phi\otimes\psi\rangle\rangle| \leq \| L_s \|_{-p} \| \phi\otimes\psi \|_p \leq C \| \phi \|_p \| \psi \|_p ,$$

and

$$\left| \int_a^t f(s)\langle\langle L_s\phi,\ \psi\rangle\rangle ds \right| \leq C \| \phi \|_p \| \psi \|_p \int_a^t |f(s)| ds, \qquad \phi,\psi \in \mathcal{W}, \quad a \leq t \leq b.$$

Namely, the right hand side of (8) is a continuous bilinear form on $\mathcal{W}$ and, therefore, a white noise operator $\Xi_{a,t}(f) \in \mathcal{L}(\mathcal{W},\mathcal{W}^*)$ is specified as in (8). Moreover, we obtain

$$|\langle\langle(\Xi_{a,t}(f) - \Xi_{a,s}(f))\phi,\ \psi\rangle\rangle| \leq C \| \phi \|_p \| \psi \|_p \int_s^t |f(u)|\, du, \quad \phi,\psi \in \mathcal{W}, \quad a \leq s < t \leq b.$$

Then for bounded subsets $B_1, B_2 \subset \mathcal{W}$ we have

$$\| \Xi_{a,t}(f) - \Xi_{a,s}(f) \|_{B_1,B_2} \leq C \| B_1 \|_p \| B_2 \|_p \int_s^t |f(s)|\, ds, \qquad a \leq s < t \leq b, \tag{9}$$

where $\| B \|_p = \sup\{\| \phi \|_p\ ;\ \phi \in B\} < \infty$ for any bounded subset $B \subset \mathcal{W}$. The continuity of $t \mapsto \Xi_{a,t}$ then follows from (9) immediately. ∎

The white noise operator $\Xi_{a,t}(f)$ defined in (8) is denoted by

$$\Xi_{a,t}(f) = \int_a^t f(s) L_s\, ds.$$

We can now mention an analogue of the fundamental theorem of calculus.

Theorem 8 *Assume that two quantum stochastic processes $\{L_t\}$ and $\{\Xi_t\}$ are related as*

$$\Xi_t = \int_a^t L_s\, ds.$$

Then, the map $t \mapsto \Xi_t \in \mathcal{L}(\mathcal{W},\mathcal{W}^*)$ is differentiable and

$$\frac{d}{dt}\Xi_t = L_t$$

holds in $\mathcal{L}(\mathcal{W},\mathcal{W}^*)$.

PROOF. For the differentiability at t it is sufficient to show that given bounded subsets $B_1, B_2 \in \mathcal{W}$

$$\lim_{h\to 0} \left\| \frac{\Xi_{t+h} - \Xi_t}{h} - L_t \right\|_{B_1,B_2} = 0. \tag{10}$$

It follows from definition that

$$\left\langle\!\!\left\langle \left(\frac{\Xi_{t+h}-\Xi_t}{h} - L_t\right)\phi,\, \psi \right\rangle\!\!\right\rangle = \frac{1}{h}\int_t^{t+h} \langle\!\langle (L_s - L_t)\phi,\, \psi\rangle\!\rangle\, ds, \qquad \phi,\psi \in \mathcal{W}.$$

Since $s \mapsto L_s$ is continuous, given $\epsilon > 0$ there exists some $\delta > 0$ such that $\| L_s - L_t \|_{B_1,B_2} < \epsilon$ for $|s-t| < \delta$. Then, for $0 < |h| < \delta$ we have

$$\left\| \frac{\Xi_{t+h}-\Xi_t}{h} - L_t \right\|_{B_1,B_2} \leq \frac{1}{h}\int_t^{t+h} \| L_s - L_t \|_{B_1,B_2}\, ds < \epsilon,$$

which proves (10). ∎

Lemma 9 *If $\{L_t\}$ is a quantum stochastic process, so are both $\{L_t a_t\}$ and $\{a_t^* L_t\}$.*

PROOF. We only prove that $t \mapsto L_t a_t \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ is continuous, for the rest is obtained by duality. To this end we fix $t \in \mathbf{R}$ and a finite interval $[a,b]$ containing t inside, and choose $p \geq 0$ and $C \geq 0$ as in the proof of Lemma 7. Let $B_1, B_2 \subset \mathcal{W}$ be bounded subsets. Then we have

$$\begin{aligned} \| L_s a_s - L_t a_t \|_{B_1,B_2} &\leq \| L_s(a_s - a_t) \|_{B_1,B_2} + \| (L_s - L_t)a_t \|_{B_1,B_2} \\ &\leq \| L_s \|_{-p} \| a_s - a_t \|_{B_1,p} \| B_2 \|_p + \| L_s - L_t \|_{a_t B_1, B_2} \\ &\leq C \| a_s - a_t \|_{B_1,p} \| B_2 \|_p + \| L_s - L_t \|_{a_t B_1, B_2}, \end{aligned} \tag{11}$$

where $\| B_2 \|_p < \infty$ and $a_t B_1 \subset \mathcal{W}$ is bounded. Then the continuity of $t \mapsto L_t a_t \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ follows immediately from (11). ∎

In Hudson–Parthasarathy calculus (see also [15], [21]) a fundamental role is played by the following three quantum stochastic processes:

$$A_t = \int_0^t a_s\, ds, \qquad A_t^* = \int_0^t a_s^*\, ds, \qquad \Lambda_t = \int_0^t a_s^* a_s\, ds,$$

which are called the *annihilation process*, the *creation process*, and the *number (gauge) process*, respectively. It follows from Proposition 4, Theorem 8 and Lemma 9 that

$$\frac{d}{dt} A_t = a_t, \qquad \frac{d}{dt} A_t^* = a_t^*, \qquad \frac{d}{dt} \Lambda_t = a_t^* a_t,$$

hold in $\mathcal{L}(\mathcal{W}, \mathcal{W})$, $\mathcal{L}(\mathcal{W}^*, \mathcal{W}^*)$ and $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$, respectively. These relations play a key role to go beyond the traditional Itô theory by means of white noise theory.

5 Initial Value Problem

We now study the initial value problem:

$$\frac{d\Xi}{dt} = F(t, \Xi), \qquad \Xi|_{t=0} = \Xi_0, \qquad 0 \leq t \leq T, \tag{12}$$

where $F : [0,T] \times \mathcal{L}(\mathcal{W}, \mathcal{W}^*) \to \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ is a continuous function and Ξ_0 is a white noise operator. A solution of (12) must be a C^1-map defined on $[0,T]$ with values in $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$, hence by Theorem 8, the initial value problem (12) is equivalent to

$$\Xi_t = \Xi_0 + \int_0^t F(s, \Xi_s)\, ds. \tag{13}$$

Since the solution depends on the "regularity property" of the initial value Ξ_0, we need to consider two weight sequences $\alpha = \{\alpha(n)\}$ and $\omega = \{\omega(n)\}$ with conditions (A1)–(A5), the generating functions of which are related in such a way that

$$G_\alpha(t) = \exp \gamma \{G_\omega(t) - 1\}, \tag{14}$$

where $\gamma > 0$ is a certain constant. In that case, we have continuous inclusions:

$$\mathcal{W}_\alpha \subset \mathcal{W}_\omega \subset L^2(E^*, \mu) \cong \Gamma(H_{\mathbf{C}}) \subset \mathcal{W}_\omega^* \subset \mathcal{W}_\alpha^*$$

and

$$\mathcal{L}(\mathcal{W}_\omega, \mathcal{W}_\omega^*) \subset \mathcal{L}(\mathcal{W}_\alpha, \mathcal{W}_\alpha^*).$$

Such a situation is abstracted from the case of Bell numbers, see (7) for definition. In fact, we have a simple recurrence formula:

$$G_{\mathrm{Bell}(k+1)}(t) = \exp \gamma_k \left\{ G_{\mathrm{Bell}(k)}(t) - 1 \right\}, \quad k \ge 1; \quad G_{\mathrm{Bell}(1)}(t) = e^t,$$

where $\gamma_{k+1} = \exp \gamma_k$ for $k \ge 1$ and $\gamma_1 = 1$.

Theorem 10 *Let $\alpha = \{\alpha(n)\}$ and $\omega = \{\omega(n)\}$ be two weight sequences with conditions (A1)–(A5) such that their generating functions are related as in (14). Let $F : [0,T] \times \mathcal{L}(\mathcal{W}_\alpha, \mathcal{W}_\alpha^*) \to \mathcal{L}(\mathcal{W}_\alpha, \mathcal{W}_\alpha^*)$ be a continuous function and assume that there exist $p \ge 0$ and a nonnegative function $K \in L^1[0,T]$ such that*

$$|\widehat{F}(s, \Xi_1)(\xi, \eta) - \widehat{F}(s, \Xi_2)(\xi, \eta)|^2 \le K(s) G_\omega(|\xi|_p^2) G_\omega(|\eta|_p^2) |\widehat{\Xi}_1(\xi, \eta) - \widehat{\Xi}_2(\xi, \eta)|^2, \tag{15}$$

and

$$|\widehat{F}(s, \Xi)(\xi, \eta)|^2 \le K(s) G_\omega(|\xi|_p^2) G_\omega(|\eta|_p^2)(1 + |\widehat{\Xi}(\xi, \eta)|^2), \tag{16}$$

for all $\xi, \eta \in E_{\mathbf{C}}$, $\Xi \in \mathcal{L}(\mathcal{W}_\omega, \mathcal{W}_\omega^*)$, and $s \in [0,T]$. Then, for any $\Xi_0 \in \mathcal{L}(\mathcal{W}_\omega, \mathcal{W}_\omega^*)$ the initial value problem (12) has a unique solution in $\mathcal{L}(\mathcal{W}_\alpha, \mathcal{W}_\alpha^*)$.

PROOF. In principle, the proof is based on the standard Picard–Lindelöf method of successive approximations (see e.g., [6]) applied to the operator symbols. We define

$$\Xi_t^{(0)} = \Xi_0,$$
$$\Xi_t^{(n)} = \Xi_0 + \int_0^t F(s, \Xi_s^{(n-1)})\, ds, \qquad n \ge 1.$$

We first prove that $\Xi_t^{(n)} \in \mathcal{L}(\mathcal{W}_\omega, \mathcal{W}_\omega^*)$ for $n = 1, 2, \cdots$. Since $\Xi_0 \in \mathcal{L}(\mathcal{W}_\omega, \mathcal{W}_\omega^*)$ by assumption, we may choose $K_0 \ge 0$ and $p_0 \ge 0$ such that

$$|\widehat{\Xi}_0(\xi, \eta)|^2 \le K_0 G_\omega(|\xi|_{p_0}^2) G_\omega(|\eta|_{p_0}^2). \tag{17}$$

Hence by (16) we have

$$|\widehat{F}(s,\Xi_0)(\xi,\eta)|^2 \le K(s)G_\omega(|\xi|_p^2)G_\omega(|\eta|_p^2)\left(1+K_0G_\omega(|\xi|_{p_0}^2)G_\omega(|\eta|_{p_0}^2)\right). \tag{18}$$

By Lemma 2 we see that

$$G_\omega(|\xi|_p^2)G_\omega(|\xi|_{p_0}^2) \le G_\omega(C_{1\omega}(|\xi|_p^2+|\xi|_{p_0}^2)) \le G_\omega(|\xi|_{p_1}^2),$$

where $p_1 \ge \max\{p,p_0\}$ is chosen in such a way that $2C_{1\omega}\| A^{-1}\|_{\mathrm{OP}}^{p_1-\max\{p,p_0\}} \le 1$, see also (5). Then (18) becomes

$$\begin{aligned}|\widehat{F}(s,\Xi_0)(\xi,\eta)|^2 &\le K(s)\left\{G_\omega(|\xi|_p^2)G_\omega(|\eta|_p^2)+K_0G_\omega(|\xi|_{p_1}^2)G_\omega(|\eta|_{p_1}^2)\right\}\\ &\le (1+K_0)K(s)G_\omega(|\xi|_{p_1}^2)G_\omega(|\eta|_{p_1}^2),\end{aligned} \tag{19}$$

and by integration,

$$\begin{aligned}|\widehat{\Xi_t^{(1)}}(\xi,\eta)|^2 &\le 2|\widehat{\Xi}_0(\xi,\eta)|^2+2\left|\int_0^t \widehat{F}(s,\Xi_0)(\xi,\eta)\,ds\right|^2\\ &\le 2|\widehat{\Xi}_0(\xi,\eta)|^2+2T\bar{K}(1+K_0)G_\omega(|\xi|_{p_1}^2)G_\omega(|\eta|_{p_1}^2),\end{aligned} \tag{20}$$

where

$$\bar{K}=\int_0^T K(s)\,ds.$$

Combining (17) and (20), we come to

$$|\widehat{\Xi_t^{(1)}}(\xi,\eta)|^2 \le K_1G_\omega(|\xi|_{p_1}^2)G_\omega(|\eta|_{p_1}^2), \qquad 0\le t\le T, \quad \xi,\eta\in E_{\mathbf{C}}, \tag{21}$$

where $K_1 = 2K_0+2T\bar{K}(1+K_0)$ is a constant. It then follows from the characterization theorem for operator symbols (Theorem 3) that $\Xi_t^{(1)} \in \mathcal{L}(\mathcal{W}_\omega,\mathcal{W}_\omega^*)$. Comparing (21) with (17), we see that the above argument can be repeated to conclude that $\Xi_t^{(n)} \in \mathcal{L}(\mathcal{W}_\omega,\mathcal{W}_\omega^*)$ for all n.

For simplicity we put

$$\Theta_n(t;\xi,\eta)=\widehat{\Xi_t^{(n)}}(\xi,\eta)=\langle\!\langle \Xi_t^{(n)}\phi_\xi,\,\phi_\eta\rangle\!\rangle, \qquad \xi,\eta\in E_{\mathbf{C}}, \quad 0\le t\le T.$$

We shall prove that the limit

$$\Theta_t(\xi,\eta)=\lim_{n\to\infty}\Theta_n(t;\xi,\eta)$$

exists. Since

$$\Theta_n(t;\xi,\eta)=\widehat{\Xi}_0(\xi,\eta)+\int_0^t \widehat{F}(s,\Xi_s^{(n-1)})(\xi,\eta)\,ds \tag{22}$$

by definition, in view of assumption (15) we have

$$\begin{aligned}|\Theta_n(t;\xi,\eta)-\Theta_{n-1}(t;\xi,\eta)|^2 &= \left|\int_0^t\left\{\widehat{F}(s,\Xi_s^{(n-1)})(\xi,\eta)-\widehat{F}(s,\Xi_s^{(n-2)})(\xi,\eta)\right\}ds\right|^2\\ &\le TG_\omega(|\xi|_p^2)G_\omega(|\eta|_p^2)\int_0^t K(s)\,|\Theta_{n-1}(s;\xi,\eta)-\Theta_{n-2}(s;\xi,\eta)|^2\,ds,\end{aligned} \tag{23}$$

and moreover, repeating this argument yields

$$
\begin{aligned}
&|\Theta_n(t;\xi,\eta)-\Theta_{n-1}(t;\xi,\eta)|^2 \\
&\quad\le \left\{TG_\omega(|\xi|_p^2)G_\omega(|\eta|_p^2)\right\}^{n-1}\int_0^t dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{n-2}}dt_{n-1} \\
&\qquad\qquad \times K(t_1)K(t_2)\cdots K(t_{n-1})|\Theta_1(t_{n-1};\xi,\eta)-\Theta_0(t_{n-1};\xi,\eta)|^2. \qquad (24)
\end{aligned}
$$

As for the last quantity, we see from (19) that

$$
\begin{aligned}
|\Theta_1(t;\xi,\eta)-\Theta_0(t;\xi,\eta)|^2 &= \left|\int_0^t \widehat{F}(s,\Xi_0)(\xi,\eta)\,ds\right|^2 \\
&\le T\int_0^T |\widehat{F}(s,\Xi_0)(\xi,\eta)|^2 ds \\
&\le T\bar{K}(1+K_0)G_\omega(|\xi|_{p_1}^2)G_\omega(|\eta|_{p_1}^2)\equiv H(\xi,\eta).
\end{aligned}
$$

Thus (24) becomes

$$
\begin{aligned}
&|\Theta_n(t;\xi,\eta)-\Theta_{n-1}(t;\xi,\eta)|^2 \\
&\quad\le \left\{TG_\omega(|\xi|_p^2)G_\omega(|\eta|_p^2)\right\}^{n-1}\times \\
&\qquad\qquad \times H(\xi,\eta)\int_0^t dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{n-2}}dt_{n-1}K(t_1)K(t_2)\cdots K(t_{n-1}) \\
&\quad\le \frac{1}{(n-1)!}\left\{T\bar{K}G_\omega(|\xi|_p^2)G_\omega(|\eta|_p^2)\right\}^{n-1}H(\xi,\eta). \qquad (25)
\end{aligned}
$$

Let $0<r<1$. Then we have

$$
\begin{aligned}
&\sum_{n=1}^{\infty}|\Theta_n(t;\xi,\eta)-\Theta_{n-1}(t;\xi,\eta)| \\
&\qquad\le \left(\frac{r^2}{1-r^2}\right)^{1/2}\left(\sum_{n=1}^{\infty}\frac{1}{r^{2n}}|\Theta_n(t;\xi,\eta)-\Theta_{n-1}(t;\xi,\eta)|^2\right)^{1/2} \\
&\qquad\le \left(\frac{H(\xi,\eta)}{1-r^2}\right)^{1/2}\exp\left\{\frac{T\bar{K}}{2r^2}G_\omega(|\xi|_p^2)G_\omega(|\eta|_p^2)\right\}. \qquad (26)
\end{aligned}
$$

This proves that

$$
\Theta_t(\xi,\eta)=\lim_{n\to\infty}\Theta_n(t;\xi,\eta)=\widehat{\Xi}_0(\xi,\eta)+\sum_{n=1}^{\infty}\left\{\Theta_n(t;\xi,\eta)-\Theta_{n-1}(t;\xi,\eta)\right\} \qquad (27)
$$

converges uniformly in t for any fixed $\xi,\eta\in E_{\mathbf{C}}$.

We next prove that that there exists a white noise operator $\Xi_t\in\mathcal{L}(\mathcal{W}_\alpha,\mathcal{W}_\alpha^*)$ such that $\Theta_t=\widehat{\Xi}_t$ for $0\le t\le T$. Condition (O1) in Theorem 3 is easily checked from (26) since the convergence (27) is also uniform in (ξ,η) running over any compact subset of $\mathcal{W}\times\mathcal{W}$. As for condition (O2) we shall estimate $|\Theta_t(\xi,\eta)|^2$. First by (26) and (27) we have

$$\begin{aligned} |\Theta_t(\xi,\eta)|^2 &\le 2|\widehat{\Xi}_0(\xi,\eta)|^2 + 2\left|\sum_{n=1}^{\infty}\left\{\Theta_n(t;\xi,\eta)-\Theta_{n-1}(t;\xi,\eta)\right\}\right|^2 \\ &\le 2|\widehat{\Xi}_0(\xi,\eta)|^2 + \frac{2H(\xi,\eta)}{1-r^2}\exp\left\{\frac{T\bar{K}}{r^2}G_\omega(|\xi|_p^2)G_\omega(|\eta|_p^2)\right\}. \end{aligned} \tag{28}$$

Using an elementary inequality: $t\,e^{\lambda t} \le e^{(\lambda+1)t}$ for $t \ge 0$, the second term of (28) becomes

$$\frac{2H(\xi,\eta)}{1-r^2}\exp\left\{\frac{T\bar{K}}{r^2}G_\omega(|\xi|_p^2)G_\omega(|\eta|_p^2)\right\} \le M_1\exp\left\{M_2G_\omega(|\xi|_{p_1}^2)G_\omega(|\eta|_{p_1}^2)\right\}, \tag{29}$$

where $|\xi|_p \le |\xi|_{p_1}$ is used and

$$M_1 = \frac{2T\bar{K}(1+K_0)}{1-r^2}, \qquad M_2 = \frac{T\bar{K}}{r^2}+1.$$

We choose $0<r<1$ in such a way that $M_2/\gamma \ge 1$, where γ is the constant defined in (14). Then by Lemmas 1 and 2 we have

$$\begin{aligned} M_2G_\omega(|\xi|_{p_1}^2)G_\omega(|\eta|_{p_1}^2) &\le M_2G_\omega\left(C_{1\omega}(|\xi|_{p_1}^2+|\eta|_{p_1}^2)\right) \\ &= \gamma\left\{\frac{M_2}{\gamma}\left[G_\omega\left(C_{1\omega}(|\xi|_{p_1}^2+|\eta|_{p_1}^2)\right)-1\right]\right\}+M_2 \\ &\le \gamma\left\{G_\omega\left(\frac{M_2}{\gamma}C_{1\omega}(|\xi|_{p_1}^2+|\eta|_{p_1}^2)\right)-1\right\}+M_2. \end{aligned} \tag{30}$$

We then take $q \ge 0$ in such a way that $(M_2/\gamma)\,C_{1\omega}\|A^{-1}\|_{\mathrm{OP}}^{2q} \le 1$. Then (30) becomes

$$M_2G_\omega(|\xi|_{p_1}^2)G_\omega(|\eta|_{p_1}^2) \le \gamma\left\{G_\omega(|\xi|_{p_1+q}^2+|\eta|_{p_1+q}^2)-1\right\}+M_2,$$

and, in view of (14) we obtain

$$\exp\left\{M_2G_\omega(|\xi|_{p_1}^2)G_\omega(|\eta|_{p_1}^2)\right\} \le e^{M_2}G_\alpha(|\xi|_{p_1+q}^2+|\eta|_{p_1+q}^2). \tag{31}$$

Consequently, combining (28), (29) and (31), we have

$$\begin{aligned} |\Theta_t(\xi,\eta)|^2 &\le 2|\widehat{\Xi}_0(\xi,\eta)|^2 + M_1e^{M_2}G_\alpha(|\xi|_{p_1+q}^2+|\eta|_{p_1+q}^2) \\ &\le 2K_0G_\omega(|\xi|_{p_0}^2)G_\omega(|\eta|_{p_0}^2) + M_1e^{M_2}G_\alpha(C_{2\alpha}|\xi|_{p_1+q}^2)G_\alpha(C_{2\alpha}|\eta|_{p_1+q}^2), \end{aligned}$$

where (17) and Lemma 2 are used. Taking $q_1 > p_1+q > p_0$ such that $C_{2\alpha}\|A^{-1}\|_{\mathrm{OP}}^{2(q_1-p_1-q)} \le 1$ and noting that $G_\omega(s) \le \gamma^{-1}e^{\gamma-1}G_\alpha(s)$ for $s \ge 0$, we come to

$$|\Theta_t(\xi,\eta)|^2 \le (2K_0\gamma^{-1}e^{\gamma-1}+M_1e^{M_2})G_\alpha(|\xi|_{q_1}^2)G_\alpha(|\eta|_{q_1}^2). \tag{32}$$

In other words, Θ_t satisfies condition (O2) in Theorem 3, and hence there exists a unique $\Xi_t \in \mathcal{L}(\mathcal{W}_\alpha,\mathcal{W}_\alpha^*)$ such that

$$\Theta_t(\xi,\eta) = \widehat{\Xi}_t(\xi,\eta), \qquad \xi,\eta \in E_{\mathbf{C}}, \quad t \in [0,T]. \tag{33}$$

We now prove that $\{\Xi_t\}$ is a solution of (12). As is already obvious, $\Theta_n(t)$ also satisfies (32) commonly, and therefore by Lemma 6 we see that $\Xi_t^{(n)} \to \Xi_t$ in $\mathcal{L}(\mathcal{W}_\alpha, \mathcal{W}_\alpha^*)$ uniformly in t. Hence, letting $n \to \infty$ in (22), we conclude that

$$\Theta_t(\xi, \eta) = \widehat{\Xi}_0(\xi, \eta) + \int_0^t \widehat{F}(s, \Xi_s)(\xi, \eta)\, ds,$$

which means that $\{\Xi_t\}$ is a solution of (13), and hence of (12).

For the uniqueness we suppose that two quantum stochastic processes $\{\Xi_t\}$ and $\{X_t\}$ satisfy the same integral equation (13). A similar argument as in the derivation of (23) yields

$$|\widehat{\Xi}_t(\xi, \eta) - \widehat{X}_t(\xi, \eta)|^2 \le TG_\omega(|\xi|_p^2)G_\omega(|\eta|_p^2) \int_0^t K(s)|\widehat{\Xi}_s(\xi, \eta) - \widehat{X}_s(\xi, \eta)|^2\, ds,$$

from which $\widehat{\Xi}_t = \widehat{X}_t$ follows by a standard argument with the Gronwall inequality. ∎

We remind that Theorem 10 covers a simple example: Let $\{L_t\}, \{M_t\} \subset \mathcal{L}(\mathcal{W}_\omega, \mathcal{W}_\omega^*)$ be two quantum stochastic processes, where t runs over $[0, T]$. Then the initial value problem

$$\frac{d}{dt}\Xi_t = L_t \diamond \Xi_t + M_t, \qquad \Xi|_{t=0} = \Xi_0 \in \mathcal{L}(\mathcal{W}_\omega, \mathcal{W}_\omega^*), \tag{34}$$

has a unique solution in $\mathcal{L}(\mathcal{W}_\alpha, \mathcal{W}_\alpha^*)$. Note that equation (34) is a considerable generalization of a traditional quantum stochastic differential equation.

References

[1] L. Accardi, Y.-G. Lu and I. Volovich: *Non-linear extensions of classical and quantum stochastic calculus and essentially infinite dimensional analysis*, in "Probability Towards 2000 (L. Accardi and C. C. Heyde, Eds.)," Lect. Notes in Stat. Vol. 128, pp. 1–33, Springer–Verlag, 1998.

[2] D. M. Chung, U. C. Ji and N. Obata: *Higher powers of quantum white noises in terms of integral kernel operators*, Infinite Dimen. Anal. Quantum Prob. **1** (1998), 533–559.

[3] D. M. Chung, U. C. Ji and N. Obata: *Normal-ordered white noise differential equations II: Regularity properties of solutions*, in "Prob. Theory and Math. Stat. (B. Grigelionis et al. Eds.)," VSP/TEV, 1999. (in press)

[4] D. M. Chung, U. C. Ji and N. Obata: *Quantum stochastic analysis via white noise operators in weighted Fock space*, preprint, 2000.

[5] W. G. Cochran, H.-H. Kuo and A. Sengupta: *A new class of white noise generalized functions*, Infinite Dimen. Anal. Quantum Prob. **1** (1998), 43–67.

[6] P. Hartman: "Ordinary Differential Equations (second edition)," Birkhäuser, 1982.

[7] T. Hida: "Analysis of Brownian Functionals," Carleton Math. Lect. Notes, no. 13, Carleton University, Ottawa, 1975.

[8] T. Hida: *Harmonic analysis on complex random systems*, in this volume, 2000.

[9] T. Hida, H.-H. Kuo, J. Potthoff and L. Streit: "White Noise: An Infinite Dimensional Calculus," Kluwer Academic Publishers, 1993.

[10] H. Holden, B. Øksendal, J. Ubøe and T. Zhang: "Stochastic Partial Differential Equations," Birkhäuser, 1996.

[11] R. L. Hudson and K. R. Parthasarathy: *Quantum Itô's formula and stochastic evolutions*, Commun. Math. Phys. **93** (1984), 301–323.

[12] Yu. G. Kondratiev and L. Streit: *Spaces of white noise distributions: Constructions, descriptions, applications I*, Rep. Math. Phys. **33** (1993), 341–366.

[13] I. Kubo and S. Takenaka: *Calculus on Gaussian white noise I*, Proc. Japan Acad. **56A** (1980), 376–380.

[14] H.-H. Kuo: "White Noise Distribution Theory," CRC Press, 1996.

[15] P.-A. Meyer: "Quantum Probability for Probabilists," Lect. Notes in Math. Vol. 1538, Springer–Verlag, 1993.

[16] N. Obata: "White Noise Calculus and Fock Space," Lect. Notes in Math. Vol. 1577, Springer–Verlag, 1994.

[17] N. Obata: *Generalized quantum stochastic processes on Fock space*, Publ. RIMS **31** (1995), 667–702.

[18] N. Obata: *Integral kernel operators on Fock space – Generalizations and applications to quantum dynamics*, Acta Appl. Math. **47** (1997), 49–77.

[19] N. Obata: *Quantum stochastic differential equations in terms of quantum white noise*, Nonlinear Analysis, Theory, Methods and Applications **30** (1997), 279–290.

[20] N. Obata: *Wick product of white noise operators and quantum stochastic differential equations*, J. Math. Soc. Japan. **51** (1999), 613–641.

[21] K. R. Parthasarathy: "An Introduction to Quantum Stochastic Calculus," Birkhäuser, 1992.

ON THE REGULARITY OF THE BERGMAN KERNEL ON THE BOUNDARY

JOE KAMIMOTO

1. Introduction

In this article, we study the regularity of the Bergman kernel and the Szegö kernel on the boundary of weakly pseudoconvex tube domains off the diagonal.

Let Ω be a domain in $\mathbb{C}^n$. The Bergman space $B(\Omega)$ is the closed subspace of $L^2(\Omega)$ consisting of holomorphic L^2-functions on Ω. The Bergman projection is the orthogonal projection $\mathbb{B} : L^2(\Omega) \to B(\Omega)$. It is known that the projection $\mathbb{B}$ can be represented by using some integral kernel:

$$\mathbb{B}f(z) = \int_\Omega B(z,w)f(w)dV(w) \quad \text{for } f \in L^2(\Omega),$$

where $B : \Omega \times \Omega \to \mathbb{C}$ is called the *Bergman kernel* of the domain Ω and dV is the Lebesgue measure on Ω.

The regularity of the Bergman kernel on the boundary off the diagonal is deeply connected with many other subjects in the $\bar{\partial}$-Neumann problem. In 1972 Kerzman [15] proved the Bergman kernel of a C^∞-smoothly bounded strictly pseudoconvex domain Ω in $\mathbb{C}^n$ is C^∞-smooth up to the boundary off the diagonal: i.e.

$$B \in C^\infty(\bar{\Omega} \times \bar{\Omega} \setminus \Delta), \tag{1.1}$$

where $\Delta = \{(z,z); z \in \partial\Omega\}$. His proof is based on a certain pseudolocal estimate of the $\bar{\partial}$-Neumann problem. Later Bell [1] and Boas [3] independently showed (1.1) in the case of domains of finite type (in the sense of Kohn or D'Angelo) by generalizing the argument of Kerzman.

Let us consider this kind of question in the real analytic category. For a set K in $\mathbb{C}^n$, $C^\omega(K)$ means the set of real analytic functions in some open neighborhood of K. In the case of C^ω-smoothly bounded strictly pseudoconvex domains in $\mathbb{C}^n$, the Bergman kernel is known in [20],[21],[22],[2] to satisfy

$$B \in C^\omega(\bar{\Omega} \times \bar{\Omega} \setminus \Delta). \tag{1.2}$$

In weakly pseudoconvex and of finite type case, it had also been expected that (1.2) always holds. Surprisingly Christ and Geller [7], in 1992, showed that the Bergman kernel does not satisfy (1.2) for the domain $\Omega_m = \{(z_1, z_2); \Im(z_2) > [\Re(z_1)]^{2m}\}$ $(m = 2, 3, \dots)$, which is a very simple weakly pseudoconvex domain of finite type. In general, necessary and sufficient conditions for (1.2) are yet to be known until now.

The following question is the first step for this problem: Find many perturbations of Ω_m whose Bergman kernels do not have the real analytic property (1.2). The following theorem partially answers this question.

Theorem 1.1. *For any weakly pseudoconvex tube domain Ω in $\mathbb{C}^2$ with real analytic boundary, there exist points on $\partial\Omega \times \partial\Omega \setminus \Delta$ where the Bergman kernel is not real analytic.*

In more detail, we can determine the set of the failure of the real analyticity and the best order of the Gevrey class (see Section 4). We remark that our theorem is established for both cases of bounded and unbounded bases of Ω.

Next let us consider an analogous problem about the Szegö kernel. Suppose that Ω has C^∞-smooth boundary equipped with a surface element $d\sigma$. The Hardy space $H^2(\Omega)$ is the subspace of $L^2(\partial\Omega)$ consisting L^2-boundary values of holomorphic functions. The Szegö projection is the orthogonal projection $\mathbb{S} : L^2(\partial\Omega) \to H^2(\Omega)$. The projection $\mathbb{S}$ can be represented by using some integral kernel:

$$\mathbb{S}f(z) = \int_{\partial\Omega} S(z, w) f(w) d\sigma(w) \quad \text{for } f \in L^2(\partial\Omega),$$

where $S : \Omega \times \Omega \to \mathbb{C}$ is called the *Szegö kernel* of the domain Ω (with respect to $d\sigma$).

There are many analogous studies about the Szegö kernel (refer to the Introduction in [7]). Christ and Geller [7] also showed the failure of the real analyticity of the Szegö kernel of Ω_m ($m = 2, 3, \dots$). We also give a similar result about the Szegö kernel.

Theorem 1.2. *For any weakly pseudoconvex tube domain Ω in $\mathbb{C}^2$ with real analytic boundary, there exist points on $\partial\Omega \times \partial\Omega \setminus \Delta$ where the Szegö kernel (with respect to some surface element) is not real analytic.*

Note that the above Szegö kernel is C^∞-smooth on $\bar{\Omega} \times \bar{\Omega} \setminus \Delta$ by [17]. The real analyticity of the Szegö kernel is deeply connected with the analytic hypoellipticity of the tangential Cauchy-Riemann operator $\bar{\partial}_b$ on the boundary. It was shown in [7] that the *CR* manifold $\partial\Omega_m$ ($m = 2, 3, \dots$) is a counterexample to the analytic hypoellipticity of $\bar{\partial}_b$ by regarding the Szegö kernel as a singular solution of $\bar{\partial}_b u = 0$. More generally Christ [4] directly constructed singular solutions for $\bar{\partial}_b \bar{\partial}_b^* u = 0$ and $\bar{\partial}_b^* u \notin C^\omega$ in the case of weakly pseudoconvex domain $\Omega_P = \{z \in \mathbb{C}^2; \Im(z_2) > P(\Re(z_1))\}$ where P is real analytic. (In [4] he mainly treated the case of bounded Reinhardt domains.) The singularity of his solutions closely resembles that of the Bergman kernel in our analysis.

Let us explain our analysis. In this ariticle we only consider the case of the Bergman kernel. Our analysis is based on integral representations of the Bergman kernel which were obtained in the case of general tube domains in [8],[19], etc. (Section 2). Christ and Geller [7] also used these representations, but their proof needed some kind of homogeneity of the domain Ω_m. In the case of general tube domains, this homogeneity cannot always be expected, so it seems difficult to apply their method directly. On the other hand the author [12] (see also [5]) computed some asymptotic expansion of the the Bergman kernel to see the situation of these singularities directly. This analysis is valid for our case. In order to apply the analysis of [5],[12], some appropriate localization of the singularity is necessary (Section 3). This property of localization implies that the failure of the real analyticity is determined by the local geometry of the boundary. After localizing integral representation, we compute some asymptotic expansion by the residue formula. In this expansion it can be directly understood that each term fails to be real analytic and the first term has the strongest singularity. Thus we can obtain Theorem 1.1 (Section 4). In the case of the Szegö kernel, similar integral representations were obtained in [16],[10],[19], etc., so Theorem 1.2 can be shown in a similar fashion.

Last we remark that Francsics and Hanges [9] obtained a very similar result to Theorem 1.1. They explain the regularity problem for the Bergman kernel by using symplectic geometry.

2. Integral representations

First let us recall an integral representation of the Bergman kernel for general tube domains. We set $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, $z_j = x_j + iy_j$ $(x_j, y_j \in \mathbb{R})$, $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$, $\bar{w} = (\bar{w}_1, \dots, \bar{w}_n) \in \mathbb{C}^n$, $t = (t_1, \dots, t_n) \in \mathbb{R}^n$ and $\langle z, t\rangle = \sum_{j=1}^n z_j t_j$.

Let $\Omega \subset \mathbb{C}^n$ be a tube domain whose base is $\omega \subset \mathbb{R}^n$; that is

$$\Omega = \mathbb{R}^n + i\omega.$$

From [8],[19], the Bergman kernel $B(z, w)$ of Ω can be expressed as follows:

$$B(z, w) = \frac{1}{(2\pi)^n} \int_{\Lambda^*} e^{i\langle z - \bar{w}, t\rangle} \frac{dt}{D(t)}, \tag{2.1}$$

with

$$D(t) = \int_\omega e^{-2\langle t, y\rangle} dy,$$

where $\Lambda^* = \{t \in \mathbb{R}^n; D(t) < \infty\}$.

Next in order to prove the theorem, we will rewrite the above representation by using appropriate transformations. From now on we assume that Ω is a pseudoconvex tube domain in $\mathbb{C}^2$ with real analytic boundary. Then it is well known that the base ω is convex in $\mathbb{R}^2$. Let $z^0 = (z_1^0, z_2^0)$ be a boundary point of Ω. By a translation of coordinate axes, we may assume that $\Im(z_1^0) = \Im(z_2^0) = 0$. Then the *maximum cone* Λ of $\omega \subset \mathbb{R}^2$ is defined by

$$\Lambda = \{y \in \mathbb{R}^2 \colon (sy_1, sy_2) \in \omega \ \text{ for any } s > 0\}.$$

and the set Λ^* becomes the *dual cone* of Λ, i.e.

$$\Lambda^* = \{t \in \mathbb{R}^2; \langle t, y\rangle \geq 0 \text{ for any } y \in \Lambda\}.$$

First we consider the case where the base ω is unbounded. By a linear transformation in $\mathbb{R}^2$, ω can be transformed into ω_f, which has the following properties: ω_f is expressed as

$$\omega_f = \{y \in \mathbb{R}^2; y_2 > f(y_1)\},$$

where $f \in C^\omega((a_-, a_+))$, with $-\infty \leq a_- < 0 < a_+ \leq \infty$, satisfying that $f(0) = f'(0) = 0$ and $f(x) \to \infty$ as $x \to a_\pm$; moreover the maximum cone of ω_f is $\Lambda_R = \{y \in \mathbb{R}^2; y_2 \geq R|y_1| > 0\}$ for $R \geq 0$ or $\Lambda_\infty := \{(0, y_2); y_2 > 0\}$. The dual cone of Λ_R is $\Lambda_R^* = \{t \in \mathbb{R}^2; t_2 \geq R^{-1}|t_1| > 0\}$ and $\Lambda_\infty^* = \{t \in \mathbb{R}^2; t_2 \geq 0\}$. From (2.1), the Bergman kernel of $\Omega_f := \mathbb{R}^2 + i\omega_f$ can be expressed as

$$B(z, w) = \frac{1}{2\pi^2}\int_0^\infty \int_{-Rt_2}^{Rt_2} e^{i(z-\bar{w}, t)} \frac{t_2}{\mathcal{D}_f(t_1, t_2)} dt_1 dt_2,$$

where

$$\mathcal{D}_f(t_1, t_2) = \int_{a_-}^{a_+} e^{-2t_2 f(\xi) - 2t_1\xi} d\xi.$$

Next we consider the case where ω is bounded. In a similar fashion, ω can be transformed into $\omega_{f,\tilde{f}}$, which has the following properties: $\omega_{f,\tilde{f}}$ is expressed as

$$\omega_{f,\tilde{f}} = \{y \in \mathbb{R}^2; f(y_1) < y_2 < \tilde{f}(y_1)\},$$

where $f, \tilde{f} \in C^\omega((a_-, a_+))$, with $-\infty < a_- < 0 < a_+ < \infty$, satisfy that $f(0) = f'(0) = 0$ and $f(a_\pm) = \tilde{f}(a_\pm)$, respectively. Here the maximum cone is $\Lambda_0 = \emptyset$ and the dual cone of Λ_0 is $\Lambda_0^* = \mathbb{R}^2$. From (2.1), the Bergman kernel of $\Omega_{f,\tilde{f}} := \mathbb{R}^2 + i\omega_{f,\tilde{f}}$ can be expressed as

$$B(z, w) = \frac{1}{2\pi^2}\iint_{\mathbb{R}^2} e^{i(z-\bar{w}, t)} \frac{t_2}{\mathcal{D}_f(t_1, t_2) - \mathcal{D}_{\tilde{f}}(t_1, t_2)} dt_1 dt_2,$$

where $\mathcal{D}_f$,$\mathcal{D}_{\tilde{f}}$ are as above.

Since linear transformations have no essential influence on the argument of regularity of the Bergman kernel, it suffices to investigate the real analyticity in the above two cases.

3. Localization

In this section we show that the singularity of the Bergman kernel at the boundary can be locally determined.

We set $\zeta = (\zeta_1, \zeta_2)$ where $\zeta_j = (z_j - \bar{w}_j)/2i$ and $\mathcal{B}(\zeta) = B(z, w)$. Let $\rho, \delta_\pm$ be constants such that $0 < \rho \leq R$ and $a_- \leq \delta_- < 0 < \delta_+ \leq a_+$. We set $\delta =$

$\min\{-\delta_-, \delta_+\}$ and $\tilde{\delta} = \max\{-\delta_-, \delta_+\}$. For $\rho, \delta_\pm$, define the function $\mathcal{B}(\zeta; \rho, \delta_\pm)$ by

$$\mathcal{B}(\zeta; \rho, \delta_\pm) = \frac{1}{2\pi^2} \int_0^\infty \int_{-\rho t_2}^{\rho t_2} e^{-2\langle \zeta, t \rangle} \frac{t_2}{\mathcal{D}_f(t_1, t_2; \delta_\pm)} dt_1 dt_2. \tag{3.1}$$

where

$$\mathcal{D}_f(t_1, t_2; \delta_\pm)(:= \mathcal{D}_f(\delta_\pm)) = \int_{\delta_-}^{\delta_+} e^{-2t_2 f(\xi) - 2t_1 \xi} d\xi.$$

Note that $\mathcal{B}(\zeta; R, a_\pm) = \mathcal{B}(\zeta)$ in the case of ω_f.

Now the singularity of the Bergman kernel $\mathcal{B}(\zeta)$ at $K_0 := \{(0, 0)\} + i\mathbb{R}^2$ is locally described as follows.

Proposition 3.1. *For any $\delta_\pm$, there exists a positive constant ρ_0 such that if $0 < \rho \le \rho_0$, then $\mathcal{B}(\zeta) - \mathcal{B}(\zeta; \rho, \delta_\pm)$ is real analytic in ζ in some neighborhood of K_0.*

The proof of this proposition is seen in [11].

4. Proof of Theorem 1.1.

4.1. **Preliminaries.** Since ω is convex and $f(x)$ is real analytic in (a_-, a_+) with $f(0) = f'(0) = 0$, there exist a natural number m and a real analytic function $g(x)$ such that $g(0) > 0$ and $f(x) = x^{2m} g(x)$ in (a_-, a_+). Note that z^0 is of type $2m$ (in the sense of D'Angelo). (If $f^{(k)}(0) = 0$ for any $k \in \mathbb{N}$, then the real analyticity of $f(x)$ implies that $\Omega_f = \{z \in \mathbb{C}^2; \Im(z_2) > 0\}$ whose Bergman space is $\{0\}$.)

We set

$$K(z, t) = \mathcal{B}((z, t + if(y)); (0, 0)).$$

Suppose that z^0 is a weakly pseudoconvex point of type $2m$ ($m \ge 2$). Now fix $\delta_\pm = \pm\delta_0$ with $0 < \delta_0 \le \min\{-a_-, a_+\}$ and set $\hat{\tau} = \tau^{1/(2m)}$. For $\tau_0, \rho_0 > 0$, define the function $K(z, t; \tau_0, \rho_0)$ by

$$K(z, t; \tau_0, \rho_0) = \frac{1}{2\pi^2} \int_{\tau_0}^\infty e^{it\tau} e^{-f(y)\tau} F(z; \hat{\tau}, \rho_0) \tau^{1+\frac{1}{m}} d\tau, \tag{4.1}$$

$$F(z; \hat{\tau}, \rho_0) = \int_{-\rho_0 \hat{\tau}^{2m-1}}^{\rho_0 \hat{\tau}^{2m-1}} \frac{e^{iz\hat{\tau}v}}{\varphi(v; \hat{\tau})} dv \tag{4.2}$$

$$\varphi(v, \hat{\tau}) = \int_{-\delta_0 \hat{\tau}}^{\delta_0 \hat{\tau}} e^{-2g(w/\hat{\tau}) w^{2m} - 2vw} dw.$$

Recalling the definitions of $\mathcal{B}(\zeta)$ and $\mathcal{B}(\zeta; \rho, \delta_\pm)$ in Section 3, we have

$$K(z, t) = \mathcal{B}(z/2i, (t + if(y))/2i),$$
$$K(z, t; 0, \rho) = \mathcal{B}(z/2i, (t + if(y))/2i; \rho, \pm\delta_0).$$

By Proposition 3.1, there exists $\rho_0 > 0$ such that if $\rho \le \rho_0$, then $K(\cdot, \cdot; 0, \rho) - K(\cdot, \cdot)$ is real analytic around $(0, 0)$. Moreover it is easy to check the real analyticity of $K(\cdot, \cdot; \tau_0, \rho_0) - K(\cdot, \cdot; 0, \rho_0)$ for any $\tau_0 \ge 0$.

In a small neighborhood of $(0,0)$, if $K(\cdot,\cdot)$ is real analytic away from $(0,0)$, then so is $K(\cdot,\cdot;\tau_0,\rho_0)$. Our goal is to show the following theorem.

Theorem 4.1. *There exist positive numbers x_0, ρ_0, τ_0 such that $K(z,t;\rho_0,\tau_0)$ is not real analytic in (z,t) on the set $\Xi(x_0) = \{(x+i0,0); 0 < |x| \le x_0\}$, moreover it belongs to s-th order Gevrey class for $s \ge 2m$, but no better, on $\Xi(x_0)$.*

Remark. If the boundary $\partial\Omega$ is locally regarded as $\mathbb{C}\times\mathbb{R}$ as above, the Bergman kernel $B((x+iy,t+if(y));(u+iv,s+if(v)))$ fails to be real analytic on the set

$$\{(x+iy,t;u+iv,s); y=v=0, t=s\} \cup \{\text{diagonal}\}$$

in some small neighborhood of $(0,0)$.

4.2. **Analysis of $\varphi(v,\hat{\tau})$.** In order to prove the theorems, it is necessary to analyze the function $\varphi(v,\hat{\tau})$. Note that a similar analysis is done in [4]. We express some positive constants depending on X by $C(X)$ or $C_j(X)$. The proofs of the lemmas blow are seen in [11].

When $\hat{\tau}$ is sufficiently large, the function $\varphi(v,\hat{\tau})$ can be well approximated by the entire function:

$$\varphi(v) = \int_{-\infty}^{\infty} e^{-2gw^{2m}-2vw}dw \quad (m = 2,3,\dots),$$

where $g := g(0)$. Indeed the Lemmas 4.2,4.3, below, show this nature. There are many studies of the properties of $\varphi(v)$ (refer to the Introduction in [13]).

First let us consider the zeros of $\varphi(\cdot,\hat{\tau})$. It is known that all zeros of φ exist on the imaginary axis ([18]) and are simple ([14]). The set of the zeros of φ is denoted by $\{\pm ia_j^*; 0 < a_j^* < a_{j+1}^* (j \in \mathbb{N})\}$ (Note that φ is an even function). For $\eta,\sigma > 0$, set $R(\eta,\sigma) = \{v \in \mathbb{C}; |\Re(v)| < \eta, |\Im(v)| < \sigma\}$. Let $\{\pm ia_{\pm j}; 0 \le \Re(a_{\pm j}) \le \Re(a_{\pm(j+1)})\}$ be the set of zeros of $\varphi(\cdot,\hat{\tau})$ in $R(\eta,\sigma)$. Note that the values of $a_{\pm j}$ depend on $\hat{\tau}$.

Lemma 4.2. *For any $\eta > 0$ and $N \in \mathbb{N}$, there exists $\hat{\tau}_0 > 0$ such that if $\hat{\tau} > \hat{\tau}_0$, then in $R(\eta,\sigma_N)$ with $\sigma_N = (a_N^* + a_{N+1}^*)/2$*

(i) *the number of zeros of $\varphi(\cdot,\hat{\tau})$ is $2N$,*
(ii) $|a_{\pm j} - a_j^*| < C_1(\eta,N)/\hat{\tau}$ *for $j = 1,\dots,N$,*
(iii) *all zeros of $\varphi(\cdot,\hat{\tau})$ are simple,*
(iv) $|\varphi_v(ia_{\pm j},\hat{\tau}) - \varphi'(ia_j^*)| < C_2(\eta,N)/\hat{\tau}$ *for $j = 1,\dots,N$, where $\varphi_v(v,\hat{\tau})$ is the partial derivative of $\varphi(v,\hat{\tau})$ in v.*

Next let us consider the behavior of $\varphi(\cdot,\hat{\tau})$ at infinity in the directions $\arg v = 0,\pi$. The following lemma shows that this behavior is similar to that of $\varphi(v)$ in these directions (see Theorem 3.1 in [12]).

Lemma 4.3. *There are positive constants α_0,ρ_0,R such that if $|v|/\hat{\tau} \le \rho_0$, $|v| > R$ and $|\arg v| < \alpha_0$ or $|\arg v - \pi| < \alpha_0$, then*

$$C_1 < |v|^{(m-1)/(2m-1)} e^{-a|v|^{2m/(2m-1)}} \cdot |\varphi(v,\hat{\tau})| < C_2,$$

where a, C_1, C_2 are positive constants independent of $\hat{\tau}, v$.

4.3. **Analysis of** $F(v; \hat{\tau}, \rho_0)$. Fix any positive integer N and set $\sigma_N = (a_N^* + a_{N+1}^*)/2$ ($\pm i a_j^*$'s are zeros of $\varphi(v)$). For the computation below, we prepare integral curves $\Gamma_\pm^{(N)}$ as follows. $\Gamma_\pm^{(N)}$ consist three parts $\Gamma_{\pm 1}^{(N)}, \Gamma_{\pm 2}^{(N)}, \Gamma_{\pm 3}^{(N)}$. First $\Gamma_{\pm 1}^{(N)}$ follow the line $\{v; \Re(v) = -\rho_0 \hat{\tau}^{2m-1}\}$ from $-\rho_0\hat{\tau}^{2m-1} + i0$ to $-\rho_0\hat{\tau}^{2m-1} \pm i\sigma_N$. Second $\Gamma_{\pm 2}^{(N)}$ follow the lines $\{v; \Im(v) = \pm\sigma_N\}$ from $-\rho_0\hat{\tau}^{2m-1} \pm i\sigma_N$ to $\rho_0\hat{\tau}^{2m-1} \pm i\sigma_N$. Third $\Gamma_{\pm 3}^{(N)}$ follow the line $\{v; \Re(v) = \rho_0\hat{\tau}^{2m-1}\}$ from $\rho_0\hat{\tau}^{2m-1} \pm i\sigma_N$ to $\rho_0\hat{\tau}^{2m-1} + i0$. (See Figure 1.)

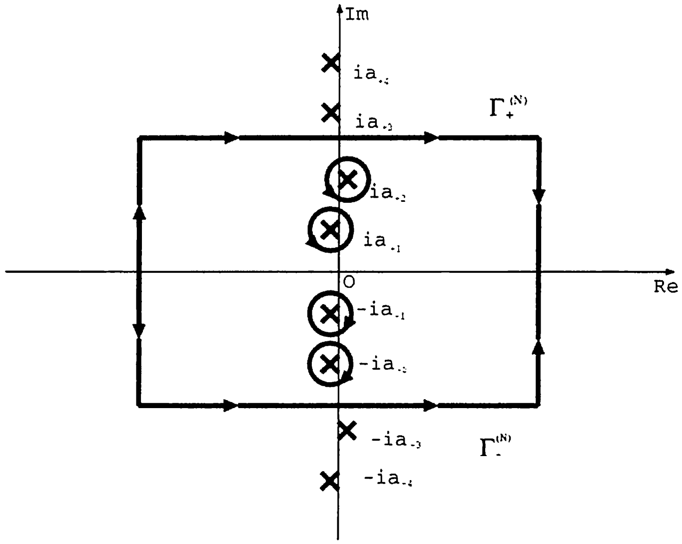


FIGURE 1. Integral contours $\Gamma_\pm^{(N)}$.

Define the functions $I_\pm^{(N)}(z; \hat{\tau}, \rho_0)$ by

$$I_\pm^{(N)}(z; \hat{\tau}, \rho_0) = \sum_{k=1}^{3} I_{\pm k}^{(N)}(z; \hat{\tau}, \rho_0),$$

where

$$I_{\pm k}^{(N)}(z; \hat{\tau}, \rho_0) = \int_{\Gamma_{\pm k}^{(N)}} \frac{e^{iz\hat{\tau}v}}{\varphi(v, \hat{\tau})} dv \quad \text{for } k = 1, 2, 3.$$

First we consider the case where $x = \Re(z) > 0$. For $N \in \mathbb{N}$, we set $\eta_N = \max\{2R, 2\sigma_N/\tan\alpha_0\}$, where R is as in Lemma 4.3. By Lemma 4.2, for any $N \in \mathbb{N}$, there exists $\hat{\tau}_N > 0$ such that if $\hat{\tau} > \hat{\tau}_N$, then in the region $R(\eta, \sigma_N)$, the number of zeros of $\varphi(\cdot, \hat{\tau})$ is $2N$, $|a_{+j} - a_j^*| < 10^{-1}\min\{a_j^* - a_{j-1}^*, a_{j+1}^* - a_j^*\}$, all zeros of $\varphi(\cdot, \hat{\tau})$ are simple and $|\varphi_v(ia_{+j}, \hat{\tau}) - \varphi'(ia_j^*)| < 10^{-1}|\varphi'(ia_j^*)|$ for $j = 1, \ldots, N$.

Suppose that $\hat{\tau} > \hat{\tau}_N$. By deforming the original integral curve in (4.2) into $\Gamma_+^{(N)}$, the residue formula implies

$$F(z; \hat{\tau}, \rho_0) = 2\pi i \sum_{j=1}^{N} \frac{e^{-a_{+j}z\hat{\tau}}}{\varphi_v(ia_{+j}, \hat{\tau})} + I_+^{(N)}(z; \hat{\tau}, \rho_0). \tag{4.3}$$

In fact the function $e^{iz\hat{\tau}v}/\varphi(v, \hat{\tau})$ in v has simple poles with residue $2\pi i e^{-a_{+j}z\hat{\tau}}/\varphi_v(ia_{+j}, \hat{\tau})$ at $v = ia_{+j}$. Hereafter we use C_N for various constants depending on N.

First $I_{+j}^{(N)}$ ($j = 1, 3$, $x > 0$) can be estimated as follows.

$$\begin{aligned} |I_{+j}^{(N)}(x + i0; \hat{\tau}, \rho_0)| &\le \int_0^{\sigma_N} \frac{e^{-x\hat{\tau}q}}{|\varphi(-\rho_0\hat{\tau}^{2m-1} + iq, \hat{\tau})|} dq \\ &\le C\hat{\tau}^{m-1} e^{-a\hat{\rho}_0\tau} \int_0^{\sigma_N} e^{-x\hat{\tau}q} dq \\ &\le C_N \hat{\tau}^{m-1} e^{-a\hat{\rho}_0\tau}, \end{aligned}$$

by using Lemma 4.3. Second $I_{+2}^{(N)}$ ($x > 0$) can be estimated as follows.

$$\begin{aligned} |I_{+2}^{(N)}(x + i0; \hat{\tau}, \rho_0)| &\le e^{-x\sigma_N\hat{\tau}} \int_{-\rho_0\hat{\tau}^{2m-1}}^{\rho_0\hat{\tau}^{2m-1}} \frac{dp}{|\varphi(p + i\sigma_N, \hat{\tau})|} \\ &\le C_N e^{-x\sigma_N\hat{\tau}}. \end{aligned}$$

In the case where $x < 0$, we can obtain the same inequality by deforming the integral curve into $\Gamma_-^{(N)}$.

Therefore if $x \neq 0$, then we have

$$\left| I_{\sigma(x)}^{(N)}(x + i0; \hat{\tau}, \rho_0) \right| \le C_N e^{-|x|\sigma_N\hat{\tau}}, \tag{4.4}$$

where $\sigma(x)$ is the sign of x.

4.4. **Proof of Theorem 1.1.** Fix any $N \in \mathbb{N}$ and suppose that $x = \Re(z) > 0$. Substituting (4.3) into (4.1), we have

$$K(z, t; \tau_N, \rho_0) = \sum_{j=1}^{N} K_j(z, t; \tau_N) + R_N(z, t; \tau_N, \rho_0). \tag{4.5}$$

where

$$K_j(z, t; \tau_N) = \frac{i}{\pi} \int_{\tau_N}^{\infty} e^{it\tau} e^{-f(y)\tau} e^{-a_{+j}z\hat{\tau}} \frac{\tau^{1+1/m}}{\varphi_v(ia_{+j}, \hat{\tau})} d\tau$$

for $j = 1, \ldots, N$ and

$$R_N(z, t; \tau_N, \rho_0) = \frac{i}{\pi} \int_{\tau_N}^{\infty} e^{it\tau} e^{-f(y)\tau} I_+^{(N)}(z; \hat{\tau}, \rho_0) \tau^{1+1/m} d\tau.$$

In the case where $x < 0$, if we replace a_{+j}, $I_+^{(N)}$ with $-a_{-j}$, $I_-^{(N)}$ respectively, then the equation (4.5) holds. Now we show the following proposition.

Proposition 4.4. *For any $N \in \mathbb{N}$, there exist $x_0 > 0$, $k_0 \in \mathbb{N}$ such that if $0 < |x| \leq x_0$ and $k \geq k_0$, then*

$$(4.6) \qquad C_j^{(1)} \frac{\Gamma(2mk + 4m + 2)}{(|x| a_j^*)^{2mk+4m+2}} \leq \left| \frac{\partial^k}{\partial t^k} K_j(x + i0, 0; \tau_N) \right| \leq C_j^{(2)} \frac{\Gamma(2mk + 4m + 2)}{(|x| a_j^*)^{2mk+4m+2}}$$

for $j = 1, \ldots, N$, where $C_j^{(1)}, C_j^{(2)} > 0$ are constants depending on j, and

$$(4.7) \qquad \left| \frac{\partial^k}{\partial t^k} R_N(x + i0, 0; \tau_N, \rho_0) \right| \leq C_N \frac{\Gamma(2mk + 4m + 2)}{(|x| \sigma_N)^{2mk+4m+2}},$$

where $C_N > 0$ is a constant depending on N.

If we admit the above proposition, each K_j does not satisfy the Cauchy inequality on the set $\Xi(x_0) = \{(x + i0, 0) : 0 < |x| \leq x_0\}$ and the singularity of K_j becomes *weaker* as j increases. Thus we can obtain Theorem 4.1, that is, K fails to be real analytic and moreover it belongs to s-th order Gevrey class for $s \geq 2m$, but no better, on $\Xi(x_0)$.

Proof of Proposition 4.4. We only consider the case where $x > 0$. There is a function $f_j(\hat{\tau})$ $(j = 1, \ldots, N)$ and a constant $c_N > 0$ such that $a_{+j}\hat{\tau} = a_j^* \hat{\tau} + f_j(\hat{\tau})$ and $|f_j(\hat{\tau})| < c_N$ for $\hat{\tau} > \hat{\tau}_N$. We take $x_0 > 0$ such that $c_N x_0 < 1/100$. Then $|e^{-f_j(\hat{\tau})x} - 1| < 1/10$. If $0 < x < x_0$, then

$$\begin{aligned}
&\left| \frac{e^{-f_j(\hat{\tau})x}}{\varphi_v(ia_{+j}, \hat{\tau})} - \frac{1}{\varphi'(ia_j^*)} \right| \\
&\leq \frac{|e^{-f_j(\hat{\tau})x} \varphi'(ia_j^*) - \varphi_v(ia_{+j}, \hat{\tau})|}{|\varphi_v(ia_{+j}, \hat{\tau})||\varphi'(ia_j^*)|} \\
&\leq \frac{10}{9} \frac{|e^{-f_j(\hat{\tau})x} - 1||\varphi'(ia_j^*)| + |\varphi'(ia_j^*) - \varphi_v(ia_{+j}, \hat{\tau})|}{|\varphi'(ia_j^*)|^2} \\
&\leq \frac{2}{9} \frac{1}{|\varphi'(ia_j^*)|}
\end{aligned}$$

Note that we took $\hat{\tau}_N$ as in Subsection 4.3. By using the above inequality, if $0 < x < x_0$, then

$$\left|\frac{\partial^k}{\partial t^k}K_j(x+i0,0;\tau_N)\right| = \frac{1}{\pi}\left|\int_{\tau_N}^{\infty}\tau^{k+1+1/m}e^{-a_j^* x\hat{\tau}}\frac{e^{-f_j(\hat{\tau})x}}{\varphi_c(ia_{+j},\hat{\tau})}d\tau\right|$$
$$\geq \frac{7}{9\pi}\frac{1}{|\varphi'(ia_j^*)|}\int_{\tau_N}^{\infty}\tau^{k+1+1/m}e^{-a_j^* x\hat{\tau}}d\tau$$
$$= \frac{7}{9\pi}\frac{1}{|\varphi'(ia_j^*)|}\left\{\frac{\Gamma(2mk+4m+2)}{(xa_j^*)^{2mk+4m+2}} - H_{j,N,k}\right\}.$$

Here it is easy to obtain

$$|H_{j,N,k}| = \left|\int_0^{\tau_N^{1/(2m)}}\tau^{2mk+4m+1}e^{-a_j^* x\tau}d\tau\right| \leq \frac{\tau_N^{k+2+1/m}}{2mk+4m+2}.$$

Therefore if k is sufficiently large, we can obtain the left inequality in (4.6) in the proposition. On the other hand, the right inequality in (4.6) can be shown as follows.

$$\left|\frac{\partial^k}{\partial t^k}K_j(x+i0,0;\tau_N)\right| \leq \frac{1}{\pi}\int_{\tau_N}^{\infty}\tau^{k+1+1/m}e^{-a_j^* x\hat{\tau}}\left|\frac{e^{-f_j(\hat{\tau})x}}{\varphi_c(ia_{+j},\hat{\tau})}\right|d\tau$$
$$\leq \frac{11}{9\pi}\frac{1}{|\varphi'(ia_j^*)|}\int_0^{\infty}\tau^{2mk+4m+1}e^{-a_j^* x\tau}d\tau$$
$$= \frac{11}{9\pi}\frac{1}{|\varphi'(ia_j^*)|}\frac{\Gamma(2mk+4m+2)}{(xa_j^*)^{2mk+4m+2}}.$$

Next by (4.4), we have

$$\left|\frac{\partial^k}{\partial t^k}R_N(x+i0,0;\tau_N,\rho_0)\right| \leq C\int_{\tau_N}^{\infty}\tau^{k+1+1/m}\left|I_+^{(N)}(x+i0;\hat{\tau},\rho_0)\right|d\tau$$
$$\leq C_N\int_0^{\infty}\tau^{k+1+1/m}e^{-x\sigma_N\hat{\tau}}d\tau$$
$$\leq C_N\frac{\Gamma(2mk+4m+2)}{(x\sigma_N)^{2mk+4m+2}}.$$

We have completed the proof of Proposition 4.4.

References

[1] S. R. Bell, *Differentiability of the Bergman kernel and pseudolocal estimates*, Math. Z. **192** (1986), 467–472.

[2] ———, *Extendibility of the Bergman kernel function*, Complex analysis, II (College Park, Md., 1985–86), 33–41, Lecture Notes in Math., **1276**, Springer, Berlin-New York, 1987.

[3] H. P. Boas, *Extension of Kerzman's theorem on differentiability of the Bergman kernel function*, Indiana Univ. Math. J. **36** (1987), 495–499.

[4] M. Christ, *Analytic hypoellipticity breaks down for weakly pseudoconvex Reinhardt domains*, International Math. Research Notices 1 (1991), 31-40.

[5] ______, *Remarks on the breakdown of analyticity for $\bar{\partial}_b$ and Szegö kernels*, Proceedings of 1990 Sendai conference on harmonic analysis (S. Igari, ed.), Lecture Notes in Math. Springer, 61-78.

[6] ______, *Remarks on analytic hypoellipticity of $\bar{\partial}_b$*, Modern method in complex analysis, Princeton Univ. Press, 41-62.

[7] M. Christ and D. Geller, *Counterexamples to analytic hypoellipticity for domains of finite type*, Ann. of Math. **235** (1992), 551-566.

[8] J. Faraut and A. Korányi, *Analysis on symmetric cones*, Oxford Univ. Press, New York (1994).

[9] G. Francsics and N. Hanges, *Analytic regularity for the Bergman kernel*, Proceedings of the Conference on Partial Differential Equations, (Saint-Jean-de-Monts, 1998), Exp. No. V, 11 pp., Univ. Nantes, Nantes, (1998).

[10] F. Haslinger, *Szegö kernels of certain unbounded domains in $\mathbb{C}^2$*, Rev. Roumaine Math. Pures Appl., **39** (1994), 939-950.

[11] J. Kamimoto, *Non-analytic Bergman and Szegö kernels for weakly pseudoconvex tube domains in $\mathbb{C}^2$*, to appear in Math. Z.

[12] ______, *On the singularities of non-analytic Szegö kernels*, J. Math. Sci. Univ. Tokyo, **6** (1999), 13-39.

[13] ______, *On an integral of Hardy and Littlewood*, Kyushu J. of Math. **52**, (1998) 249-263.

[14] J. Kamimoto, H. Ki and Y. O. Kim, *On the multiplicities of the zeros of Laguerre–Pólya functions*, Proc. of Amer. Math. Soc., **128** (2000), no. 1, 189–194.

[15] N. Kerzman, *The Bergman kernel function. Differentiability at the boundary*, Math. Ann. **195** (1972), 149–158.

[16] A. Nagel, *Vector fields and nonisotropic metrics*, Beijing Lectures in Harmonic Analysis, (E. M. Stein, ed.), Princeton University Press, Princeton, NXJ, (1986), 241-306.

[17] A. Nagel, J. P. Rosay, E. M. Stein and S. Wainger, *Estimates for the Bergman and Szegö kernels in $\mathbb{C}^2$*, Ann. of Math. **129** (1989), 113-150.

[18] G. Pólya, *Über trigonometrische Integrale mit nur reelen Nullstellen*, J. Reine Angew. Math. **58** (1927), 6-18.

[19] S. Saitoh, *Integral transforms, reproducing kernels and their applications*, Pitman Research Notes in Mathematics Series **369**, Addison Wesley Longman, UK (1997).

[20] D. Tartakoff, *Local analytic hypoellipticity for $\Box_b$ on non-degenerate Cauchy-Riemann manifolds*, Proc. Nat. Acad. Sci. U.S.A. **75** (1978), 3027-3028.

[21] ______, *The local real analyticity of solutions to $\Box_b$ and the $\bar{\partial}$-Neumann problem*, Acta Math., **145** (1980), 177-204.

[22] F. Treves, *Analytic hypoellipticity of a class of pseudodifferential operators*, Comm. in P.D.E., **3** (1978), 475-642.

DEPARTMENT OF MATHEMATICS, KUMAMOTO UNIVERSITY, KUMAMOTO 860-8555, JAPAN.
E-mail address: joe@sci.kumamoto-u.ac.jp

Series Logo
Volume 00, Number 00, Xxxx 19xx

SPECTRAL SYNTHESIS FOR L^1-ALGEBRAS AND FOURIER ALGEBRAS OF LOCALLY COMPACT GROUPS

EBERHARD KANIUTH

1. Introduction

The purpose of these notes is to report on progress that has been achieved during the past twenty years in spectral synthesis for L^1- and Fourier algebras of (non-abelian) locally compact groups. However, some of these results, in particular for Fourier algebras, are very recent.

To start with, let G be a locally compact abelian group and $L^1(G)$ the convolution algebra of integrable functions on G. Then the spectrum (or Gelfand space) of $L^1(G)$ can be identified with the dual group $\widehat{G}$ of G by means of the mapping $\alpha \to \varphi_\alpha$, where $\varphi_\alpha(f) = \widehat{f}(\alpha) = \int_G f(x)\alpha(x)dx$ for $f \in L^1(G)$ and $x \in G$. Spectral synthesis problems concern the extent to which a closed ideal I of $L^1(G)$ is determined by its hull $h(I) = \{\alpha \in \widehat{G} : \widehat{f}(\alpha) = 0 \text{ for all } f \in I\}$ in $\widehat{G}$. We refer the reader to [3] or to Section 2 for the notion of spectral set and Ditkin set for $L^1(G)$.

Since Malliavin's [20] famous discovery that, given any non-compact locally compact abelian group G (equivalently, $\widehat{G}$ is non-discrete), there exists a closed subset of $\widehat{G}$ which fails to be a spectral set for $L^1(G)$, there has been much effort in producing spectral sets and Ditkin sets. Specifically, so-called injection and projection theorems for spectral sets and Ditkin sets (see [3], [23] and [24]) as well as results about unions of such sets have been established (see [3]). As general references to spectral synthesis we mention [3], [10] and [24]. One of the major unsettled problems (even for $G = \mathbb{Z}$) is whether every spectral is actually a Ditkin set. In Sections 2 and 3 we discuss analogous problems for Fourier algebras and for L^1-algebras of (non-abelian) locally compact groups.

2. Fourier Algebras

For a locally compact group G, let $A(G)$ and $B(G)$ denote the Fourier algebra and the Fourier-Stieltjes algebra of G as introduced and first systematically studied by Eymard [5]. Recall that $B(G)$ is the linear span of all continuous positive definite functions on G and therefore is the Banach space dual of $C^*(G)$, the group C^*-algebra of G. Then $A(G)$ is the closed ideal of $B(G)$ generated by the functions in $B(G)$ with compact support. It turns out that

©0000 American Mathematical Society
0000-0000/00 $1.00 + $.25 per page

$A(G)$ consists precisely of all coefficient functions of the left regular representation λ of G on $L^2(G)$, and $A(G)$ can be identified with the predual of the von Neumann algebra $VN(G)$ generated by λ. When G is abelian and $\widehat{G}$ denotes the dual group of G, then $A(G)$ and $B(G)$ are isomorphic (by means of the Fourier transform) to $L^1(\widehat{G})$ and $M(\widehat{G})$.

$A(G)$ is a regular semisimple commutative Banach algebra with spectrum $\Delta(A(G)) = G$ [5, Théorème 3.34 and Lemme 3.2]. In fact, the mapping $x \to \varphi_x$, where $\varphi_x(u) = u(x)$ for $u \in A(G)$, provides a homeomorphism between G and $\Delta(A(G))$. Thus, associated to every closed subset E of G, is a largest and a smallest ideal, $I(E)$ and $J(E)$, of $A(G)$ with zero set equal to E. More precisely,

$$I(E) = \{u \in A(G) : u(x) = 0 \text{ for all } x \in E\}$$

and

$$J(E) = \{u \in A(G) \cap C_c(G) : u \text{ vanishes on a neighbourhood of } E\}.$$

E is called a *spectral set* or *set of synthesis* if $I(E) = \overline{J(E)}$, and E is said to be a *Ditkin set* if $u \in \overline{uJ(E)}$ for every $u \in I(E)$. Obviously, each Ditkin set is a spectral set. In addition, there are local variants of these notions (see [3, 4, 9, 16]). They are obtained by replacing $I(E)$ with $I(E) \cap C_c(G)$. When G is abelian, the local notions agree with the former ones. For any regular semisimple commutative Banach algebra A it is customary to say that spectral synthesis (respectively, local spectral synthesis) holds for A whenever every closed subset of $\Delta(A)$ is a spectral set (respectively, local spectral set).

Proposition 2.1. *Let G be an arbitrary locally compact group. Then*

(i) *Local spectral synthesis holds for $A(G)$ if and only if G is discrete.*

(ii) *Spectral synthesis holds for $A(G)$ if and only if G is discrete and $u \in \overline{uA(G)}$ for every $u \in A(G)$.*

The additional condition in (ii) is of course satisfied if $A(G)$ has an approximate identity in the weakest possible sense. It is not unlikely that this condition is fulfilled for most groups. In contrast, by a result of Leptin [15], $A(G)$ has a norm bounded approximate identity precisely when G is amenable.

The above proposition can be found in [13]. We indicate the proof of (i). Thus, suppose that local spectral synthesis holds for $A(G)$. Using the fact that this property is inherited by quotient groups and by closed subgroups, it was shown earlier (see [16] and [7]) that G must be totally disconnected (indeed, a connected Lie group is generated by its one-parameter subgroups). Fix a compact open subgroup K of G and suppose that K is infinite. Then, by a deep theorem of Zelmanov [27, Theorem 2], K contains an infinite abelian (closed) subgroup H. Now, local spectral synthesis, and hence spectral synthesis, holds for $A(H)$, contradicting Malliavin's theorem. Thus K is finite, whence G is discrete.

Proposition 2.1 and the results that have been established for $L^1(H)$, H abelian, suggest a study of (local) spectral sets and (local) Ditkin sets for

Fourier algebras. In this context, the desire to not having to treat the local variants separately, lead to the following generalization of the notions of spectral set and Ditkin set [13].

Recall that $A(G)^* = VN(G)$ and that there is natural action of $B(G)$ on $VN(G)$ given by

$$\langle u \cdot T, v\rangle = \langle T, uv\rangle,$$

$T \in VN(G), u \in B(G), v \in A(G)$. Let X be an $A(G)$-invariant linear subspace of $VN(G)$. A closed subset E of G is called an *X-spectral set* or *set of X-synthesis* for $A(G)$ if each $T \in X$ with support (in the sense of [5]) in E belongs to $I(E)^\perp$, the annihiltator of $I(E)$ in $VN(G)$. E is called an *X-Ditkin set* if for every $T \in X$ and $u \in I(E)$ there exists a net $(u_\alpha)_\alpha$ in $J(E)$ such that $\langle T, uu_\alpha\rangle \to \langle T, u\rangle$. These notions reduce to the previous ones when taking for X all of $VN(G)$ and the subspace of operators with compact support in $VN(G)$, respectively.

Returning to locally compact abelian groups, it is worthwhile to mention that while the union of two Ditkin sets is Ditkin, it is an open question whether the union of two spectral sets is again spectral. In a more general context, however, Atzmon [1] has given an example of a regular semisimple commutative Banach algebra with unit and of two sets of synthesis in $\Delta(A)$ the union of which fails to be of synthesis.

Regarding unions of spectral sets and Ditkin sets for Fourier algebras, we now have the following results [13, Theorems 2.9 amd 2.10].

Theorem 2.2. *Let G be a locally compact group and X an $A(G)$-invariant linear subspace of $VN(G)$. Suppose that E_1 and E_2 are closed subsets of G such that $E_1 \cap E_2$ is X-Ditkin. Then $E_1 \cup E_2$ is an X-spectral set if and only if both E_1 and E_2 are X-spectral sets.*

Theorem 2.3. *Let G and X be as in Theorem 2.2, and let E and F be closed subsets of G such that $E \cap F$ is an X-Ditkin set. Then $E \cup F$ is X-Ditkin if and only if both E and F are X-Ditkin sets.*

The preceding two theorems have been known before in the special case where $X = VN(G)$ [26, Theorems 1 and 4]. Such results can be used in both directions. In particular, it follows that, if $A(G)$ has an approximate identity, then each open and closed subset of G is a Ditkin set. Moreover, under the same hypothesis, it follows that finite subsets of G are spectral sets, since singletons are known to be sets of synthesis [5, Corollaire 4.10].

As pointed out in the introduction, when A is a locally compact abelian group, a second possibility to produce new sets of synthesis or Ditkin sets for $L^1(A)$ is to apply injection and projection theorems for such sets. To establish similar results for Fourier algebras turns out to be considerably more difficult and so far, as we shall outline in the sequel, there are only partial analogues due to Lohoué [16], Derighetti [4] and Kaniuth and Lau [13, 14].

We start with projection theorems. Thus, let G be a locally compact group, N a closed normal subgroup and $q : G \to G/N$ the quotient homomorphism.

The problem is whether, for a closed subset E of G/N, E is a (local) spectral set or (local) Ditkin set for $A(G/N)$ if and only if $q^{-1}(E)$ is a (local) spectral set or (local) Ditkin set for $A(G)$. The main difficulty in relating $A(G)$ and $A(G/N)$ is that, except when N is compact, there is no homomorphism from $A(G)$ onto $A(G/N)$. However, there is a natural homomorphism from $A(G) \cap C_c(G)$ onto $A(G/N) \cap C_c(G/N)$ given by $u \to T_N u$, where $T_N u(xN) = \int_N u(xn)dn, x \in G$. This homomorphism has been exploited by Lohoué to prove the following projection theorem for local spectral sets [16, Théorème].

Theorem 2.4. *Let G be a locally compact group, N a closed normal subgroup of G and $q : G \to G/N$ the quotient homomorphism. Then, for any closed subset E of G/N, E is a local spectral set for $A(G/N)$ if and only if $q^{-1}(E)$ is a local spectral set for $A(G)$.*

To prepare for the setting of injection theorems, let H be a closed subgroup of the locally compact group G, and let

$$r : A(G) \to A(H),\ u \to u|H$$

be the restriction map. r is norm decreasing and surjective. More precisely, given $v \in A(H)$, there exists $u \in A(G)$ such that $r(u) = v$ and $\|u\|_{A(G)} = \|v\|_{A(H)}$ [9, Theorem 1b; 21, Theorem 4.21]. Thus the adjoint map

$$r^* : VN(H) \to VN(G), \langle r^*(S), u\rangle = \langle S, r(u)\rangle,$$

$u \in A(G), S \in VN(H)$, is injective. The range of r^* equals $VN_H(G)$, the weak-$*$-closure of the linear span of all operators $\lambda(h), h \in H$, in $VN(G)$. Moreover, r^* maps the subspace of operators with compact support in $VN(H)$ onto the subspace of operators with compact support in $VN_H(G)$.

For any $A(G)$-invariant subspace X of $VN(G)$, let

$$X_H = r^{*^{-1}}(X),$$

an $A(H)$-invariant subspace of $VN(H)$. Now we are ready to formulate the injection theorem for X-spectral sets [13, Theorem 3.4].

Theorem 2.5. *Let X be an $A(G)$-invariant linear subspace of $VN(G)$. Let H be a closed subgroup of G and E a closed subset of H. Then E is an X-spectral set for $A(G)$ if and only if E is an X_H-spectral set for $A(H)$.*

The proof exploits properties of the map r^* as well as the fact that the subgroup H is a set of synthesis for $A(G)$ [25, Theorem 3]. Thus, as special cases, we obtain injection theorems for spectral sets and for local spectral sets. The latter has previously been shown by Derighetti [4, Proposition 8].

An injection theorem for local Ditkin sets has been proved by Derighetti [4, Théorème 12] whenever the subgroup H is normal in G. Recently, this theorem was generalized to the effect that the hypothesis that H be normal is weakened and that X-Ditkin sets, for arbitrary X, are considered.

To elaborate the condition on H, we have to introduce some more notation. Let $P(G)$ denote the set of all continuous positive definite functions on G, and, for a closed subgroup H of G, let

$$P_H(G) = \{u \in P(G) : u(h) = 1 \text{ for all } h \in H\}.$$

We say that G has the *H-separation property* if for every $x \in G, x \notin H$, there exists $u \in P_H(G)$ such that $u(x) \neq 1$. When G has the H-separation property for every closed subgroup H of G, we refer to G as a group with the *separation property.* If H is either normal, or compact, or open in G, then G has the H-separation property. Such subgroups H subsume in the class of neutral subgroups which are defined as follows. A closed subgroup H of G is called *neutral* in G if there exists a neighbourhood basis $\mathcal{V}$ of the identity of G such that $VH = HV$ for all $V \in \mathcal{V}$. Now, if G is any locally compact group and H a neutral subgroup of G, then G has the H-separation property [14, Proposition 2.2]. On the other hand, for connected groups the separation property to hold is a very restrictive condition. Indeed, by Theorem 1.1 of [14], an almost connected locally compact group G has the separation property if and only if G contains an open normal subgroup N of finite index such that N is a direct product of a compact group and a vector group.

Returning to $A(G)$, the following injection theorem for X-Ditkin sets has been proved in [14, Theorem 3.5].

Theorem 2.6. *Let G be a locally compact group and let X be an $A(G)$-invariant linear subspace of $VN(G)$. Let H be a closed subgroup of G and E a closed subset of H.*

(i) *If E is X-Ditkin for $A(G)$, then E is X_H-Ditkin for $A(H)$.*

(ii) *Suppose that G has the H-separation property and that $u \in \overline{uA(G)}$ for every $u \in I(H)$. Then, if E is X_H-Ditkin for $A(H)$, then it is also X-Ditkin for $A(G)$.*

Since, due to the regularity of $A(G)$, for each compactly supported function $u \in A(G)$ there exists $v \in A(G)$ such that $u = uv$, Theorem 2.6 includes Derighetti's injection theorem for local Ditkin sets alluded to above.

In establishing Theorem 2.6, rather than the separation property itself the following equivalent property is used. There exists a projection P from $VN(G)$ onto $VN_H(G)$ such that, in the weak-$*$-operator topology on $\mathcal{B}(VN(G))$, P is the limit of operators $T \to u \cdot T$, where $u \in P_H(G)$.

We finish this section by pointing out that the H-separation property of a locally compact group G deserves further investigation since it appears to play an important role in the ideal theory of Fourier algebras. For instance, it has been shown in [14, Theorem 3.4] that if G has the H-separation property, then the ideal $I(H)$ has an approximate identity with norm bound 2, the best possible bound whenever G/H is infinite.

3. L^1-ALGEBRAS

In this section we turn to L^1-algebras of (non-abelian) locally compact groups and discuss analogous issues as in the previous section for Fourier algebras. To start with, however, let A be an arbitrary semisimple Banach

$*$-algebra, and let $\widehat{A}$ denote the set of equivalence classes of irreducible $*$-representations of A. The primitive ideal space of A, $\mathrm{Prim}_* A$, consists of all kernels, $\ker\pi, \pi \in \widehat{A}$, and carries the hull-kernel topology. For each closed subset E of $\mathrm{Prim}_* A$, let

$$k(E) = \cap\{P : P \in E\},$$

the largest ideal of A with hull equal to E. Whenever $k(E)$ is the only closed ideal of A with hull E, then E is called a *spectral set* (or *set of synthesis*) for A. Also, we say that sepctral synthesis holds for A if every closed subset of $\mathrm{Prim}_* A$ is a spectral set.

Now, let G be a locally compact group and recall that there is a one-to-one correspondence between $\widehat{G}$, the set of equivalence classes of irreducible unitary representations of G, and $\widehat{L^1(G)}$. When G is type I and $L^1(G)$ is $*$-regular, the map $\pi \to \ker\pi$ from $\widehat{G}$ onto $\mathrm{Prim}_* L^1(G)$ is a homeomorphism and $\widehat{G}$ and $\mathrm{Prim}_* L^1(G)$ are usually identified.

It is easy to see that if G is compact, and hence $\mathrm{Prim}_* L^1(G)$ is discrete, then spectral synthesis synthesis holds for $L^1(G)$. However, it is worth mentioning that spectral synthesis may fail for a semisimple Banach $*$-algebra with discrete primitive ideal space. An example has been presented in [22]. The obvious question is whether spectral synthesis for $L^1(G)$ forces the locally compact group G to be compact. Somewhat surprising, the answer is negative. In [6] the following example was given of a non-compact locally compact group for which spectral synthesis holds.

Example 3.1. Let p be a prime and let N be the field of p-adic numbers. Let K denote the subset of elements of N of valuation 1. Then K is a compact group under multiplication. Form the semi-direct product $G = K \ltimes N$, where K acts on the additive group N by multiplication. The group G is often referred to as Fell's example of a non-compact group with countable dual. In fact,

$$\widehat{G} = \widehat{K} \cup \{\pi_j : j \in \mathbb{Z}\},$$

where each π_j is induced from some character of N. Both $\widehat{K}$ and $\{\pi_j : j \in \mathbb{Z}\}$ are discrete, $\widehat{K}$ is closed and a sequence $(\pi_{j_k})_k$ converges to some (and hence all) $\sigma \in \widehat{K}$ if and only if $j_k \to -\infty$.

Using this description of the topology of $\widehat{G}$, the projection theorem for spectral sets (see Theorem 3.5 below) and the fact that $L^1(G)$ has the so-called Wiener property (compare [17]), it is not difficult to show that every closed subset of $\widehat{G} = \mathrm{Prim}_* L^1(G)$ is a spectral set.

When looking carefully at the preceding example, an interesting problem arises. Suppose that $L^1(G)$ contains a closed ideal I such that $\mathrm{Prim}_* I$ and $\mathrm{Prim}_* L^1(G)/I$ are both discrete. Does then spectral synthesis hold for $L^1(G)$? An affirmative answer would cover Example 3.1.

Notice that the group G of Example 3.1 has an abelian normal subgroup with compact abelian quotient group. In contrast, for nilpotent locally compact groups it can be deduced from Malliavin's theorem that spectral synthesis fails for $L^1(G)$ whenever G is non-compact [12]. In the course of investigations to relate spectral synthesis to properties of certain topologies on the space of all closed ideals of the enveloping C^*-algebra $C^*(G)$, this latter result was recently generalized as follows [6, Theorem 3.7].

Theorem 3.2. *Let G be a locally compact group and suppose that G contains a compact normal subgroup K such that N/K is a finite extension of a nilpotent group. If spectral synthesis holds for $L^1(G)$, G must be compact.*

Apart from nilpotent groups this comprises, for instance, the class of Moore groups (that is, groups with finite dimensional irreducible representations).

An apparently very difficult problem for L^1-algebras of locally compact groups G is the existence of a smallest (closed) ideal $j(E)$ for a given hull $E \subseteq \mathrm{Prim}_* L^1(G)$. The next theorem is due to Ludwig [18].

Theorem 3.3. *Let G be a locally compact group of polynomial growth, and suppose that $L^1(G)$ is symmetric. Then, given a closed subset E of $\mathrm{Prim}_* L^1(G)$, there exists a smallest closed ideal whose hull is equal to E.*

We remind the reader that a locally compact group G is polynomially growing if for every compact subset K of G, the Haar measure of powers $K^n, n \in \mathbb{N}$, grows at most polynomially in n. Moreover, a Banach $*$-algebra A is called symmetric if every selfadjoint element of A has a real spectrum. Several classes of locally compact groups, among them nilpotent groups and motion groups, satisfy both of these hypotheses (see [17]). A main tool in proving Theorem 3.3 is Dixmier's functional calculus for groups of polynomial growth. Unfortunately, the ideal $j(E)$ is only described in terms of a generating set. This fact seems to be responsable for that, so far, there are no results on unions of spectral sets.

On the other hand, the existence of such smallest closed ideals turned out to be very useful in establishing injection and projection theorems for spectral sets. Naturally, for L^1-algebras of non-abelian locally compact groups, the setting is much more complicated than for Fourier algebras, and this is what we are now going to describe.

Let N be a closed normal subgroup of G, and let $q : G \to G/N$ denote the quotient homomorphism and $T : L^1(G) \to L^1(G/N)$ the corresponding homomorphism of L^1-algebras. Then there is a canonical embedding

$$i : \mathrm{Prim}_* L^1(G/N) \to \mathrm{Prim}_* L^1(G)$$

given by $i(\ker \pi) = \ker(\pi \circ q) = T^{-1}(\ker \pi)$. Then $i(\mathrm{Prim}_* L^1(G/N))$ is closed in $\mathrm{Prim}_* L^1(G)$ and i is a homeomorphism onto its range. In this situation, Hauenschild and Ludwig have proved the following injection theorem for spectral sets [8, Theorem 3.2].

Theorem 3.4. *Let N be a closed normal aubgroup of the locally compact group G, and let F be a closed subset of $\mathrm{Prim}_* L^1(G/N)$ and $E = i(F) \subseteq \mathrm{Prim}_* L^1(G)$.*

(i) *If E is a spectral set, then so is F.*

(ii) *Let F be a spectral set and suppose that G has polynomial growth and $L^1(G)$ is symmetric. Then E is a spectral set.*

In (ii), the condition that $L^1(G)$ is symmetric and G has polynomial growth can be replaced by the hypothesis that $i(\text{Prim}_* L^1(G/N))$, the hull of the kernel of T, is a spectral set for $L^1(G)$ [8]. However, the only case where $i(\text{Prim}_* L^1(G/N))$ is known to be a spectral set seems to be the indicated one.

Let us now turn to projection theorems. As before, let N be a closed normal subgroup of G. The action of G on N by inner automorphisms gives rise to actions of G on $L^1(N)$ and hence on the primitive ideal space $\text{Prim}_* L^1(N)$. Now, if π is a representation of G, then the L^1-kernel of $\pi|N$ is a G-invariant ideal of $L^1(N)$. In particular, relating spectral sets for $L^1(G)$ to spectral sets for $L^1(N)$ leads to consider G-invariant subsets of $\text{Prim}_* L^1(N)$.

Hauenschild and Ludwig have been the first to accomplish a projection theorem for spectral sets for non-abelian locally compact groups [8, Theorem 2.6]. Their result was subsequently improved by Bekka [2] as follows.

Theorem 3.5. *Let G be a locally compact group and N a closed normal subgroup of G. Let F be a closed G-invariant subset of* $\text{Prim}_* L^1(N)$ *and*

$$E = \{\ker \pi : \pi \in \widehat{G} \text{ such that } \pi|N(k(F)) = 0\}.$$

(i) *Suppose that N has polynomial growth and $L^1(N)$ is symmetric. If E is a spectral set, then so is F.*

(ii) *Suppose that G has polynomial growth and $L^1(G)$ is symmetric. If F is a spectral set, then E is a spectral set.*

Part (i) is entirely due to Hauenschild and Ludwig. For the more sophisticated part (ii), they needed an additional hypothesis which Bekka was able to remove.

To indicate the difficulty, consider a G-invariant closed ideal J of $L^1(N)$. Regarding $L^1(N)$ as a subspace of $M(G)$, naturally associated to J is a closed ideal $e(J)$ of $L^1(G)$, the extension ideal. Indeed, $e(J)$ is defined to be the closed linear span of $C_c(G) * J$ in $L^1(G)$. Retaining the notation of Theorem 3.5, if $F = h(J)$ then $E = h(e(J))$. The main problem now is to show that $e(j(F)) = j(E)$. In [8] this equality was proved when G/N is solvable, and in some other less important cases. Taking into account that groups with polynomial growth are amenable, the essential missing step was to deal with compact quotients G/N. Bekka managed this by extending Dixmier's functional calculus to matrix valued functions.

Neither part (i) nor part (ii) of the theorem holds for arbitrary G or N (see [2] and [8]).

In Example 3.1, we have already given a sample of possible applications of the projection theorem. To conclude, we mention three further examples concerning singletons in $\text{Prim}_* L^1(G)$. In treating two of them, (ii) and (iii), the projection theorem is substantial.

Example 3.6. (i) If G is a finitely generated nilpotent discrete group, then singletons in $\mathrm{Prim}_* L^1(G)$ are Ditkin sets. In fact, more generally, the so-called Helson-Reiter theorem holds for $L^1(G)$ [11].

(ii) In contrast, when G is a connected and simply connected nilpotent Lie group of nilpotence class ≥ 3, then singletons in $\mathrm{Prim}_* L^1(G)$ need not be spectral sets [19].

(iii) Let $G_n = SO(n) \ltimes \mathbb{R}^n$, $n \geq 2$, be the Euclidean motion group in dimension n. Using the two facts that the non-trivial orbits in $\widehat{\mathbb{R}^n} = \mathbb{R}^n$ are spheres and that $S^{n-1} \subseteq \mathbb{R}^n$ is a set of synthesis precisely when $n = 2$, it can be shown (see [2]) that all singletons in $\mathrm{Prim}_* L^1(G_2)$ are sets of synthesis, whereas, for $n \geq 3$, $\{\pi\} \subseteq \widehat{G_n}$ is spectral only if $\pi \in \widehat{SO(n)}$.

References

[1] A. Atzmon, *On the union of sets of synthesis and Ditkins condition in regular Banach algebras*, Bull. Amer. Math. Soc. **2** (1980), 317-320.

[2] M.B. Bekka, *The projection theorem for spectral sets*, Monatsh. Math. **101** (1986), 1-10.

[3] J. Benedetto, *Spectral synthesis*, Academic Press, 1975.

[4] A. Derighetti, *Quelques observations concernant les ensembles de Ditkin d'un groupe localement compact*, Monatsh. Math. **101** (1986). 95-113.

[5] P. Eymard, *L'algèbre de Fourier d'un groupe localement compact*, Bull. Soc. Math. France **92** (1964), 181-236.

[6] J.F. Feinstein, E. Kaniuth and D.W.B. Somerset, *Spectral synthesis and topologies on ideal spaces for Banach *-algebras*, submitted.

[7] B. Forrest, *Fourier analysis on coset spaces*, Rocky Montain J. Math. **28** (1998), 173-190.

[8] W. Hauenschild and J. Ludwig, *The injection and the projection theorem for spectral sets*, Monatsh. Math. **92** (1981), 167-177.

[9] C. Herz, *Harmonic synthesis for subgroups*, Ann. Inst. Fourier **23** (1973), 91-123.

[10] E. Hewitt and K.A. Ross, *Abstract harmonic analysis. II*, Springer, 1970.

[11] E. Kaniuth, *Ideals in group algebras of finitely generated FC-nilpotent discrete group*, Math. Ann. **243** (1980), 97-108.

[12] E. Kaniuth, *The Helson-Reiter theorem for a class of nilpotent discrete groups*, Math. Proc. Camb. Phil. Soc. **122** (1997), 95-103.

[13] E. Kaniuth and A.T. Lau, *Spectral synthesis for $A(G)$ and subspaces of $VN(G)$*, submitted.

[14] E. Kaniuth and A.T. Lau, *A separation property of positive definite functions on locally compact groups and applications to Fourier algebras*, submitted.

[15] H. Leptin, *Sur l'algèbre de Fourier d'une groupe localement compact*, C.R. Acad. Sci. Paris, Sér. A **266** (1968), 1180-1182.

[16] N. Lohoué, *Remarques sur les ensembles de synthèse des algèbre de groupe localement compact*, J. Funct. Anal. **13** (1973), 185-194.

[17] J. Ludwig, *A class of symmetric and a class of Wiener group algebras*, J. Funct. Anal. **31** (1979), 187-194.

[18] J. Ludwig, *Polynomial growth and ideals of group algebras*, Manuscr. Math. **30** (1981), 215-221.

[19] J. Ludwig, *On the spectral synthesis problem for points in the dual of a nilpotent Lie group*, Ark. Mat. **21** (1983), 127-144.

[20] P. Malliavin, *Impossibilité de la synthèse sur les groupes abéliens non compacts*, Inst. Hautes Ét. Sci. Publ. Math. **2** (1959), 61-68.

[21] J.R. McMullen, *Extensions of positive definite functions*, Mem. Amer. Math. Soc. **117**, 1972.

[22] H. Mirkil, *A counterexample to discrete spectral synthesis*, Compos. Math. **14** (1960), 269-273.

[23] H. Reiter, *Contributions to harmonic analysis. VI*, Ann. Math. (2) **77** (1963), 552-562.

[24] H. Reiter, *Classical harmonic analysis and locally compact groups*, Oxford, 1968.

[25] M. Takesaki and N. Tatsuuma, *Duality and subgroups. II*, J. Funct. Anal. **11** (1972), 184-190.

[26] C.R. Warner, *A class of spectral sets*, Proc. Amer. Math. Soc. **57** (1976), 99-102.

[27] E.I. Zelmanov, *On periodic compact groups*, Israel J. Math. **77** (1992), 83-95.

FACHBEREICH MATHEMATIK/INFORMATIK, UNIVERSITÄT PADERBORN, D-33095 PADERBORN, GERMANY

E-mail address: kaniuth@math.uni-paderborn.de

KA-wavelets on semisimple Lie groups and quasi-orthogonality of matrix coefficients

Takeshi Kawazoe
Keio University at Fujisawa
5322 Endo, Fujisawa, 252-8520 Japan
kawazoe@sfc.keio.ac.jp

§1 Introduction.

First we brief the history of continuous wavelet transforms. Originally the (continuous) wavelet transform, introduced by Morlet around 1980, was the following one. We denote by $H^2(\mathbf{R})$ the closed subspace of $L^2(\mathbf{R})$ consisting of all L^2 functions f on $\mathbf{R}$ with $\text{supp}(\hat{f}) \subset [0, \infty)$, and we fix $\psi \in H^2(\mathbf{R})$ satisfying the so-called admissible condition

$$c_\psi = \int_0^\infty \frac{|\hat{\psi}(\lambda)|^2}{\lambda} d\lambda < \infty.$$

Then the wavelet transform W_ψ associated to ψ is defined on $H^2(\mathbf{R})$ as

$$W_\psi f(u, v) = \int_{-\infty}^\infty f(x) e^{-u/2} \bar{\psi}(e^{-u}x + v) dx \quad (u, v \in \mathbf{R}).$$

Theorem 1.1. W_ψ is an isometric isomorphism from $H^2(\mathbf{R})$ onto $L^2(\mathbf{R}^2)$: For any $f \in H^2(\mathbf{R})$

$$\|f\|^2 = \frac{1}{c_\psi} \|W_\psi f\|^2.$$

Furthermore, for any $f \in H^2(\mathbf{R})$ and $x \in \mathbf{R}$ at which f is continuous,

$$f(x) = \frac{1}{c_\psi} \int_{-\infty}^\infty \int_{-\infty}^\infty (W_\psi f)(u, v) e^{-u/2} \bar{\psi}(e^{-u}x + v) du dv.$$

In [GMP] Grossmann-Morlet-Paul pointed out the group-theoretical interpretation of the wavelet transform W_ψ. Let G be the affine group $\mathbf{R}^2$ with multiplication law:

$$(u, v)(u', v') = (u + u', e^{-u'}v + v'),$$

and let $(T, H^2(\mathbf{R}))$ be an irreducible unitary representation of G defined by

$$(T(u,v)f)(x) = e^{-u/2} f(e^{-u}x + v) \quad (f \in H^2(\mathbf{R})).$$

In this scheme W_ψ can be rewritten as

$$W_\psi f(u,v) = \langle f, T(u,v)\psi\rangle,$$

where $\langle \cdot,\cdot \rangle$ is the inner product of $H^2(\mathbf{R})$. Furthermore, since $dudv$ is a left invariant Haar measure on G, Theorem 1.1 yields the square-integrability and the orthogonality of the matrix coefficients $\langle f, T(u,v)\psi\rangle$ of T on G. In this sense the theory of the continuous wavelet transform W_ψ on $H^2(\mathbf{R})$ is nothing but the one of the square-integrable representation $(T, H^2(\mathbf{R}))$ of G.

General theory of square-integrable representations of locally compact groups has been investigated by various mathematicians; Weyl [W] for compact groups, Godement [G] for unimodular locally compact groups, and Duflo-Moore [DM] for general locally compact groups. Explicit theory based on the construction of the square-integrable representations was obtained by Harish-Chandra [HC] for semisimple Lie groups and by Moore-Wolf [MW] for nilpotent groups.

How to extend the theory of square-integrable representations of locally compact groups G ? One of the ways is to replace the square-integrability on G by the one on a quotient space G/H for a closed subgroup H of G. More generally, find a representation $(T, \mathcal{H})$ of G, a measurable subset (S, ds) of G, and $\psi \in \mathcal{H}$ for which, for any $f \in \mathcal{H}$

$$(\star) \quad \|f\|^2 = \frac{1}{c_{S,\psi}} \int_S |\langle f, T(s)\psi\rangle|^2 ds.$$

Then, it is easy to see that the transform defined by $\langle f, T(s)\psi\rangle$ is an isometric isomorphism from $\mathcal{H}$ onto $L^2(S, ds)$, and each $f \in \mathcal{H}$ has an L^2 decomposition in the weak sense:

$$f = \frac{1}{c_{S,\psi}} \int_S \langle f, T(s)\psi\rangle T(s)\psi ds.$$

For the last decade researches has been done in this scheme and many wavelet transforms has been constructed on locally compact groups, for example, on $\mathbf{R}_+^* \times SO(n)$ by Murenzi [M], on $\mathbf{R}_+^* \times SO(1,n)$ by A-J. Unterberger [U], on $\mathbf{R}_+^* \times SO(1,n) \times \mathbf{R}^{n+1}$ by Bhonke [B], on $S \times V$, V is a vector space and S is a subgroup of $GL(V)$, by De Bièvre [DB], on $SO(2,1) \times \mathbf{R}^3$ by Ali, Antoine, Gazeau [AAG], on $\mathbf{R}_+^* \times SO(n) \times H_n$ by Kalisa-Toréssani [KT], Toréssani [T1,2], on $GL(n, \mathbf{R})$ by Bernier-Taylor [BT], on $SO(2,1)$ by Wu-Zhong [WZ], and on Iwasawa AN groups by Kawazoe [K3] and Liu [L].

In this paper we shall consider the case that G is a semisimple Lie group and $S = KA$, where K and A are respectively the maximal compact and abelian subgroups of G. More precisely, let G be a semisimple Lie group with finite center and $G = KAK$ the Cartan decomposition of G. dg denotes a Haar measure on G and $dg = D(a)dkdadk$ the corresponding decomposition of dg. Then we take $S = KA$ and $ds = D(a)dkda$ in the above scheme, and we try to find a representation $(T, \mathcal{H})$ of G and $\psi \in \mathcal{H}$ satisfying $(\star)$. Unfortunately, the condition $(\star)$ is very strong, so I feel that we have no answer for T and ψ. Therefore, we shall consider a weak condition; there exist constants $0 < C_1, C_2 < \infty$ such that

$$(\star\star) \quad C_1\|f\|^2 \leq \int_S |\langle f, T(s)\psi\rangle|^2 ds \leq C_2\|f\|^2$$

and we shall obtain a sufficient condition on ψ for which $\langle f, T(s)\psi\rangle$ satisfies $(\star\star)$ (see Theorem 3.1). In §4 we shall treat the case of $G = SU(1,1)$ and $(T_{1/2}, \mathcal{H}_{1/2})$ the limit of the holomorphic discrete series of G. We note that $T_{1/2}$ is not square-integrable on G. Then we shall find a $\psi \in \mathcal{H}_{1/2}$ satisfying $(\star\star)$. Moreover, we shall deduce that, if we ignore a finite dimensional subspace of $\mathcal{H}_{1/2}$, then we can find a $\psi \in \mathcal{H}_{1/2}$ satisfying $(\star)$ (see Theorem 4.4). In this process we use the facts that some differences of the matrix coefficients of $T_{1/2}$ are square-integrable on $\mathbf{R}$ with respect to $D(a)da$ and moreover, they satisfy a quasi-orthogonality. These facts are summarized in Lemmas 4.1, 4.2, and 4.3.

After the lecture, the author noticed that J.-P. Antoine and P. Vandergheynst [AV1,2] had the same idea and they obtained an example in the case of $SO(3,1)$.

§2. Notation.

Let G be a semisimple Lie group with finite center and $G = KAN$ the Iwasawa decomposition of G. Let Σ be the set of roots for (G, A) and Σ^+ the one of positive roots corresponding to N. Let A^+ denote the closed positive Weyl chamber in A and $G = KA^+K$ the Cartan decomposition of G. Let

dg denote a Haar measure on G, and dk, da, and dn ones for K, A, and, N respectively. We normalize dk as $\int_K dk = 1$. According to the Iwasawa and Cartan decompositions of G, there are decompositions of dg such that

$$dg = e^{\rho(\log a)} dkdadn = D(a) dkdadk',$$

where $\rho = \frac{1}{2}\sum_{\alpha\in\Sigma^+} m_\alpha \alpha$ and

$$D(a) = \prod_{\alpha\in\Sigma^+} \left(\sinh \alpha(\log a)\right)^{m_\alpha},$$

m_α stands for the multiplicity of α.

§3. KA-wavelets.

Let $(T, \mathcal{H})$ be a unitary representation of G and

$$\mathcal{H} = \oplus_{\tau\in\hat{K}} \mathcal{H}_\tau,$$

the K-type decomposition of $\mathcal{H}$. In the following argument we assume that

$$[T, \tau] \leq 1,$$

and we denote by $\hat{K}_T$ the set of all $\tau \in \hat{K}$ such that $[T, \tau] = 1$. Then, as a representation of K, $(T|_K, \mathcal{H}_\tau)$ is equivalent with τ for each $\tau \in \hat{K}_T$. We choose a complete orthonormal basis of $\mathcal{H}$ such that

$$\{e_n^\tau; e_n^\tau \in \mathcal{H}_\tau, 1 \leq n \leq \dim \tau,\ \tau \in \hat{K}_T\}$$

and we denote by I the set of the indexes $\{(\tau, n); 1 \leq n \leq \dim \tau,\ \tau \in \hat{K}_T\}$. For each $f \in \mathcal{H}$ the Fourier expansion of f is given by

$$f = \sum_{(\tau,n)\in I(f)} f_n^\tau e_n^\tau,$$

where $f_n^\tau = \langle f, e_n^\tau\rangle_{\mathcal{H}}$ and $I(f)$ the subset of I consisting of all (τ, n) such that $f_n^\tau \neq 0$. Here we put

$$I_A(f) = \{(\tau, n); (T(\cdot)f)_n^\tau = \langle T(\cdot)f, e_n^\tau\rangle \text{ is not identically } 0 \text{ on } A\}.$$

We say that $\psi \in \mathcal{H}$ is admissible if there exist constants $0 < C_1, C_2 < \infty$ such that, if $(\tau, n) \in I_A(\psi)$,

$$C_1 \leq c_{\psi,\tau,n} = \int_A |\langle T(a)\psi, e_n^\tau\rangle|^2 D(a)da \leq C_2.$$

We put

$$\mathcal{H}_\psi = \{f \in \mathcal{H}; I(f) \subset I_A(\psi)\}.$$

Then, by using the bounded constants $c_{\psi,\tau,n}$ we shall define a Fourier multiplier M_ψ on $\mathcal{H}_\psi$ as follows. For each $f = \sum_{(\tau,n)\in I(f)} f_n^\tau e_n^\tau$ in $\mathcal{H}_\psi$

$$M_\psi f = \sum_{(\tau,n)\in I(f)} c_{\psi,\tau,n}^{-1/2} f_n^\tau e_n^\tau.$$

Theorem 3.1. Let ψ be admissible in $\mathcal{H}$. Then for any $f \in \mathcal{H}_\psi$

(1)

$$C_1\|f\|^2 \leq \int\int_{KA} |\langle f, T(ka)\psi\rangle|^2 D(a)dkda \leq C_2\|f\|^2,$$

(2)

$$\|f\|^2 = \int\int_{KA} |\langle f, M_\psi T(ka)\psi\rangle|^2 D(a)dkda,$$

(3)

$$f = \int\int_{KA} \langle f, M_\psi T(ka)\psi\rangle M_\psi T(ka)\psi \ D(a)dkda.$$

Proof. We note that

$$T(k^{-1})f = \sum_{(\tau,n)\in I(f)} f_n^\tau T(k^{-1})e_n^\tau = \sum_{(\tau,n)\in I(f),(\tau',n')\in I} f_n^\tau \langle T(k^{-1})e_n^\tau, e_{n'}^{\tau'}\rangle e_{n'}^{\tau'}.$$

Then the orthogonality of the matrix coefficients of $T|_K$ yields that

$$\begin{aligned}
&\int\int_{KA} |\langle f, T(ka)\psi\rangle|^2 D(a)dkda \\
&= \int_A \sum_{(\tau,n)\in I(f)} |f_n^\tau|^2 |\langle T(a)\psi, e_n^\tau\rangle|^2 D(a)da \\
&= \sum_{(\tau,n)\in I(f)} |f_n^\tau|^2 \left(\int_A |\langle T(a)\psi, e_n^\tau\rangle|^2 D(a)da\right)
\end{aligned}$$

Since

$$\|f\|^2 = \sum_{(\tau,n)\in I(f)} |f_n^\tau|^2 \quad \text{and} \quad I(f) \subset I_A(\psi),$$

(1) easily follows from the definition of the admissible vector ψ. We replace f by $M_\psi f$ in the above calculation. Then $|f_n^\tau|^2$ in the last equation turns to $|f_n^\tau|^2 c_{\psi,\tau,n}^{-1}$ and then, $c_{\psi,\tau,n}^{-1}$ cancels the integral over A. Thereby (2) follows. As for (3) we put $\mathcal{H}(f) = \mathrm{Span}\{e_n^\tau; (\tau, n) \in I(f)\}$ and define an opertor Q on $\mathcal{H}(f)$ by

$$h \mapsto \int\int_{KA} \langle f, M_\psi T(ka)\psi\rangle \langle h, M_\psi T(ka)\rangle \psi \ D(a)dkda.$$

Then (2) and the Schwarz inequality yield that Q is bounded and $\|Q\| \leq \|f\|^2$, and thereby, there exists $f_0 \in \mathcal{H}(f)$ such that $Q(h) = \langle h, f_0\rangle$ and $\|f_0\| = \|Q\|$. Since $Q(f) = \langle f, f_0\rangle = \|f\|^2$ by (2), it easily follows that $f = f_0$ (cf. [K]). Clearly, $Q(h) = \langle h, f\rangle$ means (3).

Remark 3.2. When $(T, \mathcal{H})$ is an irreducible square-integrable representation of G, it is well-known that each $\psi \in \mathcal{H}$ is admissible and satisfies

$$c_{\psi,\tau,n} = d_T^{-1}\|\psi\|^2,$$

where c_T is the formal degree of T (cf.[V]). Furthermore, applying the orthogonality of the matrix coefficients on G, we can replace the integrals over KA in Theorem 3.1 by the ones over G.

§4. Example in $SU(1,1)$.

Let G be $SU(1,1)$. Then

$$K = \{k_\theta = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}; 0 \leq \theta < 4\pi\},$$

$$A = \{a_t = \begin{pmatrix} \cosh t/2 & \sinh t/2 \\ \sinh t/2 & \cosh t/2 \end{pmatrix}; t \in \mathbf{R}\},$$

and $A^+ = \{a_t; t > 0\}$. In what follows we put

$$x = \tanh t.$$

Let $(T_h, \mathcal{H}_h)$ $(h \in \mathbf{Z}/2, h \geq 1)$ be the holomorphic discrete series of G realized on the weighted Bergman space $\mathcal{H}_h$ on the unit disk $D = G/K$:

$$\mathcal{H}_h = \{f : D \to \mathbf{C}; f \text{ is holomorphic on } D \text{ and}$$

$$\|f\|_h^2 = \Gamma(2h-1)^{-1} \int_D |f(z)|^2 (1-|z|^2)^{2(h-1)} dz < \infty\},$$

and $(T_{1/2}, \mathcal{H}_{1/2})$ the limit of holomorphic discrete series of G realized on the Hardy space $\mathcal{H}_{1/2}$ on D:

$$\mathcal{H}_{1/2} = \{f : D \to \mathbf{C}; f \text{ is holomorphic on } D \text{ and}$$

$$\|f\|_{1/2}^2 = \lim_{h \to 1/2} \|f\|_h^2 < \infty\}.$$

For $h \in \mathbf{Z}/2, h \geq 1/2$ we denote by $\langle \cdot, \cdot \rangle_h$ the inner product of $\mathcal{H}_h$ and we put

$$e_n^h(z) = \left(\frac{\Gamma(2h+n)}{\Gamma(2h)\Gamma(n+1)} \right)^{1/2} z^n \quad (n \in \mathbf{N}).$$

Then $\{e_n^h; n \in \mathbf{N}\}$ is an orthonormal basis of $\mathcal{H}_h$. For simplicity we denote

$$\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{1/2} \text{ and } e_n(z) = e_n^{1/2}(z) = z^n.$$

According to this basis the matrix coefficients of T_h are given as follows (see [Sa]):

$$\begin{aligned} \langle T_h(g) e_n^h, e_m^h \rangle_h &= e^{i(n\theta + m\theta')} \langle T_h(a_t) e_n^h, e_m^h \rangle_h \quad (g = k_\theta a_t k_{\theta'}) \\ &= e^{i(n\theta + m\theta')} M(h; n, m; x), \end{aligned}$$

where for $n \geq m$,

$$M(h; n, m; x) = C_{n,m}^h (1-x^2)^h (-x)^{n-m} F(-m, n+2h, n-m+1; x^2),$$

$$C_{n,m}^h = \left(\frac{\Gamma(n+1)\Gamma(n+2h)}{\Gamma(m+1)\Gamma(m+2h)} \right)^{1/2} \frac{1}{\Gamma(n-m+1)}$$

and $F(a, b, c; x)$ is the hypergeometric function, and for $m > n$ we change n and m by m and n respectively. Since

$$D(a_t) dt = \sinh(2t) dt = \frac{2x}{(1-x^2)^2} dx,$$

$M(h;n,m;x)$ $(n,m \in \mathrm{N})$ are square-integrable on G if and only if $h > 1/2$. Here we note that for $n \geq m$,

$$\begin{aligned}
&\lim_{x\to 1}(1-x^2)^{-h}M(h;n,m;x)\\
=\ & C^h_{n,m}(-1)^n\frac{\Gamma(1-m+n)\Gamma(m+2h)}{\Gamma(2h)\Gamma(n+1)}\\
=\ & (-1)^n\frac{1}{\Gamma(2h)}\left(\frac{\Gamma(n+2h)\Gamma(m+2h)}{\Gamma(n+1)\Gamma(m+1)}\right)^{1/2}\\
=\ & (-1)^n D^h_{n,m}
\end{aligned}$$

and for $m > n$, $\lim_{x\to 1}(1-x^2)^{-h}M(h;n,m;x) = (-1)^m D^h_{m,n} = (-1)^m D^h_{n,m}$ Then we shall define the normalized matrix coefficients $NM(h;n,m,x)$ as

$$NM(h;n,m;x) = (D^h_{n,m})^{-1}M(h;n,m;x)$$

and the differences of the normalized matrix coefficients $DM(h;n,m;x)$ as

$$DM(h;n,m;x) = NM(h;n,m;x) - NM(h;n+2,m;x).$$

The key lemmas are the following.

Lemma 4.1. Let notations be as above. Then

$$DM(h;n,m;x) = \frac{(1-x^2)^{1/2}}{x}$$

$$\times\left(\frac{m}{2h}NM(h+1/2;n,m-1;x) - \frac{m+2h}{2h}NM(h+1/2;n+1,m;x)\right).$$

Proof. We realize T_h on the circle and let $z = e^{i\theta}$ $(0 \leq \theta < 2\pi)$ (see [Sa]). We first note that

$$(D^h_{n,m})^{-1}e^h_n = \left(\frac{\Gamma(2h)\Gamma(m+1)}{\Gamma(m+2h)}\right)^{1/2} z^n,$$

$$(D^h_{n+2,m})^{-1}e^h_{n+2} = \left(\frac{\Gamma(2h)\Gamma(m+1)}{\Gamma(m+2h)}\right)^{1/2} z^{n+2};$$

and moreover,

$$
\begin{aligned}
& T_h(a_t)(z^n - z^{n+2}) \\
= & \frac{1}{(-z\sinh t/2+\cosh t/2)^{2h}}\left(\frac{z\cosh t/2-\sinh t/2}{-z\sinh t/2+\cosh t/2}\right)^n \\
& \qquad \times\left(1-\left(\frac{z\cosh t/2-\sinh t/2}{-z\sinh t/2+\cosh t/2}\right)^2\right) \\
= & \frac{1}{(-z\sinh t/2+\cosh t/2)^{2h}}\left(\frac{z\cosh t/2-\sinh t/2}{-z\sinh t/2+\cosh t/2}\right)^n \\
& \qquad \times\frac{1-z^2}{(-z\sinh t/2+\cosh t/2)^2} \\
= & \frac{1}{\sinh t/2}\frac{1}{(-z\sinh t/2+\cosh t/2)^{2h+1}}\left(\frac{z\cosh t/2-\sinh t/2}{-z\sinh t/2+\cosh t/2}\right)^n \\
& \qquad \times\left(-\left(\frac{z\cosh t/2-\sinh t/2}{-z\sinh t/2+\cosh t/2}\right)+z\right).
\end{aligned}
$$

On the other hand, we easily see that

$$
\begin{aligned}
& \langle\left(\frac{\Gamma(2h)\Gamma(m+1)}{\Gamma(m+2h)}\right)^{1/2} z^{n+1}, e_m^h\rangle_h \\
= & \langle(D_{n+1,m}^{h+1/2})^{-1}e_{n+1}^{h+1/2}, e_m^{h+1/2}\rangle_h \\
= & \frac{m+2h}{2h}\langle(D_{n+1,m}^{h+1/2})^{-1}e_{n+1}^{h+1/2}, e_m^{h+1/2}\rangle_{h+1/2}
\end{aligned}
$$

and

$$
\begin{aligned}
& \langle\left(\frac{\Gamma(2h)\Gamma(m+1)}{\Gamma(m+2h)}\right)^{1/2} z^{n}, e_{m-1}^h\rangle_h \\
= & \langle(D_{n,m-1}^{h+1/2})^{-1}e_{n}^{h+1/2}, e_{m-1}^{h+1/2}\rangle_h \\
= & \frac{m}{2h}\langle(D_{n,m-1}^{h+1/2})^{-1}e_{n}^{h+1/2}, e_{m-1}^{h+1/2}\rangle_{h+1/2}.
\end{aligned}
$$

Then the desired result follows.

Lemma 4.2. Let notations be as above. Then for each $n, m \in \mathbf{N}$,

$$
0 < \int_0^1 DM(h;n,m;x)^2\frac{2x}{(1-x^2)^2}dx < \infty,
$$

and especially, for $m > n$

$$\int_0^1 DM(h;n,m;x)^2 \frac{2x}{(1-x^2)^2}dx$$

$$= \Gamma(2h)^2 2(n+h+1)\frac{\Gamma(m+1)}{\Gamma(m+2h)}\frac{\Gamma(n+1)}{\Gamma(n+2h+2)}.$$

Proof. The case of $m > n$: We note that

$$\frac{(1-x^2)^{1/2}}{x}\frac{m}{2h}NM(h+1/2;n,m-1;x)$$

$$= Ax^{m-n-2}(1-x^2)^{h+1}G_n(m-n+2h,m-n;x^2)$$

and

$$\frac{(1-x^2)^{1/2}}{x}\frac{m+2h}{2h}NM(h+1/2;n+1,m;x)$$

$$= \frac{m+2h}{n+2h+1}Ax^{m-n-2}(1-x^2)^{h+1}G_{n+1}(m-n+2h,m-n;x^2),$$

where

$$A = \frac{\Gamma(2h)\Gamma(m+1)}{\Gamma(n+2h+1)\Gamma(m-n)}$$

and $G_n(x) = G_n(\alpha,\gamma,x)$ ($\alpha = m-n+2h, \gamma = m-n$) is the Jacobi polynomial. Hence,

$$\begin{aligned} I &= \int_0^1 DM(h;n,m;x)^2 \frac{2x}{(1-x^2)^2}dx \\ &= A^2\int_0^1 x^{2(m-n-2)}(1-x^2)^{2h}\left(G_n(x^2) - \frac{m+2h}{n+2h+1}G_{n+1}(x^2)\right)^2 2xdx \\ &= A^2\int_0^1 x^{\gamma-1}(1-x)^{\alpha-\gamma}\left(G_n(x) - \frac{m+2h}{n+2h+1}G_{n+1}(x)\right)^2 \frac{dx}{x}. \end{aligned}$$

We here consider the case of $m > n+1$. Then, $\gamma - 2 = m-n-2 \geq 0$. We note that $G_n^2 = (G_n - 1)G_n + G_n$ and $(G_n - 1)/x$ is the polynomial of degree $n-1$. So the orthogonality relations for the Jacobi polynomials and the definition of $G_n(x)$;

$$G_n(x) = \frac{\Gamma(\gamma)}{\Gamma(\gamma+n)} x^{1-\gamma}(1-x)^{\gamma-\alpha}\left(\frac{d}{dx}\right)^n \left(x^{\gamma+n-1}(1-x)^{\alpha+n-\gamma}\right)$$

yield that

$$\begin{aligned}
& \int_0^1 x^{\gamma-1}(1-x)^{\alpha-\gamma} G_n(x)^2 \frac{dx}{x} \\
= & \int_0^1 x^{\gamma-1}(1-x)^{\alpha-\gamma} G_n(x) \frac{dx}{x} \\
= & \Gamma(m-n)^2 \frac{\Gamma(n+1)\Gamma(n+2h+1)}{\Gamma(m+1)\Gamma(m+2h)} \frac{m}{m-n-1} \\
= & B,
\end{aligned}$$

and similarly,

$$\begin{aligned}
& \int_0^1 x^{\gamma-1}(1-x)^{\alpha-\gamma} G_{n+1}(x)^2 \frac{dx}{x} \\
= & \int_0^1 x^{\gamma-1}(1-x)^{\alpha-\gamma} G_n(x) G_{n+1}(x) \frac{dx}{x} \\
= & \int_0^1 x^{\gamma-1}(1-x)^{\alpha-\gamma} G_{n+1}(x) \frac{dx}{x} \\
= & \frac{n+2h+1}{m+2h} \frac{n+1}{m} B.
\end{aligned}$$

Therefore,

$$\begin{aligned}
I &= A^2 B \left(1 - 2\frac{n+1}{m} + \frac{n+1}{m}\frac{m+2h}{n+2h+1}\right) \\
&= A^2 B \frac{2(m-n-1)(n+h+1)}{m(n+2h+1)}
\end{aligned}$$

and hence, the desired result follows.

In the case of $m = n+1$ we note that $(G_n(x) - G_{n+1}(x))/x$ is a polynomial of degree n and thus, the integral I is well-defined. Then the analytic continuation on γ, letting $\gamma \to 1$ in the previous case, yields the desired formula for $m = n+1$.

The case of $m \leq n$: Since $M(h+1/2; n, m-1; x)$ and $M(h+1/2; n+1, m; x)$ have the term x^{n-m+1} and $n - m + 1 \geq 1$, it easily follows from Lemma 4.1 that the desired integral is positive and finite.

This completes the proof of the lemma.

Lemma 4.3. Let notations be as above and suppose that

$$n, m \in 2\mathrm{N} \quad \text{or} \quad n, m \in 2\mathrm{N}+1.$$

Then, for $p > n, m$

$$\int_0^1 DM(h; n, p, x) DM(h; m, p, x) \frac{2x}{(1-x^2)^2} dx$$

$$= \delta_{nm} \Gamma(2h)^2 2(n+h+1) \frac{\Gamma(p+1)}{\Gamma(p+2h)} \frac{\Gamma(n+1)}{\Gamma(n+2h+2)}.$$

Proof. When $n = m$, it follows from Lemma 4.2. We may suppose that $n > m$ and hence, $n - m \geq 2$ and even. Then, applying the same argument used in the proof of Lemma 4.2, we see that the desired integral equals to

$$\int_0^1 x^{p-n-1+(n-m)/2}(1-x)^{2h}$$

$$\times \left(G_n(x) - \frac{p+2h}{n+2h+1} G_{n+1}(x) \right) \left(G_m(x) - \frac{p+2h}{m+2h+1} G_{m+1}(x) \right) \frac{dx}{x}.$$

Since $(n-m)/2$ is integer, $0 \leq (n-m)/2 - 1 \leq n-1$, and

$$\left(G_m(x) - \frac{p+2h}{m+2h+1} G_{m+1}(x) \right)$$

is a polynomial of degree $m+1 < n$, the orthogonality relations for the Jacobi polynomials yield that the integral equals to 0.

We here note that, if $h = 1/2$, then $D^h_{n,m} = 1$ and hence,

$$\begin{aligned} DM(1/2; n, m; x) &= M(1/2; n, m; x) - M(1/2; n+2, m; x) \\ &= \langle T_{1/2}(a_t)(e_n - e_{n+2}), e_m \rangle. \end{aligned}$$

Therefore, Lemma 4.2 implies that

$$0 < \int_A |\langle T_{1/2}(a_t)(e_n - e_{n+2}), e_m\rangle|^2 D(a_t)dt < \infty$$

and for $m > n$ this integral equals to

$$\frac{(2n+3)}{(n+1)(n+2)}.$$

Furthermore, these differences $\langle T_{1/2}(a_t)(e_n - e_{n+2}), e_m\rangle$ satisfy the quasi-orthogonality relations stated in Lemma 4.3 with $h = 1/2$. Thereby, as an application of Theorem 3.1, we see the following.

Theorem 4.4. Let $G = SU(1,1)$ and $(T_{1/2}, \mathcal{H}_{1/2})$ the limit of the discrete series of G.
(1) Let ψ be a finite linear combination of $e_{n+2} - e_n$. Then there exist constants $0 < C_1, C_2 < \infty$ such that for any f in $\mathcal{H}_{1/2}$

$$C_1\|f\|^2 \le \int\int_{KA} |\langle f, T_{1/2}(ka_t)\psi\rangle|^2 \sinh 2t \; dkdt \le C_2\|f\|^2.$$

(2) Let

$$\psi = \sum c_n \left(\frac{(2n+3)}{(n+1)(n+2)}\right)^{-1/2} (e_{n+2} - e_n),$$

where the sum is taken over $0 \le n \le N, n \in 2\mathrm{N}$ or $0 \le n \le N, n \in 2\mathrm{N}+1$, and let $\|\psi\|_0^2 = \sum |c_n|^2$. Then for any f in the L^2-span of $\{e_p, p \ge N+1\}$,

$$f(x) = \frac{1}{\|\psi\|_0}\int\int_{KA} \langle f, T_{1/2}(ka_t)\psi\rangle T_{1/2}(ka_t)\psi \; \sinh 2t \; dkdt.$$

References

[AAG] S. T. Ali, J.-P. Antoine and J.-P. Gazeau, *Square integrability of group representations on homogeneous spaces I, II,* Ann. Inst. H. Poincaré, Vol. 55, 1991, pp. 829-855, *ibid.*, Ann. Inst. H. Poincaré, Vol. 55, 1991, pp. 857-890.

[Ba] V. Bargmann, *On unitary ray representations for continuous groups*, Ann. Math., Vol. 48, 1947, pp. 568-640.

[BT] D. Bernier and K. F. Taylor, *Wavelets from square-integrable representations*, SIAM J. Math. Anal., Vol. 27, 1996, pp. 594-608.

[Bo] G. Bohnke, *Treillis d'ondelettes associés aux groupes de Lorentz*, Ann. Inst. H. Poincaré, Vol. 54, 1991, pp. 245-259.

[DB] De Bièvre, *coherent states over sympletic homogeneous spaces*, J. Math. Phys., Vol. 7, 1989, pp. 1401-1407.

[DM] G. B. Duflo and C.C. Moore, *On the regular representation of a nonunimodular locally compact group*, J. Funct. Anal., Vol. 21, 1976, pp. 209-243.

[G] R. Godement, *Sur les relations d'orthogonalité de Bargmann*, C. R. Acad. Sci., Vol. 255, 1947, pp. 521-523, *ibid.*, pp. 657-659.

[GMP] A. Grossmann, J. Morlet, and T. Paul, *Transforms associated to square integrable group representations. I*, J. Math. Phys., Vol. 26, 1985, pp. 2473-2479.

[HC] Harish-Chandra, *Discrete series for semisimple Lie groups I, II*, Acta Math, Vol. 113, 1965, pp. 241-318, *ibid.*, Vol. 116, 1966, pp. 1-111.

[K1] T. Kawazoe, *Wavelet transforms associated to a principal series representation of semisimple Lie groups I, II*, Proc. Japan Acad. Vol. 71, 1995, pp. 154-157, *ibid.*, Proc. Japan Acad. Vol. 71, 1995, pp. 158-160.

[K2] T. Kawazoe, *Wavelet transforms associated to an induced representation of $SL(n+2, R)$*, Ann. Inst. H. Poincaré, Vol. 65, 1996, pp. 1-13.

[K3] T. Kawazoe, *Wavelet transforms associated to the analytic continuation of the holomorphic discrete series of a semisimple Lie group*, Keio Research Report, 1998.

[KT] C. Kalisa and B. Torrésani, *N-dimensional affine Weyl-Heisenberg wavelets*, Ann. Inst. H. Poincaré, Vol. 59, 1993, pp. 201-236.

[L] H. Liu, *Wavelet transforms and symmetric tube domains,* J. Lie Theory, Vol. 8, 1998, pp. 351-366.

[M] R. Murenzi, *Ondelettes multidimensionnelles et application à l'analyse d'images,* Ph. D. Thesis, Louvain-la-Neuve, 1990.

[MW] C. C. Moore and J. A. Wolf, *Square integrable representations on nilpotent groups,* Trans. Amer. Math. Soc., Vol. 185, 1973, pp. 445-462.

[Sa] P. J. Sally, *Analytic Continuation of the Irreducible Unitary Representations of the Universal Covering Group of* $SL(2, \mathbf{R})$, Memoirs of A. M. S., Vol. 69, American Mathematical Society, Providence, Rhole Island, 1967.

[Su] M. Sugiura, *Unitary Representations and Harmonic Analysis, An Introduction,* Second Edition, North-Holland/Kodansha, Amsterdam/Tokyo, 1990.

[T1] B. Torrésani, *Wavelets associated with representations of the affine Weyl-Heisenberg group,* J. Math. Phys., Vol. 32, 1991, pp. 1273-1279.

[T2] B. Torrésani, *Time-frequency representation: wavelet packets and optimal decomposition,* Ann. Inst. H. Poincaré, Vol. 56, 1992, pp. 215-234.

[U] A. Unterberger and J. Unterberger, *A quantization of the Cartan Domain BDI ($q = 2$) and operators on the light cone,* J. Funct. Anal., Vol. 72, 1987, pp. 279-319.

[V] V. S. Varadarajan, *Harmonic Analysis on Real Reductive Groups,* Lecture Note in Math., Springer, Berlin, New York, Vol. 576, 1977.

[WZ] Y. Wu and X. Zhong, *Discrete wavelet transform on circles,* preprint.

[AV1] J-P. Antoine and P. Vandergheynst, *Wavelets on the n-sphere and other manifolds,* J. Math. Phys., Vol. 39, 1998, pp. 3987-4008.

[AV2] J-P. Antoine and P. Vandergheynst, *Wavelets on the 2-sphere: A group-theoretical approach,* Appl. Comput. Harmon. Anal., Vol. 7, 1999, pp. 262-291.

Triple Systems of Hecke Type and Hypergroups

by

Aloys Krieg

1 Introduction

One of the most important classes of hypergroups is given by double coset spaces (cf. [1]). In this note we will consider double coset spaces with different subgroups on the left and right hand side (cf. [4]) as they already appeared in the description of all normal subhypergroups arising from Hecke algebras (cf. [6], Theorem 4 c). This construction does not any longer yield an algebra in general. But we obtain an associative triple system as its algebraic structure in a natural way (cf. [7], [8]). This triple system can be embedded into a usual double coset hypergroup (cf. Theorem 2). For the sake of simplicity we only deal with discrete hypergroups arising from Hecke algebras as in [6].

2 Associative triple systems of Hecke type

We start with a multiplicative group G with unit element e. The set

$$\begin{aligned}\mathbb{C}[G] &:= \{\varphi : G \to \mathbb{C};\ \text{support}(\varphi) \text{ finite }\} \\ &= \left\{\sum_{g\in G}\varphi(g)\delta_g;\quad \varphi(g) \in \mathbb{C} \text{ non-zero for finitely many } g \in G\right\},\end{aligned}$$

where δ_g stands for the *Kronecker delta*, is a $\mathbb{C}$-vector space. Extending the product

$$\delta_g \cdot \delta_h := \delta_{gh}$$

to $\mathbb{C}[G]$ by linearity, we obtain an associative $\mathbb{C}$-algebra with unit element δ_e, the so-called *group algebra* or *group ring* of G (cf. [9]).

Now let us consider two subgroups U and V of G and *double cosets*

$$UgV := \{ugv;\ u \in U, v \in V\},\quad g \in G.$$

Two double cosets are either disjoint or equal. Let

$$K := U\backslash G/V := \{UgV;\ g \in G\}$$

stand for the space of (U, V)-double cosets in G equipped with the discrete topology.

$$\mathcal{H}(U\backslash G/V) := \{\varphi : U\backslash G/V \to \mathbb{C};\ \text{support}(\varphi)\ \text{finite}\}$$

$$= \left\{ \sum_{UgV\subset G} \varphi(UgV)\delta_{UgV};\ \varphi(UgV) \in \mathbb{C}\ \text{non-zero for finitely many}\ UgV \subset G \right\}$$

is a $\mathbb{C}$-vector space. If $V = U$ we use the abbreviation $\mathcal{H}(G/\!\!/U) = \mathcal{H}(U\backslash G/U)$ just as in [6].

For the introduction of a product we need the so-called *Hecke condition*: (G, U) is a *Hecke pair* if $[U : U \cap g^{-1}Ug] < \infty$ for every $g \in G$. Now assume additionally that V and W are subgroups of G, which are *commensurable* with U, i.e. the intersection of any two of the subgroups has finite index in both. Then (G, V) and (G, W) as well as $(G, U \cap V \cap W)$ are Hecke pairs, too. Given $a, b \in G$ we obtain finite disjoint decompositions of the double cosets

$$UaV = \bigcup_{j=1}^{m} Ua_j, \quad m = \text{ind}_U UaV, \quad VbW = \bigcup_{k=1}^{n} Vb_k, \quad n = \text{ind}_V VbW.$$

Then define

$$\delta_{UaV} \cdot \delta_{VbW} := \sum_{UcW\subset G} \mu(c)\ \delta_{UcW}, \tag{1}$$

$$\mu(c) := \sharp\{(j, k);\ Ua_jb_k = Uc\} \in \mathbb{N}_0.$$

It can be shown that the definition of $\mu(c)$ does not depend on the choice of the representatives c, a_j, b_k. This product is extended linearly. Moreover we observe

$$\text{ind}_U UaV \cdot \text{ind}_V VbW = \sum_{UcW\subset G} \mu(c)\text{ind}_U UcW. \tag{2}$$

If X is another subgroup of G, which is commensurable with U, we obtain

$$(\varphi_1 \cdot \varphi_2) \cdot \varphi_3 = \varphi_1 \cdot (\varphi_2 \cdot \varphi_3) \in \mathcal{H}(U\backslash G/X) \tag{3}$$

for all $\varphi_1 \in \mathcal{H}(U\backslash G/V)$, $\varphi_2 \in \mathcal{H}(V\backslash G/W)$, $\varphi_3 \in \mathcal{H}(W\backslash G/X)$ (cf. [4], [10]).

If $V = U$ we have the Hecke algebra $\mathcal{H}(G/\!\!/U)$ of the Hecke pair (G, U) just as in [5], [10].

In the general case again, there is a linear isomorphism

$$J = J_{U,V} : \mathcal{H}(U\backslash G/V) \to \mathcal{H}(V\backslash G/U), \quad \delta_{UaV} \mapsto \delta_{Va^{-1}U},$$

satisfying

$$J(\varphi_1 \cdot \varphi_2) = J(\varphi_2) \cdot J(\varphi_1), \quad J \circ J = \mathrm{id} \tag{4}$$

(cf. [4]).

This becomes the foundation of our algebraic structure. A $\mathbb{C}$-vector space $\mathcal{A}$ equipped with a trilinear triple product

$$\mathcal{A} \times \mathcal{A} \times \mathcal{A} \to \mathcal{A}, \quad (x, y, z) \mapsto < x, y, z >,$$

is called an *associative triple system* (of the second kind) if

$$<< u, v, w >, x, y > = < u, < x, w, v >, y > = < u, v, < w, x, y >>$$

holds for all $u, v, w, x, y \in \mathcal{A}$ (cf. [7], [8]). The notions of homomorphisms and sub-triple systems are then defined in the obvious way. Now (3) and (4) imply

Theorem 1 ([4]). *Let U and V be commensurable subgroups of a group G such that (G, U) is a Hecke pair. Then $\mathcal{H}(U\backslash G/V)$ is an associative triple system by*

$$< \varphi_1, \varphi_2, \varphi_3 > := \varphi_1 \cdot J(\varphi_2) \cdot \varphi_3.$$

The notion of associative triple systems comes from the following idea: Start with an associative $\mathbb{C}$-algebra $\mathcal{A}$ with an involution j on $\mathcal{A}$, i.e. $j : \mathcal{A} \to \mathcal{A}$ is linear and satisfies $j(xy) = j(y)j(x)$ as well as $j(j(x)) = x$ for all $x, y \in \mathcal{A}$. Then $(\mathcal{A}, j)$ becomes an associative triple system by

$$< x, y, z > := xj(y)z.$$

On the other hand Loos [7] showed that each associative triple system can be obtained as a sub-triple system of $(\mathcal{A}, j)$ for suitable $\mathcal{A}$ and j. In the case of Hecke triple systems we can simplify his construction considerably.

Theorem 2. *Let U and V be commensurable subgroups of a group G and $r := \sqrt{[U : U \cap V] \cdot [V : U \cap V]}$. Assume that (G, U) is a Hecke pair. Then*

$$\phi : (\mathcal{H}(U\backslash G/V), J) \to (\mathcal{H}(G/\!\!/(U \cap V)), J)$$
$$\varphi = \sum_{UgV \subset G} \varphi(UgV)\delta_{UgV} \mapsto \frac{1}{r} \sum_{(U\cap V)g(U\cap V) \subset G} \varphi(UgV)\delta_{(U\cap V)g(U\cap V)},$$

is an injective homomorphism of the associative triple systems.

Proof. Obviously ϕ is well-defined, linear and injective. It suffices to show that

$$\phi(\delta_{UaV}) \cdot J(\phi(\delta_{UbV})) \cdot \phi(\delta_{UcV}) = \phi(\delta_{UaV} \cdot J(\delta_{UbV}) \cdot \delta_{UcV}) \tag{5}$$

holds for all $a, b, c \in G$. Assume that

$$UaV = \bigcup_{j=1}^{\alpha} Ua_j, \quad UbV = \bigcup_{k=1}^{\beta} b_k V, \quad UcV = \bigcup_{l=1}^{\gamma} Uc_l$$
$$U = \bigcup_{\nu=1}^{s} (U \cap V) u_\nu, \quad V = \bigcup_{\mu=1}^{t} v_\mu (U \cap V)$$

are disjoint coset decompositions. Then

$$UaV = \bigcup_{j=1}^{\alpha} \bigcup_{\nu=1}^{s} (U \cap V) u_\nu a_j,$$
$$Vb^{-1}U = \bigcup_{k=1}^{\beta} \bigcup_{\mu=1}^{t} (U \cap V) v_\mu^{-1} b_k^{-1}$$
$$UcW = \bigcup_{l=1}^{\gamma} \bigcup_{\rho=1}^{s} (U \cap V) u_\rho c_l$$

are disjoint decompositions, too. In view of (1) the coefficient of $(U \cap V)g(U \cap V)$ on the left hand side of (5) is

$$\frac{1}{r^3} \sharp\{(\nu, j, \mu, k, \rho, l); \quad (U \cap V) u_\nu a_j v_\mu^{-1} b_k^{-1} u_\rho c_l = (U \cap V) g\}$$
$$= \frac{1}{r^3} \cdot \sharp\{(j, \mu, k, \rho, l); \quad Ua_j v_\mu^{-1} b_k^{-1} u_\rho c_l = Ug\}$$
$$= \frac{st}{r^3} \cdot \sharp\{(j', k', l); \quad Ua_{j'} b_{k'}^{-1} c_l = Ug\}.$$

By virtue of $st = r^2$ and (1) this is also the coefficient of $(U \cap V)g(U \cap V)$ on the right hand side of (5). Thus the claim follows. □

3 Associative Banach triple systems of Hecke type

Consider the data of section 2. Given an arbitrary mapping $\varphi : U\backslash G/V \to \mathbb{C}$ define its *norm* by

(6) $$\|\varphi\| := \sum_{(U \cap V)a \subset G} \varphi(UaV) \in [0; \infty].$$

Then

$$\hat{\mathcal{H}}(U\backslash G/V) := \{\varphi : U\backslash G/V \to \mathbb{C}; \quad \|\varphi\| < \infty\}$$

equipped with $\|\cdot\|$ is obviously a Banach space containing $\mathcal{H}(U\backslash G/V)$ as a dense subset. Extending the product form $\mathcal{H}(U\backslash G/V)$ we conclude

$$\| < \varphi_1, \varphi_2, \varphi_3 > \| \le \|\varphi_1\| \cdot \|\varphi_2\| \cdot \|\varphi_3\|$$

for all $\varphi_1, \varphi_2, \varphi_3 \in \hat{\mathcal{H}}(U\backslash G/V)$ from Theorem 1, Theorem 2 and [6], Theorem 2.

A Banach space $\mathcal{A}$, which is an associative triple system and satisfies

$$\| < x, y, z > \| \le \|x\| \cdot \|y\| \cdot \|z\| \quad \text{for all } x, y, z \in \mathcal{A}$$

is called an *associative Banach triple system* (cf. [2]). Thus we have

Corollary 1. *Let U and V be commensurable subgroups of a group G such that (G, U) is a Hecke pair. Then $\hat{\mathcal{H}}(U\backslash G/V)$ is an associative Banach triple system containing $\mathcal{H}(U\backslash G/V)$ as a dense subset.*

4 Hypergroups

Consider again the data of section 2. Let ε stand for the point measure. Given $a, b \in G$ use (1) in order to define

$$\varepsilon_{UaV} * \varepsilon_{VbW} := \sum_{UcW \subset G} \frac{\mu(c) \cdot \mathrm{ind}_U(UcW)}{\mathrm{ind}_U(UaV) \cdot \mathrm{ind}_V(VbW)} \, \varepsilon_{UcW}. \tag{7}$$

It follows from (2) that the right hand side of (7) is a probability measure again.

Recall the definition of a *hypergroup* and in particular of the discrete double coset hypergroup $(G /\!/ (U \cap V), *)$ from [1], Chapter 1.1. Thus Theorem 2, Corollary 1 and [6], Theorem 3, lead to

Theorem 3. *Let U and V be commensurable subgroups of a group G and $r := \sqrt{[U : U \cap V] \cdot [V : U \cap V]}$. Assume that (G, U) is a Hecke pair. Then*

$$\Phi : \hat{\mathcal{H}}(U\backslash G/V) \to (G /\!/ (U \cap V), *), \quad \varphi \mapsto \frac{1}{r} \sum_{(U \cap V)a \subset G} \varphi(UaV) \, \varepsilon_{(U\cap V)a(U \cap V)},$$

is an injective homomorphism of the associative triple systems.

Note that a hypergroup with the attached involution naturally defines an associative triple system. Thus we can view $(U\backslash G/V, *)$ as an *associative hypergroup triple system.*

5 Examples

The notion of Hecke algebras originates from the theory of modular forms. It should be noted that the consideration of (U, V)-double cosets there also plays an essential role when dealing with congruence subgroups (cf. [3], III.7.3, [10], section 3.4).

Next consider a Hecke pair (G, U) and a subgroup $U \subset H \subset G$ such that $H /\!/ U$ is normal in $G /\!/ U$. This means $HgH = HgU$ for all $g \in G$ due to [6], Theorem 4. In this case one can easily sharpen Theorem 2. The associative hypergroup triple systems $(H\backslash G/U, *)$ and $(G /\!/ H, *)$ are then isomorphic. An explicit example of this type is

$$G = GL_n(\mathbb{F}_q), \quad H = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \in G \right\}, \quad U = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in G \right\}$$

(cf. [6], section 3).

Now we consider finite subgroups U and V of a group G. It follows from (1) and (7) that

$$\frac{1}{\mathrm{ind}_U UaV}\,\delta_{UaV}\cdot\frac{1}{\mathrm{ind}_V Vb^{-1}U}\,\delta_{Vb^{-1}U}\cdot\frac{1}{\mathrm{ind}_U UcV}\,\delta_{UcV}$$
$$=\frac{1}{\sharp U\cdot\sharp V}\sum_{u\in U,v\in V}\frac{1}{\mathrm{ind}_U Uavb^{-1}ucV}\,\delta_{Uavb^{-1}ucV},$$
$$\varepsilon_{UaV}*\varepsilon_{Vb^{-1}U}*\varepsilon_{UcV}=\frac{1}{\sharp U\cdot\sharp V}\sum_{u\in U,v\in V}\varepsilon_{Uavb^{-1}ucV}.$$

The elements

$$c_U:=\frac{1}{\sharp U}\sum_{u\in U}\delta_u,\quad c_V:=\frac{1}{\sharp V}\sum_{v\in V}\delta_v$$

are idempotents in $\mathbb{C}[G]$. We consider the associative triple system $(\mathbb{C}[G], J)$ with $J(\delta_g)=\delta_{g^{-1}}$. In view of $J(c_U)=c_U$ and $J(c_V)=c_V$ we observe that $c_U\cdot\mathbb{C}[G]\cdot c_V$ becomes a sub-triple system of $(\mathbb{C}[G], J)$. Thus a verification (cf. [5], I(6.6), [6], Theorem 5) yields

Theorem 4. *Let U and V be finite subgroups of a group G. Then*

$$\mathcal{H}(U\backslash G/V)\to c_U\cdot\mathbb{C}[G]\cdot c_V,\quad \varphi\mapsto\frac{1}{\sqrt{\sharp U\cdot\sharp V}}\sum_{g\in G}\varphi(UgV)\delta_g,$$

is an isomorphism of the associative triple systems.

References

[1] Bloom, W.R., Heyer, H.: Harmonic analysis of probability measures on hypergroups. de Gruyter, Berlin-New York 1995.

[2] Fernandez Lopez, A., Garcia, E.: Compact associative B^*-triple systems. Quarterly J. Math. Oxford **41** (1990), 61–69.

[3] Koecher, M., Krieg, A.: Elliptische Funktionen und Modulformen. Springer-Verlag, Berlin-Heidelberg-New York 1998.

[4] Krieg, A.: Associative triple systems of Hecke type. Algebras, Groups, Geom. **5** (1988), 341–357.

[5] Krieg, A.: Hecke algebras. Memoirs Amer. Math. Soc. **435** (1990).

[6] Krieg, A.: Hecke algebras and hypergroups. In H. Heyer, J. Marion (eds.): Analysis on infinite-dimensional Lie-groups and algebras. World Scientific, Singapore-New Jersey-London 1998, 197–206.

[7] Loos, O.: Assoziative Tripelsysteme. Manuscripta Math. **7** (1972), 103–112.

[8] Meyberg, K.: Lectures on algebras and triple systems. Lecture Notes, University of Virginia, Charlottesville 1972.

[9] Passman, D.S.: The algebraic structure of group rings. J. Wiley, New York 1977.

[10] Shimura, G.: Introduction to the arithmetic theory of automorphic functions. Iwanami Shoten and Princeton University Press, Tokyo-Princeton 1971.

Author's address:
Lehrstuhl A für Mathematik, RWTH Aachen, D-52056 Aachen, Germany
e-mail: krieg@mathA.rwth-aachen.de

Irreducible Bounded Representations of Exponential Solvable Lie Groups

Jean Ludwig

Introduction

In this survey we present the theory of irreducible bounded representations of exponential solvable Lie groups. For these groups the exponential mapping from the Lie algebra $\mathfrak{g}$ of G into G is a diffeomorphism and the unitary dual is explicitly known thanks to the work of Mackey, Dixmier, Kirillov, Bernat, Pukanszky and Vergne in the years 1950 to 1970.
In the first part of the paper we recall the structure of exponential solvable Lie groups G and in the second part we explain Kirillov's theory, i.e. we give the description of the irreducible unitary representations of G using the orbit method. In the last part the algebraically irreducible (or simple) modules of the group algebra $L^1(G)$ are presented together with what is known about topologically irreducible bounded representations of G. The theory of the simple $L^1(G)$ modules, (G exponential), has been developed by Leptin and Poguntke from 1975 to 1981 and Poguntke published a classification of these modules in 1983. It turns out that irreducible unitary and simple modules can be realized in the framework of induced representations. This is no longer true for general bounded irreducible representations on Banach spaces.
In recent years, the method of Poguntke has been used to study these representations. For so called non-*-regular exponential groups, more complicated representations appear, which are not subrepresentations of induced representations and which are constructed by using irreducible non bounded representations of vector groups on Banach spaces.
Many interesting problems remain to be solved. For instance: Is it possible to characterize the separable Banach spaces, on which exponential solvable groups act irreducibly? This problem is closely related to the invariant subspace problem. Is it possible to give explicit descriptions of some of these strange representations for lower dimensional groups?
No proofs will be given in this survey article, they can be found in the literature or they will be published elsewhere.

1. The Structure of Exponential Solvable Lie Groups.

1.1 Let $\mathfrak{g}$ be a real finite dimensional Lie algebra. We let $\mathfrak{g}^1 = \mathfrak{g}$ and we define the central descending series $\mathfrak{g}^j, j = 1, 2, \cdots$, of $\mathfrak{g}$ by $\mathfrak{g}^{j+1} = [\mathfrak{g}, \mathfrak{g}^j]$. We say that $\mathfrak{g}$ is nilpotent of step k if there exists $k \in \mathbb{N}$ such that $\mathfrak{g}^{k+1} = (0)$ and $\mathfrak{g}^k \neq (0)$.

1.2. We say that $\mathfrak{g}$ is solvable if the descending series $\mathfrak{s}^1 = \mathfrak{g}, \mathfrak{s}^{j+1} = [\mathfrak{s}^j, \mathfrak{s}^j], j = 1, 2, \cdots$, stops with $\mathfrak{s}^{l+1} = (0)$ for some $l \in \mathbb{N}$.

1.3. A sequence of ideals of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{a}_1 \supset \cdots \supset \mathfrak{a}_j \supset \cdots \supset \mathfrak{a}_{m+1} = (0)$$

is called a Jordan-Hölder series or J.H. series, if for every $j = 1, \cdots, m$, the $\mathfrak{g}$-module $\mathfrak{a}_j/\mathfrak{a}_{j+1}$ is irreducible. A theorem of Lie says that for solvable Lie algebras every irreducible complex finite dimensional Lie algebra module is of dimension 1 (see [Di.3]). Hence for every J.H.-series $(\mathfrak{a}_j)_j$ of a real solvable Lie algebra the dimension of $\mathfrak{a}_j/\mathfrak{a}_{j+1}$ is equal to 1 or 2 for every j. We call these irreducible modules the roots of $\mathfrak{g}$. Let us denote by Λ the set of all the roots of $\mathfrak{g}$. If $\mathfrak{a}_j/\mathfrak{a}_{j+1}$ is one dimensional, then the corresponding root λ_j is just a real character of $\mathfrak{g}$. If $\mathfrak{a}_j/\mathfrak{a}_{j+1}$ is two dimensional then we can describe the root $\lambda_j = \lambda$ in the following way. There exist two real linear functionals l_λ and p_λ of $\mathfrak{g}$ and two vectors $X = X_j$ and $Y = Y_j$ in $\mathfrak{a}_j$, such that $\{X, Y\}$ is a basis of $\mathfrak{a}_j$ mod $\mathfrak{a}_{j+1}$ and such that

$$[U, X + iV] = (l_\lambda(U) + ip_\lambda(U))(X + iY) \text{mod } (a_{j+1})_{\mathbb{C}}, U \in \mathfrak{g},$$

(where $V_{\mathbb{C}}$ indicates the complexification of a real vector space V). In this way we may consider the roots λ of $\mathfrak{g}$ as linear functionals (a real one in the one dimensional case and as complex valued one $\lambda \simeq l_\lambda + ip_\lambda$ in the two dimensional case).

1.4. In particular $\mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}]$ is contained in the kernel of every root. Since the algebra $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is non trivial if $\mathfrak{g} \neq (0)$ and abelian we have that at least one of the roots of $\mathfrak{g}$ is 0. The roots of $\mathfrak{g}$ give us also the spectrum $\sigma(ad(X))$ of $ad(X)(X \in \mathfrak{g})$ considered as linear operator on $\mathfrak{g}_{\mathbb{C}}$. In fact $\sigma(ad(X)) = \{\lambda(X), \lambda \in \Lambda\}$.

1.5. The nilradical $\mathfrak{n}$ of $\mathfrak{g}$ is the largest nilpotent ideal of $\mathfrak{g}$. In the solvable case, the nilradical is given by

$$\mathfrak{n} = \bigcap_{\lambda \in \Lambda} \ker(\lambda) \supset [\mathfrak{g}, \mathfrak{g}].$$

From now on we will only consider solvable Lie algebras.

1.6. Let us describe the Jordan decomposition of such an algebra. If $\mathfrak{g}$ is not nilpotent, we can choose an element T of $\mathfrak{g}$ which is in general position with respect to the roots of $\mathfrak{g}$, i.e. for every pair λ and μ of roots, considered as complex linear functionals, we always have that

$$\lambda(T) - \mu(T) \neq 0.$$

We take now the Jordan decomposition of $ad(T)$ on $\mathfrak{g}_{\mathbb{C}}$:

$$\mathfrak{g}_{\mathbb{C}} = \sum_{\lambda \in \Lambda} (\mathfrak{g}_{\mathbb{C}})_\lambda,$$

where

$$(\mathfrak{g}_{\mathbb{C}})_\lambda = \{U \in \mathfrak{g}_{\mathbb{C}}, (ad(T) - \lambda(T))^k(U) = 0 \text{ for some } k > 0\}.$$

We have the classical relations

$$[(\mathfrak{g}_{\mathbb{C}})_\lambda, (\mathfrak{g}_{\mathbb{C}})_\mu] \subset (\mathfrak{g}_{\mathbb{C}})_{\lambda+\mu}, \quad \lambda, \mu \in \Lambda.$$

Since T is in general position with respect to the roots of $\mathfrak{g}$, it follows that $(\mathfrak{g}_{\mathbb{C}})_0$ is a nilpotent subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Let now $\mathfrak{g}_0 = (\mathfrak{g}_{\mathbb{C}})_0 \cap \mathfrak{g}$ and for a root $\lambda \neq 0$, let

$$\mathfrak{g}_\lambda = \left((\mathfrak{g}_{\mathbb{C}})_\lambda + \overline{(\mathfrak{g}_{\mathbb{C}})_\lambda}\right) \cap \mathfrak{g} = \left((\mathfrak{g}_{\mathbb{C}})_\lambda + (\mathfrak{g}_{\mathbb{C}})_{\overline{\lambda}}\right) \cap \mathfrak{g}.$$

Let $\mathfrak{m} = \sum_{\lambda \neq 0} \mathfrak{g}_\lambda$. Then $[\mathfrak{g}_0, \mathfrak{m}] = \mathfrak{m}$ and so $\mathfrak{m}$ is contained in $[\mathfrak{g}, \mathfrak{g}]$ whence $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{m} = \mathfrak{g}_0 + [\mathfrak{g}, \mathfrak{g}]$.

If $\mathfrak{g}$ is nilpotent, then of course every root is 0 and $\mathfrak{g} = \mathfrak{g}_0$. If not, let for $j = 1, \cdots, m$, $\mathfrak{v}_j$ be a one or two-dimensional subspace of $\mathfrak{a}_j$, such that $\mathfrak{a}_j = \mathfrak{v}_j \oplus \mathfrak{a}_{j+1}$. Then

$$\mathfrak{g} = \oplus_{j=1}^{m} \mathfrak{v}_j.$$

1.7. Let us now study simply connected solvable Lie groups. We say that a real finite dimensional connected Lie group G is nilpotent if its Lie algebra $\mathfrak{g} = Lie(G)$ is nilpotent. We can provide a nilpotent Lie algebra with a group structure using the Campbell-Baker-Hausdorff multiplication:

$$X \cdot Y = CBH(X,Y) = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] + \frac{1}{12}[Y,[Y,X]] + \cdots, \quad X, Y \in \mathfrak{g}.$$

This multiplication is a polynomial expression in X and Y, since $\mathfrak{g}$ is nilpotent. Hence $(\mathfrak{g}, CBH)$ becomes a Lie group, whose Lie algebra is $(\mathfrak{g}, [,])$. It is obvious that that for every $X \in \mathfrak{g}$, the mapping

$$E_X : \mathbb{R} \to \mathfrak{g}; t \mapsto tX,$$

is a group homomorphism from $(\mathbb{R}, +)$ to $(\mathfrak{g}, CBH)$. Hence the exponential mapping $\exp : \mathfrak{g} \to (\mathfrak{g}, CBH)$ is the identity mapping in this case and every simply connected Lie group whose Lie algebra is isomorphic to $(\mathfrak{g}, [,])$ is itself isomorphic to $(\mathfrak{g}, CBH)$.

1.8. If G is a simply connected solvable Lie group, we know (see [Di.3]), that the exponential mapping is a diffeomorphism if and only if all the roots of $\mathfrak{g} = Lie(G)$ are of the form $l_\lambda + i\omega_\lambda l_\lambda$, for some real constant ω_λ and a real valued character l_λ of $\mathfrak{g}$. More precisely, Dixmier has shown in ([Di. 3]) that for a simply connected solvable Lie group G the following conditions are equivalent:

i) The exponential mapping $\exp : \mathfrak{g} \to G$ is injective.
ii) The exponential mapping $\exp : \mathfrak{g} \to G$ is surjective.
iii) The exponential mapping $\exp : \mathfrak{g} \to G$ is a diffeomorphism.
iv) Every root λ of $\mathfrak{g}$ is of the form $\lambda = (1 + i\omega)l$ for some real linear form $l \in \mathfrak{g}^*$ and some $\omega \in \mathbb{R}$.
v) For every $X \in \mathfrak{g}$ the spectrum of the operator $ad(X)$ acting on $\mathfrak{g}_{\mathbb{C}}$ does not contain a number of the form $i\tau, \tau \in \mathbb{R} \setminus (0)$.

We call the solvable groups, which satisfy these conditions, (solvable) exponential.

Such an exponential group G can be realized on its Lie algebra $\mathfrak{g}$. The Cambell-Baker-Hausdorff multiplication, which converges on a neighbourhood of 0, extends to a unique analytic map on $\mathfrak{g} \times \mathfrak{g}$ and in this way G is isomorphic to the group $(\mathfrak{g}, CBH)$, the exponential mapping for the latter group being the identity.

1.9. A general solvable simply connected Lie group is as a variety always diffeomorphic to a vector space. Indeed, let us take a Jordan-decomposition $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{m} = \mathfrak{g}_0 + \mathfrak{n}$, for some nilpotent ideal $\mathfrak{n}$ of $\mathfrak{g}$ containing $[\mathfrak{g}, \mathfrak{g}]$. Choose a subspace $\mathfrak{t}$ of $\mathfrak{g}_0$, such that

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{n}.$$

For $S, T \in \mathfrak{t}$, we write

$$CBH(S,T) = CBH(S+T, Q(S,T)),$$

where $Q(S,T) = CBH(-S-T, CBH(S,T)) \in [\mathfrak{g}_0, \mathfrak{g}_0]$ is a polynomial expression of brackets in S and T. For a vector U in $\mathfrak{n}$ and $T \in \mathfrak{t}$, let

$$^{T}U = \exp\ (ad(-T))U = \sum_{j=0}^{\infty} \frac{ad(-T)^j}{j!}(U)$$

We obtain a group multiplication on $\mathfrak{s} = \mathfrak{t} \oplus \mathfrak{n}$ by the following rule:

$$(T,U) \cdot (T', U') = (T + T', CBH(Q(T,T'), CBH(^{T'}U, U')); T, T' \in \mathfrak{t}, U, U' \in \mathfrak{n}.$$

The Lie algebra of $(\mathfrak{s}, \cdot)$ is of course isomorphic to $\mathfrak{g}$ and so every simply connected Lie group G with a Lie algebra isomorphic to $\mathfrak{g}$ is itself isomorphic to $(\mathfrak{s}, \cdot)$. In particular

$$G = \exp(\mathfrak{t})\exp(\mathfrak{n})$$

and

$$\exp(T)\exp(U)\exp(T')\exp(U') = \exp(T+T')\exp(Q(T,T'))\exp(^{T'}U)\exp(U')$$

$(T, T' \in \mathfrak{t}, U, U' \in \mathfrak{n})$ (see [Le.Lu.]).

1.10. Let us now consider closed connected subgroups $H = \exp(\mathfrak{h})$ of the simply connected solvable Lie group G. The quotient space G/H is then diffeomorphic to the space $\mathfrak{g}/\mathfrak{h}$. We obtain coordinates on G/H in the following way:

Consider a J.H.-sequence $\mathcal{S} = (\mathfrak{a}_j)_j$ of $\mathfrak{g}$, which passes through $\mathfrak{n}$, i.e. such that $\mathfrak{a}_{j_0} = \mathfrak{n}$ for some j_0. For every j, we take a subspace $\mathfrak{w}_j$ of $\mathfrak{a}_j$, such that $\mathfrak{a}_j + \mathfrak{h} = (\mathfrak{a}_{j+1} + \mathfrak{h}) \oplus \mathfrak{w}_j$. The mapping $E_{\mathcal{S}}^{G/H} : \mathfrak{w} = \sum_j \mathfrak{w}_j \to G/H$

$$E_{\mathcal{S}}^{G/H}(\sum_j w_j) = \left(\prod_{j=1}^{m} \exp(w_j) \right) H$$

is then a diffeomorphism. In particular if $\mathfrak{h} = (0)$, then $E_{\mathcal{S}}^{G} : \mathfrak{w} = \sum_j \mathfrak{w}_j \to G$ is a diffeomorphism (see [Le.Lu.]).

1.11. We can use the mapping E_S^G to describe the left Haar measure on G. Indeed the left Haar measure dx is given by

$$\int_G \varphi(x)dx = \int_{\mathfrak{w}} \varphi(E_S^G(w))dw$$

for φ in the space $C_c(G)$ of the continuous functions with compact support on G. Associated to the Haar measure is the modular function Δ_G of G. The uniqueness of the Haar measure implies that for any $s \in G$ the left invariant measure $\varphi \mapsto \int_G \varphi(xs^{-1})dx$ is a positive multiple, denoted by $\Delta_G(s)$, of our Haar measure and so

$$\int_G \varphi(xs^{-1})dx = \Delta_G(s) \int_G \varphi(x)dx, \varphi \in C_c(G)$$

The function Δ_G is easy to compute. In fact $\Delta_G(\exp(U)) = e^{-tr\, ad(U)}, U \in \mathfrak{g}$, where $tr\, ad(U)$ denotes the trace of the operator $ad(U)$ on $\mathfrak{g}$.

1.12. We realize many of our representations on function spaces, for instance on spaces of functions which satisfy certain covariance conditions.

Let $H = \exp(\mathfrak{h})$ be a closed connected subgroup of G and let

$$\mathcal{E}(G, H) = \{\xi : G \to \mathbb{C}; \xi \text{ continuous with compact support modulo } H,$$

$$\xi(xh) = \frac{\Delta_H(h)}{\Delta_G(h)}\xi(x), x \in G, h \in H\}.$$

This space is left translation invariant and the linear mapping

$$p_{G/H} : C_c(G) \to \mathcal{E}(G, H); \quad \psi \mapsto \left(x \mapsto \int_H \psi(xh)\frac{\Delta_G(h)}{\Delta_H(h)}dh\right)$$

is surjective. The space $\mathcal{E}(G, H)$ admits a left invariant linear form, namely

$$\oint_{G/H} du : \mathcal{E}(G, H) \to \mathbb{C}, \quad \xi \mapsto \int_{\mathfrak{w}} \xi(E_S^{G/H}(w))dw.$$

Hence the linear form

$$C_c(G) \to \mathbb{C}, \quad \psi \mapsto \oint_{G/H} p_{G/H}(\psi)(u)du$$

is left translation invariant and positive and so is a multiple of our Haar measure. The uniqueness of the Haar measure implies that the positive linear form $\oint_{G/H} du$ is unique (up to a positive multiple) and so it does not depend on the choice of the J.H. sequence and not on the complementary spaces $\mathfrak{w}_j$.

1.13. The convolution algebra $L^1(G)$ of the integrable functions on G with respect to Haar measure plays a fundamental role in the theory of representations of G. The convolution of two functions φ and ψ is defined by

$$\varphi * \psi(x) = \int_G \varphi(u)\psi(u^{-1}x)du, \quad x \in G.$$

The L^1-norm on $L^1(G)$ is given by

$$\|\varphi\|_1 = \int_G |\varphi(x)|dx, \quad \varphi \in L^1(G).$$

There exists an isometric involution * on $L^1(G)$:

$$\varphi^*(x) = \Delta_G(x)^{-1}\overline{\varphi(x^{-1})}, x \in G, \varphi \in L^1(G).$$

The connection between left translation λ and convolution is the following:

$$\lambda(x)(\varphi * \psi) = (\lambda(x)\varphi) * \psi, x \in G, \varphi, \psi \in L^1(G).$$

2. The Dual Space of Exponential Solvable Lie Groups

2.1. We begin with the definitions of the different types of irreducible bounded representations.

Let G be a locally compact group. A representation (T, V) of G on a Banach space V is a strongly continuous homomorphism $T : G \to Gl(V)$ of the group G into the group $Gl(V)$ of the bounded invertible linear operators on V. Strongly continuous means that the mappings

$$G \to V, \quad x \mapsto T(x)v,$$

are continuous for every $v \in V$.

We say that the representation (T, V) is bounded, if

$$C_T = \sup_{x \in G} \|T(x)\|_{op} < \infty.$$

Here $\|a\|_{op}$ denotes the operator norm of a bounded operator a on V. Since a solvable group G is amenable, every bounded representation (T, V) on a Banach space $(V, \|\cdot\|_V)$ is in fact isometric, there exists another norm $\|\cdot\|'$ on V, which is equivalent to $\|\cdot\|_V$, such that $\|T(x)v\|' = \|v\|'$ for every $v \in V$ and $x \in G$ (see [Pi.]).

2.2. Bounded representations can be integrated to bounded representations of the Banach algebra $L^1(G)$. Indeed, for $\varphi \in L^1(G)$, the operator

$$T(\varphi) = \int_G \varphi(x)T(x)dx$$

on V is bounded and $\|T(v)\|_{op} \leq C_T\|\varphi\|_1$. We have the relations

$$T(\varphi * \psi) = T(\varphi) \circ T(\psi), T(\lambda(x)\varphi) = T(x) \circ T(\varphi), \quad x \in G, \varphi, \psi \in L^1(G).$$

Conversely, given a bounded representation (T, V) of the algebra $L^1(G)$ on a Banach space V, we have at the same time a bounded representation (T, V) of G, such that

$$T(x) \circ T(\varphi) = T(\lambda(x)\varphi)$$

for every $x \in G$ and $\varphi \in L^1(G)$ (see [Di.4]).

2.3. A closed subspace W of V is said to be G-invariant, if for every $x \in G, w \in W$, $T(x)w \in W$. The same type of definitions is valid for representations of the Banach algebra $L^1(G)$. If T is bounded, a closed subspace W of V is G-invariant if and only if it is $L^1(G)$-invariant.

2.4. We say that a representation (T, V) is (topologically) irreducible, if the two trivial spaces (0) and V are the only closed G- invariant subspaces of V.

A Banach module (T, V) of $L^1(G)$ is said to be simple or algebraically irreducible if the trivial spaces (0) and V are the only $L^1(G)$-invariant subspaces of V.

2.5. We say that a representation $(\pi, \mathcal{H})$ is unitary if the Banach space $\mathcal{H}$ is in fact a Hilbert space (with scalar product $\langle , \rangle$) and if $\pi(x)$ is a unitary operator for any $x \in G$. A unitary operator being isometric, every unitary representation of G is bounded and the corresponding representation of $L^1(G)$ has the property that $\pi(\varphi)^* = \pi(\varphi^*)$ for any $\varphi \in L^1(G)$.

2.6. Two representations (T, V) and (T', V') are called equivalent if there exists a bounded linear bijection $u : V \to V'$, which intertwines T and T', i.e. such that

$$T'(x) \circ u = u \circ T(x), \forall x \in G.$$

We write $T \simeq T'$ for two equivalent representations. In particular if $T \simeq T'$, then T is irreducible if and only T' is.

2.7. By Schur's lemma, we know that a unitary representation $(\pi, \mathcal{H})$ is irreducible if and only if every bounded operator $a \in L(\mathcal{H})$, which commutes with π, i.e. for which $\pi(x) \circ a = a \circ \pi(x)$ for every $x \in G$, is a multiple of the identity operator $\mathbb{I}_{\mathcal{H}}$. Hence for two equivalent irreducible unitary representations $(\pi, \mathcal{H})$ and $(\pi', \mathcal{H}')$ there exists a unique (up to scalar multiple) interwining operator $u : \mathcal{H} \to \mathcal{H}'$, which is even unitary .

We write $[\pi]$ for the equivalence class of the representation π, i.e. for the set $\{(\pi', \mathcal{H}'), \pi \simeq \pi'\}$.

We denote by $\widehat{G}$ the family of all the equivalence classes of irreducible unitary representations of G.

By the theorem of Gelfand-Naimark, the irreducible unitary representations separate the points of G (see [Di.4]).

2.8. In 1931 Stone and von Neumann determined the unitary dual of the Heisenberg group. In the late fourties Mackey proved his imprimitivity theorem, the fundamental tool to compute irreducible unitary representations in the solvable case. Dixmier proved in 1957 (see [Di.5]), that every irreducible unitary representation of a connected nilpotent Lie group is monomial, i.e. is induced from a unitary character. The breakthrough came with Kirillov's orbit picture of the dual space of nilpotent Lie groups in 1962 (see [Ki.]). Kirillov's orbit method also works for exponential groups. Bernat, Pukanszky and Vergne determined the dual space of these groups in the years 1965-1970 with the orbit method (see [Ber.], [Puk.1,2], [Ve.1,2,3]).

2.9. The irreducible representations of exponential groups are induced from characters. Let us describe briefly induced representations. Let H be a closed subgroup of the group G and let $(\rho, \mathcal{F})$ be a unitary representation of H. We realize the induced representation $\tau = \tau_\rho$ of ρ by left translation on a space of mappings $\mathcal{E}(\rho)$ from G into $\mathcal{H}$. The space $\mathcal{E}(\rho)$ is the space

$$\mathcal{E}(\rho) = \{\xi : G \to \mathcal{F}; \xi \text{ continuous with compact support modulo } H,$$

$$\xi(xh) = (\frac{\Delta_H(h)}{\Delta_G(h)})^{1/2}\rho(h)^{-1}\xi(x), x \in G, h \in H\}.$$

This space of mappings is left translation invariant and we observe that for $\xi \in \mathcal{E}(\rho)$, the function $x \to \|\xi(x)\|^2$ is contained in $\mathcal{E}(G, H)$. Hence the scalar product

$$(\xi, \eta) \to \langle \xi, \eta \rangle_{\mathcal{H}} = \oint_{G/H} \langle \xi(x), \eta(x) \rangle_{\mathcal{F}} dx$$

is G-invariant, positive and hermitian and so left translation is isometric on the prehilbert space $(\mathcal{E}(\rho), \langle , \rangle)$. The completion $\mathcal{H}$ of the space $\mathcal{E}(\rho)$ with respect to the norm $\|\cdot\|_{\mathcal{H}}$ is a Hilbert space on which the group G acts by left translation, i.e.

$$\tau(x)\xi(s) = \xi(x^{-1}s), x, s \in G, \xi \in \mathcal{H}.$$

We take now the special case where ρ is a unitary character of H. Then $\mathcal{H}$ is a space of complex valued functions and we see that the operators $\tau(\varphi), \varphi \in C_c(G)$, are kernel operators with continuous kernels. Indeed, for $\xi \in \mathcal{E}(\rho)$,

$$\tau(\varphi)\xi(s) = \int_G \varphi(x)\xi(x^{-1}s)dx = \int_G \varphi(sx^{-1})\Delta_G(x)^{-1}\xi(x)dx$$

$$= \oint_{G/H} \int_H \varphi(sh^{-1}x^{-1})\Delta_G(xh)^{-1}\frac{\Delta_G(h)}{\Delta_H(h)}(\frac{\Delta_H(h)}{\Delta_G(h)})^{1/2}\overline{\chi(h)}\xi(x)dhdx$$

$$= \oint_{G/H} \Delta_G(x)^{-1} \left(\int_H \varphi(shx^{-1})(\frac{\Delta_G(h)}{\Delta_H(h)})^{1/2}\chi(h)dh\right) \xi(x)dx.$$

Hence the kernel $\varphi_{H,\chi}$ of the operator $\tau(\varphi)$ is the function

$$(s, x) \to \Delta_G(x)^{-1} \int_H \varphi(shx^{-1})\chi(h)(\frac{\Delta_G(h)}{\Delta_H(h)})^{1/2}dh.$$

2.10. Let $H = \exp(\mathfrak{h})$ be a closed connected subgroup of G. Every unitary character χ of H is of the form

$$\chi(\exp(T)) = \chi_f(\exp(T)) = e^{-if(T)}, T \in \mathfrak{h},$$

where f is a real linear functional on $\mathfrak{g}$, such that

$$f([\mathfrak{h}, \mathfrak{h}]) = (0).$$

We remark that for every $t \in G$, the representations $\tau_{H,\chi}$ and $\tau_{tHt^{-1},{}^t\chi}$ are equivalent. Here ${}^t\chi$ is the unitary character of the group tHt^{-1} defined by ${}^t\chi(p) = \chi(t^{-1}pt), p \in tHt^{-1}$. An intertwining operator u between these two representations is given by right translation:

$$u(\xi)(s) = \xi(st), \quad \xi \in \mathcal{E}(\chi), s \in G.$$

We define the coadjoint representation Ad^* of G on the dual vector space $\mathfrak{g}^*$ of $\mathfrak{g}$ by:

$$Ad^*(x)f(U) = f(Ad(x^{-1})U), U \in \mathfrak{g}, x \in G, f \in \mathfrak{g}^*.$$

Hence the induced representations τ_{H,χ_f} and $\tau_{tHt^{-1},\chi_{Ad^*(t)f}}$ are equivalent, since $\chi_{Ad^*(t)f} = {}^t\chi, t \in G$.

2.11. A subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is called a polarisation at $f \in \mathfrak{g}^*$, if $\mathfrak{p}$ is subordinated to f (i.e. if $f([\mathfrak{p}, \mathfrak{p}]) = (0)$), and if $\mathfrak{p}$ has maximal dimension with this property. This maximal dimension is equal to $\frac{1}{2}(\dim \mathfrak{g} + \dim \mathfrak{g}(f))$. Here $\mathfrak{g}(f)$ denotes the stabilizer of f in $\mathfrak{g}$, i.e. $\mathfrak{g}(f) = \{U \in \mathfrak{g}; f([U, \mathfrak{g}]) = (0)\}$. For a polarisation $\mathfrak{p}$ at f we always have that $Ad^*(H)f$ is open in $f + \mathfrak{p}^{\perp}$. We say that $\mathfrak{p}$ is a Pukanszky polarisation, if $Ad^*(H)f = f + \mathfrak{p}^{\perp}$.

2.12. We can now describe the unitary dual of an exponential group G. The theory of Kirillov-Bernat-Vergne-Pukanszky says that the induced representation τ_{H,χ_f} is irreducible if and only if $\mathfrak{h}$ is a Pukanszky polarisation at f. Furthermore, given $f \in \mathfrak{g}^*$, there always exists a Pukanszky polarisation $\mathfrak{p}$ at f and for two Pukanszky polarisations $\mathfrak{p}$, resp. $\mathfrak{p}'$ at f, resp. at f', the representations τ_{P,χ_f} and $\tau_{P',\chi_{f'}}$ are unitarily equivalent, if and only if the coadjoint orbits of f and f' are the same. Finally, by Mackey's imprimitivity theorem, every irreducible unitary representation π of G is equivalent to some induced representation τ_{P,χ_f}. We obtain in this way a bijection (the orbit picture) between the space of the coadjoint orbits $\mathfrak{g}^*/G$ and the dual space of G:

$$\mathcal{K} : \mathfrak{g}^*/G \to \widehat{G}, \quad Ad^*(G)f \to [\tau_{P,\chi_f}], (P = \exp(\mathfrak{p}), \mathfrak{p} \text{ any Pukanszky polarisation at } f).$$

2.13. We can construct Pukanszky polarisations at $f \in \mathfrak{g}^*$ in the following way. Let as before $\mathfrak{n}$ denote the nilradical or any nilpotent ideal of $\mathfrak{g}$ containing $[\mathfrak{g}, \mathfrak{g}]$. Take a J.H. sequence $(\mathfrak{a}_j)_{j=s}^{m}$ for the action of $\mathfrak{g}$ on $\mathfrak{n}$ and let $\mathfrak{g}(q)$ be the stabilizer of $q = f_{|\mathfrak{n}}$ in $\mathfrak{g}$. The subspace

$$\mathfrak{p}_0 = \sum_{j=m}^{s} \mathfrak{a}_j(f_{|\mathfrak{a}_j})$$

is then a polarisation at q in $\mathfrak{n}$ (see [Ve.1,2]. The stabilizer $\mathfrak{g}(q)$ of q in $\mathfrak{g}$ is a subalgebra of $\mathfrak{g}$ containing $\mathfrak{g}(f)$ and the quotient algebra $\mathfrak{g}(q)/\ker(f)\cap\mathfrak{n}(q)$ is either abelian or isomorphic to a Heisenberg algebra. Furthermore we have that $[\mathfrak{g}(q),\mathfrak{p}_0]\subset\mathfrak{p}_0$. Let $\mathfrak{p}_1$ be a polarisation at $f_{|\mathfrak{g}(q)}$. Then $\mathfrak{p}=\mathfrak{p}_1+\mathfrak{p}_0$ is a Pukanszky polarisation at f (see also [Le.Lu]).

The Heisenberg algebra

$$\mathfrak{h}_n = \text{ span } \{X_1\cdots,X_n,Y_1,\cdots,Y_n,Z\},(n\in\mathbb{N})$$

has the bracket relations:

$$[X_i,Y_j]=\delta_{i,j}Z,[X_i,X_j]=[Y_i,Y_j]=0=[U,Z],\quad 1\le i,j\le n, U\in\mathfrak{h}_n.$$

For a linear functional f on $\mathfrak{h}_n$, we see that the stabilizer $\mathfrak{h}_n(f)$ at f is equal to $\mathfrak{h}_n$ if $f(Z)=0$ and $\mathfrak{h}_n(f)=\mathbb{R}Z$ if $f(Z)\neq 0$. In the latter case we have many polarisations. For instance the subspaces $\text{span}\{X_1+\alpha_1Y_1,\cdots,X_n+\alpha_nY_n,Z\}$, where $\alpha_1,\cdots,\alpha_n$ are any real numbers, give us an infinity of polarisations at f.

2.14. The irreducible representations $\pi=\tau_{p,\chi_f}$ of an exponential group G have the following property. The subspace $\mathcal{H}^1$ of all the vectors ξ in the space $\mathcal{H}$ of π, for which there exists an element $\varphi=\varphi_\xi\in L^1(G)$, such that the operator $\pi(\varphi)$ is the orthogonal projection P_ξ onto $\mathbb{C}\xi$ is different from (0), and hence is dense in $\mathcal{H}$ since π is irreducible. There exist even non zero elements ξ in $\mathcal{H}^1$, such that φ_ξ is rapidly decreasing, which means that $\nu\varphi_\xi$ is also in $L^1(G)$ for every real character ν of G. This was proved by Howe (see [Ho.]) in the nilpotent case, by Ludwig (see [Lu.2]) and by Poguntke (see [Po.1]) in the exponential case.

3. Algebraically and topologically irreducible Representations.

3.1. Let A be a Banach algebra and (T,V) an algebraically irreducible A-module. For any $v\in V, v\neq 0$, the annihilator $A_v=\{a\in A;T(a)v=0\}$ is a maximal modular left ideal, which is automatically closed, and so the representation (T,V) is equivalent to the left module $(\lambda,A/A_v)$. In particular (T,V) is a Banach module of A. (see [Bo. Du.])

3.2. Let now (T,V) be a topologically irreducible representation of A. We can again fix a non zero vector of V and consider the annihilator A_v of v in A, which is a closed left ideal. We have an injection

$$i:A/A_v\to V, i(a \bmod A_v)=T(a)v,$$

and the image of the mapping i is dense in V since T is irreducible. We transfer the norm $\|\cdot\|_V$of V to the space A/A_v via i and so we can replace the Banach space V by the completion of A/A_v and realize T by left translation on the space A/A_v and on its completion. In this way, the module (T,V) is determined by the closed left ideal A_v and a certain module norm $\|\cdot\|$ on A/A_v which satisfies the following inequality:

$$\|ab \bmod A_v\|\le\|a\|_A\|b \bmod A_v\|, a,b\in A.$$

3.3. Let A^f be the ideal in A, consisting of all the a's in A, such that $T(a)$ is an operator of finite rank. Suppose that $A^f\neq(0)$. Then the submodule $V^1=\text{ span }\{T(a)v,a\in A^f,v\in V\}$ is dense in V and defines a simple A-module.

3.4. The simple $L^1(G)$-modules in the nilpotent case have been determined by Dixmier (see [Di.1]), Leptin (see [Le.2]), Poguntke (see [Po.4]), Jenkins (see [Je.]) and Ludwig (see [Lu.3]) from 70 to 77 and Leptin and Poguntke studied the exponential case in some papers from 76-81 (see for instance [Le.Po.]) and finally Poguntke (see [Po.2]) gave a complete description of these modules in 1983. It turns out that every simple $L^1(G)$-module is of the form (T, V^1) for some topologically irreducible Banach representation (T, V) of $L^1(G)$. We will describe them in (3.14).

3.5. Let us analyse such a topologically irreducible $L^1(G)$-module (T, V), for an exponential group G. Then T is also a G-irreducible module and we can restrict T to the nilradical $N = \exp(\mathfrak{n})$ of G. The group G acts on N by conjugation and so also on the functions of N and in particular on the elements of $L^1(N)$. Whence an ideal $I \subset L^1(N)$ is G-invariant if for every $\varphi \in I$ the function

$$n \mapsto \Delta_G(t)\varphi(t^{-1}nt) = {}^t\varphi(n), n \in N,$$

is also in I for every $t \in G$. The restriction of T to N is no longer irreducible, but the kernel $\ker_{L^1(N)}(T)$ of T in $L^1(N)$ is a closed G-prime ideal. A G- prime ideal I in $L^1(N)$ is by definition a twosided G-invariant ideal, which has the property that for every pair I_1, I_2 of twosided G-invariant ideals in $L^1(N)$, such that $I_1 * I_2 \subset I$, necessarily one of the two ideals I_1 and I_2 is contained in I. It has been shown by Molitor-Braun in 1996 (see [Mo.1] and [Lu.Mo.3]), that every closed G-prime ideal I in $L^1(N)$ is the kernel of a G-orbit in $\widehat{N}$, i.e.

$$I = \bigcap_{t\in G} \ker_{L^1(N)}({}^t\tau) = \ker({}^G\tau)$$

for some $\tau \in \widehat{N}$. The representation τ of N is associated to its Kirillov-orbit $Ad^*(N)q$ for some $q \in \mathfrak{n}^*$. Let $f \in \mathfrak{g}^*$ be an extension of q. We take a subspace $\mathfrak{t}$ of $\mathfrak{g}(f)$, such that $\mathfrak{g}(f) = \mathfrak{t} \oplus (\mathfrak{g}(f) \cap \mathfrak{n})$. Let $\mathfrak{h}$ be a subspace of $\mathfrak{g}$ containing $\mathfrak{n}$, such that $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{h}$. Then $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{n} \subset \mathfrak{h}$ and so $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. Let $p = f_{|\mathfrak{h}} \in \mathfrak{h}^*$. Let us choose a Pukanszky polarisation $\mathfrak{p}$ at f, such that $\mathfrak{p}_0 = \mathfrak{p} \cap \mathfrak{n}$ is a polarisation at q as in (2.13). Then $\mathfrak{p} \cap \mathfrak{h}$ is a Pukansky polarisation at p and the restriction of the representation $\pi = \tau_{P,\chi_f}$ of G to $H = \exp(\mathfrak{h})$ is irreducible and equivalent to $\sigma = \tau_{P\cap H,\chi_p}$. Our choice of $\mathfrak{h}$ implies that the H-orbit of p is saturated with respect to $\mathfrak{n}$, i.e. $Ad^*(H)p + \mathfrak{n}^\perp = Ad^*(H)p$. As a consequence, (see [Ha.Lu] and [Lu.Mo.3]),

$$\ker_{L^1(H)}(\sigma) = \ker_{L^1(H)}(T).$$

Hence the representation T annihilates the twosided ideal

$$I_T = \overline{\text{span } (L^1(G) * \ker_{L^1(H)}(\sigma))} = \overline{\text{span } (L^1(G) * \ker_{L^1(N)}(\tau))}$$

of $L^1(G)$ (here $\overline{(--)}$ denotes closure in $L^1(G)$). Thus the representation T factorizes through I_T and defines an irreducible representation $\tilde{T}$ of $A = L^1(G)/I_T$. The algebra A is itself a generalized L^1-algebra. As Banach space A is isometrically isomorhic to $L^1(\mathcal{T}, (L^1(H)/\ker_{L^1(H)}(\sigma)))$, where

$$\mathcal{T} = \exp(\mathfrak{t}) \simeq G(f)/G(f) \cap N \simeq G/H$$

and the algebra $L^1(H)$ acts by convolution on the right and on the left on A and so A has many idempotent multipliers (see [Po.2]). Indeed, we can choose exponentially decreasing elements $\varphi = \varphi_\lambda$ in $L^1(H)$, such that $\sigma(\varphi)$ is the orthogonal projector P_λ onto $\mathbb{C}\lambda$. Hence $\alpha \mapsto \varphi * \alpha \mod I_T$ defines an idempotent multiplier on A, since $\varphi * \varphi = \varphi$ modulo $\ker_{L^1(H)}(T)$. We take for every $t \in \mathcal{T} \subset G(f)$ the element $v(t) \in L^1(H)/\ker_{L^1(H)}(\sigma)$ for which $\sigma(v(t)) = \pi(t)^{-1} \circ P_\lambda$. The norm $\omega(t)$ of $v(t)$ in the quotient space $L^1(H)/\ker_{L^1(H)}(\sigma)$ is a measurable submultiplicative function which is constant on $G(f) \cap N$ and defines a weight on $G(f)/G(f) \cap N$. It follows from this that the subspace $B = B_\varphi = \varphi * A * \varphi$ is a closed subalgebra of A. Furthermore we have for $a \in A$ that

$$\varphi * a * \varphi(t) = h(t)v(t) \in L^1(H)/\ker_{L^1(H)}(\sigma), t \in \mathcal{T},$$

where $t \mapsto h(t)$ is a measurable function defined on $\mathcal{T}$ and in fact on $G(f)/G(f) \cap N$, such that

$$\|\varphi * a * \varphi\|_A = \int_{\mathcal{T}} |h(t)| \|v(t)\| dt.$$

It turns out that the mapping $\varphi * a * \varphi \mapsto h$ is even an isometric isomorphism of the algebra B onto the weighted convolution Banach algebra $L^1(G(f)/G(f) \cap N, \omega)$ (see [Po.2]). Since $G(f)/G(f) \cap N$ is commutative, it follows that B itself is commutative. Let now $W = T(\varphi_\lambda)V \subset V$. Since $T(\varphi_\lambda)$ is a projector we have that W is a closed subspace of V and W is an irreducible B-submodule of V. Let us denote by S the restriction of $\tilde{T}$ to W.

3.7. If T is algebraically irreducible, then (S, W) is also a simple B-module and B being abelian, it follows that W is one dimensional and S is a character of the algebra B, which we denote by χ_ν. We can describe this character by a linear form (denoted by ν) on $\mathfrak{g}(f)$:

$$\chi_\nu(\varphi * a * \varphi) = \int_{G(f)/G(f) \cap N} h(t) e^{-i\nu(log(t))} dt, a \in A.$$

3.8. If T is only topologically irreducible, the space W need not be one dimensional. The commutative algebra $L^1(G(f)/G(f) \cap N, \omega)$ has infinite dimensional irreducible representations, if the weight ω is exponential. It suffices in that case for instance to take a real linear functional ν on $\mathfrak{g}(f)/\mathfrak{g}(f) \cap \mathfrak{n}$, such that $e^{\nu(T)} \leq \omega(log(T)), T \in \mathfrak{g}(f)$, and to choose any infinite dimensional Banach space W, which admits a bounded operator u, which has no closed invariant subspaces except the trivial ones (see [Be.]). The representation S defined by

$$S(\varphi * a * \varphi) = \int_{G(f)/G(f) \cap N} h(t) e^{-\nu(log(t))u} dt, a \in A,$$

is then irreducible on W.

3.9. Conversely, every irreducible Banach space representation (S, W) of the algebra B allows us to define a family of topologically irreducible representations of G in the following way. Choose a non-zero vector $w \in W$ and let

$$B_w = \{b \in B; S(b)w = 0\}, \quad A_w = \{a \in A, \quad S(\varphi * b * a * \varphi)w = 0, \forall b \in A.\}$$

Define the function $\|\cdot\|_{min}$ on A/A_w by

$$\|a \bmod A_w\|_{min} = \inf_{\|b\|_A=1} \|S(\varphi * b * a * \varphi)w\|_W, a \in A.$$

It turns out that $\|\cdot\|_{min}$ is a norm on A/A_w for which

$$\|\psi * a \bmod A_w\|_{min} \leq \|\psi\|_1 \|a \bmod A_w\|_{min}, a \in A, \psi \in L^1(G).$$

Furthermore the restriction of $\|\cdot\|_{min}$ to $(B+A_w)/A_w \simeq B/B_w$ is equivalent to the norm $b \mapsto \|S(b)w\|_W$ of B. Hence we obtain a Banach space V^{min}, the completion of A/A_w with respect to $\|\cdot\|_{min}$ of A, such that convolution on the left on A/A_w extends to a bounded representation T^{min} of $L^1(G)$ on V^{min}. Furthermore the subspace $W^{min} = T^{min}(\varphi_\lambda)V^{min}$ is isomorphic to W and the representation S^{min} of B is equivalent to the representation (S, W). We say that (T^{min}, V^{min}) is an extension of (S, W). It is easy to show that T^{min} is even irreducible (see [Lu.Mo.3]).

3.10. There may be other extensions. For instance if S is character of B, then we may take as extension norm the quotient norm on A/A_w, since now B/B_w is one dimensional. The left ideal A_w is now modular and a modular left unit is given by any element of B, on which S has the value 1. It is not difficult to see that A_w is even maximal and so A/A_w is an algebraically irreducible submodule of the module V^{min}. We see also that two simple modules $\tilde{T}$ and $\tilde{T}'$ of A are equivalent, if and only if the corresponding characters of the algebra B coincide (see [Po.2]).

3.11. We say that a norm $\|\cdot\|$ on A/A_w is an extension norm, if

$$\|\psi * a \bmod A_w\| \leq C_{\|\cdot\|}\|\psi\|_1\|a \bmod A_w\|$$

for any $a \in A$ and $\psi \in L^1(G)$ (for some constant $C_{\|\cdot\|}$) and if the restriction of $\|\cdot\|$ to $B/B_w \simeq (B+A_w)/A_w$ is equivalent to the norm $b \mapsto \|S(b)w\|$ of B. It turns out that every extension norm $\|\cdot\|$ dominates the minimal norm, i.e. we have that $\|a\|_{min} \leq C\|a\|, a \in A$, (for some constant C) and that the completion of A/A_w with respect to the norm $\|\cdot\|$, considered as a subspace of the Banach space V^{min}, is also an irreducible $L^1(G)$ module. Hence there are as many equivalence classes of irreducible extensions of a given (S, W) module as there are equivalence classes of extension norms (see [Lu.Mo.3]).

3.12. In the case where S is a character, there are in general an infinity of such extensions. For instance, if G is nilpotent every closed prime ideal I of $L^1(G)$ is the kernel of an element π of $\widehat{G}$. Hence every irreducible bounded irreducible module (T, V) with $\ker_{L^1(G)}(T) = \ker_{L^1(G)}(\pi)$ contains as simple submodule a copy of $(\pi, \mathcal{H}^1)$. Let us realise π as $\mathrm{ind}_P^G \chi_f$ for a polarisation $P = \exp(\mathfrak{p})$ at f. Instead of taking the Hilbert space $\mathcal{H}$ we may take the Banach spaces

$$L^p(G/P, \chi_f) = \{\xi : G \to \mathbb{C}; \xi \text{ measurable }, \xi(xp) = \chi_f(p)^{-1}\xi(x), x \in G, p \in P,$$

$$\int_{G/H} |\xi(x)|^p dx = \|\xi\|_p^p < \infty, \}$$

$(1 \le p < \infty)$. For $p = \infty$, we can take the space

$$C_\infty(G/P, \chi_f) = \{\xi : G \to \mathbb{C}; \xi(xp) = \chi_f(p)^{-1}\xi(x), x \in G, p \in P,$$
$$\xi \text{ continuous, tending to 0 at } \infty\}.$$

The group G acts by left translation on all these spaces and we write $\tau_{(P,\chi_f,p)}$ for these representations. Since the spaces $L^p(G/P, \chi_f)$ are not isomorphic, the representations $\tau_{(P,\chi_f,p)}$ cannot be equivalent. The operators $\tau_{(P,\chi_f,p)}(\varphi), \varphi \in L^1(G)$, are kernel operators whose kernels $\varphi_{(p,\chi_f,p)}$ do not depend on p. In fact

$$\varphi_{(P,\chi_f,p)}(u,v) = \int_P \varphi(upv^{-1})\chi_f(p)dp = \varphi_{P,\chi_f,2}(u,v), u, v \in G,$$

and so $\ker_{L^1(G)}(\tau_{(P,\chi_f,p)}) = \ker_{L^1(G)}(\pi)$ and the representations $\tau_{(P,\chi_f,p)}$ are irreducible and all contained in the corresponding V^{min}.

3.13. Let us sum up what has been said above. For every G-orbite $Ad^*(G)q$ in $\mathfrak{n}^*$, we have the commutative subalgebras $B_{\varphi_\lambda} \simeq \varphi_\lambda * L^1(G)/\overline{L^1(G) * \ker_{L^1(N)}(\tau_q)} * \varphi_\lambda$ which are all isomorhic to $L^1(\mathcal{T}, \omega) \simeq L^1(G(l)/G(l) \cap N, \omega)$, for some weight independent of λ. Having fixed one of the φ_λ's, every irreducible bounded module (T, V) defines an irreducible bounded module (S, W) of B, where for $h \in L^1(\mathcal{T}, \omega)$,

$$S(h) = \int_{\mathcal{T}} h(t)T(t)T(v_\lambda(t))dt.$$

The representations (T, V) and (T', V') are equivalent if and only if their $Ad^*(G)$ orbits in $\mathfrak{n}^*$ coincide, if the modules (S, W) and (S', W') are equivalent and if the extension norms on $A/A_w = A/A'_w$ are equivalent.

3.14. Let us finish this exposition with a characterisation of the simple modules of $L^1(G)$. We have seen that every simple module is determined by its orbit $Ad^*(G)q$ in $\mathfrak{n}^*$ and a character $\chi_T = \chi_\nu$ of $B = L^1(G(l)/G(l) \cap N, \omega) \simeq L^1(\mathcal{T}, \omega)$.

Poguntke has given a description of the weight ω (see [Po.2]). Choose a J.H. sequence $(\mathfrak{b}_j)_{j=1}^m$ of the $\mathfrak{g}(f)$-module $\mathfrak{n}/\mathfrak{p}_0$, where $\mathfrak{p}_0$ is a $\mathfrak{g}(f)$-invariant polarisation of q (see 2.13). Let for $T \in \mathfrak{g}(f)$,

$$\mu(T) = \mu_q(T) = \frac{1}{2}\sum_{j=1}^m |tr\, ad_{\mathfrak{b}_j/\mathfrak{b}_{j+1}}(T)|.$$

Then the weight ω satisfies the following inequalities:

$$e^{\mu(T)} \le \omega(exp(T)) \le e^{\mu(T)}R(T), T \in \mathfrak{g}(f),$$

for some polynomially bounded expression R of T. Hence the characters χ_ν of B are of the following form:

$$\chi_\nu(h) = \int_{\mathcal{T}} h(t)e^{-i(\nu(log(t)))}dt, h \in L^1(\mathcal{T}, \omega),$$

where ν is any complex linear functional of $\mathfrak{g}(f)$, for which $|Im(\nu)| \leq \mu$. We see thus that B has exponentially increasing characters, if and only if one of the modules $\mathfrak{b}_j/\mathfrak{b}_{j+1}$ is not trivial. In that case the group G is not *-regular in the sense of Boïdol (see [Boi.]).

3.15. We shall show now that for a simple module (T, V) of $L^1(G)$, there exists a topologically irreducible module $(T_{\overline{p}}, V_{\overline{p}})$ of G such that (T, V) is equivalent to $(T_{\overline{p}}, V_{\overline{p}}^1)$. Let $q \in \mathfrak{n}^*$ and let $f \in \mathfrak{g}^*$ be an extension of q. Let $\mathfrak{b} = \mathfrak{g}(q) + \mathfrak{n}$, which is an ideal of $\mathfrak{g}$ and which contains our Pukanszky polarisation $\mathfrak{p} = \mathfrak{p}_1 + \mathfrak{p}_0$ at f.

We choose a J.H. sequence

$$\mathfrak{n} = \mathfrak{a}_s \supset \cdots \supset \mathfrak{a}_m \supset \mathfrak{a}_{m+1} = \mathfrak{p}_0$$

of the $\mathfrak{b}$-module $\mathfrak{n}/\mathfrak{p}_0$. Let $\mathfrak{x}$ be a subspace of $\mathfrak{p}$ such that $\mathfrak{b} = \mathfrak{x} \oplus (\mathfrak{p} + \mathfrak{n})$ and let $\mathfrak{s}$ be a subspace of $\mathfrak{g}$ such that $\mathfrak{s} \oplus \mathfrak{b} = \mathfrak{g}$. Let us also choose for every j a subspace $\mathfrak{w}_j$ of $\mathfrak{a}_j$ such that $\mathfrak{a}_j + \mathfrak{p}_0 = \mathfrak{w}_j \oplus (\mathfrak{a}_{j+1} + \mathfrak{p}_0)$. We let $\overline{p} = (p_1, \cdots, p_m) \in [1, \infty]^m$ and for $T \in \mathfrak{g}(q)$ we set

$$\delta_{\overline{p}}(T) = \sum_{j=1}^{m} \frac{tr(ad(T))_{\mathfrak{a}_j/\mathfrak{a}_{j+1}}}{p_j}.$$

Let $\Delta_{\overline{p}}(\exp(T)) = e^{\delta_{\overline{p}}(T)}, T \in \mathfrak{p}$, and let

$$L^{\overline{p}}(G/P, \chi_f) = \{\xi : G \to \mathbb{C}; \xi \text{ measurable }, \xi(xp) = \Delta_{\overline{p}}(h)\chi_f(p)^{-1}\xi(x), x \in G, p \in P,$$

$$\|\xi\|_{\overline{p}} = \Big(\int_{\mathfrak{x}}\Big(\int_{\mathfrak{s}}\Big(\int_{\mathfrak{w}_1}\Big(\cdots\Big(\int_{\mathfrak{w}_m} |\xi(\exp(S)\exp(X)\exp(U_1)\cdots\exp(U_m))|^{p_m} dU_m\Big)^{\frac{1}{p_m}}$$

$$\cdots\Big)^{p_1} dU_1\Big)^{\frac{1}{p_1}}\Big)^2 dXdS\Big)^{\frac{1}{2}} < \infty\}.$$

It is easy to verify that this norm $\|\cdot\|_{\overline{p}}$ is translation invariant and that for $\overline{p} = (2, \cdots, 2) = \overline{2}$, we obtain the Hilbert space of the induced representation $\mathrm{ind}_P^G \chi_f$. Left translation defines thus an isometric representation denoted by $\tau_{(P,\chi_f,\overline{p})}$ on $L^{\overline{p}(G/P,\chi_f)}$. For every $\varphi \in L^1(G)$, the operator $\tau_{(P,\chi_f,\overline{p})}(\varphi)$ is a kernel operator, whose kernel $\varphi_{(P,\chi_f,\overline{p})}$ is equal to the kernel of the operator $\tau_{H,\chi_f}(\Delta_{\overline{p}}\Delta_{\overline{2}}^{-1}\varphi)$, if φ is exponentially decreasing. This observation tells us that $\tau_{(P,\chi_f,\overline{p})}$ is irreducible and that there exist many $\varphi \in L^1(G)$, for which $\tau_{(P,\chi_f,\overline{p})}(\varphi)$ is of rank one. The character $\chi_{\nu_{f,\overline{p}}}$ of the commutative algebra B defined by the simple module $(\tau_{(P,\chi_f,\overline{p})}, L^{\overline{p}}(G/P, \chi_f)^1)$ is given by

$$\chi_{\nu_{f,\overline{p}}}(h) = \int_F h(t) e^{\sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{2}) tr ad_{\mathfrak{a}_j/\mathfrak{a}_{j+1}}(\log t)} dt.$$

It turns out that every real linear functional $\nu = \nu_T$ on $\mathfrak{g}(f)$, for which $|\nu(T)| \leq \mu_q(T), T \in \mathfrak{g}(f)$, is of the form

$$\nu = \sum_{j=1}^{m} (\frac{1}{p_j} - \frac{1}{2}) tr\, ad_{\mathfrak{a}_j/\mathfrak{a}_{j+1}}$$

for some $(p_1, \cdots, p_m)$. This shows that any simple module (T, V) of $L^1(G)$ is equivalent to

$$(T_{\overline{p}}, V_{\overline{p}}) = (\tau_{(P,\chi_f,\overline{p})}, L^{\overline{p}}(G/H, \chi_f)^1)$$

for some $f \in \mathfrak{g}^*$ and some $\overline{p}$.

We obtain finally the following description of the space $\widetilde{G}$ of the equivalence classes of simple $L^1(G)$ modules.

Let $\mathfrak{g}^*_{prim}$ be the collection of all pairs $(f, \nu) \in \mathfrak{g}^* \times \mathfrak{g}(f)^*$, such that $|\nu| \leq \mu_{f_{|\mathfrak{n}}}$. The group G acts on $\mathfrak{g}^*_{prim}$ by Ad^*. Let $\mathfrak{g}^*_p/G$ be the corresponding quotient space. The mapping

$$\mathfrak{g}^*_{prim}/G \to \widetilde{G}, \quad [(f, \nu)] \mapsto [(\tau_{(P,\chi_f,\overline{p})}, L^{\overline{p}}(G/H, \chi_f)^1)], \quad \nu = \sum_{j=1}^{m} (\frac{1}{p_j} - \frac{1}{2}) tr\, ad_{\mathfrak{a}_j/\mathfrak{a}_{j+1}},$$

is a bijection (see [Po.2],[Lu.Mi.Mo.])

References:

[Be.] Beauzamy, B., *Introduction to Operator Theory and Invariant Subspaces,* North-Holland Mathematical Library (North-Holland, Amsterdam, New York, Oxford, Tokyo, 1988).

[Ber.] Bernat, P., Sur les représentations unitaires des groupes de Lie résolubles, *Ann. Ec. Norm. Sup.* **82** (1965), 37-99.

[Ber.Co.] Bernat, P., Conze, N., Duflo, M., Lévy-Nahas, M., Rais, M., Renouard, P., Vergne, M., *Représentations des groupes de Lie résolubles* (Dunod, Paris, 1972).

[Bo.Du.] Bonsall, F.F. , Duncan, J., *Complete Normed Algebras,* (Springer, 1973).

[Boi.] Boidol, J., *-Regularity of Exponential Lie Groups, *Invent. math.* **56** (1980), 231-238.

[Boi.Le.] Boidol, J., Leptin, H., Schürman, J., Vahle, D., Räume primitiver Ideale von Gruppenalgebren, *Math. Ann.* **236** (1978),1-13.

[Cor.Gr.] Corwin, L., Greenleaf, F.P., *Representations of nilpotent Lie groups and their applications,* Cambridge University Press (Cambridge, 1990).

[Di.1] Dixmier, J., Opérateurs de rang fini dans les représentations unitaires, *Inst. Hautes Etudes Sci. Publi. Math.* **6** (1960), 305-317.

[Di.2] Dixmier, J., *Algèbres enveloppantes* (Gauthiers-Villard, Paris, 1969).

[Di.3] Dixmier, J., L'application exponentielle dans les groupes de Lie résolubles, *Bull. Soc. Math. France* **85** (1957), 113-121.

[Di.4] Dixmier, J., *Les C^*-algèbres et leurs représentations* (Gauthiers-Villard, Paris, 1974).

[Di.5] Dixmier, J., Sur les représentations unitaires des groupes de Lie nilpotents , *Bull. Soc. Math. France* **85** (1957), 325-388.

[Fe.Do.] Fell, J.M.G., Doran, R.S., *Representations of *- Algebras, Locally Compact Groups and Banach *-Algebraic Bundles,* Volume **2** (Academic Press, Inc., San Diego, London 1988).

[Ha.Lu.] Hauenschild, W., Ludwig, J., The injection and the projection theorem for spectral sets, *Monatsh. Math.* **92** (1981), 167-177.

[Ho.] Howe, R., On a connection between nilpotent groups and oscillatory integrals associated to singularities, *Pacific. J. Math.* **73** (1977), 329-363.

[Hu.] Hulanicki, A., A functional calculus for Rockland operators on nilpotent Lie groups, *Studia Math.* **78** (1984), 253-266.

[Je.] Jenkins. J.W., Representations of exponentially bounded groups, *Amer. J. Math.* **98** No 1 (1976), 29-38.

[Ki.] Kirillov, A.A., Unitary representations of nilpotent Lie groups, *Uspekhi Mat. Nauk.* **17** (1962), 53-104.

[Le.1] Leptin, H., Ideal Theory in Group Algebras of Locally Compact Groups, *Invent.math.* **31** (1976), 259-278.

[Le.2] Leptin, H., Lokal kompakte Gruppen mit symmetrischen Algebren, Symposia Mat. **22** (1979).

[Le.Po.] Leptin, H., Poguntke, D, Symmetry and nonsymmetry for locally compact groups, *J. Funct. Anal.* **33** (1979), 119-134.

[Le.Lu.] Leptin, H., Ludwig, J., *Unitary Representation Theory of Exponential Lie Groups*, De Gruyter Expositions in Mathematics **18** (De Gruyter, Berlin, New York, 1994).

[Lu.1] Ludwig, J., Irreducible representations of exponential solvable Lie groups and operators with smooth kernels, *J. Reine Angew. Math.* **339** (1983), 1-26.

[Lu.2] Ludwig, J., A Class of symmetric and a class of Wiener group algebras, *J. Funct. Anal.* **31** (1979), 187-194.

[Lu.3] Ludwig, J., Minimal C^*-dense ideals and algebraically irreducible representations of the Schwartz-algebra of a nilpotent Lie group, *Harmonic Analysis*, Springer Verlag (1987), 209-217.

[Lu.Mo.1] Ludwig, J., Molitor-Braun, C., L'algèbre de Schwartz d'un groupe de Lie nilpotent, *Travaux math.* **VII**, *Publications du C.U. de Luxembourg* (1995), 25-67.

[Lu.Mo.2] Ludwig, J., Molitor-Braun, C., Exponential actions, orbits and their kernels, *Bull. Austral. Math. Soc.* **57** (1998), 497-513.

[Lu.Mo.3] Ludwig, J., Molitor-Braun, C., Représentations irréductibles bornées des groupes de Lie exponentiels, preprint.

[Lu.Mi.Mo.] Ludwig, J., Mint Elhacen, S., Molitor-Braun, C., Characterization of the simple $L^1(G)$-modules for exponential Lie groups, preprint.

[Mo.1] Molitor-Braun, C., *Actions exponentielles et idéaux premiers*, Thèse (Metz, 1996).

[Mo.2] Molitor-Braun, C., Exponential actions and maximal $\mathfrak{D}$- invariant ideals, *Manuscr. math.* **96** (1998), 23-35.

[Pa.] Palmer, T.W., *Banach Algebras and the General Theory of $*$-Algebras*, Volume I, *Algebras and Banach Algebras*, Encyclopedia of mathematics and its applications, Vol. **49**, Cambridge University Press (Cambridge, 1994).

[Pi.] Pier, J.P., *Amenable Locally Compact Groups*, J. Wiley and sons (New York, 1984).

[Po.1] Poguntke, D., Operators of Finite Rank in Unitary Representations of Exponential Lie Groups, *Math. Ann.* **259** (1982), 371-383.

[Po.2] Poguntke, D., Algebraically irreducible representations of L^1-algebras of exponential Lie groups, *Duke math. J.*, Vol. **50**, N. **4** (1983), 1077-1106.

[Po.3] Poguntke, D., Symmetry and Non Symmetry for a class of exponential Lie groups, *J. reine angew. Math.* **315** (1980), 127-138.

[Po.4] Poguntke, D., Nilpotente Liesche Gruppen haben symmetrische Gruppenalgebren, *Math. Ann.* **227** (1980), 51-59.
[Pu.1] Pukanszky, L., On the unitary representations of exponential groups, *J. Funct. Anal.* **2** (1968), 73-113.
[Pu.2] Pukanszky, L., On the theory of exponential groups, *Trans. Amer. Math. Soc.,* **126** (1967), 487-507.
[So.] Soergel, W., An irreducible not admissible Banach representation of $SL(2,\mathbb{R})$, *Proc. Amer. Math. Soc.*, Vol. **104**, N. **4** (1988), 1322-1324.
[Ve.1] Vergne, M., Constructions de sous-algèbres subordonnées à un élément du dual d'une algèbre de Lie résoluble, *C. R. Acad. Sci. Paris,* **270** (1970), 173-175.
[Ve.2] Vergne, M., Constructions de sous-algèbres subordonnées à un élément du dual d'une algèbre de Lie résoluble, *C. R. Acad. Sci. Paris,* **270** (1970), 704-707.
[Ve.3] Vergne, M., Etude de certaines représentations induites d'un groupe de Lie résoluble exponentiel, *Ann. Ec. Norm. Sup.,* **3** (1970), 353-384.

Jean Ludwig
Département de Mathématiques
Université de Metz
Ile du Saulcy
F-57045 Metz CEDEX
France
e-mail: ludwig@poncelet.sciences.univ-metz.fr

Theta lifting of two-step nilpotent orbits for the pair $O(p,q) \times Sp(2n,\mathbb{R})$

Kyo Nishiyama

Faculty of Integrated Human Studies
Kyoto University
Sakyo, Kyoto 606-8501, Japan
E-mail: kyo@math.h.kyoto-u.ac.jp

Introduction

Let G be a linear reductive Lie group which is a subgroup in its complexification $G_{\mathbb{C}}$. We denote the Lie algebra of G by $\mathfrak{g}_0$, and its complexification by $\mathfrak{g} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}_0$. We will use the similar notation for any linear Lie group L; thus, $L_{\mathbb{C}}$ denotes its complexification, $\mathfrak{l}_0$ its Lie algebra, and $\mathfrak{l}$ the complexification of $\mathfrak{l}_0$.

Take a maximal compact subgroup K of G. Then K determines a Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{s}_0$ and its complexification $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$. The adjoint action of $K_{\mathbb{C}}$ preserves $\mathfrak{s}$, and the set of all nilpotent elements $\mathcal{N}_{\mathfrak{s}}$ in $\mathfrak{s}$. It is well known that $\mathcal{N}_{\mathfrak{s}}$ is a normal variety and that it has finitely many $K_{\mathbb{C}}$-orbits ([2]).

Now consider a dual pair $(G, G') = (O(p,q), Sp(2n,\mathbb{R}))$ (see [1] for the properties of dual pairs). In this note, we define certain double fibration maps of nilpotent varieties for $O(p,q)$ and $Sp(2n,\mathbb{R})$. We use the double fibration maps to get a correspondence between nilpotent $K_{\mathbb{C}}$-orbits in $\mathfrak{s}$ and nilpotent $K'_{\mathbb{C}}$-orbits in $\mathfrak{s}'$, which is called a "theta lift". We describe the theta lifts of two-step nilpotent orbits in $\mathcal{N}_{\mathfrak{s}'}$, where $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{s}'$ is a Cartan decomposition for $G' = Sp(2n,\mathbb{R})$ (Proposition 1.3).

If a nilpotent $K_{\mathbb{C}}$-orbit $\mathcal{O} \subset \mathfrak{s}$ is the theta lift of a nilpotent $K'_{\mathbb{C}}$-orbit $\mathcal{O}' \subset \mathfrak{s}'$, it is interesting to describe the regular function ring $\mathbb{C}[\overline{\mathcal{O}}]$ by means of $\mathbb{C}[\overline{\mathcal{O}'}]$. Our main results are descriptions of the $K_{\mathbb{C}}$-module structure of $\mathbb{C}[\overline{\mathcal{O}}]$ in terms of the double fibration maps (Theorem 2.4 and Proposition 3.4). In the course of the proof, we realize the closure $\overline{\mathcal{O}}$ of the orbit as a geometric quotient of the fiber of $\overline{\mathcal{O}'}$ (Proposition 3.3). As an application of these results, we get a formula of branching coefficients between different kind of classical groups (Corollary 3.5).

The $K_{\mathbb{C}}$-module structures of nilpotent orbits may reflect the K-type decompositions of the corresponding admissible representation of G via orbit method (or geometric quantization). Thus we can expect to extract information on the admissible representations from the geometry of nilpotent orbits. This will be treated elsewhere.

Acknowledgments. The author thanks Cheng-Bo Zhu for the discussions on the double fibration maps. Some of the ideas used here are due to him. He also thanks Takuya Ohta and Hiroshi Yamashita for stimulating conversations.

1 Double fibration of nilpotent varieties

Let $G = O(p,q)$ be an orthogonal group of signature (p,q). Then a maximal compact subgroup K is isomorphic to $O(p) \times O(q)$. We realize them as follows.

$$G = O(p,q) = \{g \in GL(p+q,\mathbb{R}) \mid {}^t g 1_{p,q} g = 1_{p,q}\}, \quad 1_{p,q} = \begin{pmatrix} 1_p & 0 \\ 0 & -1_q \end{pmatrix},$$
$$K = O(p) \times O(q) = \begin{pmatrix} O(p) & 0 \\ 0 & O(q) \end{pmatrix}.$$

Then the corresponding (complexified) Cartan decomposition is given by

$$\mathfrak{g} = \left\{ \begin{pmatrix} \alpha & \beta \\ {}^t\beta & \gamma \end{pmatrix} \mid \begin{matrix} \alpha \in \mathrm{Alt}_p(\mathbb{C}) \\ \gamma \in \mathrm{Alt}_q(\mathbb{C}) \end{matrix}, \beta \in M_{p,q}(\mathbb{C}) \right\}$$
$$= \begin{pmatrix} \mathrm{Alt}_p(\mathbb{C}) & 0 \\ 0 & \mathrm{Alt}_q(\mathbb{C}) \end{pmatrix} \oplus \begin{pmatrix} 0 & M_{p,q}(\mathbb{C}) \\ {}^t M_{p,q}(\mathbb{C}) & 0 \end{pmatrix} = \mathfrak{k} \oplus \mathfrak{s}.$$

Hence we identify $\mathfrak{s}$ with $M_{p,q}(\mathbb{C})$ via

$$M_{p,q}(\mathbb{C}) \ni \beta \leftrightarrow \begin{pmatrix} 0 & \beta \\ {}^t\beta & 0 \end{pmatrix} \in \mathfrak{s}.$$

Denote the set of nilpotent elements in $\mathfrak{s}$ by $\mathcal{N}_\mathfrak{s}$. Then, by the above identification, $\beta \in M_{p,q}(\mathbb{C})$ belongs to $\mathcal{N}_\mathfrak{s}$ if and only if ${}^t\beta\beta$ is a nilpotent matrix, if and only if $\beta\,{}^t\beta$ is so.

Next we consider the symplectic group $G' = Sp(2n,\mathbb{R})$ of rank n. A maximal compact subgroup K' is isomorphic to the unitary group $U(n)$ of size n. To realize K' in a simple way, we define $Sp(2n,\mathbb{R})$ in a slightly different manner from the usual one. Namely, we put

$$\begin{aligned} G' &= U(n,n) \cap Sp(2n,\mathbb{C}) \\ &= \{g \in GL(2n,\mathbb{C}) \mid {}^t\overline{g} 1_{n,n} g = 1_{n,n}, {}^t g J g = J\}, \end{aligned}$$

where

$$1_{n,n} = \begin{pmatrix} 1_n & 0 \\ 0 & -1_n \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}.$$

Then G' is isomorphic to $Sp(2n,\mathbb{R})$, and

$$K' = \left\{ \begin{pmatrix} k & 0 \\ 0 & {}^t k^{-1} \end{pmatrix} \mid k \in U(n) \right\} \subset G'$$

is a maximal compact subgroup. The corresponding Cartan decomposition is given by

$$\mathfrak{g}' = \left\{ \begin{pmatrix} H & 0 \\ 0 & -{}^tH \end{pmatrix} \mid H \in \mathfrak{gl}_n(\mathbb{C}) \right\} \oplus \left\{ \begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix} \mid C, D \in \mathrm{Sym}_n(\mathbb{C}) \right\} = \mathfrak{k}' \oplus \mathfrak{s}'.$$

We identify $\mathfrak{s}'$ with $\mathrm{Sym}_n(\mathbb{C}) \oplus \mathrm{Sym}_n(\mathbb{C})$ via

$$\mathrm{Sym}_n(\mathbb{C}) \oplus \mathrm{Sym}_n(\mathbb{C}) \ni (C, D) \leftrightarrow \begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix} \in \mathfrak{s}'.$$

Then (C, D) belongs to the nilpotent variety $\mathcal{N}_{\mathfrak{s}'}$ if and only if $C \cdot D$ is nilpotent, if and only if $D \cdot C$ is so.

Now we shall define the double fibration maps. Let $W = M_{p+q,n}(\mathbb{C})$ be the space of all the $(p+q) \times n$-matrices. We express a matrix Z in W as

$$Z = \begin{pmatrix} A \\ B \end{pmatrix} \in W; \qquad A \in M_{p,n}(\mathbb{C}), \quad B \in M_{q,n}(\mathbb{C}).$$

We define two maps φ and ψ by

$$\begin{aligned} \varphi &: W \ni Z \longmapsto A\,{}^tB \in M_{p,q}(\mathbb{C}) = \mathfrak{s}, \\ \psi &: W \ni Z \longmapsto ({}^tAA, {}^tBB) \in \mathrm{Sym}_n(\mathbb{C}) \oplus \mathrm{Sym}_n(\mathbb{C}) = \mathfrak{s}'. \end{aligned}$$

Put

$$\begin{aligned} M_{\mathbb{C}} &= GL_p(\mathbb{C}) \times GL_q(\mathbb{C}) \supset O(p, \mathbb{C}) \times O(q, \mathbb{C}) = K_{\mathbb{C}}, \\ M'_{\mathbb{C}} &= GL_n(\mathbb{C}) \times GL_n(\mathbb{C}) \supset \Delta GL_n(\mathbb{C}) = K'_{\mathbb{C}}, \end{aligned}$$

and define $M_{\mathbb{C}} \times M'_{\mathbb{C}}$-action on W by

$$(m, m') \cdot Z = \begin{pmatrix} m_1 A\,{}^tm'_1 \\ m_2 B {m'_2}^{-1} \end{pmatrix} \quad \text{for } Z = \begin{pmatrix} A \\ B \end{pmatrix} \in W,$$

where

$$\begin{aligned} m &= (m_1, m_2) \in M_{\mathbb{C}} = GL_p(\mathbb{C}) \times GL_q(\mathbb{C}), \\ m' &= (m'_1, m'_2) \in M'_{\mathbb{C}} = GL_n(\mathbb{C}) \times GL_n(\mathbb{C}). \end{aligned}$$

We introduce $M_{\mathbb{C}}$-action on $\mathfrak{s}$ (resp. $M'_{\mathbb{C}}$-action on $\mathfrak{s}'$) so that $\varphi : W \to \mathfrak{s}$ is an $M_{\mathbb{C}} \times K'_{\mathbb{C}}$-equivariant map (resp. $\psi : W \to \mathfrak{s}'$ is a $K_{\mathbb{C}} \times M'_{\mathbb{C}}$-equivariant map). Note that the induced action is compatible with the adjoint $K_{\mathbb{C}}$-action on $\mathfrak{s}$ (resp. $K'_{\mathbb{C}}$-action on $\mathfrak{s}'$). As a $GL_n(\mathbb{C})$-module, the second component $\mathrm{Sym}_n(\mathbb{C})$ of $\mathfrak{s}'$ is regarded as the contragredient of the first component. By this reason, sometimes we will write $\mathfrak{s}' = \mathrm{Sym}_n(\mathbb{C}) \oplus \mathrm{Sym}_n(\mathbb{C})^*$.

Our first observation is the following.

Lemma 1.1 *$\varphi \circ \psi^{-1}$ and $\psi \circ \varphi^{-1}$ preserve nilpotent elements:*

$$\varphi(\psi^{-1}(\mathcal{N}_{\mathfrak{s}'})) \subset \mathcal{N}_{\mathfrak{s}}, \qquad \psi(\varphi^{-1}(\mathcal{N}_{\mathfrak{s}})) \subset \mathcal{N}_{\mathfrak{s}'}.$$

PROOF. This is an easy consequence of direct calculations. Q.E.D.

Definition 1.2 Let $\mathcal{O}$ (resp. $\mathcal{O}'$) be a nilpotent $K_{\mathbb{C}}$-orbit in $\mathfrak{s}$ (resp. $K'_{\mathbb{C}}$-orbit in $\mathfrak{s}'$). If $\overline{\mathcal{O}} = \varphi(\psi^{-1}(\overline{\mathcal{O}'}))$ holds, we say that $\mathcal{O}$ is the *theta lift* of $\mathcal{O}'$.

Note that $\varphi(\psi^{-1}(\overline{\mathcal{O}'}))$ is an affine closed cone.

Proposition 1.3 *Assume that $2n < \min(p,q)$. Let $\mathcal{O}'_{[r,s]} = \mathcal{O}'_{\lambda_{r,s}} \subset \mathcal{N}_{\mathfrak{s}'}$ be a nilpotent $K'_{\mathbb{C}}$-orbit through*

$$\lambda_{r,s} = \left(\begin{array}{c|c} 0 & \begin{matrix} 1_r & \\ & 0 \end{matrix} \\ \hline \begin{matrix} 0 & \\ & 1_s \end{matrix} & 0 \end{array}\right) \quad (r+s \le n).$$

Then there exists a nilpotent $K_{\mathbb{C}}$-orbit $\mathcal{O} \subset \mathcal{N}_{\mathfrak{s}}$ for which $\varphi(\psi^{-1}(\overline{\mathcal{O}'_{[r,s]}})) = \overline{\mathcal{O}}$ holds, i.e., the theta lift of $\mathcal{O}'_{[r,s]}$ exists. We denote $\mathcal{O} = \mathcal{O}_{[n;r,s]}$.

Remark 1.4 We allow $r = s = 0$, which means that $\mathcal{O}'_{[0,0]} = \{0\}$. Note that $\mathcal{O}'_{[r,s]}$ exhausts all the two-step nilpotent orbits in $\mathfrak{s}'$.

PROOF. We will specify the nilpotent $K_{\mathbb{C}}$-orbit $\mathcal{O} = \mathcal{O}_{[n;r,s]}$ in the end of the proof.

To prove the proposition, it suffices to prove that $\psi^{-1}(\overline{\mathcal{O}'_{[r,s]}})$ is irreducible. In fact, if it is irreducible, then $\varphi(\psi^{-1}(\overline{\mathcal{O}'_{[r,s]}}))$ is an irreducible closed set, and is $K_{\mathbb{C}}$-stable in $\mathcal{N}_{\mathfrak{s}}$. Since $\mathcal{N}_{\mathfrak{s}}$ contains only a finite number of $K_{\mathbb{C}}$-orbits, it must be the closure of a single orbit.

Let us see that $\psi^{-1}(\overline{\mathcal{O}'_{[r,s]}})$ is irreducible. We call

$$\mathfrak{N}_{p,k} = \{A \in M_{p,k}(\mathbb{C}) \mid {}^tAA = 0\}$$

a *null cone* of size (p,k). It is known to be irreducible if $2k < p$. Thus, if we put

$$N_{r,s} = \left\{ Z = \begin{pmatrix} A \\ B \end{pmatrix} \in W \mid A = \left(\begin{array}{c|c} 1_r & 0 \\ \hline 0 & E \end{array}\right), B = \left(\begin{array}{c|c} 0 & 1_s \\ \hline F & 0 \end{array}\right), \right.$$
$$\left. \text{where } E \in \mathfrak{N}_{p-r,n-r} \text{ and } F \in \mathfrak{N}_{q-s,n-s} \right\}$$
$$\simeq \mathfrak{N}_{p-r,n-r} \times \mathfrak{N}_{q-s,n-s},$$

then $N_{r,s}$ is irreducible and is contained in the fiber of $\lambda_{r,s}$. Moreover, under the condition that $2n < \min(p,q)$, it is easy to check that the exact fiber of $\lambda_{r,s}$ is given by

$$K_{\mathbb{C}}^{\circ} \cdot N_{r,s} = \psi^{-1}(\lambda_{r,s}),$$

where $K_{\mathbb{C}}^{\circ} \simeq SO(p,\mathbb{C}) \times SO(q,\mathbb{C})$ is the identity component of $K_{\mathbb{C}}$. Now we see that

$$(K_{\mathbb{C}}^{\circ} \times K_{\mathbb{C}}') \cdot N_{r,s} = \psi^{-1}(\mathcal{O}'_{[r,s]}),$$

is irreducible, and hence $\psi^{-1}(\overline{\mathcal{O}'_{[r,s]}})$ is irreducible.

We can take the following matrix as a representative of a generic $K_{\mathbb{C}}^{\circ} \times K_{\mathbb{C}}'$-orbit in $\psi^{-1}(\mathcal{O}'_{[r,s]})$.

$$Z = \begin{pmatrix} A \\ B \end{pmatrix} \in W;$$

$$A = \left(\begin{array}{c|c} 1_r & 0 \\ \hline 0 & 1_{n-r} \\ \hline 0 & 0 \\ \hline 0 & i1_{n-r} \end{array}\right) \in M_{p,n}(\mathbb{C}), \quad B = \left(\begin{array}{c|c} 1_{n-s} & 0 \\ \hline 0 & 1_s \\ \hline 0 & 0 \\ \hline i1_{n-s} & 0 \end{array}\right) \in M_{q,n}(\mathbb{C}). \tag{1.1}$$

By the above arguments, we know that the theta lift of $\mathcal{O}'_{[r,s]}$ should be exactly the $K_{\mathbb{C}}$-orbit through $\varphi(Z)$, where Z is given in (1.1). Q.E.D.

By the above proof, we conclude that the theta lift $\mathcal{O}_{[n;r,s]}$ of $\mathcal{O}'_{[r,s]}$ consists of at most three-step nilpotents. It is two-step nilpotent if and only if $r = s = 0$. Thus, we see that the theta lift of a k-step nilpotent orbit is a $(k+1)$-step nilpotent orbit.

2 Regular function ring of nilpotent orbits

In this section, we always assume that $2n < \min(p,q)$.

Let $\mathcal{O}'_{[r,s]} = \mathcal{O}'_{\lambda_{r,s}}$ $(r + s \leq n)$ be a nilpotent $K_{\mathbb{C}}'$-orbit in $\mathcal{N}_{\mathfrak{s}'}$ given in Proposition 1.3. We denote the corresponding theta lift by $\mathcal{O}_{[n;r,s]}$, which is a nilpotent $K_{\mathbb{C}}$-orbit in $\mathcal{N}_{\mathfrak{s}}$.

We consider the case $s = 0$ in the following. Then we have

$$\mathcal{O}'_{[r,0]} = \{(C,0) \in \mathfrak{s}' \mid C \in \operatorname{Sym}_n(\mathbb{C}), \operatorname{rank} C = r\},$$

and it is known that $\overline{\mathcal{O}'_{[r,0]}}$ is the associated variety of an irreducible unitary highest weight representation of $Sp(2n,\mathbb{R})$ (or its metaplectic double cover). In particular, $\overline{\mathcal{O}'_{[n,0]}} \simeq \operatorname{Sym}_n(\mathbb{C})$ is the associated variety of a holomorphic discrete series representation of $Sp(2n,\mathbb{R})$.

Since $\mathcal{O}'_{[r,0]}$ is a $K_{\mathbb{C}}'$-orbit, the regular function ring $\mathbb{C}[\overline{\mathcal{O}'_{[r,0]}}]$ carries a natural $K_{\mathbb{C}}'$-module structure. Note that $K_{\mathbb{C}}' = GL_n(\mathbb{C})$. We denote by $\mathcal{P}_k$ the set of all partitions of length $\leq k$, i.e., $\mathcal{P}_k = \{\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{Z}^k \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0\}$.

Theorem 2.1 *The regular function ring $\mathbb{C}[\overline{\mathcal{O}'_{[r,0]}}]$ is decomposed as*

$$\mathbb{C}[\overline{\mathcal{O}'_{[r,0]}}] \simeq \sum_{\lambda \in \mathcal{P}_r}^{\oplus} \tau_{2\lambda}{}^{*} \qquad (\textit{as a } GL_n(\mathbb{C})\textit{-module}),$$

where τ_μ denotes an irreducible finite dimensional representation of $GL_n(\mathbb{C})$ with highest weight μ, and $\tau_\mu{}^*$ is its contragredient.

PROOF. See [5], for example. Q.E.D.

Note that the fibration map

$$\psi : W = M_{p,n}(\mathbb{C}) \times M_{q,n}(\mathbb{C}) \longrightarrow \mathrm{Sym}_n(\mathbb{C}) \times \mathrm{Sym}_n(\mathbb{C})^* = \mathfrak{s}'$$

is a product of two maps of the same kind,

$$\psi_p : M_{p,n} \ni A \longmapsto {}^tAA \in \mathrm{Sym}_n(\mathbb{C}) \quad \text{and}$$
$$\psi_q : M_{q,n} \ni B \longmapsto {}^tBB \in \mathrm{Sym}_n(\mathbb{C})^*.$$

Since $Sp(2n, \mathbb{R})/U(n)$ is a Hermitian symmetric space, $\mathfrak{s}'$ decomposes into two pieces of $K'_{\mathbb{C}}$-stable subspaces $\mathfrak{s}' = \mathfrak{s}'_+ \oplus \mathfrak{s}'_-$, which we can identify with the decomposition $\mathfrak{s}' = \mathrm{Sym}_n(\mathbb{C}) \oplus \mathrm{Sym}_n(\mathbb{C})^*$. Our orbit $\mathcal{O}'_{[r,0]}$ lives in $\mathfrak{s}'_+$ alone. Therefore, if we put $\Xi_{[r,0]} = \psi^{-1}(\overline{\mathcal{O}'_{[r,0]}})$, it is decomposed as a product of closed affine cones

$$\Xi_{[r,0]} = \psi_p^{-1}(\overline{\mathcal{O}'_{[r,0]}}) \times \psi_q^{-1}(\{0\}) = \Xi_r^{(p)} \times \mathfrak{N}_{q,n},$$

where $\mathfrak{N}_{q,n}$ denotes the null cone given in the proof of Proposition 1.3, and

$$\begin{aligned} \Xi_r^{(p)} = \psi_p^{-1}(\overline{\mathcal{O}'_{[r,0]}}) &= \{A \in M_{p,n}(\mathbb{C}) \mid {}^tAA \in \overline{\mathcal{O}'_{[r,0]}}\} \\ &= \{A \in M_{p,n}(\mathbb{C}) \mid \mathrm{rank}\, {}^tAA \le r\}. \end{aligned}$$

Recall that

$$K'_{\mathbb{C}} = \Delta GL_n(\mathbb{C}) \subset GL_n(\mathbb{C}) \times GL_n(\mathbb{C}) = M'_{\mathbb{C}}.$$

The following lemma is now clear.

Lemma 2.2 *The fiber $\Xi_{[r,0]} = \psi^{-1}(\overline{\mathcal{O}'_{[r,0]}})$ is a product $\Xi_r^{(p)} \times \mathfrak{N}_{q,n}$, and hence it is $K_{\mathbb{C}} \times M'_{\mathbb{C}}$-stable. The regular function ring breaks up into*

$$\mathbb{C}[\Xi_{[r,0]}] \simeq \mathbb{C}[\Xi_r^{(p)}] \boxtimes \mathbb{C}[\mathfrak{N}_{q,n}]$$

as an $(O(p, \mathbb{C}) \times GL_n(\mathbb{C})) \times (O(q, \mathbb{C}) \times GL_n(\mathbb{C}))$-module.

The regular function ring $\mathbb{C}[\mathfrak{N}_{q,n}]$ consists of precisely the $O(q, \mathbb{C})$-harmonic polynomials in $\mathbb{C}[M_{q,n}]$ (see [4], for example). As a consequence, it decomposes in a multiplicity-free manner,

$$\mathbb{C}[\mathfrak{N}_{q,n}] \simeq \sum_{\mu \in \mathcal{P}_n}^{\oplus} \sigma_\mu^{(q)} \boxtimes \tau_\mu \quad (\text{as an } O(q, \mathbb{C}) \times GL_n(\mathbb{C})\text{-module}), \tag{2.1}$$

where $\sigma_\mu^{(q)}$ denotes an irreducible finite dimensional representation of $O(q, \mathbb{C})$ with highest weight μ. Let us decompose $\mathbb{C}[\Xi_r^{(p)}]$ as an $O(p,\mathbb{C}) \times GL_n(\mathbb{C})$-module,

$$\mathbb{C}[\Xi_r^{(p)}] \simeq \sum_{\lambda,\eta}^{\oplus} m(\lambda,\eta)\, \sigma_\eta^{(p)} \boxtimes \tau_\lambda{}^* \quad (\text{as an } O(p,\mathbb{C}) \times GL_n(\mathbb{C})\text{-module}), \tag{2.2}$$

where $m(\lambda, \eta)$ denotes the multiplicity.

For $\lambda \in \mathcal{P}_n$, decompose an irreducible representation $\tau_\lambda^{(p)}$ of $GL_p(\mathbb{C})$ restricted to $O(p,\mathbb{C})$,

$$\tau_\lambda^{(p)}\big|_{O(p,\mathbb{C})} \simeq \sum_{\eta\in\mathcal{P}_n}^{\oplus} b_\eta^\lambda\, \sigma_\eta^{(p)}, \tag{2.3}$$

where b_η^λ denotes the branching coefficient. Note that η is also a partition of length $\leq n$.

Lemma 2.3 *The summation in* (2.2) *is taken over $\lambda, \eta \in \mathcal{P}_n$; and the multiplicity $m(\lambda,\eta)$ satisfies the following inequality,*

$$\delta_{\lambda,\eta} \leq m(\lambda,\eta) \leq b_\eta^\lambda, \tag{2.4}$$

where $\delta_{\lambda,\eta}$ denotes Kronecker's delta. Moreover, we have a decomposition

$$\mathbb{C}[\Xi_{[r,0]}] \simeq \sum_{\lambda,\mu,\eta\in\mathcal{P}_n}^{\oplus} m(\lambda,\eta)\, (\sigma_\eta^{(p)} \boxtimes \sigma_\mu^{(q)}) \boxtimes (\tau_\lambda{}^* \boxtimes \tau_\mu) \tag{2.5}$$

as an $(O(p,\mathbb{C}) \times O(q,\mathbb{C})) \times (GL_n(\mathbb{C}) \times GL_n(\mathbb{C}))$-module, where $m(\lambda,\eta)$ denotes the multiplicity given above.

PROOF. Since $\Xi_r^{(p)}$ is a closed subvariety of $M_{p,n}$, $\mathbb{C}[\Xi_r^{(p)}]$ is a quotient of $\mathbb{C}[M_{p,n}]$. On the other hand, it is well known that $\mathbb{C}[M_{p,n}]$ decomposes as

$$\mathbb{C}[M_{p,n}] \simeq \sum_{\lambda\in\mathcal{P}_n}^{\oplus} \tau_\lambda^{(p)*} \boxtimes \tau_\lambda^{(n)*} \quad (\text{as a } GL_p(\mathbb{C}) \times GL_n(\mathbb{C})\text{-module}).$$

Therefore, we have

$$\mathbb{C}[M_{p,n}] \simeq \sum_{\lambda,\eta\in\mathcal{P}_n}^{\oplus} b_\eta^\lambda\, \sigma_\eta^{(p)} \boxtimes \tau_\lambda^{(n)*} \quad (\text{as an } O(p,\mathbb{C}) \times GL_n(\mathbb{C})\text{-module}).$$

Now the second inequality in (2.4) is clear. The first inequality follows from the fact that $\mathfrak{N}_{p,n} \subset \Xi_r^{(p)}$ (cf. (2.1)). Q.E.D.

Theorem 2.4 *We assume that $2n < \min(p, q)$. Then the regular function ring of the theta lift $\mathcal{O}_{[n;r,0]}$ decomposes as*

$$\mathbb{C}[\overline{\mathcal{O}_{[n;r,0]}}] \simeq \sum_{\lambda,\eta\in\mathcal{P}_n}^{\oplus} m(\lambda,\eta)\,\sigma_\eta^{(p)} \boxtimes \sigma_\lambda^{(q)} \tag{2.6}$$

as a $K_\mathbb{C} = O(p,\mathbb{C}) \times O(q,\mathbb{C})$-module, where the multiplicity $m(\lambda,\eta)$ is given in (2.2) *(cf. Lemma 2.3).*

We shall prove Theorem 2.4 in the next section.

Corollary 2.5 (1) *We have a multiplicity-free decomposition*

$$\mathbb{C}[\overline{\mathcal{O}_{[n;0,0]}}] \simeq \sum_{\lambda\in\mathcal{P}_n}^{\oplus} \sigma_\lambda^{(p)} \boxtimes \sigma_\lambda^{(q)} \qquad \text{(cf. [4])}.$$

(2) *If we denote the branching coefficient of the restriction $GL_p(\mathbb{C}) \downarrow O(p,\mathbb{C})$ by b_η^λ (see* (2.3)*), the following decomposition holds.*

$$\mathbb{C}[\overline{\mathcal{O}_{[n;n,0]}}] \simeq \sum_{\lambda,\eta\in\mathcal{P}_n}^{\oplus} b_\eta^\lambda\,\sigma_\eta^{(p)} \boxtimes \sigma_\lambda^{(q)}.$$

3 Harmonic polynomials and geometric quotient

In this section, we always assume that $2n < \min(p, q)$ as in the former section.

To prove Theorem 2.4, we study the induced algebra homomorphisms

$$\varphi^* : \mathbb{C}[\mathfrak{s}] \longrightarrow \mathbb{C}[W], \quad \text{and} \quad \psi^* : \mathbb{C}[\mathfrak{s}'] \longrightarrow \mathbb{C}[W].$$

Let us introduce a coordinate on $\mathfrak{s}'$. Take $(C, D) \in \mathfrak{s}'_+ \oplus \mathfrak{s}'_- = \mathfrak{s}'$, where $C = (C_{ij})$ and $D = (D_{ij})$ are symmetric matrices. We use $\{C_{ij} \mid 1 \le i \le j \le n\} \cup \{D_{ij} \mid 1 \le i \le j \le n\}$ as a coordinate on $\mathfrak{s}'$. Then ψ^* is given explicitly by

$$\psi^*(C_{ij}) = \sum_{k=1}^{p} A_{ki}A_{kj}; \quad \psi^*(D_{ij}) = \sum_{l=1}^{q} B_{li}B_{lj},$$

where $\{A_{ij} = Z_{ij} \mid 1 \le i \le p, 1 \le j \le n\} \cup \{B_{ij} = Z_{p+i,j} \mid 1 \le i \le q, 1 \le j \le n\}$ is considered as a system of coodinate functions on W which extracts the (i,j)-th element of $Z = \begin{pmatrix} A \\ B \end{pmatrix} \in M_{p+q,n}(\mathbb{C}) = W$. Note that the image of the coodinate functions via ψ^* is precisely the fundamental invariants for $K_\mathbb{C} = O(p,\mathbb{C}) \times O(q,\mathbb{C})$, which generate all the $K_\mathbb{C}$-invariants in $\mathbb{C}[W]$. Thus

$$\psi^* : \mathbb{C}[\mathfrak{s}'] \longrightarrow \mathbb{C}[W]^{K_\mathbb{C}}$$

is surjective. Moreover, we have

Lemma 3.1 *Assume that $2n < \min(p,q)$. Then the map $\psi^* : \mathbb{C}[\mathfrak{s}'] \longrightarrow \mathbb{C}[W]^{K_\mathbb{C}}$ is an isomorphism.*

Similarly, if we introduce a coordinate on $\mathfrak{s}$ by the (k,l)-th element of $X = (X_{kl}) \in M_{p,q}(\mathbb{C}) = \mathfrak{s}$, we see that

$$\varphi^*(X_{kl}) = \sum_{i=1}^{n} A_{ki}B_{li},$$

which is a fundamental invariant for $K'_\mathbb{C} = GL_n(\mathbb{C})$. Thus $\varphi^* : \mathbb{C}[\mathfrak{s}] \to \mathbb{C}[W]^{K'_\mathbb{C}}$ is surjective by the similar arguments as above. Let $\mathfrak{s}_{[n]} = \{X \in M_{p,q}(\mathbb{C}) \mid \operatorname{rank} X \leq n\}$ be the determinantal variety of rank n.

Lemma 3.2 *Assume that $2n < \min(p,q)$. Then the image of φ is precisely the determinantal variety: $\varphi(W) = \mathfrak{s}_{[n]}$. Thus the induced algebra homomorphism $\varphi^* : \mathbb{C}[\mathfrak{s}_{[n]}] \longrightarrow \mathbb{C}[W]^{K'_\mathbb{C}}$ is an isomorphism.*

The proofs of the above two lemmas are almost immediate. We omit them.

Proposition 3.3 *Let $\mathcal{O}_{[n;r,s]}$ be the theta lift of $\mathcal{O}'_{[r,s]}$. Then $\overline{\mathcal{O}_{[n;r,s]}}$ is the geometric quotient of the fiber $\Xi_{[r,s]} = \psi^{-1}(\overline{\mathcal{O}'_{[r,s]}})$ by $K'_\mathbb{C}$, i.e., $\overline{\mathcal{O}_{[n;r,s]}} = \Xi_{[r,s]}//K'_\mathbb{C}$. In particular, we have*

$$\mathbb{C}[\overline{\mathcal{O}_{[n;r,s]}}] \simeq \mathbb{C}[\Xi_{[r,s]}]^{K'_\mathbb{C}}.$$

PROOF. Let $J = \mathbf{I}(\Xi_{[r,s]})$ be the defining ideal of $\Xi_{[r,s]} \subset W$. Then, $I = (\varphi^*)^{-1}(J)$ is the defining ideal of $\overline{\mathcal{O}_{[n;r,s]}}$, since $\varphi(\Xi_{[r,s]}) = \overline{\mathcal{O}_{[n;r,s]}}$. Recall that $\varphi^* : \mathbb{C}[\mathfrak{s}] \to \mathbb{C}[W]^{K'_\mathbb{C}}$ is surjective.

$$\begin{array}{ccc} \mathbb{C}[\mathfrak{s}] & \xrightarrow{\varphi^* \,:\, \text{surjection}} & \mathbb{C}[W]^{K'_\mathbb{C}} \\ \text{projection}\downarrow & & \downarrow\text{projection} \\ \mathbb{C}[\overline{\mathcal{O}_{[n;r,s]}}] = \mathbb{C}[\mathfrak{s}]/I & \xrightarrow{\ \sim\ } & \mathbb{C}[W]^{K'_\mathbb{C}}/J^{K'_\mathbb{C}} \end{array}$$

Therefore, we get $\mathbb{C}[\mathfrak{s}]/I \simeq \mathbb{C}[W]^{K'_\mathbb{C}}/J^{K'_\mathbb{C}}$. Note that $\mathbb{C}[\Xi_{[r,s]}]^{K'_\mathbb{C}} = (\mathbb{C}[W]/J)^{K'_\mathbb{C}} \simeq \mathbb{C}[W]^{K'_\mathbb{C}}/J^{K'_\mathbb{C}}$. Thus, the proposition is proved. Q.E.D.

Let us consider the case where $s = 0$, and recall the decomposition (2.5). By the proposition above, we get

$$\begin{aligned} \mathbb{C}[\overline{\mathcal{O}_{[n;r,0]}}] &\simeq \mathbb{C}[\Xi_{[r,0]}]^{K'_\mathbb{C}} \\ &= \sum_{\lambda,\mu,\eta\in\mathcal{P}_n}^{\oplus} m(\lambda,\eta)\,(\sigma_\eta^{(p)} \boxtimes \sigma_\mu^{(q)}) \boxtimes (\tau_\lambda{}^* \boxtimes \tau_\mu)^{\Delta GL_n(\mathbb{C})}. \end{aligned}$$

By Schur's lemma, we have

$$({\tau_\lambda}^* \boxtimes \tau_\mu)^{\Delta GL_n(\mathbb{C})} \simeq \begin{cases} 0 & \text{if } \lambda \neq \mu, \\ \mathbb{C} & \text{if } \lambda = \mu. \end{cases}$$

Therefore, the above formula becomes

$$\mathbb{C}[\overline{\mathcal{O}_{[n;r,0]}}] \simeq \sum_{\lambda,\eta\in\mathcal{P}_n}^{\oplus} m(\lambda,\eta)\, \sigma_\eta^{(p)} \boxtimes \sigma_\lambda^{(q)},$$

which finishes the proof of Theorem 2.4.

Finally, let us assume that $r = n$, and express the multiplicity $m(\lambda,\eta)$ by the Littlewood-Richardson coefficient $c_{\mu,\nu}^{\lambda}$, which is defined by the following formula

$$\tau_\mu \otimes \tau_\nu = \sum_{\lambda}^{\oplus} c_{\mu,\nu}^{\lambda} \tau_\lambda.$$

Proposition 3.4 *Let $\mathcal{O}_{[n;n,0]}$ be the theta lift of the open $K_{\mathbb{C}}$-orbit $\mathcal{O}'_{[n,0]}$ in $\mathfrak{s}'_+$. Then we get a $K_{\mathbb{C}}$-type decomposition*

$$\mathbb{C}[\overline{\mathcal{O}_{[n;n,0]}}] \simeq \sum_{\lambda,\eta\in\mathcal{P}_n}^{\oplus} \left(\sum_{\mu\in\mathcal{P}_n} c_{\eta,2\mu}^{\lambda} \right) \sigma_\eta^{(p)} \boxtimes \sigma_\lambda^{(q)}.$$

Therefore, the multiplicity $m(\lambda,\eta)$ in Theorem 2.4 is given by

$$m(\lambda,\eta) = \sum_{\mu\in\mathcal{P}_n} c_{\eta,2\mu}^{\lambda},$$

for $r = n$.

Proof. In this case, we have $\Xi_n^{(p)} = M_{p,n}$. Let $\mathcal{H}$ be the space of all $O(p,\mathbb{C})$-harmonics in $\mathbb{C}[M_{p,n}]$. Then we have an isomorphism

$$\mathcal{H} \otimes \mathbb{C}[M_{p,n}]^{O(p,\mathbb{C})} \xrightarrow{\sim} \mathbb{C}[M_{p,n}]$$

given by the multiplication map. Thus we get

$$\mathbb{C}[\Xi_n^{(p)}] = \mathbb{C}[M_{p,n}] \simeq \mathcal{H} \otimes \mathbb{C}[M_{p,n}]^{O(p,\mathbb{C})} \simeq \mathcal{H} \otimes \mathbb{C}[\mathfrak{s}'_+].$$

From the following two decompositions,

$$\mathcal{H} \simeq \mathbb{C}[\mathfrak{N}_{p,n}] \simeq \sum_{\eta\in\mathcal{P}_n}^{\oplus} \sigma_\eta^{(p)} \boxtimes {\tau_\eta}^* \quad (\text{as an } O(p,\mathbb{C}) \times GL_n(\mathbb{C})\text{-module}),$$

$$\mathbb{C}[\mathfrak{s}'_+] \simeq \sum_{\mu\in\mathcal{P}_n}^{\oplus} {\tau_{2\mu}}^* \quad (\text{as a } GL_n(\mathbb{C})\text{-module}),$$

we conclude that

$$\begin{aligned}
\mathbb{C}[\Xi_n^{(p)}] &\simeq \mathcal{H} \otimes \mathbb{C}[\mathfrak{s}'_+] \\
&\simeq \sum_{\eta,\mu\in\mathcal{P}_n}^{\oplus} \sigma_\eta^{(p)} \boxtimes ({\tau_\eta}^* \otimes {\tau_{2\mu}}^*) \\
&\simeq \sum_{\eta,\mu\in\mathcal{P}_n}^{\oplus} \sigma_\eta^{(p)} \boxtimes \sum_{\lambda\in\mathcal{P}_n}^{\oplus} c_{\eta,2\mu}^{\lambda} {\tau_\lambda}^* \\
&\simeq \sum_{\lambda,\eta\in\mathcal{P}_n}^{\oplus} \left(\sum_{\mu\in\mathcal{P}_n} c_{\eta,2\mu}^{\lambda} \right) \sigma_\eta^{(p)} \boxtimes {\tau_\lambda}^*.
\end{aligned}$$

Q.E.D.

As an application of the above proposition, we get an interesting formula for the branching coefficient b_η^λ (see (2.3) for definition).

Corollary 3.5 *If* $2n < \min(p, q)$, *then we have*

$$b_\eta^\lambda = \sum_{\mu\in\mathcal{P}_n} c_{\eta,2\mu}^{\lambda} \qquad \textit{for } \lambda, \eta \in \mathcal{P}_n.$$

Remark 3.6 The branching coefficient b_η^λ is naturally identified with the multiplicity of the K-type τ_λ in the holomorphic discrete series of $Sp(2n, \mathbb{R})$ with the minimal K-type τ_η. Thus, it does not depend on the particular value p, but only depends on $\lambda, \eta \in \mathcal{P}_n$.

PROOF. This follows from Corollary 2.5 (2). Q.E.D.

4 Further results and comments

Let us briefly discuss generalizations of the results above.

First, we note that we can develop the similar theory interchanging the role of the pair (G, G'), if $p + q \le n$ holds. So, if one of the pair is very small (i.e., if the pair is in the stable range), we can define the theta lifting from the smaller member of the pair to the larger one.

Almost all the arguments and results above are also valid for the other type I dual pairs with appropriate modifications. However, we must develop a new, unified language to describe them in general. For example, at present, we have to construct double fibration maps based on the case-by-case analysis. See the arguments in [6] for the pair $U(p, q) \times U(n, n)$.

Though the double fibration maps defined here might seem quite ad hoc, we have a natural interpretation for them, using the kernels and the images of nilpotent elements (cf. [7], [3]). Also there may be another interpretation by using moment maps. These interpretations will be useful for a general theory.

Our correspondence of nilpotent orbits is intimately related to the theta lifts of representations of $Sp(2n, \mathbb{R})$ to $O(p, q)$. The orbits $\mathcal{O}'_{[r,0]}$ treated in this note are associated to the unitary highest weight representations of $Sp(2n, \mathbb{R})$ (or its metaplectic double cover). In particular, $\mathcal{O}'_{[n,0]}$ corresponds to a holomorphic discrete series representation. Therefore, the theta lift $\mathcal{O}_{[n;n,0]}$ should be associated to the theta lift of a holomorphic discrete series. See [8] for the theta lift of the trivial representation, which is associated to the trivial orbit $\mathcal{O}'_{[0,0]} = \{0\}$.

Detailed discussions on the subjects commented above will appear elsewhere.

References

[1] R. Howe, Reciprocity laws in the theory of dual pairs. Progr. Math., 40, pp. 159–175, Birkhäuser Boston, Boston, Mass., 1983.

[2] B. Kostant and S. Rallis, Orbits and representations associated with symmetric spaces. Amer. J. Math., 93 (1971), 753 - 809.

[3] H. Kraft and C. Procesi, Closures of conjugacy classes of matrices are normal. Invent. Math. 53 (1979), no. 3, 227–247.

[4] Kyo Nishiyama, Multiplicity-free actions and the geometry of nilpotent orbits. Preprint, 1999.

[5] Kyo Nishiyama, Hiroyuki Ochiai, and Kenji Taniguchi, Bernstein degree and associated cycles of Harish-Chandra modules (Hermitian symmetric case), Kyushu University Preprint Series in Mathematics, 1999-16.

[6] Kyo Nishiyama and Chen-Bo Zhu, Theta lifting of the trivial representation and the associated nilpotent orbit — the case of $U(p, q) \times U(n, n)$ —. In "Proceedings of Symposium on Representation Theory 1999" in Tateyama, Chiba, pp. 188 - 206.

[7] Takuya Ohta, The closures of nilpotent orbits in the classical symmetric pairs and their singularities. Tôhoku Math. J. (2) 43 (1991), no. 2, 161–211.

[8] C.-B. Zhu and J.-S. Huang, On certain small representations of indefinite orthogonal groups, Represent. Theory **1** (1997), 190–206 (electronic).

One-parameter Semigroups related to abstract Quantum Models of Calogero Type

Margit Rösler
Zentrum Mathematik, Technische Universität München
Arcisstr. 21, D-80290 München, Germany
e-mail: roesler@mathematik.tu-muenchen.de

Abstract

We study various classes of strongly continuous one-parameter semigroups which are generated by abstract versions of linear Calogero-Moser-Sutherland Hamiltonians for arbitrary root systems. These Hamiltonians contain modifications by exchange terms and can be written in terms of Dunkl operators. The semigroups under consideration include the generalized heat semigroup and the Schrödinger semigroup related with the free abstract Calogero Hamiltonian, as well as the semigroup generated by the Calogero Hamiltonian with harmonic confinement. The latter one is closely related with a Dunkl-type generalization of the classical Ornstein-Uhlenbeck semigroup.

1 Introduction

In recent years, quantum many particle models of Calogero-Moser-Sutherland (CMS) type have gained considerable interest in theoretical physics. These models describe systems of identical particles on a circle or line which interact pairwise through long range potentials of inverse square type. They are exactly solvable and are therefore of great interest for the understanding of quantum many body physics. CMS models have in particular attracted some attention in conformal field theory, and they are being used to test the ideas of fractional statistics ([Ha], [Hal]). While explicit spectral resolutions of such models were already obtained by Calogero and Sutherland ([Ca], [Su]), a new aspect in the understanding of their algebraic structure and quantum integrability was only recently initiated by [Po] and [He]. The Hamiltonian under consideration is hereby modified by certain exchange operators, which allow to write it in a decoupled form. These exchange modifications can be expressed in terms of Dunkl operators of type A_{N-1}. Dunkl operators, as introduced and first studied by C.F. Dunkl ([D1], [D2]), are parametrized differential-reflection operators associated with root systems. They extend the usual partial derivatives by additional reflection terms. Besides their important role in the context of quantum integrable many particle systems, Dunkl operators provide a key tool in the analysis of special functions related with root systems. In the present paper, we study several classes of one-parameter semigroups which are generated by second order Dunkl operators. These operators can be seen as abstract versions of linear

CMS operators which are associated with arbitrary root systems and are modified by exchange terms in the sense of [Po]. After a brief survey on Dunkl operators in Section 2, the connection of these operators with quantum Calogero models is described in Section 3. We then turn to the basic one-parameter semigroup in the Dunkl setting, namely the generalized heat semigroup introduced in [R1]; it is discussed in Section 4 on various function spaces besides $(C_0(\mathbb{R}^N), \|.\|_\infty)$. When considered for imaginary times, the Dunkl-type heat semigroup in a suitably weighted L^2-space leads to the solution of the time-dependent Schrödinger equation for the free quantum Calogero model. This is contained in Section 5. Finally, the last section is devoted to the semigroup generated by the Calogero Hamiltonian with harmonic confinement. It can be interpreted as the Dunkl-type version of the classical oscillator semigroup, and is closely related with the Ornstein-Uhlenbeck semigroup studied in [R-V].

2 Some basic facts from the theory of Dunkl operators

Let R be a (reduced, not necessarily crystallographic) root system in $\mathbb{R}^N$, i.e. a finite subset of $\mathbb{R}^N \setminus \{0\}$ with $R \cap \mathbb{R}\cdot\alpha = \{\pm\alpha\}$ and $\sigma_\alpha(R) = R$ for all $\alpha \in R$. Here σ_α denotes the reflection in the hyperplane orthogonal to α, which is given by $\sigma_\alpha(x) = x - \langle\alpha, x\rangle \cdot \alpha$, with $\langle .,. \rangle$ denoting the standard Euclidean scalar product. We hereby assume that the root system R is normalized, i.e. $|\alpha|^2 = 2$ for all $\alpha \in R$, where $|\,.\,|$ is the Euclidean norm. We further denote by G the finite reflection group generated by $\{\sigma_\alpha,\ \alpha \in R\}$. A function $k : R \to \mathbb{C}$ is called a multiplicity function on the root system R, if it invariant under the natural action of G on R. We fix some multiplicity-function k on R, which is throughout this paper assumed to be non-negative, i.e. $k(\alpha) \geq 0$ for all $\alpha \in R$. The Dunkl operators on $\mathbb{R}^N$ associated with G and k are defined by

$$T_i f(x) := \partial_i f(x) + \sum_{\alpha \in R_+} k(\alpha)\alpha_i \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x\rangle},$$

where R_+ is an (arbitrary) positive subsystem of R, i.e. $\langle\alpha, \beta\rangle > 0$ for some $\beta \in \mathbb{R}^N$ and all $\alpha \in R_+$. The operators T_i can be considered as a perturbation of the usual partial derivatives in the parameter k, and many properties of the usual partial derivatives carry over to them ([D1], [D2], [dJ]); here we mention only the following ones:

(i) The set $\{T_i,\ i = 1, \dots, N\}$ generates a commutative algebra of differential-reflection operators on $\mathbb{R}^N$.

(ii) The operators T_i are homogeneous of degree -1 on the space $\Pi^N := \mathbb{C}[\mathbb{R}^N]$ of polynomial functions in N variables, i.e. if $p \in \Pi^N$ has total degree k, then $T_i p$ has total degree $k-1$.

(iii) If $f \in C^k(\mathbb{R}^N)$ with $k \geq 1$, then $T_i f \in C^{k-1}(\mathbb{R}^N)$; moreover, if f belongs to the Schwartz space $\mathscr{S}(\mathbb{R}^N)$ of rapidly decreasing functions on $\mathbb{R}^N$, then also $T_i f \in \mathscr{S}(\mathbb{R}^N)$.

Of particular importance in our context is the generalized Laplacian, which is defined by $\Delta_k := \sum_{i=1}^N T_i^2$. It is given explicitly by

$$\Delta_k = \Delta + \sum_{\alpha\in R} k(\alpha)\delta_\alpha \tag{2.1}$$

with

$$\delta_\alpha f(x) = \frac{\langle \nabla f(x), \alpha\rangle}{\langle \alpha, x\rangle} - \frac{f(x) - \sigma_\alpha f(x)}{\langle \alpha, x\rangle^2};$$

here Δ and ∇ denote the usual Laplacian and gradient respectively.

2.1 Example. (Dunkl operators of type A_{N-1}). These belong to the symmetric group $G = S_N$, which acts in a canonical way on $\mathbb{R}^N$ by permuting the standard basis vectors $e_1, \ldots, e_N$. Each transposition (ij) acts as a reflection σ_{ij}, sending $e_i - e_j$ to its negative. On $C^1(\mathbb{R}^N)$, σ_{ij} acts by transposing the coordinates x_i and x_j with respect to the standard basis. The attached root system, of type A_{N-1}, is given by $R = \{e_i - e_j,\ 1 \leq i,j \leq N,\ i \neq j\}$. Since all transpositions are conjugate in S_N, the vector space of multiplicity functions on R is one-dimensional. The Dunkl operators associated with the multiplicity parameter $k \in \mathbb{C}$ are given by

$$T_i^S = \partial_i + k \cdot \sum_{j\neq i} \frac{1-\sigma_{ij}}{x_i - x_j} \quad (i = 1, \ldots, N),$$

and the generalized Laplacian is

$$\Delta_k^S = \Delta + 2k \sum_{1\leq i<j\leq N} \frac{1}{x_i - x_j}\Big[(\partial_i - \partial_j) - \frac{1-\sigma_{ij}}{x_i - x_j}\Big].$$

The Dunkl theory provides also a counterpart to the usual exponential function, called the Dunkl kernel $E_k(x,y)$. For each fixed $y \in \mathbb{R}^N$, the function $x \mapsto E_k(x,y)$ can be characterized as the unique solution of the system $T_i f = y_i f$ $(i = 1, \ldots, N)$ with $f(0) = 1$; see [O]. The kernel $E_k(x,y)$ is symmetric in its arguments and has a unique holomorphic extension to $\mathbb{C}^N \times \mathbb{C}^N$. It satisfies $E_k(z,0) = 1$ and $E_k(\lambda z, w) = E_k(z, \lambda w)$ for all $z, w \in \mathbb{C}^N$ and all $\lambda \in \mathbb{C}$. Moreover, E_k has a Bochner-type representation of the form

$$E_k(x,z) = \int_{\mathbb{R}^N} e^{\langle \xi, z\rangle} d\mu_x^k(\xi), \qquad \text{for all } z \in \mathbb{C}^N,$$

where μ_x^k is a compactly supported probability measure on $\mathbb{R}^N$ with $\operatorname{supp}\mu_x^k$ being contained in the convex hull of the orbit $\{gx, g \in G\}$, see [R2]. It follows that $|E_k(x, iy)| \leq 1$ for all $x, y \in \mathbb{R}^N$, and that

$$\min_{g\in G} e^{\langle gx, y\rangle} \leq E_k(x,y) \leq \max_{g\in G} e^{\langle gx, y\rangle}. \tag{2.2}$$

In particular, $E_k(x,y) > 0$ for all $x, y \in \mathbb{R}^N$. We mention that this positivity result was first deduced in [R1] from the positivity of the associated heat semigroup. The Dunkl kernel gives rise to a corresponding integral transform on $\mathbb{R}^N$ with respect to the weight function

$$w_k(x) = \prod_{\alpha\in R_+} |\langle \alpha, x\rangle|^{2k(\alpha)}.$$

Notice that w_k is G-invariant and homogeneous of degree 2γ, with the index

$$\gamma := \gamma(k) = \sum_{\alpha \in R_+} k(\alpha).$$

The Dunkl transform on $L^1(\mathbb{R}^N, w_k)$ is defined by

$$\widehat{f}^k(\xi) := c_k^{-1} \int_{\mathbb{R}^N} f(x) E_k(-i\xi, x)\, w_k(x) dx,$$

where c_k is the Mehta-type constant

$$c_k := \int_{\mathbb{R}^N} e^{-|x|^2/2} w_k(x) dx.$$

This integral transform has many properties which are completely analogous to those of the classical Fourier transform. A thorough investigation is given in [dJ]. We recall from there that the Dunkl transform is a homeomorphism of $\mathscr{S}(\mathbb{R}^N)$, satisfying $(T_j f)^{\wedge k}(\xi) = i\xi_j \widehat{f}^k(\xi)$. Moreover, it has a unique Plancherel-type extension to an isometric isomorphism of $L^2(\mathbb{R}^N, w_k)$, which is also denoted by $f \mapsto \widehat{f}^k$. The inverse transform is given by $f^{\vee k}(x) = \widehat{f}^k(-x)$.

3 Quantum Calogero models

We continue with a short explanation of linear Calogero-Moser-Sutherland models and the relevance of Dunkl operators in their algebraic description. The Hamiltonian of the so-called quantum Calogero model with harmonic confinement in $L^2(\mathbb{R}^N)$ is given by

$$\mathcal{H}_C = -\Delta + \omega^2 |x|^2 + 2k(k-1) \sum_{1 \le i < j \le N} \frac{1}{(x_i - x_j)^2}; \tag{3.1}$$

here $\omega > 0$ is a frequency parameter and $k \ge 0$ is a coupling constant. In case $\omega = 0$, (3.1) describes the free Calogero model. The study of this Hamiltonian was initiated by Calogero ([Ca]); he computed its spectrum and determined the structure of the eigenfunctions and scattering states in the confined and free case, respectively. Perelomov [Pe] observed that (3.1) is completely quantum integrable, i.e. there exist N commuting, algebraically independent symmetric linear operators in $L^2(\mathbb{R}^N)$ including $\mathcal{H}_C$. We mention that the complete integrability of the classical Hamiltonian systems associated with (3.1) goes back to Moser [Mo]. There exist generalizations of the classical Calogero-Moser-Sutherland models in the context of abstract root systems, see e.g. [O-P1], [O-P2]. In particular, if R is an arbitrary root system on $\mathbb{R}^N$ and k is a nonnegative multiplicity function on it, then the corresponding abstract Calogero Hamiltonian with harmonic confinement is given by

$$\widetilde{\mathcal{H}}_k = -\widetilde{\mathcal{F}}_k + \omega^2 |x|^2$$

with the formal expression

$$\widetilde{\mathcal{F}}_k = \Delta - 2 \sum_{\alpha \in R_+} k(\alpha)(k(\alpha) - 1) \frac{1}{\langle \alpha, x \rangle^2}.$$

If R is of type A_{N-1}, then $\widetilde{\mathcal{H}}_k$ just coincides with $\mathcal{H}_C$. For both the classical and the quantum case, partial results on the integrability of this model are due to Olshanetsky and Perelomov [O-P1], [O-P2]. A new aspect in the understanding of the algebraic structure and the quantum integrability of CMS systems was later initiated by Polychronakos [Po] and Heckman [He]. The underlying idea is to construct quantum integrals for CMS models from differential-reflection operators. Polychronakos introduced them in terms of an "exchange-operator formalism" for (3.1). He thus obtained a complete set of commuting observables for (3.1) in an elegant way. In [He] it was observed in general that the complete algebra of quantum integrals for free, abstract Calogero models is intimately connected with the corresponding algebra of Dunkl operators. Since then, there has been an extensive and ongoing study of CMS models and explicit operator solutions for them via differential-reflection operator formalisms; among the broad literature, we refer to [L-V], [K], [BHKV], [BF], and [U-W]. Let us briefly describe the connection of abstract Calogero models with Dunkl operators: Consider the following modification of $\widetilde{\mathcal{F}}_k$, involving reflection terms:

$$\mathcal{F}_k = \Delta - 2\sum_{\alpha\in R_+} \frac{k(\alpha)}{\langle\alpha, x\rangle^2}\,(k(\alpha) - \sigma_\alpha). \tag{3.2}$$

In order to avoid singularities in the reflecting hyperplanes, it is suitable to carry out a gauge transform by $\sqrt{w_k}$. One obtains (c.f. Lemma 3.1. of [R3]) that $\mathcal{F}_k$ is essentially self-adjoint when considered as a linear operator in $L^2(\mathbb{R}^N)$ with domain $\mathcal{D}(\mathcal{F}_k) := \{w_k^{1/2} f : f \in \mathscr{S}(\mathbb{R}^N)\}$. Moreover,

$$\mathcal{F}_k = w_k^{1/2}\Delta_k\, w_k^{-1/2},$$

where Δ_k is the Dunkl Laplacian in $L^2(\mathbb{R}^N, w_k)$ with domain $\mathscr{S}(\mathbb{R}^N)$. Consider now the algebra of G-invariant polynomials on $\mathbb{R}^N$:

$$(\Pi^N)^G = \{p \in \Pi^N : g\cdot p = p \;\text{ for all }\; g \in G\}.$$

It follows easily from equivariance properties of the Dunkl operators (c.f. [dJ]) that for every $p \in (\Pi^N)^G$, the Dunkl operator $p(T)$ leaves $(\Pi^N)^G$ invariant. For such p we denote the restriction of $p(T)$ to $(\Pi^N)^G$ by $\mathrm{Res}\,(p(T))$. Then, as observed in [He], the family

$$\{\mathrm{Res}\,(p(T)) : p \in (\Pi^N)^G\}$$

is a commutative algebra of differential operators, containing the operator

$$\mathrm{Res}\,(\Delta_k) = w_k^{-1/2}\widetilde{\mathcal{F}}_k\, w_k^{1/2}.$$

This implies the integrability of the free Calogero Hamiltonian $\widetilde{\mathcal{F}}_k$. Polychronakos [Po] also succeeded to determine a complete set of quantum integrals for the classical, i.e. S_N-type Calogero Hamiltonian with harmonic confinement - at least in the physically relevant bosonic and fermionic subspaces of $L^2(\mathbb{R}^N)$. He constructed the integrals by a Lax formalism involving suitable lowering and raising operators. For the abstract Calogero operator $\widetilde{\mathcal{H}}_k$ with harmonic confinement, the general question of how to obtain an algebra of quantum integrals is, to the author's knowledge, still open. It is, however, easy to achieve a complete spectral analysis of $\widetilde{\mathcal{H}}_k$. We again work with the gauge-transformed version with reflection terms,

$$\mathcal{H}_k := w_k^{-1/2}(-\mathcal{F}_k + \omega^2|x|^2)\, w_k^{1/2} \;=\; -\Delta_k + \omega^2|x|^2.$$

This operator is symmetric and densely defined in $L^2(\mathbb{R}^N, w_k)$ with domain $\mathcal{D}(\mathcal{H}_k) := \mathscr{S}(\mathbb{R}^N)$. Notice that in case $k = 0$, $\mathcal{H}_k$ is just the Hamiltonian of the N-dimensional isotropic harmonic oscillator. We further consider the Hilbert space $L^2(\mathbb{R}^N, m_k^\omega)$, where m_k^ω is the probability measure

$$m_k^\omega(x) \;:=\; c_k^{-1}(2\omega)^{\gamma+N/2}\, e^{-\omega|x|^2} w_k(x)dx \;\in M^1(\mathbb{R}^N) \quad (\omega > 0). \tag{3.3}$$

Moreover, we introduce the operator

$$\mathcal{J}_k := \;-\Delta_k + 2\omega \sum_{j=1}^{N} x_j \partial_j$$

in $L^2(\mathbb{R}^N, m_k^\omega)$, with the dense domain $\mathcal{D}(\mathcal{J}_k) := \Pi^N$ (the polynomials in N variables). The following connection between $\mathcal{H}_k$ and $\mathcal{J}_k$ is established in the same way as part (2) of Theorem 3.4.(2) in [R1].

3.1 Lemma. *On $\mathcal{D}(\mathcal{J}_k) = \Pi^N$,*

$$\mathcal{J}_k \;=\; e^{\omega|x|^2/2}\left(\mathcal{H}_k - (2\gamma + N)\omega\right) e^{-\omega|x|^2/2}.$$

In particular, $\mathcal{J}_k$ is symmetric in $L^2(\mathbb{R}^N, m_k^\omega)$.

We conclude with a complete description of the spectral properties of $\mathcal{H}_k$ and $\mathcal{J}_k$; these results generalize well-known facts for the corresponding classical operators. In the following, $\mathcal{P}_n^N$ denotes the space of polynomials from Π^N which are homogeneous of degree n. Notice also that by the homogeneity of Δ_k, the operator $e^{c\Delta_k}$ is well defined on polynomials and preserves the total degree.

3.2 Theorem. *For $\omega > 0$ and $n \in \mathbb{Z}_+$ define*

$$V_n^\omega := \{e^{-\Delta_k/4\omega}\, p : p \in \mathcal{P}_n^N\} \subset \Pi^N \quad \text{and} \quad W_n^\omega := \{e^{-\omega|x|^2/2}\, q(x),\, q \in V_n^\omega\} \subset \mathscr{S}(\mathbb{R}^N).$$

Then the following assertions hold:

(1) *The spaces $L^2(\mathbb{R}^N, m_k^\omega)$ and $L^2(\mathbb{R}^N, w_k)$ admit the orthogonal Hilbert space decompositions*

$$L^2(\mathbb{R}^N, m_k^\omega) \;=\; \bigoplus_{n\in\mathbb{Z}_+} V_n^\omega \quad \text{and} \quad L^2(\mathbb{R}^N, w_k) \;=\; \bigoplus_{n\in\mathbb{Z}_+} W_n^\omega\,;$$

here V_n^ω is the eigenspace of $\mathcal{J}_k$ corresponding to the eigenvalue $2n\omega$, and W_n^ω is the eigenspace of $\mathcal{H}_k$ corresponding to the eigenvalue $(2n + 2\gamma + N)\omega$.

(2) *The operators $\mathcal{H}_k$ and $\mathcal{J}_k$ are essentially self-adjoint; the spectra of their closures are discrete and given by $\sigma(\overline{\mathcal{H}_k}) = \{(2n + 2\gamma + N)\omega,\, n \in \mathbb{Z}_+\}$ and $\sigma(\overline{\mathcal{J}_k}) = \{2n\omega,\, n \in \mathbb{Z}_+\}$ respectively.*

Proof. (1) It was shown in Theorem 3.4.(1) of [R1] that in case $\omega = 1$, each function from V_n^ω is an eigenfunction of $\mathcal{J}_k$ corresponding to the eigenvalue $2n\omega$. For arbitrary ω, the corresponding result is obtained by rescaling. Moreover, $V_n^\omega \perp V_m^\omega$ for $n \neq m$ by the symmetry of $\mathcal{J}_k$. This proves the statements for $\mathcal{J}_k$, because $\Pi^N = \bigoplus V_n^\omega$ is dense in $L^2(\mathbb{R}^N, m_k^\omega)$. The statements for $\mathcal{H}_k$ are then immediate by the previous Lemma.
(2) follows from (1) by a well-known criterion for self-adjointness of symmetric operators on a Hilbert space which have a complete set of orthogonal eigenfunctions within their domain (Lemma 1.2.2 of [Da3]). □

By the G-equivariance of Δ_k, the spectral resolution of the Calogero Hamiltonian $\widetilde{\mathcal{H}}_k$ in the bosonic subspace $L^2(\mathbb{R}^N)^G$ is now an easy consequence of Theorem 3.2.

3.3 Corollary. *For $n \in \mathbb{Z}_+$, put $W_n^{\omega,G} = \{e^{-\omega|x|^2/2}e^{-\Delta_k/4\omega}p : p \in \mathcal{P}_n^N \cap (\Pi^N)^G\}$. Then*

$$L^2(\mathbb{R}^N)^G = \bigoplus_{n\in\mathbb{Z}_+} W_n^{\omega,G},$$

and $W_n^{\omega,G}$ is the eigenspace of $\widetilde{\mathcal{H}}_k$ in $L^2(\mathbb{R}^N)^G$ corresponding to the eigenvalue $(2n+2\gamma+N)\omega$.

4 Heat semigroups associated with finite reflection groups

This section deals with the Dunkl-type analogues of the classical heat semigroup on several Banach spaces. These semigroups are generated by the Dunkl Laplacian, and they are governed by a generalized heat kernel which was introduced in [R1] and replaces the usual Gaussian kernel in the Dunkl setting.

4.1 Definition. The generalized heat kernel Γ_k associated with the reflection group G and the multiplicity function k is defined by

$$\Gamma_k(t,x,y) := \frac{M_k}{t^{\gamma+N/2}}\, e^{-(|x|^2+|y|^2)/4t}\, E_k\Big(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\Big), \quad x,y \in \mathbb{R}^N,\ t>0$$

with $M_k = (2^{\gamma+N/2}c_k)^{-1}$.

The strict positivity of E_k for real arguments implies that Γ_k is strictly positive as well. In the following, we collect some further important properties of this kernel.

4.2 Lemma. (1) $\displaystyle \frac{M_k}{t^{\gamma+N/2}} \min_{g\in G} e^{-|gx-y|^2/4t} \leq \Gamma_k(t,x,y) \leq \frac{M_k}{t^{\gamma+N/2}} \max_{g\in G} e^{-|gx-y|^2/4t}.$

(2) $\displaystyle \int_{\mathbb{R}^N} \Gamma_k(t,x,y)\, w_k(y)dy = 1.$

(3) *For fixed t and x, the function $y \mapsto \Gamma_k(t,x,y)$ belongs to $\mathscr{S}(\mathbb{R}^N)$, with $\Gamma_k(t,x,.)^{\wedge k}(\xi) = c_k^{-1}e^{-t|\xi|^2}E_k(-ix,\xi)$.*

(4) $\displaystyle \Gamma_k(t+s,x,y) = \int_{\mathbb{R}^N} \Gamma_k(t,x,z)\,\Gamma_k(s,y,z)\, w_k(z)dz.$

(5) *For fixed $y \in \mathbb{R}^N$, the function $u(t,x) := \Gamma_k(t,x,y)$ solves the generalized heat equation $\Delta_k u = \partial_t u$ on $(0,\infty)\times\mathbb{R}^N$.*

Proof. The estimates (1) are immediate from the bounds (2.2) on E_k. Properties (2) and (5) have been shown in [R1]. The first part of (3) is easily deduced from (1), while the second statement follows from the reproducing identity for E_k (c.f. [D3]),

$$\int_{\mathbb{R}^N} E_k(x,z)\, E_k(x,w)\, e^{-|x|^2/2} w_k(x)dx \;=\; c_k\, e^{(\langle z,z\rangle+\langle w,w\rangle)/2} E_k(z,w) \quad (z,w \in \mathbb{C}^N). \quad (4.1)$$

For the proof of (4), we use (3) and the Plancherel theorem for the Dunkl transform to obtain

$$\int_{\mathbb{R}^N} \Gamma_k(t,x,z)\, \Gamma_k(s,y,z)\, w_k(z)dz \;=\; c_k^{-1} \int_{\mathbb{R}^N} e^{-t|\xi|^2} E_k(ix,\xi) \Gamma_k(s,y,.)^{\wedge k}(\xi) w_k(\xi)d\xi$$

$$= c_k^{-2} \int_{\mathbb{R}^N} e^{-(s+t)|\xi|^2} E_k(ix,\xi)\, E_k(-iy,\xi) w_k(\xi)d\xi \;=\; \Gamma_k(t+s,x,y).$$

□

We next introduce the generalized heat operators associated with the kernel Γ_k.

4.3 Definition. For $f \in L^p(\mathbb{R}^N, w_k)$ $(1 \le p \le \infty)$ and $t \ge 0$ define

$$H_k(t)f(x) := \begin{cases} \displaystyle\int_{\mathbb{R}^N} \Gamma_k(t,x,y) f(y)\, w_k(y)dy & \text{if } t > 0, \\ f(x) & \text{if } t = 0 \end{cases}$$

Notice that the decay properties of Γ_k assure that the integral defining $H_k(t)f(x)$ converges for all $t > 0$, $x \in \mathbb{R}^N$. We recall the following properties of the operators $H_k(t)$ on the Schwartz space $\mathscr{S}(\mathbb{R}^N)$ from[R1]:

4.4 Theorem. *Let $f \in \mathscr{S}(\mathbb{R}^N)$. Then $u(t,x) := H_k(t)f(x)$ belongs to $C_b([0,\infty) \times \mathbb{R}^N) \cap C^2((0,\infty) \times \mathbb{R}^N)$ and solves the Cauchy problem*

$$\begin{cases} (\Delta_k - \partial_t)\, u = 0 & \text{on } (0,\infty) \times \mathbb{R}^N, \\ u(0,.) = f. \end{cases}$$

Moreover, $H_k(t)f$ has the following properties:

1. *$H_k(t)f \in \mathscr{S}(\mathbb{R}^N)$ for all $t > 0$.*
2. *$H_k(t+s)\, f \;=\; H_k(t)H_k(s)f$ for all $s, t \ge 0$.*
3. *$\|H_k(t)f - f\|_\infty \to 0$ with $t \to 0$.*

4.5 Lemma. *For every $t > 0$, $H_k(t)$ defines a continuous linear operator on each of the Banach spaces $L^p(\mathbb{R}^N, w_k)$ $(1 \le p \le \infty)$, $(C_b(\mathbb{R}^N), \|.\|_\infty)$ and $(C_0(\mathbb{R}^N), \|.\|_\infty)$, with norm $\|H_k(t)\| \le 1$.*

Proof. The estimates for the kernel Γ_k in Lemma 4.2(3) and its normalization ensure that for every $f \in L^\infty(\mathbb{R}^N, w_k)$, we have $H(t)f \in C_b(\mathbb{R}^N)$ with $\|H_k(t)f\|_\infty \;\le\; \|f\|_\infty$. Moreover, if $f \in L^p(\mathbb{R}^N, w_k)$, then Jensen's inequality implies that

$$|H_k(t)f(x)|^p \;\le\; \int_{\mathbb{R}^N} \Gamma_k(t,x,y)\, |f(y)|^p\, w_k(y)dy,$$

and therefore $\|H_k(t)f\|_{p,w_k} \;\le\; \|f\|_{p,w_k}$. Finally, the invariance of $C_0(\mathbb{R}^N)$ under $H_k(t)$ follows from part (1) of the previous theorem, together with the density of $\mathscr{S}(\mathbb{R}^N)$ in $C_0(\mathbb{R}^N)$. □

In the following, X is one of the Banach spaces $L^p(\mathbb{R}^N, w_k)(1 \le p < \infty)$ or $(C_0(\mathbb{R}^N), \|.\|_\infty)$. We consider the Dunkl Laplacian Δ_k as a linear operator in X with dense domain $\mathcal{D}(\Delta_k) := \mathscr{S}(\mathbb{R}^N)$.

4.6 Theorem. (1) *$(H_k(t))_{t\geq 0}$ is a strongly continuous, positivity-preserving contraction semigroup on X.*

(2) *Δ_k is closable, and its closure $\overline{\Delta}_k$ is the generator of the semigroup $(H_k(t))_{t\geq 0}$ on X.*

In view of this result, we call $(H_k(t))_{t\geq 0}$ the generalized Gaussian or heat semigroup on X.

Proof. (1) Theorem 4.4(2), together with Lemma 4.5 and the density of $\mathscr{S}(\mathbb{R}^N)$ in X, ensures that $(H_k(t))_{t\geq 0}$ forms a semigroup of continuous linear operators on X. Its positivity is clear by the positivity of Γ_k. Moreover, in case $X = (C_0(\mathbb{R}^N), \|\,.\,\|_\infty)$, its strong continuity follows from part (3) of Theorem 4.4. It remains to check strong continuity in the case $X = L^p(\mathbb{R}^N, w_k)$, $1 \le p < \infty$. In view of Lemma 4.5, and as $C_c(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N, w_k)$, it suffices to show that $\lim_{t\downarrow 0} \|H(t)f - f\|_{p,w_k} = 0$ for all $f \in C_c(\mathbb{R}^N)$; hereby we may further assume that $f \geq 0$. We then obtain

$$\|H_k(t)f\|_{1,w_k} = \int_{\mathbb{R}^N} H_k(t)f(x)\, w_k(x)dx = \int_{\mathbb{R}^N} f(x) w_k(x)dx = \|f\|_{1,w_k} \quad \text{for } t > 0.$$

As $\lim_{t\downarrow 0} \|H_k(t)f - f\|_\infty = 0$, a well-known convergence criterion (see for instance Theorem (13.47) of [H-St]) implies that $\lim_{t\downarrow 0} \|H_k(t)f - f\|_{1,w_k} = 0$. The estimation

$$\|H_k(t)f - f\|_{p,w_k}^p \le \|H_k(t)f - f\|_{1,w_k} \cdot \|H_k(t)f - f\|_{\infty,w_k}^{p-1}$$

then entails that $\lim_{t\downarrow 0} \|H_k(t)f - f\|_{p,w_k} = 0$ as well.
(2) Let A be the generator of the semigroup $(H_k(t))_{t\geq 0}$ on X. As A is closed, it suffices to prove that $A|_{\mathscr{S}(\mathbb{R}^N)} = \Delta_k$, and that $A = \overline{A|_{\mathscr{S}(\mathbb{R}^N)}}$, i.e. $\mathscr{S}(\mathbb{R}^N)$ is a core of A. The proof of these statements is similar to the classical case. To begin with, let $f \in \mathscr{S}(\mathbb{R}^N)$. Then by Theorem 4.4(1), $H_k(t)f \in \mathscr{S}(\mathbb{R}^N)$ for all $t > 0$, and application of the Dunkl transform yields

$$\left[\frac{1}{t}(H_k(t) - id)\, f\right]^{\wedge k}(\xi) = \frac{1}{t}\left(e^{-t|\xi|^2} - 1\right)\widehat{f}^k(\xi).$$

It is easily checked that with $t \downarrow 0$, this tends to $-|\xi|^2\widehat{f}^k(\xi)$ in the topology of $\mathscr{S}(\mathbb{R}^N)$. The Dunkl transform being a homeomorphism of $\mathscr{S}(\mathbb{R}^N)$, we therefore obtain

$$\lim_{t\downarrow 0} \frac{1}{t}(H_k(t) - id)f = (-|\xi|^2\widehat{f}^k)^{\vee k} = \Delta_k f$$

in the topology of $\mathscr{S}(\mathbb{R}^N)$, and therefore in $\|\,.\,\|_{p,w_k}$ as well. It follows that f belongs to the domain $\mathcal{D}(A)$ of A. Thus $\mathscr{S}(\mathbb{R}^N) \subset \mathcal{D}(A)$, and $A|_{\mathscr{S}(\mathbb{R}^N)} = \Delta_k$. Moreover, $\mathscr{S}(\mathbb{R}^N)$ is dense in X and invariant under $(H_k(t))_{t\geq 0}$. A well-known characterization of cores for the generators of strongly continuous semigroups (see, for instance, Theorem 1.9 of [Dal]) now implies that $\mathscr{S}(\mathbb{R}^N)$ is a core of A. □

The above theorem says in particular that $(H_k(t))_{t\geq 0}$ is a symmetric Markov semigroup on $L^2(\mathbb{R}^N, w_k)$ in the following sense:

4.7 Definition. ([Da2]) Let $\mu \in M^+(\mathbb{R}^N)$ be a positive Radon measure on $\mathbb{R}^N$. A strongly continuous contraction semigroup $(T(t))_{t\geq 0}$ on $L^2(\mathbb{R}^N, \mu)$ is called a symmetric Markov semigroup, if it satisfies the following conditions:

(1) The generator A of $(T(t))_{t\geq 0}$ is self-adjoint and non-positive, i.e. $\langle Af, f\rangle \leq 0$ for all $f \in \mathcal{D}(A)$;

(2) $(T(t))_{t\geq 0}$ is positivity-preserving for all $t \geq 0$, i.e. $T(t)f \geq 0$ for $f \geq 0$;

(3) If $f \in L^\infty(\mathbb{R}^N, \mu) \cap L^2(\mathbb{R}^N, \mu)$, then $\|T(t)f\|_{\infty,\mu} \leq \|f\|_{\infty,\mu}$ for all $t \geq 0$.

Theorem 1.4.2 of [Da2] implies the following

4.8 Corollary. *For $1 < p < \infty$, the semigroup $(H_k(t))_{t\geq 0}$ on $L^p(\mathbb{R}^N, w_k)$ is a bounded holomorphic semigroup (in the sense of [Da1]) in the sector*

$$\Big\{z \in \mathbb{C} : |arg(z)| < \pi \cdot \min\Big(\frac{1}{p}, \frac{1}{q}\Big)\Big\},$$

where q is the conjugate index defined by $\frac{1}{p} + \frac{1}{q} = 1$.

Remarks. 1. For $X = (C_0(\mathbb{R}^N), \|\cdot\|_\infty)$, Theorem 4.6 just says that the generalized heat semigroup is a Feller-Markov semigroup, i.e. a (strongly continuous) positive contraction semigroup on $C_0(\mathbb{R}^N)$. This observation was the starting point in [R-V] for the construction of an associated semigroup of Markov kernels on $\mathbb{R}^N$. It leads to a Markov process in $\mathbb{R}^N$ which admits a càdlàg version (i.e., there exists an equivalent process whose paths are right-continuous and have limits from the left), and which obeys a modified notion of translation-invariance. For a detailed study of this Dunkl-type Brownian motion we refer to [R-V].

2. It is a basic fact from semigroup theory that for given initial data $f \in \mathcal{D}(\overline{\Delta}_k) \subset X$, the function $u(t) := H_k(t)f$ provides the unique classical solution of the abstract Cauchy problem

$$\begin{cases} \dfrac{d}{dt}u(t) = \overline{\Delta}_k u(t) & \text{for } t > 0, \\ u(0) = f; \end{cases}$$

here "classical" means $u \in C^1([0,\infty), X)$ with $u(t) \in \mathcal{D}(\overline{\Delta}_k)$ for all $t \geq 0$. We refer to [R1] for the solution of the classical initial-boundary value problem for the Dunkl-type heat equation, with initial data taken from $C_b(\mathbb{R}^N)$.

5 The free, time-dependent Schrödinger equation

Consider again the self-adjoint Dunkl Laplacian $\overline{\Delta}_k$ in $L^2(\mathbb{R}^N, w_k)$. By Stone's Theorem, the skew-adjoint operator $i\overline{\Delta}_k$ generates a strongly continuous unitary semigroup $\big(e^{it\overline{\Delta}_k}\big)_{t\geq 0}$ on $L^2(\mathbb{R}^N, w_k)$. The explicit determination of this semigroup can be achieved by standard arguments, see for instance Chapter IX. 1.8 of [Kat] for the classical case. First, notice that the heat kernel Γ_k extends naturally to complex "time" arguments, by

$$\Gamma_k(z, x, y) = \frac{M_k}{z^{\gamma+N/2}}\, e^{-(|x|^2+|y|^2)/4z}\, E_k\Big(\frac{x}{2z}, y\Big)$$

for $x, y \in \mathbb{R}^N$ and $z \in \mathbb{C}_- := \mathbb{C}\backslash\{w \in \mathbb{R} : w \leq 0\}$; here $z^{\gamma+N/2}$ is the holomorphic branch in $\mathbb{C}_-$ with $1^{\gamma+N/2} = 1$. We next determine the Schrödinger semigroup on a sufficiently large subset of $\mathscr{S}(\mathbb{R}^N)$.

5.1 Lemma. *If $f(x) = e^{-b|x|^2}$ with a parameter $b > 0$, then*

$$e^{it\overline{\Delta}_k} f = \int_{\mathbb{R}^N} \Gamma_k(it, ., y)\, f(y)\, w_k(y) dy \quad \textit{for all } t > 0. \tag{5.1}$$

Proof. Consider the function

$$u(t,x) := \frac{1}{(1+4ibt)^{\gamma+N/2}}\, e^{-b|x|^2/(1+4ibt)} \quad (t \geq 0,\ x \in \mathbb{R}^N).$$

The same calculation as in Lemma 4.3. of [R1] shows that u satisfies the generalized Schrödinger equation

$$\partial_t u = i\Delta_k u \quad \text{on } (0,\infty) \times \mathbb{R}^N,$$

with $u(0,x) = e^{-b|x|^2}$. It is also easily verified that the function $t \mapsto u(t, .)$ belongs to $C^1([0,\infty), L^2(\mathbb{R}^N, w_k))$. This shows that $e^{it\overline{\Delta}_k} f = u(t, .)$ for $t \geq 0$. Finally, the reproducing identity (4.1) for E_k implies that for $t \geq 0$,

$$\frac{1}{(1+4bt)^{\gamma+N/2}}\, e^{-b|x|^2/(1+4bt)} = \int_{\mathbb{R}^N} \Gamma_k(t,x,y)\, e^{-b|y|^2} w_k(y) dy.$$

By analytic continuation, this identity remains true if t is replaced by it. This completes the proof. □

In the following, we shall need the notion of a generalized translation on the Schwartz space $\mathscr{S}(\mathbb{R}^N)$, c.f. [R1]. Its definition is natural:

$$L_k^y f(x) := c_k^{-1} \int_{\mathbb{R}^N} \widehat{f}^k(\xi)\, E_k(ix,\xi) E_k(iy,\xi)\, w_k(\xi) d\xi \quad (x,\, y \in \mathbb{R}^N,\ f \in \mathscr{S}(\mathbb{R}^N)). \tag{5.2}$$

Notice that that for $k = 0$, we just have $L_0^y f(x) = f(x+y)$. Important properties of the usal group translation on $\mathbb{R}^N$ carry over to the generalized translation for arbitrary k. It is, for example, easily checked that $L_k^y f$ belongs to $\mathscr{S}(\mathbb{R}^N)$ again with $(L_k^y f)^{\wedge k}(\xi) = E_k(iy,\xi)\, \widehat{f}^k(\xi)$. Moreover, $L_k^y f(x) = L_k^x f(y)$ for all $x, y \in \mathbb{R}^N$, and the operators L_k^y commute with the corresponding Dunkl operators T_i on $\mathscr{S}(\mathbb{R}^N)$. The following statement is obtained exactly as its classical analogue in [Kat], by using the Plancherel formula and the injectivity of the Dunkl transform.

5.2 Lemma. *The $\mathbb{C}$-linear hull $\langle M \rangle$ of the set*

$$M := \{x \mapsto L_k^a e^{-b|x|^2}, \quad a \in \mathbb{R}^N, b > 0\}$$

is dense in $L^2(\mathbb{R}^N, w_k)$.

We thus have shown that on the dense subspace $\langle M \rangle$ of $L^2(\mathbb{R}^N, w_k)$, the linear operators

$$S_k(t) f := \int_{\mathbb{R}^N} \Gamma_k(it, ., y) f(y)\, w_k(y) dy, \quad t > 0,$$

coincide with the unitary operators $e^{it\overline{\Delta}_k}$. They can therefore be extended uniquely to unitary operators on $L^2(\mathbb{R}^N, w_k)$, which are written in the same way, the integral now being understood in the L^2-sense. In this sense, we have for all $f \in L^2(\mathbb{R}^N, w_k)$,

$$e^{it\overline{\Delta}_k} f = \begin{cases} \int_{\mathbb{R}^N} \Gamma_k(it, \,.\,, y) f(y)\, w_k(y) dy & \text{if } t > 0, \\ f & \text{if } t = 0. \end{cases} \tag{5.3}$$

6 The semigroup of the Calogero Hamiltonian with harmonic confinement

For a fixed parameter $\omega > 0$, consider the Hamiltonian

$$\mathcal{J}_k = -\Delta_k + 2\omega \sum_{j=1}^{N} x_j \partial_j$$

with domain $\mathcal{D}(\mathcal{J}_k) := \Pi^N$ in the weighted Hilbert space $L^2(\mathbb{R}^N, m_k^\omega)$ (c.f. Section 3). Notice that $\mathcal{J}_k$ can be interpreted as the Dunkl-type generalization of the classical oscillator Hamiltonian in $L^2(\mathbb{R}^N)$. In the following, we shall work with generalized Hermite polynomials with respect to the measure m_k^ω. Generalized Hermite polynomials were introduced in [R1] (for $\omega = 1$) by means of homogeneous orthogonal systems with respect to a certain bilinear form on polynomials. We give an equivalent definition, which is more convenient on the basis of Theorem 3.2:

6.1 Definition. A family $\{H_\nu = H_\nu(\omega, \,.\,), \nu \in \mathbb{Z}_+^N\} \subset \Pi^N$ of real-valued polynomials is called a system of generalized Hermite polynomials (associated with the reflection group G, the multiplicity parameter k and the frequency parameter ω), if the following are satisfied:

(i) $\{H_\nu, |\nu| = n\}$ is a $\mathbb{C}$-basis of V_n^ω for every $n \in \mathbb{Z}_+$.

(ii) The H_ν, $\nu \in \mathbb{Z}_+^N$ are orthogonal with respect to the probability measure m_k^ω on $\mathbb{R}^N$.

We now consider a fixed system $\{H_\nu, \nu \in \mathbb{Z}_+^N\}$ of generalized Hermite polynomials associated with G and k. We assume in addition that the H_ν are even orthonormal with respect to m_k^ω. By definition, they form a basis of eigenfunctions of $\mathcal{J}_k$ in $L^2(\mathbb{R}^N, m_k^\omega)$ with

$$\mathcal{J}_k H_\nu = 2|\nu|\omega \cdot H_\nu. \tag{6.1}$$

We shall need the following Mehler formula, which was shown in [R1] for $\omega = 1$ and is obtained for general ω by rescaling:

6.2 Lemma. *(Mehler-formula for the generalized Hermite polynomials.) The polynomials $H_\nu = H_\nu(\omega;\,.\,)$ satisfy*

$$\sum_{\nu \in \mathbb{Z}_+^N} H_\nu(x) H_\nu(y)\, r^{|\nu|} = M_k(r, x, y) \tag{6.2}$$

with the generalized Mehler kernel

$$M_k(r,x,y) = \frac{1}{(1-r^2)^{\gamma+N/2}} \exp\left\{-\frac{\omega r^2(|x|^2+|y|^2)}{1-r^2}\right\} E_k\Big(\frac{2\omega r x}{1-r^2}, y\Big).$$

The sum on the left hand side of (6.2) converges absolutely for all $x, y \in \mathbb{R}^N$ *and* $0 < r < 1$.

According to Theorem 3.2, $\mathcal{J}_k$ is essentially self-adjoint in $L^2(\mathbb{R}^N, m_k^\omega)$. Let $\langle\, .\, ,.\rangle$ denote the scalar product in $L^2(\mathbb{R}^N, m_k^\omega)$. Then the closure of $\mathcal{J}_k$ is given by

$$\overline{\mathcal{J}_k}(f) = \sum_{\nu\in\mathbb{Z}_+^N} 2|\nu|\omega\,\langle f, H_\nu\rangle\, f\,,$$

with domain

$$\mathcal{D}(\overline{\mathcal{J}_k}) = \{f \in L^2(\mathbb{R}^N, m_k^\omega) : \sum_{\nu\in\mathbb{Z}_+^N} |\nu|^2 |\langle f, H_\nu\rangle|^2 < \infty\}.$$

The spectral resolution of $\overline{\mathcal{J}_k}$ directly implies that $-\overline{\mathcal{J}_k}$ generates a strongly continuous contraction semigroup on $L^2(\mathbb{R}^N, m_k^\omega)$, namely

$$e^{-t\overline{\mathcal{J}_k}} f = \sum_{\nu\in\mathbb{Z}_+^N} e^{-2|\nu|\omega t}\,\langle f, H_\nu\rangle H_\nu \quad \text{for all } t \geq 0.$$

According to (6.2), we have

$$\sum_{\nu\in\mathbb{Z}_+^N} e^{-2|\nu|\omega t} H_\nu(x) H_\nu(y) = M_k(e^{-2t}, x, y)$$

for all $t > 0$. It is easily seen from the absolute convergence of the sum on the left, together with the orthogonality of the generalized Hermite polynomials, that the function $y \mapsto M_k(e^{-2t}, x, y)$ belongs to $L^2(\mathbb{R}^N, m_k^\omega)$ for each fixed $x \in \mathbb{R}^N$. This shows that for $t > 0$,

$$e^{-t\overline{\mathcal{J}_k}} f(x) = \int_{\mathbb{R}^N} M_k(e^{-2t}, x, y) f(y)\, m_k^\omega(y) \quad a.e.$$

6.3 Proposition. $\big(e^{-t\overline{\mathcal{J}}_k}\big)_{t\geq 0}$ *is a symmetric Markov semigroup on* $L^2(\mathbb{R}^N, m_k^\omega)$ *in the sense of Definition 4.7.*

Proof. $\overline{\mathcal{J}}_k$ is self-adjoint and non-negative, and the semigroup $\big(e^{-t\overline{\mathcal{J}_k}}\big)_{t\geq 0}$ is positivity-preserving on $L^2(\mathbb{R}^N, m_k^\omega)$, because the kernel M_k is strictly positive. The $\{H_\nu\,,\, \nu \in \mathbb{Z}_+^N\}$ being orthonormal with $H_0 = 1$, we further have

$$\int_{\mathbb{R}^N} M_k(e^{-2t}, x, y)\, dm_k^\omega(y) = 1 \quad \text{for all } t > 0,\ x \in \mathbb{R}^N\,. \tag{6.3}$$

This implies that the operators $e^{-t\overline{\mathcal{J}}_k}$, $t \geq 0$ are also contractive with respect to $\|\,.\,\|_\infty$.
□

As a consequence, the generalized oscillator semigroup $\left(e^{-t\mathcal{J}_k}\right)_{t\geq 0}$ also allows an extension to a strongly continuous contraction semigroup on each of the Banach spaces $L^p(\mathbb{R}^N, m_k^\omega)$. We introduce the following notation:

6.4 Definition. For $f \in L^1(\mathbb{R}^N, m_k^\omega)$ and $t \geq 0$ set

$$O_k(t)f(x) := \begin{cases} \int_{\mathbb{R}^N} M_k(e^{-2t}, x, y) f(y)\, dm_k^\omega(y) & \text{if } t > 0, \\ f(x) & \text{if } t = 0 \end{cases} \tag{6.4}$$

6.5 Corollary. *$(O_k(t))_{t\geq 0}$ is a strongly continuous, positivity-preserving contraction semigroup on each of the Banach spaces $L^p(\mathbb{R}^N, m_k^\omega)$, $1 \leq p < \infty$. For $p > 1$ it is a bounded holomorphic semigroup in the sector*

$$\left\{z \in \mathbb{C} : |arg(z)| < \pi \cdot \min\left(\frac{1}{p}, \frac{1}{q}\right)\right\},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. This follows from Proposition 6.3 together with Theorems 1.4.1 and 1.4.2 of [Da2]. □

Direct inspection shows that the Mehler kernel is related to the Gaussian kernel Γ_k via

$$M_k(e^{-2t}, x, y)\, m_k^\omega(y) = \Gamma_k\left(\frac{1 - e^{-4\omega t}}{4\omega}, e^{-2\omega t}x, y\right) w_k(y)dy \quad (t > 0,\, x \in \mathbb{R}^N). \tag{6.5}$$

The operators $O_k(t)$ can be expressed in terms of the heat operators $H_k(t)$:

$$O_k(t)f(x) = H_k\left(\frac{1 - e^{-4\omega t}}{4\omega}\right) f\left(e^{-2\omega t}x\right) \tag{6.6}$$

for all $f \in C_0(\mathbb{R}^N)$ and all $t > 0$. This implies that $(O_k(t))_{t\geq 0}$ leaves both $C_0(\mathbb{R}^N)$ and $\mathscr{S}(\mathbb{R}^N)$ invariant. It provides in fact a Feller-Markov semigroup on $(C_0(\mathbb{R}^N), \|.\|_\infty)$, which is a generalization of the classical Ornstein-Uhlenbeck semigroup to the Dunkl setting. The essential parts of the following result are contained in Section 10 of [R-V]:

6.6 Proposition. *$(O_k(t))_{t\geq 0}$ defines a strongly continuous, positivity-preserving contraction semigroup on $(C_0(\mathbb{R}^N), \|\,.\,\|_\infty)$. The Schwartz space $\mathscr{S}(\mathbb{R}^N)$ is a core of its generator A, and $A|_{\mathscr{S}(\mathbb{R}^N)} = \Delta_k - 2\omega\sum_{j=1}^N x_j\partial_j$.*

Proof. The first part of the statement has been shown in [R-V]. The proof given there implies also that $\mathscr{S}(\mathbb{R}^N)$ is contained in the domain of A, and that $A|_{\mathscr{S}(\mathbb{R}^N)} = \Delta_k - 2\omega\sum_{j=1}^N x_j\partial_j$. Since $\mathscr{S}(\mathbb{R}^N)$ is invariant under $(O_k(t))_{t\geq 0}$, it is in fact a core of A. □

Remark. It is also shown in [R-V] that for each $f \in C_b(\mathbb{R}^N)$, the function $u(t,x) := O_k(t)f(x)$ belongs to $C_b([0,\infty) \times \mathbb{R}^N) \cap C^2((0,\infty) \times \mathbb{R}^N)$ and solves the initial value problem

$$\begin{cases} \partial_t u = (\Delta_k - 2\omega\sum_{j=1}^N x_j\partial_j)\, u & \text{on } (0,\infty) \times \mathbb{R}^N, \\ u(0,.) = f. \end{cases}$$

References

[BF] Baker, T.H., Forrester, P.J., Non-symmetric Jack polynomials and integral kernels. *Duke Math. J.* **95** (1998), 1–50.

[BHKV] Brink, L., Hansson, T.H., Konstein, S., Vasiliev, M.A., The Calogero model - anyonic representation, fermionic extension and supersymmetry. *Nucl. Phys. B* 401 (1993), 591-612.

[Ca] Calogero, F., Solution of the one-dimensional N-body problems with quadratic and/or inversely quadratic pair potentials. *J. Math. Phys.* 12 (1971), 419–436.

[Da1] Davies, E.B., *One-Parameter Semigroups.* L.M.S. Monographs 15, Academic Press, 1980.

[Da2] Davies, E.B., *Heat Kernels and Spectral Theory.* Cambridge University Press, 1989.

[Da3] Davies, E.B., *Spectral Theory and Differential Operators.* Cambridge University Press, 1995.

[D1] Dunkl, C.F., Differential-difference operators associated to reflection groups. *Trans. Amer. Math. Soc.* 311 (1989), 167–183.

[D2] Dunkl, C.F., Integral kernels with reflection group invariance. *Canad. J. Math.* 43 (1991), 1213 - 1227.

[D3] Dunkl, C.F., Hankel transforms associated to finite reflection groups. In: *Proc. of the special session on hypergeometric functions on domains of positivity, Jack polynomials and applications.* Proceedings, Tampa 1991, Contemp. Math. 138 (1992), pp. 123–138.

[Ha] Ha, Z.N.C., Exact dynamical correlation functions of the Calogero-Sutherland model and one dimensional fractional statistics in one dimension: View from an exactly solvable model. *Nucl. Phys. B* 435 (1995), 604–636.

[Hal] Haldane, D., Physics of the ideal fermion gas: Spinons and quantum symmetries of the integrable Haldane-Shastry spin chain. In: A. Okiji, N. Kamakani (eds.), *Correlation effects in low-dimensional electron systems.* Springer, 1995, pp. 3–20.

[He] Heckman, G.J., A remark on the Dunkl differential-difference operators. In: Barker, W., Sally, P. (eds.) *Harmonic analysis on reductive groups.* Progress in Math. **101**, Birkhäuser, 1991. pp. 181 - 191.

[H-St] Hewitt, E., Stromberg, K., *Real and Abstract Analysis.* Springer, 1975.

[dJ] Jeu, de, M.F.E., The Dunkl transform. *Invent. Math.* 113 (1993), 147 - 162.

[K] Kakei, S., Common algebraic structure for the Calogero-Sutherland models. *J. Phys. A* 29 (1996), L619–L624.

[Kat] Kato, T. *Perturbation Theory for Linear Operators.* Springer, 1966.

[L-V] Lapointe L., Vinet, L., Exact operator solution of the Calogero-Sutherland model. *Commun. Math. Phys.* 178 (1996), 425–452.

[Mo] Moser, J., Three integrable Hamiltonian systems connected with isospectral deformations, *Adv. in Math.* 16 (1975), 197–220.

[O] Opdam, E.M., Dunkl operators, Bessel functions and the discriminant of a finite Coxeter group. *Compositio Math.* 85 (1993), 333–373.

[O-P1] Olshanetsky, M.A., Perelomov, A.M., Completely integrable Hamiltonian systems connected with semisimple Lie algebras. *Invent. Math.* 37 (1976), 93–108

[O-P2] Olshanetsky, M.A., Perelomov, A.M., Quantum systems related to root systems, and radial parts of Laplace operators. *Funct. Anal. Appl.* 12 (1978), 121–128.

[Pe] Perelomov, A.M., Algebraical approach to the solution of a one-dimensional model of N interacting particles. *Teor. Mat. Fiz.* 6 (1971), 364–391.

[Po] Polychronakos, A.P., Exchange operator formalism for integrable systems of particles. *Phys. Rev. Lett.* 69 (1992), 703–705.

[R1] Rösler, M., Generalized Hermite polynomials and the heat equation for Dunkl operators. *Commun. Math. Phys.* 192 (1998), 519–542.

[R2] Rösler, M., Positivity of Dunkl's intertwining operator. *Duke Math. J.* **98**, 445–463 (1999).

[R3] Rösler, M., Short time estimates for heat kernels associated with root systems. Submitted.

[R-V] Rösler, M., Voit, M., Markov Processes related with Dunkl operators. *Adv. Appl. Math.* 21 (1998), 575–643.

[Su] Sutherland, B., Exact results for a quantum many-body problem in one dimension. *Phys. Rep.* A5 (1972), 1372–1376.

[U-W] Ujino, H., Wadati., M., Rodrigues formula for the nonsymmetric multivariable Hermite polynomial. *J. Phys. Soc. Japan* 68 (1999), 391–395.

The Lévy Laplacian and Stochastic Processes

Kimiaki Saitô
Department of Mathematics
Meijo University
Nagoya 468-8502, Japan

Abstract

In this paper, we give infinite dimensional stochastic processes generated by functions of the Lévy Laplacian. Moreover we introduce an operator to connect the Lévy Laplacian with the Number operator and also give a relationship between a (C_0)-semigroup generated by the Lévy Laplacian and an infinite dimensional Ornstein-Uhlenbeck process.

1. Introduction

An infinite dimensional Laplacian, the Lévy Laplacian, was introduced by P. Lévy [17]. This Laplacian was introduced into the framework of white noise analysis initiated by T. Hida [4]. L. Accardi et al. [1] obtained an important relationship between this Laplacian and the Yang-Mills equations. It has been studied by many authors (see [1, 2, 3, 5, 7, 8, 13, 15, 16, 18, 21, 22, 23, 24 etc]).

In the previous papers [25,26] we obtained stochastic processes generated by the powers of an extended Lévy Laplacian and also in [29] we obtained stochastic processes generated by some functions of the Laplacian.

The purpose of this paper is to present recent developments on stochastic processes generated by functions of the Lévy Laplacian acting on white noise distributions based on the idea in [29] and to give a stochastic expression of an equi-continuous semigroup of class (C_0) generated by the Laplacian related to an infinite dimensional Ornstein-Uhlenbeck process following [27].

The paper is organized as follows. In Section 2 we summarize some basic definitions and results in white noise analysis. In Section 3 we introduce a Hilbert space as a domain of the extended Lévy Laplacian which is self-adjoint on the domain following our previous paper [27], and we give an equi-continuous semigroup of class (C_0) generated by some functions of the extended Lévy Laplacian. In Section 4 we give infinite dimensional stochastic processes generated by those functions of the Lévy Laplacian. In the last section we give a relationship between the semigroup generated by the Lévy Laplacian and an infinite dimensional Ornstein-Uhlenbeck process.

2. Preliminaries

In this section we assemble some basic notations of white noise analysis following [7, 12, 15, 19].

We take the space $E^* \equiv \mathcal{S}'(\mathbf{R})$ of tempered distributions with the standard Gaussian measure μ which satisfies

$$\int_{E^*} \exp\{i\langle x, \xi\rangle\}\, d\mu(x) = \exp\left(-\frac{1}{2}|\xi|_0^2\right), \quad \xi \in E \equiv \mathcal{S}(\mathbf{R}),$$

where $\langle \cdot, \cdot \rangle$ is the canonical bilinear form on $E^* \times E$.

Let $A = -(d/du)^2 + u^2 + 1$. This is a densely defined self-adjoint operator on $L^2(\mathbf{R})$ and there exists an orthonormal basis $\{e_\nu; \nu \geq 0\}$ for $L^2(\mathbf{R})$ such that $Ae_\nu = 2(\nu+1)e_\nu$. We define the norm $|\cdot|_p$ by $|f|_p = |A^p f|_0$ for $f \in E$ and $p \in \mathbf{R}$, where $|\cdot|_0$ is the $L^2(\mathbf{R})$- norm, and let E_p be the completion of E with respect to the norm $|\cdot|_p$. Then E_p ia a real separable Hilbert space with the norm $|\cdot|_p$ and the dual space E_p' of E_p is the same as E_{-p} (see [10]).

Let E be the projective limit space of $\{E_p; p \geq 0\}$ and E^* the dual space of E. Then E becomes a nuclear space with the Gel'fand triple $E \subset L^2(\mathbf{R}) \subset E^*$. We denote the complexifications of $L^2(\mathbf{R})$, E and E_p by $L^2_{\mathbf{C}}(\mathbf{R})$, $E_{\mathbf{C}}$ and $E_{\mathbf{C},p}$, respectively.

The space $(L^2) = L^2(E^*, \mu)$ of complex-valued square-integrable functionals defined on E^* admits the well-known Wiener-Itô decomposition:

$$(L^2) = \bigoplus_{n=0}^{\infty} H_n,$$

where H_n is the space of multiple Wiener integrals of order $n \in \mathbf{N}$ and $H_0 = \mathbf{C}$. Let $L^2_{\mathbf{C}}(\mathbf{R})^{\hat{\otimes} n}$ denote the n-fold symmetric tensor product of $L^2_{\mathbf{C}}(\mathbf{R})$. If $\varphi \in (L^2)$ has the representation $\varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n)$, $f_n \in L^2_{\mathbf{C}}(\mathbf{R})^{\hat{\otimes} n}$, then the (L^2)-norm $\|\varphi\|_0$ is given by

$$\|\varphi\|_0 = \left(\sum_{n=0}^{\infty} n!|f_n|_0^2\right)^{1/2},$$

where $|\cdot|_0$ is the norm of $L^2_{\mathbf{C}}(\mathbf{R})^{\hat{\otimes} n}$.

For $p \in \mathbf{R}$, let $\|\varphi\|_p = \|\Gamma(A)^p \varphi\|_0$, where $\Gamma(A)$ is the second quantization operator of A. If $p \geq 0$, let $(E)_p$ be the domain of $\Gamma(A)^p$. If $p < 0$, let $(E)_p$ be the completion of (L^2) with respect to the norm $\|\cdot\|_p$. Then $(E)_p$, $p \in \mathbf{R}$, is a Hilbert space with the norm $\|\cdot\|_p$. It is easy to see that for $p > 0$, the dual space $(E)_p^*$ of $(E)_p$ is given by $(E)_{-p}$. Moreover, for any $p \in \mathbf{R}$, we have the decomposition

$$(E)_p = \bigoplus_{n=0}^{\infty} H_n^{(p)},$$

where $H_n^{(p)}$ is the completion of $\{\mathbf{I}_n(f); f \in E_{\mathbf{C}}^{\hat{\otimes} n}\}$ with respect to $\|\cdot\|_p$. Here $E_{\mathbf{C}}^{\hat{\otimes} n}$ is the n-fold symmetric tensor product of $E_{\mathbf{C}}$. We also have $H_n^{(p)} = \{\mathbf{I}_n(f); f \in E_{\mathbf{C},p}^{\hat{\otimes} n}\}$ for any $p \in \mathbf{R}$, where

$E_{\mathbf{C},p}^{\hat{\otimes}n}$ is also the n-fold symmetric tensor product of $E_{\mathbf{C},p}$. The norm $\|\varphi\|_p$ of $\varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n) \in (E)_p$ is given by

$$\|\varphi\|_p = \left(\sum_{n=0}^{\infty} n! |f_n|_p^2 \right)^{1/2}, \quad f_n \in E_{\mathbf{C},p}^{\hat{\otimes}n},$$

where the norm of $E_{\mathbf{C},p}^{\hat{\otimes}n}$ is denoted also by $|\cdot|_p$.

The projective limit space (E) of spaces $(E)_p$, $p \in \mathbf{R}$ is a nuclear space. The inductive limit space $(E)^*$ of spaces $(E)_p, p \in \mathbf{R}$ is nothing but the dual space of (E). The space $(E)^*$ is called the space of *generalized white noise functionals.* We denote by $\ll \cdot, \cdot \gg$ the canonical bilinear form on $(E)^* \times (E)$. Then we have

$$\ll \Phi, \varphi \gg = \sum_{n=0}^{\infty} n! \langle F_n, f_n \rangle$$

for any $\Phi = \sum_{n=0}^{\infty} \mathbf{I}_n(F_n) \in (E)^*$ and $\varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n) \in (E)$, where the canonical bilinear form on $(E_{\mathbf{C}}^{\hat{\otimes}n})^* \times (E_{\mathbf{C}}^{\hat{\otimes}n})$ is denoted also by $\langle \cdot, \cdot \rangle$.

Since $\exp\langle \cdot, \xi \rangle \in (E)$, the *S-transform* is defined on $(E)^*$ by

$$S[\Phi](\xi) = \exp\left(-\frac{1}{2} \langle \xi, \xi \rangle \right) \ll \Phi, \exp\langle \cdot, \xi \rangle \gg, \qquad \xi \in E_{\mathbf{C}}.$$

3. An equi-continuous semigroup of class (C_0) generated by a function of the Lévy Laplacian

Let Φ be in $(E)^*$. Then the S-transform $S[\Phi]$ of Φ is *Fréchet differentiable*, i.e.

$$S[\Phi](\xi + \eta) = S[\Phi](\xi) + S[\Phi]'(\xi)(\eta) + o(\eta),$$

where $o(\eta)$ means that there exists $p \geq 0$ depending on ξ such that $o(\eta)/|\eta|_p \to 0$ as $|\eta|_p \to 0$.

We fix a finite interval T in $\mathbf{R}$. Take an orthonormal basis $\{\zeta_n\}_{n=0}^{\infty} \subset E$ for $L^2(T)$ satisfying the equally dense and uniform boundedness property (see [7,15,16,18,24, etc]). Let $\mathcal{D}_L$ denote the set of all $\Phi \in (E)^*$ such that the limit

$$\tilde{\Delta}_L S[\Phi](\xi) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} S[\Phi]''(\xi)(\zeta_n, \zeta_n)$$

exists for any $\xi \in E_{\mathbf{C}}$ and is in $S[(E)^*]$. The Lévy Laplacian Δ_L is defined by

$$\Delta_L \Phi = S^{-1} \tilde{\Delta}_L S \Phi$$

for $\Phi \in \mathcal{D}_L$. We denote the set of all functionals $\Phi \in \mathcal{D}_L$ such that $S[\Phi](\eta) = 0$ for all $\eta \in E$ with $\text{supp}(\eta) \subset T^c$ by $\mathcal{D}_L^T$.

A generalized white noise functional

$$\Phi = \int_{\mathbf{R}^n} f(u_1, \dots, u_n) : e^{ia_1 x(u_1)} \dots e^{ia_n x(u_n)} : d\mathbf{u} \in \mathcal{D}_L^T, \tag{3.1}$$

$$f \in L^1_{\mathbf{C}}(\mathbf{R})^{\hat{\otimes} n} \cap L^2_{\mathbf{C}}(\mathbf{R})^{\hat{\otimes} n}, a_k \in \mathbf{R}, k = 1, 2, \dots, n,$$

is equal to

$$\int_{T^n} f(u_1, \dots, u_n) : e^{ia_1 x(u_1)} \dots e^{ia_n x(u_n)} : d\mathbf{u}$$

and the S-transform $S[\Phi]$ of Φ is given by

$$S[\Phi](\xi) = \int_{T^n} f(\mathbf{u}) e^{ia_1 \xi(u_1)} \dots e^{ia_n \xi(u_n)} d\mathbf{u}. \tag{3.2}$$

This functional is important as an eigenfunction of the operator Δ_L. In fact, we have the following result:

Theorem 1.[27] *A generalized white noise functional Φ as in (3.1) satisfies the equation*

$$\Delta_L \Phi = -\frac{1}{|T|} \sum_{k=1}^{n} a_k^2 \Phi. \tag{3.3}$$

We set

$$\mathbf{D}_n = \left\{ \int_{T^n} f(\mathbf{u}) : \prod_{\nu=1}^{n} e^{ix(u_\nu)} : d\mathbf{u} \in \mathcal{D}_L^T ; f \in E_{\mathbf{C}}(\mathbf{R})^{\hat{\otimes} n} \right\}$$

for each $n \in \mathbf{N}$ and set $\mathbf{D}_0 = \mathbf{C}$. Then $\mathbf{D}_n$ is a linear subspace of $(E)_{-p}$ for any $p \geq 1$, and Δ_L is a linear operator from $\mathbf{D}_n$ into itself such that $\|\Delta_L \Phi\|_{-p} = \frac{n}{|T|} \|\Phi\|_{-p}$ for any $\Phi \in \mathbf{D}_n$. We define a space $\overline{\mathbf{D}}_n$ by the completion of $\mathbf{D}_n$ in $(E)_{-p}$ with respect to $\|\cdot\|_{-p}$. Then for each $n \in \mathbf{N} \cup \{0\}$, $\overline{\mathbf{D}}_n$ becomes a Hilbert space with the inner product of $(E)_{-p}$. For each $n \in \mathbf{N} \cup \{0\}$, the operator Δ_L can be extended to a continuous linear operator $\overline{\Delta_L}$ from $\overline{\mathbf{D}}_n$ into itself satisfying

$$\|\overline{\Delta_L} \Phi\|_{-p} = \frac{n}{|T|} \|\Phi\|_{-p} \text{ for any } \Phi \in \overline{\mathbf{D}}_n.$$

The operator $\overline{\Delta_L}$ is a self-adjoint operator on $\overline{\mathbf{D}}_n$ for each $n \in \mathbf{N} \cup \{0\}$.

Proposition 2. [27] *Let $\Phi = \sum_{n=0}^{\infty} \Phi_n, \Psi = \sum_{n=0}^{\infty} \Psi_n$ be generalized white noise functionals such that Φ_n and Ψ_n are in $\overline{\mathbf{D}}_n$ for each $n \in \mathbf{N} \cup \{0\}$. If $\Phi = \Psi$ in $(E)^*$, then $\Phi_n = \Psi_n$ in $(E)^*$ for each $n \in \mathbf{N} \cup \{0\}$.*

Proposition 2 says that $\sum_{n=0}^{\infty} \Phi_n, \Phi_n \in \overline{\mathbf{D}}_n$, is uniquely determined as an element of $(E)^*$. Therefore, for any $\ell \in \mathbf{R}$, we can define a space $\mathbf{E}_{-p,\ell}$ by

$$\mathbf{E}_{-p,\ell} = \left\{ \sum_{n=0}^{\infty} \Phi_n \in (E)^*; \sum_{n=0}^{\infty} \left(1 + \frac{n}{|T|}\right)^{2\ell} \|\Phi_n\|_{-p}^2 < \infty, \Phi_n \in \overline{\mathbf{D}}_n, n = 0, 1, 2, \dots \right\}$$

with the norm $|||\cdot|||_{-p,\ell}$ given by

$$|||\Phi|||_{-p,\ell} = \left(\sum_{n=0}^{\infty} \left(1 + \frac{n}{|T|}\right)^{2\ell} \|\Phi_n\|_{-p}^2 \right)^{1/2}, \quad \Phi = \sum_{n=0}^{\infty} \Phi_n \in \mathbf{E}_{-p,\ell}$$

for each $\ell \in \mathbf{R}$ and $p \geq 1$. For any $\ell \in \mathbf{R}$ and $p \geq 1$, $\mathbf{E}_{-p,\ell}$ is a Hilbert space with the norm $|||\cdot|||_{-p,\ell}$.

Put $\mathbf{E}_{-p,\infty} = \bigcap_{\ell \geq 1} \mathbf{E}_{-p,\ell}$ with the projective limit topology and put $\mathbf{E}_{-p,-\infty} = \bigcup_{\ell \geq 1} \mathbf{E}_{-p,-\ell}$ with the inductive limit topology. Then, for any $\ell \geq 1$, we have the following inclusion relations:

$$\mathbf{E}_{-p,\infty} \subset \mathbf{E}_{-p,\ell+1} \subset \mathbf{E}_{-p,\ell} \subset \mathbf{E}_{-p,1} \subset (E)_{-p} \subset \mathbf{E}_{-p,-1} \subset \mathbf{E}_{-p,-\ell} \subset \mathbf{E}_{-p,-\ell-1} \subset \mathbf{E}_{-p,-\infty}.$$

The space $\mathbf{E}_{-p,\infty}$ includes $\overline{\mathbf{D}}_n$ for any $n \in \mathbf{N} \cup \{0\}$. The operator $\overline{\Delta_L}$ can be extended to a continuous linear operator defined on $\mathbf{E}_{-p,-\infty}$, denoted by the same notation $\overline{\Delta_L}$, satisfying $|||\overline{\Delta_L}\Phi|||_{-p,\ell} \leq |||\Phi|||_{-p,\ell+1}$, $\Phi \in \mathbf{E}_{-p,\ell+1}$, for each $\ell \in \mathbf{R}$. Any restriction of $\overline{\Delta_L}$ is also denoted by the same notation $\overline{\Delta_L}$. With these properties, we have the following:

Theorem 3. *The operator $\overline{\Delta_L}$ restricted on $\mathbf{E}_{-p,\ell+1}$ is a self-adjoint operator densely defined on $\mathbf{E}_{-p,\ell}$ for each $\ell \in \mathbf{R}$ and $p \geq 1$.*

Proof: We can apply the same proof of Theorem 2 in [27] to this theorem. □

Let $\{X_t; t \geq 0\}$ be a stochastic process and $c_{X_t}(z)$ be a characteristic function of X_t. For each $t \geq 0$ we consider an operator $G[X_t]$ on $\mathbf{E}_{-p,-\infty}$ defined by

$$G[X_t]\Phi = \sum_{n=0}^{\infty} c_{X_t}\left(\frac{n}{|T|}\right) \Phi_n$$

for $\Phi = \sum_{n=0}^{\infty} \Phi_n \in \mathbf{E}_{-p,-\infty}$. For any $\Phi = \sum_{n=0}^{\infty} \Phi_n$ in $\mathbf{E}_{-p,-\infty}$, there exists $\ell \in \mathbf{R}$ such that $\Phi \in \mathbf{E}_{-p,\ell}$. Then, for any $t \geq 0, p \geq 1$, the norm $|||G[X_t]\Phi|||_{-p,\ell}$ is estimated as follows:

$$\begin{aligned} |||G[X_t]\Phi|||_{-p,\ell}^2 &= \sum_{n=0}^{\infty} \left(1 + \frac{n}{|T|}\right)^{2\ell} \left\| c_{X_t}\left(\frac{n}{|T|}\right) \Phi_n \right\|_{-p}^2 \\ &\leq \sum_{n=0}^{\infty} \left(1 + \frac{n}{|T|}\right)^{2\ell} \|\Phi_n\|_{-p}^2 = |||\Phi|||_{-p,\ell}^2. \end{aligned}$$

Thus the operator $G[X_t]$ is a continuous linear operator from $\mathbf{E}_{-p,-\infty}$ into itself. Moreover we have the following:

Proposition 4. *Let $\{X_t; t \geq 0\}$ be a stochastic process. Then the family $\{G[X_t]; t \geq 0\}$ is an equi-continuous semigroup of class (C_0) if and only if there exists a complex-valued continuous function $h(z)$ of $z \in \mathbf{R}$ such that $h(0) = 0$ and $c_{X_t}(z) = e^{h(z)t}$ for all $t \geq 0$.*

Proof: If there exists a complex-valued continuous function $h(z)$ of $z \in \mathbf{R}$ such that $c_{X_t}(z) = e^{h(z)t}$, then it is easily checked that $G[X_0] = I$, $G[X_t]G[X_s] = G[X_{t+s}]$ for each $t, s \geq 0$. Moreover we can estimate that

$$|||G[X_t]\Phi - G[X_{t_0}]\Phi|||_{-p,\ell}^2 = \sum_{n=0}^{\infty}\left(1+\frac{n}{|T|}\right)^{2\ell}\left|c_{X_t}\left(\frac{n}{|T|}\right) - c_{X_{t_0}}\left(\frac{n}{|T|}\right)\right|^2 ||\Phi_n||_{-p}^2$$
$$\leq 4\sum_{n=0}^{\infty}\left(1+\frac{n}{|T|}\right)^{2\ell} ||\Phi_n||_{-p}^2 = 4|||\Phi|||_{-p,\ell}^2 < \infty$$

for each $t, t_0 \geq 0, \ell \in \mathbf{R}$ and $\Phi = \sum_{n=0}^{\infty} \Phi_n \in \mathbf{E}_{-p,\ell}$. Therefore, by the Lebesgue convergence theorem, we get that

$$\lim_{t \to t_0} G[X_t]\Phi = G[X_{t_0}]\Phi \text{ in } \mathbf{E}_{-p,\infty}$$

for each $t_0 \geq 0$ and $\Phi \in \mathbf{E}_{-p,-\infty}$. Thus the family $\{G[X_t]; t \geq 0\}$ is an equi-continuous semigroup of class (C_0). Conversely, if $\{G[X_t]; t \geq 0\}$ is an equi-continuous semigroup of class (C_0), then it is easily checked that $c_{X_0}\left(\frac{n}{|T|}\right) = 1$, $c_{X_t}\left(\frac{n}{|T|}\right) c_{X_s}\left(\frac{n}{|T|}\right) = c_{X_{t+s}}\left(\frac{n}{|T|}\right)$ for any $t, s \geq 0$ and $\lim_{t \to t_0} c_{X_t}\left(\frac{n}{|T|}\right) = c_{X_{t_0}}\left(\frac{n}{|T|}\right)$ for any $t_0 \geq 0$ and $n \in \mathbf{N}$. Therefore, by the continuity of $c_{X_t}(z)$ of z, we have that $c_{X_0} = 1$, $c_{X_t}c_{X_s} = c_{X_{t+s}}$ for any $t, s \geq 0$ and $\lim_{t \to t_0} c_{X_t} = c_{X_{t_0}}$ for any $t_0 \geq 0$. Consequently, there exists a complex-valued function $h(z)$ of $z \in \mathbf{R}$ such that $h(0) = 0$ and $c_{X_t}(z) = e^{h(z)t}$. Since $c_{X_t}(z)$ is a characteristic function, the function $h(z)$ is continuous. □

For any $p \geq 1$ and complex-valued continuous function $h(z), z \in \mathbf{R}$ satisfying the condition:

(P) there exists a polynomial $r(z)$ of $z \in \mathbf{R}$ such that $|h(z)| \leq r(|z|)$ for all $z \in \mathbf{R}$,

the operator $h(-\overline{\Delta_L})$ on $\mathbf{E}_{-p,-\infty}$ is given by

$$h(-\overline{\Delta_L})\Phi = \sum_{n=0}^{\infty} h\left(\frac{n}{|T|}\right)\Phi_n, \text{ for } \Phi = \sum_{n=0}^{\infty} \Phi_n \in \mathbf{E}_{-p,-\infty}.$$

Theorem 5. *If $h(z)$ in Proposition 4 satisfies the condition* **(P)**, *then the infinitesimal generator of $\{G[X_t]; t \geq 0\}$ is given by $h(-\overline{\Delta_L})$.*

Proof: Let $p \geq 1$ and let $\Phi = \sum_{n=0}^{\infty} \Phi_n \in \mathbf{E}_{-p,-\infty}$. Then, there exists $\ell \in \mathbf{R}$ such that $\Phi \in \mathbf{E}_{-p,\ell}$. Let d_r be the degree of the polynomial r in the condition **(P)**. Then we note that

$$\left|\left|\left|\frac{G[X_t]\Phi - \Phi}{t} - h(-\overline{\Delta_L})\Phi\right|\right|\right|_{-p,\ell-d_r}^2 = \sum_{n=0}^{\infty}\left(1+\frac{n}{|T|}\right)^{2(\ell-d_r)}\left|\left|\left(\frac{e^{h\left(\frac{n}{|T|}\right)t}-1}{t} - h\left(\frac{n}{|T|}\right)\right)\Phi_n\right|\right|_{-p}^2 \tag{3.4}$$

Since $e^{h(z)t}$ is a characteristic function, we note that $Re[h(z)] \leq 0$. By the mean value theorem, for any $t > 0$ there exists a constant $\theta \in (0, 1)$ such that

$$\left|\frac{e^{h\left(\frac{n}{|T|}\right)t}-1}{t}\right| = \left|h\left(\frac{n}{|T|}\right)\right| e^{Re\left[h\left(\frac{n}{|T|}\right)\right]t\theta} \leq r\left(\frac{n}{|T|}\right).$$

Therefore we get that

$$\left\| \frac{e^{h\left(\frac{n}{|T|}\right)t} - 1}{t} \Phi_n - h\left(\frac{n}{|T|}\right)\Phi_n \right\|_{-p}^2 = \left| \frac{e^{h\left(\frac{n}{|T|}\right)t} - 1}{t} - h\left(\frac{n}{|T|}\right) \right|^2 ||\Phi_n||_{-p}^2$$
$$\leq 4r\left(\frac{n}{|T|}\right)^2 ||\Phi_n||_{-p}^2.$$

Since there exists a positive constant C_r depending on r such that $\left(1+\frac{n}{|T|}\right)^{2(\ell-d_r)} r\left(\frac{n}{|T|}\right)^2 \leq C_r\left(1+\frac{n}{|T|}\right)^{2\ell}$, we have

$$\sum_{n=0}^{\infty}\left(1+\frac{n}{|T|}\right)^{2(\ell-d_r)} r\left(\frac{n}{|T|}\right)^2 ||\Phi_n||_{-p}^2 < \infty. \tag{3.5}$$

By (3.4), (3.5) and

$$\lim_{t\to 0}\left| \frac{e^{h\left(\frac{n}{|T|}\right)t} - 1}{t} - h\left(\frac{n}{|T|}\right) \right| = 0,$$

the Lebesgue convergence theorem admits

$$\lim_{t\to 0}\left|\left|\left| \frac{G[X_t]\Phi - \Phi}{t} - h(-\overline{\Delta_L})\Phi \right|\right|\right|_{-p,\ell-d_r}^2 = 0.$$

Thus the proof is completed. □

4. Stochastic processes generated by functions of the Lévy Laplacian

In this section, we give stochastic processes generated by functions of the extended Lévy Laplacian by considering the stochastic expression of the operator $G[X_t]$.

Let $\{X_t; t \geq 0\}$ be a stochastic process such that $\{G[X_t]; t \geq 0\}$ is an equi-continuous semigroup of class (C_0) and satisfies the condition of Theorem 5. Take a smooth function $\eta_T \in E$ with $\eta_T = \frac{1}{|T|}$ on T. Put $\widetilde{G[X_t]} = SG[X_t]S^{-1}$ on $S[\mathbf{E}_{-p,\infty}]$ with the topology induced from $\mathbf{E}_{-p,\infty}$ by the S-transform. Then by Theorem 5, $\{\widetilde{G[X_t]}; t \geq 0\}$ is an equi-continuous semigroup of class (C_0) generated by the operator $h(-\widetilde{\Delta}_L)$, where $\widetilde{\Delta}_L$ means $S\overline{\Delta_L}S^{-1}$.

Theorem 6. *Let F be the S-transform of a generalized white noise functional in $\mathbf{E}_{-p,\infty}$. Then it holds that*

$$\widetilde{G[X_t]}F(\xi) = \mathrm{E}[F(\xi + X_t\eta_T)], \quad \xi \in E.$$

Proof. Put $F(\xi) = \int_{T^n} f(\mathbf{u})e^{i\xi(u_1)}\cdots e^{i\xi(u_n)}d\mathbf{u}$ with $f \in E_{\mathbf{C}}^{\hat{\otimes}n}$. Then we have

$$\mathrm{E}[F(\xi + X_t\eta_T)] = \int_{T^n} f(\mathbf{u})e^{i\xi(u_1)}\cdots e^{i\xi(u_n)}\mathrm{E}[e^{i\frac{n}{|T|}X_t}]d\mathbf{u}$$

$$= e^{h\left(\frac{n}{|T|}\right)t}F(\xi) = \widetilde{G[X_t]}F(\xi).$$

Let $F = \sum_{n=0}^{\infty} F_n \in S[\mathbf{E}_{-p,\infty}]$. Then for any $n \in \mathbf{N} \cup \{0\}$, F_n is expressed in the following form:

$$F_n(\xi) = \lim_{N\to\infty} \int_{T^n} f^{[N]}(\mathbf{u}) e^{i\xi(u_1)} \dots e^{i\xi(u_n)} d\mathbf{u},$$

where $(f^{[N]})_N$ is a sequence of functions in $E_{\mathbf{C}}^{\hat{\otimes} n}$. Hence we have

$$\begin{aligned}
& \sum_{n=0}^{\infty} \mathrm{E}[|F_n(\xi + X_t\eta_T)|] \\
= & \sum_{n=0}^{\infty} \mathrm{E}\left[\lim_{N\to\infty} \left| \int_{T^n} f^{[N]}(\mathbf{u}) e^{i\xi(u_1)} \dots e^{i\xi(u_n)} e^{iX_t\eta_T(u_1)} \dots e^{iX_t\eta_T(u_n)} d\mathbf{u} \right|\right] \\
= & \sum_{n=0}^{\infty} \lim_{N\to\infty} \left| \int_{T^n} f^{[N]}(\mathbf{u}) e^{i\xi(u_1)} \dots e^{i\xi(u_n)} d\mathbf{u} \right| \\
= & \sum_{n=0}^{\infty} |F_n(\xi)|.
\end{aligned}$$

Since $F_n \in S[\mathbf{E}_{-p,\infty}]$, there exists some $\Phi_n \in \mathbf{E}_{-p,\infty}$ such that $F_n = S[\Phi_n]$ for any n. By the characterization theorem of the U-functional (see [12,20,21, etc]), we see that

$$\begin{aligned}
\sum_{n=0}^{\infty} |F_n(\xi)| & \leq \sum_{n=0}^{\infty} ||\Phi_n||_{-p} ||\varphi_\xi||_p \\
& \leq \left\{ \sum_{n=0}^{\infty} \left(1 + \frac{n}{|T|}\right)^{-2\ell} \right\}^{1/2} \left\{ \sum_{n=0}^{\infty} \left(1 + \frac{n}{|T|}\right)^{2\ell} ||\Phi_n||_{-p}^2 \right\}^{1/2} ||\varphi_\xi||_p < \infty,
\end{aligned}$$

for all $\xi \in E$ and some $\ell \geq 1$, where $\varphi_\xi(x) =: \exp\{\langle x, \xi \rangle\} :$. Therefore by the continuity of $\widetilde{G[X_t]}$ we get that

$$\begin{aligned}
\mathrm{E}[F(\xi + X_t\eta_T)] & = \sum_{n=0}^{\infty} \mathrm{E}[F_n(\xi + X_t\eta_T)] \\
& = \sum_{n=0}^{\infty} \widetilde{G[X_t]} F_n(\xi) \\
& = \widetilde{G[X_t]} F(\xi).
\end{aligned}$$

Thus we obtain the assertion. □

Theorem 6 says that the infinite dimensional stochastic process $\{\xi + X_t\eta_T; t \geq 0\}$ is generated by $h(-\widetilde{\Delta_L})$.

For any $\Phi \in (E)^*$ and $\eta \in E$, the translation $\tau_\eta\Phi$ of Φ by η is defined as a generalized white noise functional $\tau_\eta\Phi$ whose S-transform is given by $S[\tau_\eta\Phi](\xi) = S[\Phi](\xi + \eta)$, $\xi \in E_{\mathbf{C}}$. Then we can translate Theorem 6 to be in words of generalized white noise functionals.

Corollary 7. *Let Φ be a generalized white noise functional in $\mathbf{E}_{-p,\infty}$. Then it holds that*

$$G[X_t]\Phi(x) = \mathrm{E}[\tau_{X_t\eta_T}\Phi(x)].$$

By Corollary 7 we can see that $\{\tau_{X_t\eta_T}; t \geq 0\}$ is an operator-valued stochastic process and $\{E[\tau_{X_t\eta_T}]; t \geq 0\}$ is an equi-continuous semigroup of class (C_0) generated by $h(-\overline{\Delta_L})$.

Example: Let $\{X_t; t \geq 0\}$ be an additive process with the characteristic function $c_{X_t}(z)$ of X_t for each $t \geq 0$ given by

$$c_{X_t}(z) = \exp\left[t\left\{imz - \frac{v}{2}z^2 + \int_{|u|<1}\left(e^{izu} - 1 - izu\right)d\nu(u) + \int_{|u|\geq 1}\left(e^{izu} - 1\right)d\nu(u)\right\}\right],$$

where $m \in \mathbf{R}, v \geq 0$ and ν is a measure on $\mathbf{R}$ satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbf{R}}(1 \wedge |u|^2)d\nu(u) < \infty$. Then the function

$$h(z) = imz - \frac{v}{2}z^2 + \int_{|u|<1}\left(e^{izu} - 1 - izu\right)d\nu(u) + \int_{|u|\geq 1}\left(e^{izu} - 1\right)d\nu(u)$$

satisfies conditions of Proposition 5 and the condition (**P**). Therefore $\{G[X_t]; t \geq 0\}$ is an equi-continuous semigroup of class (C_0) generated by $h(-\overline{\Delta_L})$. The stochastic process $\{\xi + X_t\eta_T; t \geq 0\}$ is also generated by $h(-\widetilde{\Delta_L})$.

In particular, if $\{X_t^\gamma; t \geq 0\}$, $0 < \gamma \leq 2$, is a strictly stable process with the characteristic function $c_{X_t^\gamma}(z)$ of X_t^γ given by $c_{X_t^\gamma}(z) = e^{-t|z|^\gamma}$, then $\{\xi + X_t^\gamma\eta_T; t \geq 0\}$ is generated by $-(-\widetilde{\Delta_L})^\gamma$.

5. A relationship to an infinite dimensional Ornstein-Uhlenbeck process

Put

$$[E]_{q,\ell} = \left\{\varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n) \in (E); \sum_{n=0}^{\infty}\left(1 + \frac{n}{|T|}\right)^\ell e^{n^2/2}|f_n|_q^2 < \infty,\ \mathrm{supp}(f_n) \subset T, n = 0, 1, 2, \ldots\right\}$$

for $q \geq 0$ and $\ell \geq 0$. Define a space $\overline{[E]_{q,\ell}}$ by the completion of $[E]_{q,\ell}$ with respect to the norm $\|\cdot\|_{\overline{[E]_{q,\ell}}}$ given by

$$\|\varphi\|_{\overline{[E]_{q,\ell}}} = \left(\sum_{n=0}^{\infty}\left(1 + \frac{n}{|T|}\right)^\ell e^{n^2/2}|f_n|_q^2\right)^{1/2}$$

for $\varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n) \in (E)^*$. Then $\overline{[E]_{q,\ell}}$ is a Hilbert space with norm $\|\cdot\|_{\overline{[E]_{q,\ell}}}$. It is easily checked that $\overline{[E]_{q,\ell}} \subset (E)_q$ for any $q \geq 0$. Put $\overline{[E]_{\infty,\ell}} = \bigcap_{q\geq 0} \overline{[E]_{q,\ell}}$ with the projective limit topology and also put $\overline{[E]_{\infty,\infty}} = \bigcap_{\ell\geq 1} \overline{[E]_{\infty,\ell}}$ with the projective limit topology.

Define an operator K on $\overline{[E]_{\infty,\infty}}$ by

$$K[\Phi] = S^{-1}[S[\Phi](e^{i\xi})].$$

Then we have the following:

Proposition 8. *Let $p \geq 1$. Then the operator K is a continuous linear operator from $\overline{[E]_{\infty,\infty}}$ into $\mathbf{E}_{-p,\infty}$.*

Proof. Let $p \geq 1$. Then for each $\ell \geq 1$ we can calculate the norm $|||K[\varphi]|||_{-p,\ell}^2$ of $K[\varphi]$ for $\varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n) \in \overline{[E]_{\infty,\infty}}$ as follows:

$$\begin{aligned} |||K[\varphi]|||_{-p,\ell}^2 &= \sum_{n=0}^{\infty} \left(1+\frac{n}{|T|}\right)^{\ell} \|\langle :(e^{ix})^{\otimes n}:, f_n\rangle\|_{-p}^2 \\ &\leq \sum_{n=0}^{\infty} \left(1+\frac{n}{|T|}\right)^{\ell} \sum_{\ell=0}^{\infty} \ell! \sum_{k_1,\ldots,k_\ell=0}^{\infty} \prod_{j=1}^{\ell} (2k_j+2)^{-2p} \left| \sum_{|\nu|=\ell} \frac{1}{\nu!} \langle F_\nu, e_{k_1} \otimes \cdots \otimes e_{k_\ell}\rangle \right|^2, \end{aligned}$$

where $\nu = (\nu_1, \ldots, \nu_n) \in \mathbf{N}\cup\{0\}$, $|\nu| = \nu_1 + \cdots + \nu_n$, $\nu! = \nu_1! \ldots \nu_n!$ and $F_\nu = \int_{\mathbf{R}^n} f(\mathbf{u}) \hat{\otimes}_{j=1}^{n} \delta_{u_j}^{\hat{\otimes}\nu_j} d\mathbf{u}$. Since there exists $q \geq 0$ such that

$$\begin{aligned} &\sum_{k_1,\ldots,k_\ell=0}^{\infty} \prod_{j=1}^{\ell} (2k_j+2)^{-2p} \left| \sum_{|\nu|=\ell} \frac{1}{\nu!} \langle F_\nu, e_{k_1} \otimes \cdots \otimes e_{k_\ell}\rangle \right|^2 \\ &\leq |f_n|_q^2 n^{2\ell} \left(\sum_{|\nu|=\ell} \frac{1}{\nu!} \right)^2 \left(\sum_{k=0}^{\infty} (2k+2)^{-2p} |e_k|_{-q}^2 \right)^{\ell}, \end{aligned}$$

we get that

$$\begin{aligned} |||K[\varphi]|||_{-p,\ell}^2 &\leq \sum_{n=0}^{\infty} \left(1+\frac{n}{|T|}\right)^{\ell} e^{n^2 \sum_{k=0}^{\infty} (2k+2)^{-2(p+q)}} |f_n|_q^2 \\ &\leq \sum_{n=0}^{\infty} \left(1+\frac{n}{|T|}\right)^{\ell} e^{n^2/2} |f_n|_q^2. \end{aligned}$$

This is nothing but the inequality:

$$|||K[\varphi]|||_{-p,\ell} \leq \|\varphi\|_{\overline{[E]_{q,\ell}}}.$$

Thus the proof is completed. □

The operator K implies a relationship between $\overline{\Delta_L}$ and the number operator $\mathcal{N}$ on $(E)^*$ given by

$$\mathcal{N}\Phi = \sum_{n=0}^{\infty} n\mathbf{I}_n(f_n) \quad \text{for} \quad \Phi = \textstyle\sum_{n=0}^{\infty} \mathbf{I}_n(f_n) \in (E)^*.$$

The operator K implies also a relationship between the semigroup $\{G[X_t^1]; t \geq 0\}$ and the E^*-valued Ornstein-Uhlenbeck process:

$$U_t^x = e^{-t}x + \sqrt{2}\int_0^t e^{-(t-s)}d\mathbf{B}(s), \quad t \geq 0,$$

where $\{\mathbf{B}(t); t \geq 0\}$ is a standard E^*-valued Wiener process starting at 0. Since $\overline{[E]_{\infty,\infty}}$ is in (E), we can apply the same proofs of Proposition 5 and Theorem 6 in [27] to get the following results.

Proposition 9. *For any* $\varphi \in \overline{[E]_{\infty,\infty}}$ *we have*

$$\overline{\Delta_L}K[\varphi] = -\frac{1}{|T|}K[\mathcal{N}[\varphi]].$$

Theorem 10. *For any* $\varphi \in \overline{[E]_{\infty,\infty}}$ *we have*

$$G[X_t^1]K[\varphi](x) = K[E[\varphi(U_{t/|T|}^x)]].$$

Acknowledgements

The author wishes to express his deep gratitude to Professors H. Heyer (Universität Tübingen), T. Hirai (Kyoto University) and N. Obata (Nagoya University) for their hard work in organizing this highly stimulating symposium " Infinite Dimensional Harmonic Analysis ". This work was partially supported by the Research Project " Quantum Information Theoretical Approach to Life Science " for the Academic Frontier in Science promoted by the Ministry of Education in Japan and JMESSC Grant-in-Aid for Scientific Research (C)(2) 11640139. The author is grateful for their supports.

References

[1] Accardi, L., Gibilisco, P. and Volovich, I.V.: The Lévy Laplacian and the Yang-Mills equations, Rendiconti dell'Accademia dei Lincei, (1993).

[2] Accardi,L., Smolyanov, O. G.: Trace formulae for Levy-Gaussian measures and their application, *Proc. The IIAS Workshop "Mathematical Approach to Fluctuations*, **Vol. II** World Scientific (1995) 31 -47.

[3] Chung, D. M., Ji, U. C. and Saitô, K.: Cauchy problems associated with the Lévy Laplacian in white noise analysis, *Infinite Dimensional Analysis, Quantum Probability and Related Topics* Vol.**2**, No.**1** (1999), 131-153.

[4] Hida, T.: "Analysis of Brownian Functionals", Carleton Math. Lecture Notes, No.13, Carleton University, Ottawa, 1975.

[5] Hida, T.: A role of the Lévy Laplacian in the causal calculus of generalized white noise functionals, " Stochastic Processes A Festschrift in Honour of G. Kallianpur " (S. Cambanis et al. Eds.) Springer-Verlag, 1992.

[6] Hida, T., Kuo, H. - H. and Obata, N.: Transformations for white noise functionals, *J. Funct. Anal.* (1990)

[7] Hida, T., Kuo, H. - H., Potthoff, J. and Streit, L.: "White Noise: An Infinite Dimensional Calculus", Kluwer Academic, 1993.

[8] Hida, T. and Saitô, K.: White noise analysis and the Lévy Laplacian, "Stochastic Processes in Physics and Engineering" (S. Albeverio et al. Eds.), 177-184, 1988.

[9] Hida, T., Obata, N. and Saitô, K.: Infinite dimensional rotations and Laplacian in terms of white noise calculus, *Nagoya Math. J.* **128** (1992), 65-93.

[10] Itô, K.: Stochastic analysis in infinite dimensions, *in* " Proc. International conference on stochastic analysis ", Evanston, Academic Press, 187-197, 1978.

[11] Kubo, I.: A direct setting of white noise calculus, in:*Stochastic analysis on infinite dimensional spaces, Pitman Research Notes in Mathematics Series,* **310** (1994) 152-166.

[12] Kubo, I. and Takenaka, S.: Calculus on Gaussian white noise I, II, III and IV, *Proc. Japan Acad.* **56A** (1980) 376-380; **56A** (1980) 411-416; **57A** (1981) 433-436; **58A** (1982) 186-189.

[13] Kuo, H. - H.: On Laplacian operators of generalized Brownian functionals, *in* " Lecture Notes in Math." **1203**, Springer-Verlag, 119-128, 1986.

[14] Kuo, H. - H.: Lectures on white noise calculus, Soochow J. (1992), 229-300.

[15] Kuo, H. - H.: White noise distribution theory, CRC Press (1996).

[16] Kuo, H. - H., Obata, N. and Saitô, K.: Lévy Laplacian of generalized functions on a nuclear space, *J. Funct. Anal.* **94** (1990), 74-92.

[17] Lévy, P.: "Lecons d'analyse fonctionnelle", Gauthier-Villars, Paris 1922.

[18] Obata, N.: A characterization of the Lévy Laplacian in terms of infinite dimensional rotation groups, *Nagoya Math. J.* **118** (1990), 111-132.

[19] Obata, N.: "White Noise Calculus and Fock Space," Lecture Notes in Mathematics 1577, Springer-Verlag, 1994.

[20] Potthoff, J. and Streit, L.: A characterization of Hida distributions, *J. Funct. Anal.* **101** (1991), 212-229.

[21] Saitô, K.: Itô's formula and Lévy's Laplacian I and II, *Nagoya Math. J.* **108** (1987), 67-76, **123** (1991), 153-169.

[22] Saitô, K.: A group generated by the Lévy Laplacian and the Fourier-Mehler transform, *Stochastic analysis on infinite dimensional spaces, Pitman Research Notes in Mathematics Series*, **310** (1994) 274-288.

[23] Saitô, K.: A (C_0)-group generated by the Lévy Laplacian, *Journal of Stochastic Analysis and Applications* **16, No. 3** (1998) 567-584.

[24] Saitô, K.: A (C_0)-group generated by the Lévy Laplacian II, *Infinite Dimensional Analysis, Quantum Probability and Related Topics* **Vol. 1, No. 3** (1998) 425-437.

[25] Saitô, K.: A stochastic process generated by the Lévy Laplacian, Volterra International School "White Noise Approach to Classical and Quantum Stochastic Calculi and Quantum Probability ", Trento, Italy, July 19-23, 1999.

[26] Saitô, K.: The Lévy Laplacian and stable processes, to appear in *Proceedings of the Les Treilles International Meeting* (1999).

[27] Saitô, K., Tsoi, A.H.: The Lévy Laplacian as a self-adjoint operator, *Proceedings of the First International Conference on Quantum Information*, World Scientific (1999) 159-171.

[28] Saitô, K., Tsoi, A.H.: The Lévy Laplacian acting on Poisson Noise Functionals, to appear in *Infinite Dimensional Analysis, Quantum Probability and Related Topics* **Vol. 2** (1999) 503-510.

[29] Saitô, K., Tsoi, A.H.: Stochastic processes generated by functions of the Lévy Laplacian, to appear in *Proceedings of the Second International Conference on Quantum Information*, World Scientific (2000).

[30] Sato, K.-I.: "Lévy Processes and Infinitely Divisible Distributions", Cambridge, 1999.

[31] Yosida, K.: "Functional Analysis 3rd Edition", Springer-Verlag, 1971.

UNITARY REPRESENTATIONS AND DIFFERENTIAL REPRESENTATIONS OF THE GROUP OF DIFFEOMORPHISMS AND ITS APPLICATIONS

BY
HIROAKI SHIMOMURA

DEPARTMENT OF MATHEMATICS FUKUI UNIVERSITY FUKUI 910-8507 JAPAN

1. INTRODUCTION

Let M be a d-dimensional paracompact C^∞-manifold and Diff(M) be the group of all C^∞-diffeomorphisms on M. Among the subgroups of Diff(M), we take here the group $\mathrm{Diff}_0(M)$ which consists of all $g \in \mathrm{Diff}(M)$ with compact supports, that is the set $\{P \in M|\ \ g(P) \neq P\}$ is relatively compact. Up to the present time, unitary representations $(U, \mathcal{H})$ of $\mathrm{Diff}_0(M)$ or of its subgroups ($\mathcal{H}$ is the representation Hilbert space of U) are constructed and considered by many authors. A purpose of this report is a trial to construct some differential method to analyze these representations $(U, \mathcal{H})$ of $\mathrm{Diff}_0(M)$ or of its subgroups. Roughly speaking, we wish to consider a differential representation of a given one.

So the first step we should do is to define a suitable Lie algebra $\mathcal{G}_0$ of $\mathrm{Diff}_0(M)$, regarding it as an infinite dimensional Lie group. For the case of compact manifold, it is well known for a pretty long time ago that $\mathrm{Diff}(M) = \mathrm{Diff}_0(M)$ is an infinite dimensional Lie group whose modeled space is a nuclear Fréchet space called strong inductive limit of Hilbert spaces by a few authors, especially by H.Omori.(cf.[12]) So after them, we are naturally derived that we should take a set $\Gamma_0(M)$ of all C^∞-vector fields X with compact supports as the Lie algebra $\mathcal{G}_0$, and it is appropriate to take a map Exp(X) as the exponential map from $\Gamma_0(M)$ to $\mathrm{Diff}_0(M)$, where $\{\mathrm{Exp}(tX)\}_{t\in\mathbf{R}}$ is an integral curve along a vector field $X \in \Gamma_0(M)$.

Thus formally we have self adjoint operators $dU(X)$ on $\mathcal{H}$ by Stone theorem,

$$U(\mathrm{Exp}(tX)) = \exp(\sqrt{-1}t\,dU(X)) \qquad \text{for all} \quad t \in \mathbf{R},$$

and simaltaneouly there arise many problems for such $dU(X)$ and for Exp(X). Among them the following questions are fundamental.

(1) Is a common domain of $\{dU(X)\}_{X\in\Gamma_0(M)}$ rich such one like Gårding space ?
(2) Does $\sqrt{-1}dU$ become a linear representation under suitable restriction of the domain of each $dU(X)$?
(3) Is a subgroup generated by Exp(X), $X \in \Gamma_0(M)$ dense in $\mathrm{Diff}_0(M)$?

It is easily expected that the linearity of $\sqrt{-1}dU$ mostly depends on a formula which is similar with one derived from usual Campbell-Hausdorff formula, listed as the following theorem and actually it was made sure in [19]. (2) is affirmative.

Theorem 1.1. *Let $X, Y \in \Gamma_0(M)$. Then as n tends to $+\infty$,*

(1) $\{\mathrm{Exp}(\frac{tX}{n}) \circ \mathrm{Exp}(\frac{tY}{n})\}^n$ *converges to* $\mathrm{Exp}(t(X+Y))$, *and*

(2) $\{\mathrm{Exp}(-\frac{tX}{\sqrt{n}}) \circ \mathrm{Exp}(-\frac{tY}{\sqrt{n}}) \circ \mathrm{Exp}(\frac{tX}{\sqrt{n}}) \circ \mathrm{Exp}(\frac{tY}{\sqrt{n}})\}^n$ *converges to* $\mathrm{Exp}(-t^2[X,Y])$

in τ_K uniformly on every compact interval of t, respectively, where K is any compact set containig supp$X$ *and* supp$Y$, *and τ_K is a toplogy of uniform convergence on K together with its every derivative.*

Proof. It is carried out by using C^1-hair theory on regular Fréchet group. For details see [13] and [19].

Now the theory of product integral works so effectively on (3). It turns that the above subgroup is dense in the connected component $\mathrm{Diff}_0^*(M)$ of id, where id is the identity map and the topology τ on $\mathrm{Diff}_0(M)$ is the inductive limit topology of $\{\mathrm{Diff}(K), \tau_K\}_{K:\mathrm{cpt}}$, where $\mathrm{Diff}(K) := \{g \in \mathrm{Diff}_0(M) |\ \ \mathrm{supp} g \subseteq K\}$. It is noteworthy that τ never gives a group topology, unless M is compact (cf. [21], [22]), so we must take care of topological group operations on $\mathrm{Diff}_0(M)$. Nevertheless $\mathrm{Diff}_0^*(M)$ is normal and it is also arcwise connected. In other words, any element in $\mathrm{Diff}_0^*(M)$ is homotopic to the identity as a map, and vice versa.

Theorem 1.2. *A subgroup generated by* $\mathrm{Exp}(X), X \in \Gamma_0(M)$ *is dense in the arcwise connected subgroup* $\mathrm{Diff}_0^*(M)$.

Proof is omitted. (cf.[19])

Now as a direct cosequence of (2) and (3), for example, we have that there is no continuous finite dimensional representations of $\mathrm{Diff}_0^*(M)$ except for a trivial one.

However for almost all parts concerning the questions (2) and (3), I have already reported at several places (cf. [19] and [20]). What I wish to discuss in this paper are problems for the first question. Thus in what follows I will write this report fully placing the focus on the matters for the first question. The last section is briefly devoted to an application of these reults to 1-cocycles.

2. C^∞-VECTORS AND QUASI-INVARIANT MEASURES ON THE GROUP OF DIFFEOMORPHISMS

Now to the first question the following is a partial answer which is a main theorem of this issue.

Main theorem. *Assume that*

(1) *M is a compact Riemannian manifold and*

(2) *$(U, \mathcal{H})$, which is a unitary representation of* $\mathrm{Diff}_0^*(M)$ *at first, has a continuous extension to a larger group* $\mathrm{Diff}^{*K}(M)$, *which consists of all C^K-diffeomorphisms g being homotopic to* id. *Then a set of C^∞-vectors is dense in $\mathcal{H}$.*

Let us show first an idea of the proof and next follow the proof itself. The idea comes from the usual locally compact Lie group theory.

Idea of the proof. For any $h \in \mathcal{H}$, put

$$w_h := \int_{\xi(V)} Q(f)U(f)h\, \mu(df),$$

where $\xi(V)$ is a neighbourhood of id in $\mathrm{Diff}^{*k+\gamma}(M)$ $(0 < \gamma < 1)$ (later, this group will be explained exactly), Q is a non negative function such that

$$\mathrm{supp}Q \subset \xi(V) \quad \text{and} \quad \int_{\xi(V)} Q(f)\mu(df) = 1,$$

and finally μ is a $\mathrm{Diff}^{*k+m}(M)$-quasi-invariant measure on $\mathrm{Diff}^{*k+\gamma}(M)$ which was first considered by Shavgulidze. Of course m must be taken so largely. In the papers [14], [15] and [16], Shavgulidze constructed such a measure. His idea is nice, but there needs some corrections to his proofs. So a definite proof of the existence of such a measure is desired. Now I'll justify it by the following successive 8-steps.

2.1. Construction of quasi-invariant measures on the group of diffeomorphisms.

1-step Let $U \subseteq \mathbf{R}^d$ be an open set and f be a C^k-diffeomorphism defined on U. Take $m, \ell, k \in \mathrm{N}$ such that $3m \leq \ell \leq k$. Shavgulidze defined a map $A_{U,\ell,m}(f)$ for each $h_1, \cdots, h_\ell \in \mathbf{R}^d$ as follows.

$$A_{U.\ell,m}(f)(x)(h_1,\cdots,h_\ell) := \frac{1}{\ell!}\sum_{\sigma\in\mathfrak{S}_\ell}\sum_{i=0}^{m}\alpha_i\partial_{h_{\sigma(1)}}\cdots\partial_{h_{\sigma(i)}}df_x^{-1}(\partial_{h_{\sigma(i+1)}}\cdots\partial_{h_{\sigma(\ell)}}f(x)),$$

where α_i $(i = 0, \cdots, m)$ is a real number which satisfies the following equations,

$$\sum_{i=0}^{m}\alpha_i = 1, \qquad \sum_{i=0}^{m} {}_{\ell-i}C_{p-j}\ {}_iC_j\alpha_i = 0 \quad (0 \leq {}^\forall j < p, \ \ 1 \leq {}^\forall p \leq m).$$

Of course ${}_iC_j$ is the combinatorial number, if $i \geq j$ and it is equal to 0, if $j > i$. Further ∂_h is a directional derivative along h and df_x is a differential of the map f at x. Needless to say, here all tangent spaces are identified with each other. Note that $A_{U.\ell,m}(f)(x)(h_1, \cdots, h_\ell)$ defines a $C^{k-\ell}$-vector field on U for each fixed $h_1, \cdots, h_\ell$.

Theorem 2.1. *If ϕ is a C^{k+m}-diffeomorphism on $f(U)$,*

$$A_{U.\ell,m}(\phi \circ f)(x)(h_1,\cdots,h_\ell) - A_{U.\ell,m}(f)(x)(h_1,\cdots,h_\ell)$$

is a vector field of $C^{k+m-\ell}$-class.

Proof is derived from the usual chain rure and Leibniz formula. (cf. [16])

2-step Let us consider a group $\mathrm{Diff}^{k+\gamma}(M)$ $(k \in \mathrm{N},\ \ 0 < \gamma < 1)$. The definition is as follows : $g \in \mathrm{Diff}^{k+\gamma}(M)$ if and only if $g \in \mathrm{Diff}^k(M)$ and every derivative of order k is Lipshitz continuous of order γ. Making a parallel definition of the vector field, we obtain a Banach space space $\Gamma^{k+\gamma}(M)$ with the natural norm and a Banach manifold $\mathrm{Diff}^{k+\gamma}(M)$ via a coordinate map ξ on an open neighbourhood U of $0 \in \Gamma^{k+\gamma}(M)$ given by Omori,

$$\xi(u)(x) := \exp_x u(x) \qquad (u \in \Gamma^{k+\gamma}(M)),$$

where $\exp_x u(x)$ is a terminal point of a unit geodisic starting at x along the direction $u(x)$.

3-step In what follows we always assume that

$$3m \leq 2\ell \leq k.$$

According to Shavgulidze, we extend the previous map $A_{U,2\ell,m}$ to a global one as $A_{2\ell,m}$ from $\mathrm{Diff}^{k+\gamma}(M)$ to $\Gamma^{k+\gamma-2\ell}(M)$ such that

$$A_{2\ell,m}(f)(x) = \sum_{i_1=1}^{d} \cdots \sum_{i_\ell=1}^{d} \sum_{i,j=1}^{n} \rho_j(f(x))\rho_i(x)(d\psi_i)_{\psi_i^{-1}(x)} A_{U_i\cap\psi_i^{-1}(f^{-1}(V_j)),2\ell,m}(\psi_j^{-1}\circ f\circ\psi_i)$$
$$(\psi_i^{-1}(x))(h_{i.i_1}, h_{i.i_1}, h_{i.i_2}, h_{i.i_2}, \cdots, h_{i.i_\ell}, h_{i.i_\ell}),$$

where $\{(V_i,\psi_i)\}_{i=1}^{n}$ is an atras of M, $\{\rho_i\}_{i=1}^{n}$ is a partition of unity such that $\mathrm{supp}\rho_i \subset V_i$, $U_i := \psi_i^{-1}(V_i)$ and finally $(d\psi_i)_{\psi_i^{-1}(x)}(h_{i,1}), \cdots, (d\psi_i)_{\psi_i^{-1}(x)}(h_{i,d})$ is a linear base in a tangent space $T_x(M)$.

Theorem 2.2. (1) *$A_{2\ell,m}$ is a C^∞-map from* $\mathrm{Diff}^{k+\gamma}(M)$ *to* $\Gamma^{k+\gamma-2\ell}(M)$.
(2) $A_{2\ell,m}(\phi\circ f) - A_{2\ell,m}(f) \in \Gamma^{k+m-2\ell}(M)$, *whenever* $\phi \in \mathrm{Diff}^{k+m}(M)$.
(3) *Put* $L := dA_{2\ell,m}|_{f=\mathrm{id}}$. *Then L is a differential operator of elliptic type with C^∞-coefficient on the vector field.*

Proof. It is not hard to see the properties (1) and (2). Let us check the third property. Set

$$L(u) := dA_{2\ell,m}|_{f=\mathrm{id}}(u) \qquad (u \in \Gamma^{k+\gamma}(M)).$$

In a little while let us use notations as below for simplicity.

$$y := \psi_i^{-1}(x), f_t(y) := \psi_j^{-1}\circ\xi(tu)\circ\psi_i(y), \quad U := U_i\cap\psi_i^{-1}(V_j), \quad \text{and} \quad k_s := h_{i,i_s}(1 \leq s \leq \ell).$$

Then it is easy to see that $\left.\dfrac{d}{dt}\right|_{t=0} A_{U,2\ell,m}(f_t)(y)(k_1,k_1,\ldots,k_\ell,k_\ell)$ is a differential operator with respect to u with C^∞-coefficients and the term of order 2ℓ, which is the highest part, is given by

$$\frac{1}{(2\ell)!}\sum_{\sigma\in\mathfrak{S}_{2\ell}}\sum_{s=0}^{m}\alpha_s(df_0^{-1})_y(\partial_{k_{\sigma(1)}}\cdots\partial_{k_{\sigma(2\ell)}}d\psi_j^{-1}(u(x))) = d\psi_i^{-1}\circ d\psi_j(\partial_{k_1}\partial_{k_1}\cdots\partial_{k_\ell}\partial_{k_\ell}d\psi_j^{-1}(u(x))$$

Hence

$$dA_{2\ell,m}(u)(x) = \sum_{i_1=1}^{d}\cdots\sum_{i_\ell=1}^{d}\sum_{i,j=1}^{n}\rho_j(x)\rho_i(x)(d\psi_j)_{\psi_j^{-1}(x)}\partial_{k_1}\partial_{k_1}\cdots\partial_{k_\ell}\partial_{k_\ell}d\psi_j^{-1}(u(x))$$
$$+ \text{ terms of order less than } 2\ell.$$

Now take any $u \in \Gamma^{k+\gamma}(M)$ and $\varphi \in C^\infty(M)$ with properties, $u(x) \neq 0, \varphi(x) = 0$ and $d\varphi(x) \neq 0$. Then it follows from an equality,

$$\partial_{k_1}\partial_{k_1}\cdots\partial_{k_\ell}\partial_{k_\ell}d\psi_j^{-1}((\varphi^{2\ell}u)(x)) = (2\ell)!\prod_{s=1}^{\ell}\{d\varphi_x\circ d\psi_i(k_s)\}^2(d\psi_j^{-1})_x(u(x)),$$

that we have

$$L(\varphi^{2\ell}u)(x) = (2\ell)!\sum_{i=1}^{n}\sum_{i_1=1}^{d}\cdots\sum_{i_\ell=1}^{d}\rho_i(x)\prod_{s=1}^{\ell}\{d\varphi_x\circ d\psi_i(h_{i,i_s})\}^2u(x).$$

The linear independence of $d\psi_i(h_{i,j})$ $(j = 1, \cdots d)$ and the choice of φ lead to that $d\varphi_x \circ d\psi_i(h_{i,i_0}) \neq 0$ for some i_0, and so a term corresponding to $i_1 = i_2 = \cdots = i_\ell = i_0$ is positive. Thus, we get $L(\varphi^{2\ell}u)(x) \neq 0$. □

4-step Generalized Hodge theorem. Let $E_p^{k+\gamma}(M)$ be a collection of all p-forms of class C^k together with all kth derivatives having Lipshitz continuity of order γ, and let L be a differential operator of elliptic type of order ℓ with C^∞-coefficients on the space of p-forms.

Theorem 2.3.

$$E_p^{k+\gamma}(M) = L(E_p^{k+\ell+\gamma}(M)) \oplus \ker L^*$$

$$E_p^{k+\gamma}(M) = L^*(E_p^{k+\ell+\gamma}(M)) \oplus \ker L,$$

where $\oplus$ means an orthgonal decomposition defined by the L^2-norm, in the orientable case, with respect to the volume form on the compact Riemannian manifold M. While in the non orientable case, it is defined by an inner product on $E_p^{k+\gamma}(M)$ defined by

$$< \omega_1, \omega_2 >_M := < \delta\pi\omega_1, \delta\pi\omega_2 >_{\tilde{M}},$$

where $(\tilde{M}, \pi)$ is the the double covering of M, π is a natural projection, and $< \cdot, \cdot >_{\tilde{M}}$ is an inner product which defines the L^2-structure on $\tilde{M}$. Further L^* is a formal adjoint operator of L with respect to these inner products.

Proof. It is derived from theorem 4.1 in p84 in [17] concerning with interior Shauder estimates.

Note that $\ker L$ and $\ker L^*$ have finite dimensions, respectively, so $L(E^{k+\ell+\gamma})$ is also a Banach space with the induced normed topology.

Remark 2.1. *According to an example* 4.1 *in p* 85 *in* [17], *the above theorem is no longer true, even for Laplace-Beltrami operator for the case $\gamma = 0$. This is the reason why the γ-factor is added to the regularity of diffeomorphisms.*

In what follows, I use the above result for the 1-form and identify $E_1^{k+\gamma}(M)$ with $\Gamma^{k+\gamma}(M)$ by the Riemannian metric on M.

5-step This step is devoted to a definition of a fundamental map A. So let

$$\pi_1^{k+\gamma-2\ell} : \Gamma^{k+\gamma-2\ell}(M) \longmapsto L(\Gamma^{k+\gamma}(M)), \qquad \pi_2^{k+\gamma} : \Gamma^{k+\gamma}(M) \longmapsto \ker L$$

be natural projections, respectively, and put

$$Z^{k+\gamma} := L(\Gamma^{k+\gamma}(M)) \times \ker L.$$

Now define $A : \xi(U) \longmapsto Z^{k+\gamma}$ by

$$A(f) := (\pi_1^{k+\gamma-2\ell}(A_{2\ell,m}(f)), \pi_2^{k+\gamma}(\xi^{-1}(f))).$$

Theorem 2.4. *There exists a neighbourhood $U_1 (\subseteq U)$ of 0 such that A is a C^∞-diffeomorphism from $\xi(U_1)$ to $Z^{k+\gamma}$.*

Proof. It is straightforward to check that $dA|_{f=\mathrm{id}}(u) = (Lu, \pi_2^{k+\gamma}(u))$, and that it is a continuous bijection from $\Gamma^{k+\gamma}(M)$ to $Z^{k+\gamma}$. So the inverse function theorem on Banach manifold assures its validity. □

Of course we may assume that the relations

$$\xi(U_1)\xi(U_1) \subseteq \xi(U), \qquad \xi(U_1)^{-1} = \xi(U_1)$$

holds good, if necessary, taking a sufficiently small neighbourhood of 0.

6-step Here we make preparations from a category of Sobolev spaces. Put $d* = \left[\frac{d}{2}\right] + 1$ and $m = 3d^* + 2$. So the relation of m, ℓ and k now becomes,

$$9d^* + 6 = 3m \leq 2\ell \leq k.$$

Consider a Sobolev space $H^s(M)$ of all vector fields with square summable derivatives of order less than or equal to s equipped with the natural Hilbertian norm. Then we have

$$\Gamma^{k+3d^*+1-2\ell}(M) \subset H^{k+3d^*+1-2\ell}(M) \subset H^{k+d^*+1-2\ell}(M) \subset \Gamma^{k+1-2\ell}(M) \subset \Gamma^{k+\gamma-2\ell}(M),$$

where the second inclusion map is nuclear and the third one is actually imbedding due to the choice of d^*. Next let us put

$$E^s(M) := Cl(L(\Gamma^{s+2\ell}(M)) \quad \text{in} \quad H^s(M).$$

Then

$$L(\Gamma^{k+3d^*+1}(M)) \subset E^{k+3d^*+1-2\ell}(M) \subset E^{k+d^*+1-2\ell}(M) \subset L(\Gamma^{k+\gamma}(M)),$$

where the third set is actually a subset of the last one. For, given any $f \in E^{k+d^*+1-2\ell}(M)$, choose $\{f_n\}_n \subset L((\Gamma^{k+d^*+1}(M))$ such that $f_n \longrightarrow f \ (n \longrightarrow \infty)$ in $H^{k+d^*+1-2\ell}(M)$. Since $f_n \in (\ker L^*)^\perp$ for all n, the same holds for f, which together with Theorem 2.3 assures $f \in L(\Gamma^{k+\gamma}(M))$.

For the topologies on these spaces, we give the natural Banach topologies on the series of L-image of Γ-spaces and give the Hilbertian topologies on the series of E-spaces. Then the all injections are continuous. Now put,

$$X \equiv X^{k,\ell} := E^{k+d^*+1-2\ell}(M) \times \ker L,$$

which is a subspace of $Z^{k+\gamma}$, and consider a transformation

$$A_\phi := A \circ L_\phi \circ A^{-1} \qquad \text{on} \quad X^{k,\ell}$$

for all $\phi \in \xi(U_1) \cap \mathrm{Diff}^{k+m}(M)$. For any $(\eta, r) \in X^{k,\ell} \cap A(\xi(U_1))$, let us write down A_ϕ explicitly using its components,

$$A_\phi(\eta, r) := (\eta + F_\phi^1(\eta, r), F_\phi^2(\eta, r)).$$

Theorem 2.5. (1) *For* $(\eta, r) \in X^{k,\ell} \cap A(\xi(U_1))$, $F_\phi^1(\eta, r)$ *belongs to* $L(\Gamma^{k+3d^*+1}(M))$ *and for the map* F_ϕ^1, *regarding it as* $L(\Gamma^{k+3d^*+1}(M))$*-valued map from* $X^{k,\ell} \cap A(\xi(U_1))$, *it is continuously differentiable.*
(2) A_ϕ *is a local* C^1*-diffeomorphism on* $X^{k,\ell}$.

Proof. The most part of them are derived from Theorem 2.2. □

7-step Now we shall introduce a basic measure for our arguments. As we have seen, $H_1 := E^{k+3d^*+1-2\ell}$ is nuclearly imbedded into $H := E^{k+d^*+1-2\ell}$. Let ι be the imbedding map and decompose it into T and U, $\iota = T \circ U$, where $U : H_1 \longmapsto H$ is an onto isometric operator and T is a strictly positive-definite nuclear operator on H. It is well known that there exists a Gaussian measure g_T with mean 0 and variance operator T on H,

$$\hat{g}_T(x) \left(:= \int_h \exp(\sqrt{-1} < x, y >_H) g_T(dy) \right) = \exp(-\frac{1}{2} < Tx, x >_H).$$

The following is a transformation formula for variable change.

Theorem 2.6. *Let $X := H \times \mathbf{R}^s$ ($\ni (\eta, r)$), where H is a real separable Hilbert space and $s \in \mathbf{N}$. Suppose that*

$$F(\eta, r) = (\eta + Tf_1(\eta, r), f_2(\eta, r))$$

is a C^1-diffeomorphism from an open set U in X to $F(U)$, where f_1 is a C^1-map from X to H and T is a strictly positive-definite nuclear operator on H. Then for any Borel set $B \subseteq U$,

$$g_T \otimes \lambda(F(B)) = \int_B \exp(- < \eta, f_1(\eta, r) >_H - \frac{1}{2} < Tf_1(\eta, r), f_1(\eta, r) >_H) \cdot \\ |\det(dF_{(\eta,r)})| g_T \otimes \lambda(d\eta, dr),$$

where λ is Lebesgue measure on $\mathbf{R}^s$ and

$\det(dF_{(\eta,r)}) := \lim_{n\to\infty} \det(P_n dF_{(\eta,r)}|X_n)$ *(the limit surely exists at every point in U), P_n is a natural projection from X to $X_n := Sp\{\eta_k \times \mathbf{R}^s \ \ (k = 1, \cdots, n)\}$ and finally η_k is an eigen-vector of T*

$$T\eta = \sum_{k=1}^{\infty} \tau_k < \eta, \eta_k >_H \eta_k, \qquad (\tau_1 \geq \tau_2 \geq \cdots \geq \tau_n \geq \cdots > 0).$$

Of course there are more fundamental formulas for variable change, namely without finite dimensional component λ. They are also actively now studied by many mathematicians. A particular one of these theorems is due to [16]. The above theorem is a simple version of this result.

Now let us return to our case. That is,

$$H = E^{k+d^*+1-2\ell}(M), \quad \mathbf{R}^s = \ker L \quad \text{and} \quad F = A_\phi.$$

Then settling the above arguments, we find that

Theorem 2.7. *For any Borel set $B \subseteq X \cap A(\xi(U_1))$,*

$$g_T \otimes \lambda(A_\phi(B)) = \int_B \exp(- < \eta, UF_\phi^1(\eta, r) >_H - \frac{1}{2} < F_\phi^1(\eta, r), UF_\phi^1(\eta, r) >_H) \cdot \\ |\det((dA_\phi)_{(\eta,r)})| g_T \otimes \lambda(d\eta, dr). \tag{2.1}$$

8-step Now we are in a position to define a desired mesure. Define

$$\mu(E) := g_T \otimes \lambda(A(E) \cap X) \qquad (E \subseteq \xi(U_1)).$$

Theorem 2.8. *There exists a neighbourhood $U_2(\subseteq U_1)$ of 0 in $\Gamma^{k+\gamma}(M)$ such that for any Borel set $E \subseteq \xi(U_2)$,*

$$\mu(E \ominus L_\phi(E)) \longrightarrow 0, \quad \textit{whenever} \quad \phi \longrightarrow \mathrm{id} \quad \textit{in} \quad \mathrm{Diff}^{k+3d^*+2}(M).$$

Proof. It is done by long but elementary calculations, using standard techniques in measure theory and subgaussian poperty described , for example, in p79 in [5].

The detailed proof is as follows. First we state the following lemma which is an immediate consequence of Theorm 2.6 without finite dimensional component λ.

Lemma 2.1. *Let H be a real separable Hilbert space, B be a bounded operator on H, and T be a strictly positive-definite nuclear operator on H, which has a form,*

$$Tx = \sum_{n=1}^{\infty} \tau_n < x, h_n > h_n, \quad \tau_1 \geq \cdots \geq \tau_n \geq \cdots > 0, \quad \textit{and} \quad \sum_{n=1}^{\infty} \tau_n < \infty.$$

Further let us assume that $\mathrm{Id} + TB$ *is invertible. Then a limit*

$$\det(\mathrm{Id} + TB) := \lim_{n\to\infty} \det(\mathrm{Id} + P_n TB|H_n)$$

exists, where $H_n := \mathrm{Sp}\{h_1, \cdots, h_n\}$ and $P_n : H \longmapsto H_n$ is the natural projection. Moreover the following formula holds good for Gaussian measure g_T on H and for any but fixed continuous non negative bounded function $s \not\equiv 0$ with bounded support.

$$(2.2) \quad \int_H s((\mathrm{Id} + TB)^{-1}x) g_T(dx) = |\det(\mathrm{Id} + TB)| \cdot \int_H s(x) \exp(- < Bx, x >_H - \frac{1}{2} < TBx, Bx >_H) g_T(dx).$$

Lemma 2.2. *Under the same notation as in* Lemma 2.1,

(1) $\det(\mathrm{Id} + TB)$ *is bounded on a domain $\|B\| \leq r$ for any but fixed $r > 0$.*

(2) $|\det(\mathrm{Id} + TB)|$ *is a continuous function of B with respect to the operator norm.*

Proof. They follow easily from (2.2).

Returning to our case, we find that by Lemma 2.1, $\det((dA_\phi)_{(\eta,r)})$ has the following explicit form, using Gaussian measure $g_{\tilde{T}}$ on X, where $\tilde{T}$ is a nuclear operator defined by $\tilde{T}(\eta, r) := (T\eta, r)$, and using a continuous non negative bounded function $s \not\equiv 0$ on X with bounded support.

$$(2.3) \qquad |\det((dA_\phi)_{(\eta,r)})| = I_1 \cdot I_2^{-1},$$

$$I_1 := \int_X s((dA_\phi)^{-1}_{(\eta,r)})(\eta', r')) g_{\tilde{T}}(d\eta', dr'),$$

$$I_2 := \int_X s(\eta', r') \exp(- < U(dF^1_\phi)_{(\eta,r)}(\eta', r'), \eta' >_H - < (dF^2_\phi)_{(\eta,r)}(\eta', r') - r', r' >_{\ker L}) \cdot \exp(-\frac{1}{2} < (dF^1_\phi)_{(\eta,r)}(\eta', r'), U(dF^1_\phi)_{(\eta,r)}(\eta', r') >_H -\frac{1}{2}\|(dF^2_\phi)_{(\eta,r)}(\eta', r') - r'\|^2_{\ker L}) g_{\tilde{T}}(d\eta', dr').$$

Hereafter we will denote the integrand in (2.1) by $\rho_\phi(\eta, r)$. Note that $F^1_\phi(\eta, r)$ is a map of C^1 class from $X \times \mathrm{Diff}^{k+3d^*+2}(M)$ to $L(\Gamma^{k+3d^*+1}(M))$ and that $F^2_\phi(\eta, r)$ is a map of C^{3d^*+1} class on $X \times \mathrm{Diff}^{k+3d^*+2}(M)$. Hence there exists a neighbourhood $\xi(W)$ of id in

$\mathrm{Diff}^{k+3d^*+2}(M)$ $(W \subset U_1)$ and a neighbourhood $\xi(U_1^{(1)})$ of id in $\mathrm{Diff}^{k+\gamma}(M)$ $(U_1^{(1)} \subseteq U_1)$ such that the followings hold good with a positive constant K_1,

$$\|F_\phi^1(\eta, r)\|_{E^{k+3d^*+1-2\ell}} \leq K_1, \quad \|(dF_\phi^1)_{(\eta,r)}\|_{op} \leq K_1 \quad \text{and} \quad \|(dF_\phi^2)_{(\eta,r)}\|_{op} \leq K_1,$$

for all $\phi \in \xi(W)$ and $(\eta, r) \in A(\xi(U_1^{(1)}))$. Thus it follows from Lemma 2.2 and (2.3) that the second term in the integrand in (2.1), that is, $|\det(dA_\phi)_{(\eta,r)}|$ is bounded on $A(\xi(U_1^{(1)})) \times \xi(W)$. Further by the following elementary estimate of the first term,

$$\exp(-<\eta, UF_\phi^1(\eta,r)>_H - \frac{1}{2} < F_\phi^1(\eta,r), UF_\phi^1(\eta,r)>_H) \leq \exp(\frac{1}{2}K_1^2)\exp(K_1\|\eta\|_{E^{k+d^*+1-2\ell}}),$$

we get

$$|\rho_\phi(\eta, r)| \leq {}^\exists M \exp(K_1\|\eta\|_{E^{k+d^*+1-2\ell}})$$

on this region, and the later function is summable with respect to $g_T(d\eta)$. (cf.[5]) As (2) in Lemma 2.2 leads us to

$$\rho_\phi(\eta, r) \longrightarrow 1, \qquad \text{whenever} \quad \phi \longrightarrow \text{id} \quad \text{in} \quad \mathrm{Diff}^{k+3d^*+2}(M),$$

it follows from the bounded convergence theorem that

$$\int_{X\cap A(\xi(U_1^{(1)}))} |\rho_\phi(\eta, r) - 1| g_T(d\eta)\lambda(dr) \longrightarrow 0,$$

whenever $\phi \longrightarrow$ id in $\mathrm{Diff}^{k+3d^*+2}(M)$.

Next we take a sufficiently small neighbourhoods $U_1^{(2)}, U_1^{(3)}$ of 0 in $\Gamma^{k+\gamma}(M)$ such that

$$U_1^{(3)} \subseteq U_1^{(2)} \subseteq U_1^{(1)}, \ \xi(U_1^{(2)})\xi(U_1^{(2)}) \subseteq \xi(U_1^{(1)}), \ \xi(U_1^{(3)})\xi(U_1^{(3)}) \subseteq \xi(U_1^{(2)}), \ \xi(U_1^{(3)})^{-1} = \xi(U_1^{(3)}).$$

Moreover from now on till the end of this proof, let us assume that ϕ belongs to $\mathrm{Diff}^{k+3d^*+2}(M) \cap \xi(U_1^{(3)})$. Then for any Borel set $E \subseteq \xi(U_1^{(3)})$,

$$\begin{aligned} \mu(E \ominus L_\phi(E)) &= g_T \otimes \lambda(A(E \ominus L_\phi(E)) \cap X) \\ &= g_T \otimes \lambda(A(E) \cap X \ominus A_\phi A(E) \cap X) \\ &= g_T \otimes \lambda(A(E) \cap X \ominus A_\phi(A(E) \cap X)). \end{aligned}$$

Given $\epsilon > 0$, take a closed set F and an open set G in X which fulfills,

$$F \subseteq A(E) \cap X \subseteq G \subseteq A(\xi(U_1^{(3)})) \cap X \quad \text{and} \quad g_T \otimes \lambda(G \backslash F) < \epsilon,$$

and take a continuous function σ on X such that

$$0 \leq \sigma \leq 1, \quad \sigma = 1 \ \text{ on } \ F \ \text{ and } \ \sigma = 0 \ \text{ on } \ G^c.$$

Then

(2.4)

$$\mu(E \ominus L_\phi(E)) \leq \int_X |\chi_{A(E)\cap X}(\eta, r) - \sigma(\eta, r)| g_T \otimes \lambda(d\eta, dr) + \int_X |\sigma(\eta, r) - \sigma_\phi(\eta, r)| g_T \otimes \lambda(d\eta, dr) + \int |\sigma_\phi(\eta, r) - \chi_{A_\phi(A(E)\cap X)}(\eta, r)| g_T \otimes \lambda(d\eta, dr),$$

where a function σ_ϕ is defined by

$$\sigma_\phi(\eta, r) = \begin{cases} \sigma(A_{\phi^{-1}}(\eta, r)), & \text{if} \quad (\eta, r) \in A_\phi(A(\xi(U_1^{(2)})) \cap X) \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that

$$|\sigma(\eta,r)-\sigma_\phi(\eta,r)| \le \chi_{A(\xi(U_1^{(2)}))\cap X}(\eta,r)|\sigma(\eta,r)-\sigma(A_{\phi^{-1}}(\eta,r))|,$$

so the second term in the right hand side in (2.4) converges to 0 according to $\phi \longrightarrow \mathrm{id}$ in $\mathrm{Diff}^{k+3d^*+2}(M)$. Further a sum of the remainder terms in that inequality is dominated by

$$\epsilon + \int_{A(\xi(U_1^{(1)}))\cap X} |\rho_\phi(\eta,r)-1| g_T \otimes \lambda(d\eta,dr),$$

by virtue of an obvious inequality,

$$|\chi_{A(E)\cap X}(\eta,r)-\sigma(\eta,r)| \le \chi_G(\eta,r)-\chi_F(\eta,r).$$

Consequently, for any Borel set E in $\xi(U_2)$, where $U_2 := U_1^{(3)}$, we see that $\mu(E \ominus L_\phi(E)) \longrightarrow 0$, whenever $\phi \longrightarrow \mathrm{id}$ in $\mathrm{Diff}^{k+3d^*+2}(M)$. □

Next take a countable dense set $\{\phi_i\}_i$ from $\mathrm{Diff}^{*k+3d^*+2}(M)$ and define

$$\tilde{\mu}(B) := \sum_{i=1}^{\infty} \alpha_i \mu(L_{\phi_i}(B) \cap \xi(U_2)) \qquad (B \subseteq \mathrm{Diff}^{*k+\gamma}(M)),$$

where $\alpha_i > 0 \ (i = 1,2,\cdots)$, and $\sum_{i=1}^{\infty} \alpha_i = 1$.

Theorem 2.9. *$\tilde{\mu}$ is a $\mathrm{Diff}^{*k+m}(M)$-quasi-invariant and continuous measure on $\mathrm{Diff}^{*k+\gamma}(M)$, where $m = 3d^* + 2$, and $3m \le (2\ell) \le k$.*

Proof. It is evident that $\tilde{\mu}(B) = 0$ if and only if $\mu(L_{\phi_i}(B) \cap \xi(U_2)) = 0$ for all i. Now given any $\phi \in \mathrm{Diff}^{*k+m}(M)$, take a sequence ϕ_{i_j} converging to ϕ and put $\phi_{i_j} = \varphi_j \phi$. Then,

$$\begin{aligned}|\mu(L_{\phi_{i_j}}(B)\cap\xi(U_2)) - \mu(L_\phi(B)\cap\xi(U_2))| &\le \mu((L_{\phi_{i_j}}(B)\ominus L_\phi(B))\cap\xi(U_2)) \\ &= \mu((L_{\varphi_j}(L_\phi(B))\ominus L_\phi(B))\cap\xi(U_2)) \\ &\le \mu(L_{\varphi_j}(L_\phi(B)\cap\xi(U_2))\ominus L_\phi(B)\cap\xi(U_2)) \\ &\quad + \mu(L_{\varphi_j}(\xi(U_2))\ominus\xi(U_2)) \longrightarrow 0, \quad (j\longrightarrow\infty),\end{aligned}$$

due to Theorem 2.8. Therefore $\mu(L_\phi(B)\cap\xi(U_2)) = 0$, whenever $\tilde{\mu}(B) = 0$. This shows the quasi-invariance. For the continuity it is enough to show that

$$^\forall B,\ ^\forall i, \quad \mu(L_{\phi_i}(B \ominus L_\phi(B)) \cap \xi(U_2)) \longrightarrow 0,$$

whenever $\phi \longrightarrow \mathrm{id}$ in $\mathrm{Diff}^{k+m}(M)$. Put

$$E := L_{\phi_i}(B) \qquad \text{and} \qquad \psi := \phi_i \phi \phi_i^{-1}.$$

Then,

$$\begin{aligned}\mu(L_{\phi_i}(B\ominus L_\phi(B))\cap\xi(U_2)) &= \mu((E\ominus L_\psi(E))\cap\xi(U_2)) \\ &\le \mu(L_\psi(E\cap\xi(U_2))\ominus E\cap\xi(U_2)) \\ &\quad + \mu(L_\psi(E\cap\xi(U_2))\ominus L_\psi(E)\cap\xi(U_2)) \\ &\le \mu(L_\psi(E\cap\xi(U_2))\ominus E\cap\xi(U_2)) \\ &\quad + \mu(L_\psi(\xi(U_2))\ominus\xi(U_2)) \longrightarrow 0 \quad (\phi\longrightarrow\mathrm{id}).\end{aligned}$$

□

2.2. Existence and denseness of C^∞-vectors. Let $(U,\mathcal{H})$ be a unitary representation of $\mathrm{Diff}^*(M)$ on a compact Riemannian manifold M. Suppose that our unitary representation $(U,\mathcal{H})$ has a continuous extension to a larger group $\mathrm{Diff}^{\bullet K}(M)$. Take k so large that $k \geq K$.

Further take a C^∞-function $\rho \equiv \rho_{a,b}$ $(0 < a < b)$ on $[0,\infty)$ such that

$$0 \leq \rho \leq 1, \quad \rho = 1 \quad \text{on} \quad [0,a], \quad \rho = 0 \quad \text{on} \quad [b,\infty),$$

and define a function $\tilde{Q}$ on $Z^{k+\gamma}$ by

$$\tilde{Q}(\eta,r) := \rho(\|(\eta,r) - A(\mathrm{id})\|_X^2)\chi_X(\eta,r)/C,$$

where C is a normalizing constant such that $\int_X \tilde{Q}(\eta,r)g_T(d\eta)\lambda(dr) = 1$, and χ_X is an indicator function of X, and $\|\cdot\|_X$ is the natural norm. Finally put

$$Q(f) \equiv Q_{a,b}(f) := \tilde{Q}(Af) \qquad\qquad (f \in \xi(U_2)).$$

Then after long calculations we have the following announced result.

Theorem 2.10. *For any $h \in \mathcal{H}$ define*

$$w_h \equiv w_h^{a,b} := \int_{\xi(U_2)} Q_{a,b}(f)U(f)h\,\mu(df).$$

Then $w_h^{a,b}$ is a C^∞-vector and $w_h^{a,b}$ converges to h, whenever a,b tend to 0.

Proof. Needless to say,

$$dU(X)h = \left.\frac{d}{d\tau}\right|_{\tau=0} U(\mathrm{Exp}(tX))h \qquad (X \in \Gamma(M) \text{ and } h \in \mathcal{H}),$$

and h is said to be a C^∞-vector of $(U,\mathcal{H})$, if and only if $dU(X_1)(\cdots(dU(X_n)h))$ exists for every n and $X_1,\cdots,X_n \in \Gamma(M)$. Thus for the proof it is enough to see that for any n and any $s(\leq n)$, $U(\mathrm{Exp}(t_1X_1)\cdots\mathrm{Exp}(t_nX_n))$ is s-times continuously differentiable on a neighbourhood of $t := (t_1,\cdots,t_n) = (0.\cdots,0)$. Hereafter we always assume that $\mathrm{supp}Q_{a,b} \subset \xi(U_2)$. Put

$$\phi_t := \mathrm{Exp}(t_1X_1)\circ\cdots\circ\mathrm{Exp}(t_nX_n), \quad \text{and} \quad \psi_t := \mathrm{Exp}(-t_nX_n)\circ\cdots\circ\mathrm{Exp}(-t_1X_1).$$

Then

$$U(\psi_t)w_h = \int_{A(\xi(U_2))\cap X} Q(A^{-1}(\eta,r))U(\psi_t\circ A^{-1}(\eta,r))h\,g_T\otimes\lambda(d\eta,dr),$$

and for sufficiently small $|t| := |t_1| + \cdots + |t_n|$,

$$U(\psi_t)w_h = \int_{A(\xi(U_3))\cap X} Q(A^{-1}A_{\phi_t}(\eta,r))\rho_{\phi_t}(\eta,r)U(A^{-1}(\eta,r))h\,g_T\otimes\lambda(d\eta,dr), \tag{2.5}$$

where $U_3 := U_1^{(2)}$ which was already given in the proof of Theorem 2.8. Thus for the proof we must check differentials of $Q(A^{-1}A_{\phi_t}(\eta,r))$ and $\rho_{\phi_t}(\eta,r)$ with respect to t.

First note that by the definition of $\tilde{Q}$ and ρ, the integration in (2.5) is actually carried out over a set of (η,r) satisfying $\|A_{\phi_t}(\eta,r) - A(\mathrm{id})\|_X^2 \leq b$. Next, since the map $A_\phi(\eta,r) : \mathrm{Diff}^{k+3d^*+2}(M)\times X \longmapsto X$ is continuous, so for a sufficiently small $|t|$ and for such a b, the above inequality implies that $\|A_{\psi_t}(A_{\phi_t}(\eta,r)) - A(\mathrm{id})\|_X \leq 1$. In other words, an actual integral domain D in (2.5) may be assumed to be bounded.

Now let us consider first the differentials of $\rho_{\phi_t}(\eta,r)$, and so recall the definition of $F^1_{\phi_t}$ and $F^2_{\phi_t}$. Namely,

$$F^1_{\phi_t}(\eta,r) = \pi_1^{k+3d^*+1-2\ell}(A_{2\ell,m}(\phi_t\circ f) - A_{2\ell,m}(f)), \tag{2.6}$$

$$(2.7) \qquad F^2_{\phi_t}(\eta, r) = \pi_2^{k+\gamma}\xi^{-1}(\phi_t \circ f),$$

where $f := A^{-1}(\eta, r)$. Further let us denote a terminal point of unit geodisic starting at x along a direction u by $K(x,u)$ and denote a tangent vector at x of unit geodisic with an initial point x and a terminal point y by $J(x,y)$. Then K and J are C^∞-maps on the tangent bundle on M and on $M \times M$, respectively. Since for the maps $f =: \xi(u)$ and $\phi_t =: \xi(v_{t_1,\cdots,t_n})$ we have,

$$\begin{aligned}
\phi_t \circ f(x) &= K(K(x, u(x)), v_{t_1,\cdots,t_n}(K(x, u(x))), \\
v_{t_1,\cdots,t_n}(x) &= J(x, \mathrm{Exp}(t_1 X_1) \circ \cdots \circ \mathrm{Exp}(t_n X_n)(x)), \\
\xi^{-1}(\phi_t \circ f)(x) &= J(x, K(K(x, u(x)), v_{t_1,\cdots,t_n}(K(x, u(x)))),
\end{aligned}$$

so $F^1_{\phi_t}(\eta, r)$ and $F^2_{\phi_t}(\eta, r)$ are infinitely differentiable maps with respect to t. Further somewhat long and complicated calculations lead us to that there exists $\delta_1 > 0$ such that

$$(2.8) \qquad \|\partial_t^s F^1_{\phi_t}(\eta, r)\|_{E^{k+3d^*+1-2\ell}} \quad \text{and} \quad \|\partial_t^s F^2_{\phi_t}(\eta, r)\|_{\ker L} \quad \text{are bounded}$$

for any $|t| < \delta_1$ and $(\eta, r) \in D$. Thus the derivatives of the first term of $\rho_{\phi_t}(\eta, r)$, that is,

$$\exp(- < \eta, UF^1_{\phi_t}(\eta, r) >_{E^{k+d^*+1-2\ell}} - \frac{1}{2} < F^1_{\phi_t}(\eta, r), UF^1_{\phi_t}(\eta, r) >_{E^{k+d^*+1-2\ell}})$$

are also bounded and continuous. While for the second term in that function, namely $|\det((dA)_{\phi_t}(\eta, r))|$, we take, in the present case, $\sigma(\eta, r) := \rho(\|(\eta, r)\|^2_X)$ as the function s in (2.3) and write it down as follows.

$$I_1(t, \eta, r) := \int_X \sigma(\eta' + (dF^1_{\psi_t})_{(\eta,r)}(\eta', r'), (dF^2_{\psi_t})_{(\eta,r)}(\eta', r')) g_{\tilde{T}}(d\eta', dr'),$$

(Since $(\eta, r) \in A(\xi(U_3))$, we see that a support of the above integrand is bounded as far as $|t|$ is sufficiently small)

$$\begin{aligned}
I_2(t, \eta, r) := \int_X &\exp(- < U(dF^1_{\phi_t})_{(\eta,r)}(\eta', r'), \eta' >_{E^{k+d^*+1-2\ell}} - < (dF^2_{\phi_t})_{(\eta,r)}(\eta', r') - r', r' >_{\ker L}) \cdot \\
&\exp(-\frac{1}{2} < (dF^1_{\phi_t})_{(\eta,r)}(\eta', r'), U(dF^1_{\phi_t})_{(\eta,r)}(\eta', r') >_{E^{k+d^*+1-2\ell}} - \frac{1}{2}\|(dF^2_{\phi_t})_{(\eta,r)}(\eta', r') - r'\|^2_{\ker L}) \cdot \\
&\sigma(\eta', r') g_{\tilde{T}}(d\eta', dr').
\end{aligned}$$

Then by virtue of the previous arguments, $I_1(t, \eta, r)$ and $I_2(t, \eta, r)$ are bounded for any $(\eta, r) \in A(\xi(U_2)) \cap X$ and for any $|t| < {}^\exists\delta_2$.

Next let us observe $\partial_t^s(dF^1_{\phi_t})_{(\eta,r)}$ and $\partial_t^s(dF^2_{\phi_t})_{(\eta,r)}$. Since

$$(dF^1_{\phi_t})_{(\eta,r)}(\eta', r') = \frac{d}{d\tau}\Big|_{\tau=0} \pi^{k+3d^*+1-2\ell}(A_{2\ell,m}(\phi_t \circ A^{-1}(\eta + \tau\eta', r + \tau r')) - A_{2\ell,m}(A^{-1}(\eta + \tau\eta', r + \tau r'))),$$

so changing $f = A^{-1}(\eta, r)$ to $f_\tau = A^{-1}(\eta + \tau\eta', r + \tau r')$, together changing $u := \xi^{-1}(f)$ to $u_\tau := \xi^{-1}(f_\tau)$, and proceeding in the same manner as before, we have

$$\|\partial_t^s(dF^1_{\phi_t})_{(\eta,r)}(\eta', r')\|_{E^{k+3d^*+1-2\ell}} \quad \text{is bounded}$$

for any $(\eta, r) \in A(\xi(U_3)) \cap X$ (if necessary, taking a smaller neighbourhood U_3' in place of U_3), for any $|t| < {}^\exists\delta_3$ and for any (η', r') in any but fixed bounded domain. The same estimate holds for $\|\partial_t^s(dF^2_{\phi_t})_{(\eta,r)}(\eta', r')\|_{\ker L}$. By the above, $\partial_t^s|\det(dA_{\phi_t}(\eta, r))|$ surely exists and it is bounded and continuos on the integral domain. Therefore the same conclusion for $\rho_{\phi_t}(\eta, r)$ follows directly.

Lastly for the function $Q(A^{-1}A_{\phi_t}(\eta, r))$, we have

$$Q(A^{-1}A_{\phi_t}(\eta,r)) = \tilde{Q}(A_{\phi_t}(\eta,r)) = C^{-1}\rho(\|(\eta + F^1_{\phi_t}(\eta,r), F^2_{\phi_t}(\eta,r)) - A(\mathrm{id})\|^2_X).$$

So there follows from (2.8) that $\partial_t^s Q(A^{-1}A_{\phi_t}(\eta, r))$ is continuous and bounded for the same region.

Consequently the s-th derivative of the integrand is continuous and bounded for any $|t| < \min(t_1, t_2, t_3)$ on the integral domain. Therefore $w_h^{a,b}$ is a C^∞-vector. The rest of the proof is obvious. □

3. Application to 1-cocycles on the group of diffeomorphisms

The rest of this issue is devoted to an application of these results to 1-cocycles. So let us introduce the notions of them briefly.

Assume that a subgroup G of $\mathrm{Diff}_0^s(M)$ acts on a measurable space $(X, \mathfrak{B})$ from left $(g, x) \in G \times X \longrightarrow gx \in X$.

A $U(H)$-valued function θ on $X \times G$, $U(H)$ is the unitary group of a complex Hilbert space H, is said to be 1-cocycle, if

$$^\forall g_1, g_2 \in G, \quad ^\forall x \in X, \qquad \theta(x, g_1)\theta(g_1^{-1}x, g_2) = \theta(x, g_1g_2). \quad \text{(cocycle equality)}$$

For regularity of 1-cocycles, several notions have been considered. Some of them are as follows.

Definition 3.1. (1) *θ is said to be precontinuous $\Longleftrightarrow$ $^\forall x_0$: fixed, $\theta(x_0, g)$ is continuous on a stabilizer group, $G(x_0) := \{g \in G |\ gx_0 = x_0\}$.*
(2) *θ is said to be continuous $\Longleftrightarrow$ $^\forall x_0$: fixed, $\theta(x_0, g)$ is continuous on the whole group, G.*
(3) *θ is said to be measurable $\Longleftrightarrow$ $^\forall g_0$: fixed, $\theta(x_0, g)$ is $\mathfrak{B}$-measurable.*

We remark that someimes (3) implies (1), for example, under an assumption of denseness of C^∞-vectors. (cf. p138-140 in [9])

Now for the present discussions, I pick up the following two spaces as X, since they are standard for the representation theory on the group of diffeomorphisms.

Finite configuration space B_M^n which is a collection of all n-point subsets in M. It is also a quotient space of $\hat{M}^n$, where $\hat{M}^n := \{\hat{P} = (P_1, \cdots, P_n) \in M^n |\ ^\forall P_i \neq P_j\}$, and the equivalence relation is defined in an obvious way.

Infinite configuration space Γ_M which is also a quotient space of $\hat{M}^\infty := \{\hat{P} = (P_1, \cdots, P_n, \cdots) \in M^\infty |\ ^\forall P_i \neq P_j,$ and $\{P_n\}_n$ has no accumulation points$\}$, and the equivalence relation is similar with the above one. In this case we should assume that M is non compact. Of course $\mathrm{Diff}_0^s(M)$ acts on these spaces diagonally as, $\hat{g}(\hat{P}) := (g(P_1), \cdots, g(P_n), \cdots)$.

Now let θ be a 1-cocycle on the finite or infinite configuration space. Then there correspondes one to one a symmetrical cocycle on the product space to θ. Thus it is reasonable to observe a cocycle form on the product space $\hat{M}^n$ or $\hat{M}^\infty$. Also for the sake of limit of pages and for simplicity, we will confine ourself to these situations.

Then the differential methods which we have seen lead us to the following theorem determining a local form of 1-cocycles.

Theorem 3.1. (*Local form of precontinuous* 1*-cocycle*)
Let θ *be a* $U(H)$*-valued precontinuous* 1*-cocycle on* $\hat{M}^n \times \mathrm{Diff}_0^*(M)$*, and assume that* $\dim(H) < \infty$*. Take an arbitrary finite Euclidean smooth measure* μ *on* M*. Then for any* $\hat{Q} \in \hat{M}^n$ *there exist a relatively compact open neighbourhood of* $V(\hat{Q})$ *of* $\hat{Q}$*, a* $U(H)$*-valued map* C *defined on* $V(\hat{Q})$ *and a commutative system of self-adjoint operators* $\{H_k\}_{1\leq k\leq n}$ *on* H *such that*

$$\theta(\hat{P},g) = C(\hat{P})^{-1} \prod_{k=1}^{n} \left(\frac{d\mu_g}{d\mu}(P_k)\right)^{\sqrt{-1}H_k} C(\hat{g}^{-1}(\hat{P})), \tag{3.1}$$

provided that $(\hat{P}, g)$ *satisfies the following condition.*

$(*)$ *There exists a continuous path* $\{g_t\}_{0\leq t\leq 1} \subset \mathrm{Diff}_0^*(M)$ *such that* $g_0 = \mathrm{id}, g_1 = g$ *and* $^\forall t, \hat{g}_t^{-1}(\hat{P}) \in V(\hat{Q})$.

If moreover θ *is continuous, then so is the map* C.

Of course a global form of 1-cocycle will be obtained by patching up these local results. However difficulties arise because of non uniqueness of the above map C, which forms so called coboundary term. Roughly speaking we will meet a similar situation with many valuedness problem to analytic continuation. So some geometrical conditions on M are required in order to obtain a global result. One direction is as follows. (cf. [20])

Theorem 3.2. (*Global form of precontinuous* 1*-cocycle*)
Under the same notation in the above theorem and under the assumption that $\hat{M}^n$ *is simply connected,* (3.1) *gives a general form of precontinuous* 1*-cocycle.*

Remark 3.1. (1) *In oder that* $\hat{M}^n$ *is simply connected, it is sufficient that* M *is simply connected and* $\dim M \geq 3$ *, thanks to dimension theory.*
(2) Theorem 3.2 *is no longer trure, unless* $\hat{M}^n$ *is simply connected.* (*cf.* [19], [20])

A cocycle form on $\hat{M}^\infty$, in a special case that M is simply connected, is described in the following last theorem.

Theorem 3.3. (1) *Suppose that* M *is simply connected,* $\dim(M) \geq 3$. *and* $\dim H < \infty$.
Then the general form of precontinuous $U(H)$*-valued* 1*-cocycles on* $\hat{M}^\infty \times \mathrm{Diff}_0^*(M)$ *is as follows.*

$$\theta(\hat{P},g) = C(\hat{P})^{-1} \prod_{k=1}^{\infty} \left(\frac{d\mu_g}{d\mu}(P_k)\right)^{\sqrt{-1}H_k^{[P]}} C(\hat{g}^{-1}(\hat{P})), \tag{3.2}$$

where C *is a* $U(H)$*-valued map on* $\hat{M}^\infty$*, and* $\{H_k^{[P]}\}_k$ *is a commutative system of self-adjoint operators on* H *depending on the residue class* $[P]$ *defined by* $[P] := \{\hat{Q} \in \hat{M}^\infty |\ Q_n = P_n$ *except finite numbers of* $n\}$.

Finally I wish to mension a few words about natural representations formed by measures and 1-cocycles. Their irreducibility and equivalence are also examined by similar methods established here and they are characterized by the above theorems.

Acknowledgement I express my thanks to Professor N. Shimakura at Tohoku University for giving me impotant facts for generalized Hodge theorem.

REFERENCES

1. G.A.Goldin,J.Grodrick,R.T.Powers, and D.H.Sharp, Non relativisitic current algebra in the N/V limit, J.Math.Phys., **15** (1974), 217-228.
2. T.Hirai, Construction of irreducible unitary representations of the infinite symmstric group $\mathfrak{S}_\infty$, J.Math.Kyoto Univ., **31** (1991), 495-541.
3. T.Hirai, Irreducible unitary representations of the group of diffeomorphisms of a non-compact manifold, ibid., **33** (1993), 827-864.
4. T.Hirai and H.Shimomura, Relations between unitary representations of diffeomorphism groups and those of the infinite symmetric group or of related permutation groups, ibid., **37** (1997), 261-316.
5. J.Hoffmann-Jϕrgensen, Probability in B-spaces, Lecture notes series, bf 48 (1977).
6. R.S.Ismagilov, Unitary representations of the group of diffeomorphisms of a circle, Funct.Anal.Appl., **5** (1971), 45-53 (= Funct.Anal., **5** (1971), 209-216 (English Translation)).
7. R.S.Ismagilov, On unitary representations of the group of diffeomorphisms of a compact manifold, Math.USSR Izvestija, **6** (1972) 181-209.
8. R.S.Ismagilov, Unitary representations of the group of diffeomorphisms of the space $\mathbf{R}^n$, $n \geq 2$, Funct.Anal.Appl., **9** (1975), 71-72 (= Funct.Anal.,**9** (1975), 154-155 (English Translation)).
9. R.S.Ismagilov, Representations of infinite-dimensional groups, Trans.Math. Monographs Amer.Math.Soc., **152** (1996).
10. J.A.Leslie, On a differentiable structure for the group of diffeomorphisms, Topology, **6** (1967) 263-271.
11. Yu.A.Neretin, The complementary series of representations of the group of diffeomorphisms of the circle, Russ.Math.Surv., **37** (1982), 229-230.
12. H.Omori, Infinite dimensional Lie groups, Trans.Math.Monographs, **158** Amer.Math.Soc. (1997).
13. H.Omori,Y.Maeda,A.Yoshioka and O.Kobayashi, On regular Fréchet Lie groups IV, Tokyo J.Math., **5** (1982), 365-398.
14. E. Shavgulidze, On a measure that is quasi-invariant with respect to the action of a group of diffeomorphisms of a finite-dimensional manifold, Dokl.Acad.Nauk, **303** (1998) (= Soviet Math.Dokl., **38** (1989) 622-625).
15. E. Shavgulidze, Mesures quasi-invariantes sur les groupes de difféomorphismes des variétés riemaniennes, C.R.Acad.Sci., **321** (1995) 229-232.
16. E.Shavgulidze, Quasi-invariant measures on groups of diffeomorphisms, Trudy Mathematicheskogo Instituta im V.A.Steklova, **217** (1997) 189-208.
17. N.Shimakura, Partial differential operators of elliptic type, Trans.Math.Monographs, **99** Amer.Math.Soc. (1992).
18. H.Shimomura, Poisson measures on the configuration space and unitary representations of the group of diffeomorphisms, J.Math.Kyoto Univ., **34** (1994) 599-614.
19. H.Shimomura, 1-cocycles on the group of diffeomorphisms, ibid., **38** (1998) 695-725.
20. H.Shimomura, 1-cocycles on the group of diffeomorphisms II, ibid., **39** (1999) 493-527.
21. H.Shimomura and T.Hirai, On group topologies on the group of diffeomorphisms, RIMS kōkyūroku, **1017** (1997) 104-115.
22. N.Tatsuuma,H.Shimomura and T.Hirai, On group topologies and unitary representations of inductive limits of topological groups and the case of the group of diffeomorphisms, J.Math.Kyoto Univ., **38** (1998) 551-578.
23. A.M.Vershik,I.M.Gel'fand and M.I.Graev, Representations of the group of diffeomorphism, Usp.Mat.Nauk., **30** (1975), 3-50 (= Russ. Math.Surv., **30** (1975), 3-50).

FREE PROBABILITY THEORY AND FREE DIFFUSION

ROLAND SPEICHER*

1. Introduction

Free probability theory was introduced and developed by Dan Voiculescu in an operator algebraic context, but has since then turned out to possess links to a lot of quite different fields of mathematics and physics. I will give a short general introduction into the basics of free probability and illuminate certain aspects of that theory (in particular, the analogy between classical and free probability theory) by a closer look at free diffusion.

An extensive presentation of the basic theory of free probability is given in the monograph [VDN], whereas for getting an impression of the diversity of this field one should consult [V2, V3].

2. Free probability theory

Free probability theory was introduced by Dan Voiculescu around 1985 as a tool for investigating the structure of special von Neumann algebras. Voiculescu separated from that concrete context the following abstract concept of 'freeness' and found it worth to be investigated on its own sake. The definition and the main properties of freeness do not require an operator algebraic frame, but can be formulated on the level of unital algebras and unital linear functionals.

Definition 2.1. Let $\mathcal{A}$ be a unital algebra and $\varphi : \mathcal{A} \to \mathbb{C}$ a linear functional with $\varphi(1) = 1$.
1) Let $\mathcal{A}_1, \dots, \mathcal{A}_m \subset \mathcal{A}$ be unital subalgebras. The subalgebras $\mathcal{A}_1, \dots, \mathcal{A}_m$ are called **free**, if $\varphi(a_1 \cdots a_k) = 0$ for all $k \in \mathbb{N}$ and all $a_i \in \mathcal{A}_{j(i)}$ $(1 \le j(i) \le m)$ whenever $\varphi(a_i) = 0$ for all $i = 1, \dots, k$, and neighbouring elements are from different subalgebras, i.e., $j(1) \neq j(2) \neq \cdots \neq j(k)$.
2) Elements $a_1, \dots, a_m \in \mathcal{A}$ are called free, if $\mathcal{A}_1, \dots, \mathcal{A}_m$ are free, where, for $i = 1, \dots, m$, $\mathcal{A}_i := \mathrm{alg}(1, a_i)$ is the unital algebra generated by a_i.

*Supported by a Heisenberg-Fellowship of the DFG.

Voiculescu chose the name 'free' because the basic example where such situations occur are von Neumann algebras which are constructed from free groups (the so-called free group factors).

The basic philosophy for the investigation of the concept 'freeness' is to consider it as an analogue of the concept 'independence' from classical probability theory. Hence we are using a probabilistic kind of language and are usually guided by concepts and ideas from classical probability theory. In this sense, the theory of freeness can be considered as a part of non-commutative probability theory and it is usually referred to as 'free probability theory'.

Let us first introduce some general notions from non-commutative probability theory.

Notations 2.2. A pair $(\mathcal{A}, \varphi)$ consisting of a unital algebra $\mathcal{A}$ and a unital linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ is called a **(non-commutative) probability space**, elements $a_1, \dots, a_m$ from the given algebra $\mathcal{A}$ are called **random variables** and expressions like $\varphi(a_{j(1)} \cdots a_{j(k)})$ are called **moments**. The collection of all moments, for all $k \in \mathbb{N}$ and all $1 \leq j(1), \dots, j(k) \leq m$, is called the **(joint) distribution** of the random variables $a_1, \dots, a_m$.

Remark 2.3. One should note that in the case of one self-adjoint bounded random variable $a = a^* \in B(\mathcal{H})$, one can identify the so-defined distribution of a indeed with a probability measure μ on $\mathbb{R}$ by the requirement that the moments of a coincide with the moments of μ, i.e.

$$\varphi(a^n) = \int x^n d\mu(x) \qquad \text{for all } n \in \mathbb{N}. \tag{1}$$

In that case we will denote this probability measure also with $\text{distr}(a)$. In general, the distribution of random variables cannot be identified with some kind of probability measure, but is just a collection of numbers.

Examples 2.4. Let us now give some examples of probability spaces and distributions in this general algebraic sense – in order to become familiar with this kind of notations and to introduce some basic frame for our later investigations.

1) Classical probability spaces. Classical probability spaces $(\Omega, \mathcal{Q}, P)$ – consisting of a set Ω, a σ-algebra $\mathcal{Q}$ of measurable subsets of Ω and a probability measure P on Ω – can be treated in this frame by setting, e.g., $\mathcal{A} = L^{\infty-}(\Omega) := \cup_{p=1}^{\infty} L^p(\Omega)$ and where $\varphi = E$ is the expectation

$$\varphi(X) = \int_\Omega X(\omega) dP(\omega) \qquad (X \in \mathcal{A}). \tag{2}$$

2) Matrices. Let, for $n \in \mathbb{N}$, $\mathcal{A} = M_n$ be equal to the $n \times n$-matrices. A canonical state on this is given by the normalized trace $\varphi = \text{tr}$, i.e., for $a = (a_{ij})_{i,j=1}^n \in \mathcal{A}$ we have

$$\varphi(a) = \frac{1}{n} \sum_{i=1}^n a_{ii}. \tag{3}$$

One should note that for self-adjoint matrices $a = a^*$ the distribution $\text{distr}(a)$ is nothing but the eigenvalue distribution of a, i.e., if $\lambda_1, \dots, \lambda_n$ are the (real) eigenvalues of a, then $\text{distr}(a)$ is that probability measure on $\mathbb{R}$ which puts mass $1/n$ on each of the eigenvalues, i.e.

$$\text{distr}(a) = \frac{1}{n} \sum_{i=1}^n \delta_{\lambda_i}. \tag{4}$$

3) Random matrices. Random matrices are a combination of (1) and (2), namely matrices whose entries are classical random variables: $\mathcal{A} = M_n \otimes L^{\infty -}(\Omega)$ and $\varphi = \text{tr} \otimes E$, i.e., $a \in \mathcal{A}$ are of the form $a = (a_{ij})_{i,j=1}^n$, where the entries $a_{ij} \in L^{\infty -}(\Omega)$, and

$$\varphi(a) = E[\frac{1}{n} \sum_{i=1}^n a_{ii}] = \frac{1}{n} \sum_{i=1}^n \int_\Omega a_{ii}(\omega) dP(\omega). \tag{5}$$

In the case of a self-adjoint random matrix $a = a^*$, the distribution $\text{distr}(a)$ is the averaged eigenvalue distribution of a.

To enrich the general frame of non-commutative probability theory by some substance one has to add additional structure. In free probability theory this is the concept of 'freeness'. In analogy with the concept 'independence' it should be considered as a rule for calculating mixed moments in free random variables. This might not be directly clear from the definition, so let us present some examples to get familiar with the concept of freeness.

Examples 2.5. Let x and y be free random variables (with respect to a given unital functional φ). We want to calculate some mixed moments in x and y.

1) The simplest mixed moment is $\varphi(xy)$. The definition of freeness tells us immediately that $\varphi(xy) = 0$, if $\varphi(x) = 0$ and $\varphi(y) = 0$. But we can also reduce the general case to the definition by going over to centered variables: since $\hat{x} := x - \varphi(x)1$ is an element from the unital

algebra generated by x with the property $\varphi(\hat{x}) = 0$, and similarly for $\hat{y} := y - \varphi(y)1$, we have that $\varphi(\hat{x}\hat{y}) = 0$; however, by linearity, we also have

$$0 = \varphi(\hat{x}\hat{y}) = \varphi\big((x - \varphi(x))(y - \varphi(y))\big) = \varphi(xy) - \varphi(x)\varphi(y).$$

Hence we have in general for free variables x and y that

$$\varphi(xy) = \varphi(x)\varphi(y). \tag{6}$$

2) The mixed moment $\varphi(xxyy)$ calculates in the same way by going over to the centered variables:

$$\varphi\big((x^2 - \varphi(x^2))(y^2 - \varphi(y^2))\big) = 0$$

yields

$$\varphi(xxyy) = \varphi(xx)\varphi(yy). \tag{7}$$

3) Let us also consider a more complicated mixed moment:

$$\varphi\big((x - \varphi(x))(y - \varphi(y))(x - \varphi(x))(y - \varphi(y))\big) = 0$$

leads to

(8)

$$\varphi(xyxy) = \varphi(xx)\varphi(y)\varphi(y) + \varphi(x)\varphi(x)\varphi(yy) - \varphi(x)\varphi(y)\varphi(x)\varphi(y).$$

Remarks 2.6. 1) The last example shows that freeness gives a different result than independence. Although both concepts are analogous, they provide different rules for calculating mixed moments. In particular, freeness is not a non-commutative generalization of independence.
2) If x and y are classical random variables, then, in particular, they commute, i.e. we have in this case that $\varphi(xxyy) = \varphi(xyxy)$. However, for x and y free we have quite different expressions for these two mixed moments and one can easily see that they can only agree if at least one of the two variables is a constant. Thus classical random variables are, apart from trivial cases, never free. Freeness is really a concept for non-commuting variables.
3) As the last example above indicates the formulas for mixed moments in free variables are more complicated than the corresponding formulas for independent variables and it is not clear from the definition of freeness how the structure of a general mixed moment can be described. However, there is a nice combinatorial structure behind these formulas. I have shown that their structure is (via so-called free cumulants) governed by the lattice of non-crossing partitions (see, e.g., the survey [Sp2]). This description is totally analogous to the description in classical probability theory via cumulants and the lattice of all partitions and it provides an alternative approach (compared to the analytical approach of Voiculescu) to the theory of free random variables.

Let me end this short introduction into the generalities of free probability theory by pointing out that there are two fundamental types of examples for free variables: The definition of freeness is modeled according to the situation occurring in free group factors, thus it is not very surprising that special operators in free group factors (or more concretely, special operators on full Fock spaces) are free. But there is also a totally different context where free variables arise, namely it is one of basic results of Voiculescu [V1] that special $n \times n$-random matrices become free in the limit $n \to \infty$. I will be more concrete on such types of examples when I present the free Brownian motion.

3. Free Diffusion

As pointed out before one of the basic philosophies in free probability theory is to consider freeness as an analogue of independence. Thus one tries to develop a free theory which goes parallel to classical probability theory. Astonishingly, this analogy is very far reaching and there exist a lot of (non-trivial) free counterparts of classical results.

In the following I want to illuminate this general statement by a recent joint work [BSp1, BSp2] with Philippe Biane on free diffusion.

3.1. **Classical diffusion.** Let me first explain what I mean with the corresponding classical notion. If $V : \mathbb{R} \to \mathbb{R}$ is a sufficiently nice function (called potential in the following), one can consider the classical diffusion in this potential. On one side there is a probabilistic construction of this object, namely it is a stochastic process $(X_t)_{t\geq 0}$ which is given as the solution of a special stochastic differential equation. What I call here 'diffusion in the potential V' is the solution of

$$dX_t = -\frac{1}{2}V'(X_t)dt + dB_t, \tag{9}$$

where B_t is classical Brownian motion.

There exists also an analyical aspect of this diffusion, namely if we denote, for fixed $t \geq 0$, by $\operatorname{distr}(X_t)$ the distribution of the random variable X_t, then this is a probability measure on $\mathbb{R}$ which has a density with respect to Lebesgue measure. Denote this density by p_t. Then one can write down a differential equation for the time evolution of this density, namely

$$\frac{\partial p_t(x)}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x}\big[(\frac{\partial}{\partial x} + V'(x))p_t(x)\big]. \tag{10}$$

This linear partial differential equation is usually called the Fokker-Planck equation of the corresponding diffusion and, from an analytical point of view, one can consider the diffusion also as a solution of that equation. Furthermore, there exist also connections between such diffusions and classical entropy.

The problem which I want to address in the following is whether there exist a free counterpart of these statements, i.e., can we define a free diffusion as a solution of a free stochastic differential equation and is there a corresponding free Fokker-Planck equation. In order to speak about free stochastic differential equations, we first have to introduce free Brownian motion.

3.2. **Free Brownian motion.** In analogy with classical Brownian motion one could define free Brownian motion [Sp1] abstractly as a (non-commutative) stochastic process, i.e. a collection $(S_t)_{t\geq 0}$ of random variables, which have the properties that their increments are free and that the distribution of the increments is given by the free analogue of the Gaussian distribution (which is what one gets as the limit distribution in a free central limit theorem). It is easy to verify that, by abstract reasons, such an object exists and that its distribution is uniquely determined. Fortunately, there are also nice concrete realizations of free Brownian motion.

Examples 3.2.1. In the spirit of the last statement in Sect. 2 there exist two such realizations, a functional analytic one by concrete operators on Fock spaces and a probabilistic one by random matrices.
1) Realization on full Fock space. Denote by $\mathcal{H}$ the Hilbert space $\mathcal{H} := L^2(\mathbb{R}_+)$ and let

$$\mathcal{F}(\mathcal{H}) := \mathcal{H}^{\otimes 0} \oplus \mathcal{H}^{\otimes 1} \oplus \mathcal{H}^{\otimes 2} \oplus \dots \tag{11}$$

be the full Fock space over $\mathcal{H}$, where $\mathcal{H}^{\otimes 0}$ is a one-dimensional Hilbert space which we write in the form $\mathcal{H}^{\otimes 0} = \mathbb{C}\Omega$ for a distinguished vector Ω of norm 1. Ω is also called vacuum. For each vector $f \in \mathcal{H}$, we define on $\mathcal{F}(\mathcal{H})$ a creation operator $l(f)$ and an annihilation operator $l^*(f)$ by linear extension of

$$l(f) f_1 \otimes \dots \otimes f_n = f \otimes f_1 \otimes \dots \otimes f_n \tag{12}$$

and

$$l^*(f) f_1 \otimes \dots \otimes f_n = \langle f, f_1 \rangle f_2 \otimes \dots \otimes f_n \tag{13}$$

$$l^*(f)\Omega = 0. \tag{14}$$

The operators $l(f)$ and $l^*(f)$ are bounded and adjoints of each other. Now put

$$S_t := l(1_{[0,t)}) + l^*(1_{[0,t)}), \tag{15}$$

where $1_{[0,t)}$ is the characteristic function of the interval $[0,t)$. Then it is quite easy to check that $(S_t)_{t\geq 0}$ is with respect to the vacuum expectation state φ, given by

$$\varphi(a) := \langle \Omega, a\Omega \rangle, \tag{16}$$

indeed a free Brownian motion.

The von Neumann algebra generated by all S_t $(t \geq 0)$ is isomorphic to a free group factor, and this example comes from the original context of Voiculescu's investigations on the free group factors. Thus the appearance of freeness in this context is not very surprising.

2) Realization by random matrices. Let, for $1 \leq i \leq j < \infty$, $B_{ij}(t)$ be independent classical real-valued Brownian motions, and put $B_{ji}(t) = B_{ij}(t)$ for $j > i$. We put now these Brownian motions as entries in a matrix, i.e. we consider the selfadjoint random matrices

$$X_t^{(n)} := \frac{1}{\sqrt{n}}\big(B_{ij}(t)\big)_{i,j=1}^{n} \tag{17}$$

in the probability space $(M_n \otimes L^{\infty-}(\Omega), \varphi^{(n)} = \mathrm{tr} \otimes E)$. (These special random matrices are usually called Gaussian random matrices.) Then the basic result of Voiculescu [V1] on the connection between freeness and random matrices tells us that the processes $(X_t^{(n)})_{t\geq 0}$ converge in distribution, for $n \to \infty$, towards the free Brownian motion $(S_t)_{t\geq 0}$. This means that

$$\lim_{n\to\infty} \varphi^{(n)}(X_{t_1}^{(n)} \cdots X_{t_k}^{(n)}) = \varphi(S_{t_1} \cdots S_{t_k}) \tag{18}$$

for all $k \in \mathbb{N}$ and all $t_1, \dots, t_k \geq 0$. Thus, in a sense, free Brownian motion can be considered as an $\infty \times \infty$-random matrix. However, one should note that this is not just an infinite array of entries, but the crucial information lies in the state. There exists no normalized trace on infinite arrays, and freeness is the mathematical structure which survives under taking this limit.

Remark 3.2.2. The realization of free Brownian motion by random matrices gives us an interesting connection with systems of interacting particles. Namely, for fixed t, we know that the distribution $\mathrm{distr}(X_t^{(n)})$ is the averaged eigenvalue distribution of these $n \times n$-random matrices and thus free Brownian motion describes in particular also the behaviour of the eigenvalues of Gaussian $n \times n$-random matrices in the limit $n \to \infty$. However, it is well known that the eigenvalues of such Gaussian random matrices are not independent, but they behave like electrically charged particles in two dimensions, i.e. like particles with a special type of pair-interaction. In a probabilistic language, the eigenvalues of the random matrices $X_t^{(n)}$ obey the stochastic differential equation

$$d\lambda_i(t) = \frac{1}{\sqrt{n}} dB_i(t) + \frac{1}{n} \sum_{\substack{1\leq j\leq n \\ j\neq i}} \frac{1}{\lambda_i - \lambda_j} dt \qquad (i = 1, \dots, n), \tag{19}$$

where $B_i(t)$ $(i = 1, \dots, n)$ are independent classical Brownian motions.

In the limit $n \to \infty$, the diffusive term can be neglected compared to the deterministic term and thus this limit corresponds to a system of infinitely many particles which interact with each other by a special type of pair interaction. Free Brownian motion provides thus in particular the description for such a system of infinitely many interacting particles.

3.3. **Free stochastic differential equations.** The next step is to develop a stochastic calculus with respect to free Brownian motion in order to be able to define and deal effectively with corresponding stochastic differential equations. By integration the meaning of a stochastic differential equation is reduced to the meaning of the corresponding stochastic integrals. In our case, this means that we have to define objects like $\int A_t dS_t B_t$, where dS_t is the increment of the free Brownian motion and where $(A_t)_{t\geq 0}$ and $(B_t)_{t\geq 0}$ are adapted processes. ($(A_t)_{t\geq 0}$ adapted means that, for each $t \geq 0$, A_t is an element of the von Neumann algebra generated by all S_s with $s \leq t$.) In contrast to the classical case, our processes and the increments do not commute, so one should really consider this bilinear integral in (A_t, B_t) instead just a one-sided integral. Such stochastic integrals are defined as usual, namely for elementary processes, which are constant on time intervals I_i and take there a fixed value A_i or B_i, the integral is defined as

$$\int A_t dS_t B_t := \sum_i A_i S(I_i) B_i, \tag{20}$$

where $S(I_i)$ is the increment of the free Brownian motion over the interval I_i. Then one has to prove estimates for such integrals in some suitable norms and extend the definiton of the integral to the closure of elementary functions under the involved norms. The easiest norm estimate is an L^2-estimate which works in the same way as for other stochastic calculi and which yields the usual Ito-isometry. Results of Pisier and Xu [PX] on non-commutative martingales can be used to obtain L^p-estimates for $p < \infty$. Whereas such kind of estimates are also true for other kind of stochastic calculi, a very specific feature of the free calculus is that one can also derive L^∞-estimates, i.e. one can estimate the integrals in operator norm.

Theorem 3.3.1. ([BSp1]) *Let* $(A_t)_{t\geq 0}$ *and* $(B_t)_{t\geq 0}$ *be adapted processes. Then we have*

$$\| \int A_t dS_t B_t \| \leq 2\sqrt{2} \Big(\int \|A_t\|^2 \cdot \|B_t\|^2 dt \Big)^{1/2}. \tag{21}$$

Having established the existence of the free stochastic integrals in nice topologies one can continue to investigate the corresponding stochastic calculus. There exists also a free Ito formula [KSp, BSp1], which, on a formal differential level, states that

$$dS_t A dS_t = \varphi(A) dt \qquad \text{for } A \text{ adapted.} \tag{22}$$

This should be compared to the classical Ito formula $dB_t A dB_t = Adt$. The differences between the usual stochastic calculus and the free stochastic calculus can, on a formal level, be reduced to this difference between the corresponding Ito formulas.

One can also derive free analogues of classical stochastic analysis. In [BSp1] we treated, e.g., iterated stochastic integrals, which give rise to a chaos decompositon of the L^2-space of the free Brownian motion and allow to prove a representation theorem for martingales or to extend the free Ito integral to a free Skorohod integral for non-adapted processes.

3.4. Free diffusion.

Definition 3.4.1. We will consider the free stochastic differential equation

$$dX_t = -\frac{1}{2} V'(X_t) dt + dS_t \tag{23}$$

We call the solution of (23), if it exists, the **free diffusion** in the potential V.

Remark 3.4.2. In the same way as free Brownian motion describes the behaviour of infinitely many particles which interact with a special pair-interaction, the free diffusion in the potential V describes the behaviour of such particles if we put them in addition into a potential V.

Theorem 3.4.3. ([BSp2]) *Let* X_0 *be free from the free Brownian motion* $(S_t)_{t\geq 0}$ *and* V' *be sufficiently smooth (e.g.,* $V' \in C^2$*).*
1) Then there exists a unique solution $(X_t)_{t\geq 0}$ *of the equation (23). Furthermore, we have that* X_t *lies in the* C^**-algebra generated by* X_0 *and all* S_s *with* $s \leq t$ *and that the mapping* $t \mapsto X_t$ *is* $\|\cdot\|$*-continuous.*
2) The distribution of X_t *is absolutely continuous with respect to Lebesgue measure,* $distr(X_t) = p_t(x)dx$*, where the density* p_t *is bounded*

(but not smooth in general) and a weak solution of the following **free Fokker-Planck equation**

$$\frac{\partial p_t(x)}{\partial t} = -\frac{\partial}{\partial x}\left[(Hp_t(x) - \frac{1}{2}V'(x))p_t(x)\right], \tag{24}$$

where H is (up to a constant) the Hilbert transform, i.e.

$$Hp(x) := \int \frac{p(y)}{x-y}dy. \tag{25}$$

Remarks 3.4.4. 1) Note that the free Fokker-Planck equation (24) is compatible with the picture of infinitely many interacting particles in the potential V: the particles at position x feel a force coming via the pair-interaction from the other particles at all possible positions y and in addition the force $V'(x)$ coming from the potential.
2) The structure of the free Fokker-Planck equation is on a formal level very similar to the classical Fokker-Planck equation (10); the only difference is that the second derivative is replaced by the Hilbert transform Hp_t; however, this changes of course totally the nature of the considered equation; instead of a second-order linear we have now a first-order non-linear partial differential equation. The non-linearity reflects the fact that we are dealing with interacting particles; in contrast, classical free diffusion can be thought of as infinitely many diffusing particles in the potential V without any interaction.

3.5. **Free diffusion and free entropy.** The above mentioned results show a formal analogy between classical diffusion and free diffusion. But this analogy goes much further. As mentioned in Sect. 2, there exists a relation between classical diffusion and classical entropy. There is also a free counterpart of that. Voiculescu introduced free analogues of the classical notions of entropy and Fisher information [V4, V5]. A relative version (with respect to V) of these are as follows. ($V = 0$ corresponds to the original definition of Voiculescu).

Notations 3.5.1. The **relative free entropy** and the **relative free Fisher information** are given by

$$\Sigma_V(\mu) := \int\int \log|x-y|\mu(dx)\mu(dy) - \int V(x)\mu(dx) \tag{26}$$

and (for $\mu(dx) = p(x)dx$)

$$I_V(\mu) := 4\int \left(Hp(x) - \frac{1}{2}V'(x)\right)^2 p(x)dx, \tag{27}$$

respectively.

With these notations we have the following theorem.

Theorem 3.5.2. ([BSp2]) *Let $(X_t)_{t\geq 0}$ be the solution of the free diffusion equation (23). Then we have*

$$\frac{d}{dt}\Sigma_V(X_t) = \frac{1}{2}I_V(X_t). \tag{28}$$

In particular, $\Sigma_V(X_t)$ is increasing with t.

If we replace Σ_V and I_V by their classical counterparts then the same theorem is true for classical diffusion.

3.6. **Conclusion.** Formally there exists a very far reaching analogy between the theory of free diffusion and the theory of classical diffusion. However, free diffusion and classical diffusion describe quite different situations. Whereas the latter provides a theory for diffusing particles without interaction the former describes particles with a special type of pair-interaction. It is very surprising (but also exciting and promising) that a special type of interaction behaves in a very probabilistic way. Free probability theory seems to be the right tool for dealing with this kind of interaction.

References

[BSp1] P. Biane and R. Speicher, *Stochastic calculus with respect to free Brownian motion and analysis on Wigner space*, Probab. Theory Relat. Fields **112** (1998), 373–409.

[BSp2] P. Biane and R. Speicher, *Free diffusions, free entropy and free Fisher information*, Preprint (1999).

[KSp] B. Kümmerer and R. Speicher, *Stochastic integration on the Cuntz algebra O_∞*, J. Funct. Anal. **103** (1992), 372–408.

[PX] G. Pisier and Q. Xu, *Non commutative martingale inequalities* Comm. Math. Phys. **189** (1997), 667–698.

[Sp1] R. Speicher, *A new example of 'independence' and 'white noise'*, Probab. Theory Relat. Fields **84** (1990), 141–159.

[Sp2] R. Speicher, *Free probability theory and non-crossing partitions*, Seminaire Lotharingien de Combinatoire **B39c** (1997); also available under `http://www.rzuser.uni-heidelberg.de/~L95`.

[V1] D. Voiculescu, *Limit laws for random matrices and free products*, Invent. math. **104** (1991), 201–220.

[V2] D. Voiculescu (ed.), *Free Probability Theory* (Fields Institute Communications, vol. 12), AMS, 1997.

[V3] D. Voiculescu, *Lectures on free probability theory. Notes for a course at the Saint-Flour Summer School on Probability Theory*, Preprint (1998)

[V4] D. Voiculescu, *The analogues of entropy and Fisher's information in free probability theory, I*, Comm. Math. Phys. **155** (1993), 71–92.

[V5] D. Voiculescu, *The analogues of entropy and Fisher's information in free probability theory, V: Noncommutative Hilbert transforms*, Invent. Math. **132** (1998), 182–227.

[VDN] D.V. Voiculescu, K.J. Dykema, and A. Nica, *Free Random Variables* (CRM Monograph Series, vol. 1), AMS, 1993.

UNIVERSITÄT HEIDELBERG, INSTITUT FÜR ANGEWANDTE MATHEMATIK, IM NEUENHEIMER FELD 294, D-69120 HEIDELBERG, GERMANY
E-mail address: roland.speicher@urz.uni-heidelberg.de

A Girsanov-type Formula for Lévy Processes on Commutative Hypergroups

Michael Voit
Mathematisches Institut, Universität Tübingen
Auf der Morgenstelle 10, 72076 Tübingen, Germany

Abstract

In this note we present a Girsanov-type formula which turns (central) Bessel processes on $[0,\infty[$ of arbitary indices into non-central ones. It will be shown that this result may be seen as a special case of a general Girsanov formula for Lévy processes on commutative hypergroups which connects Lévy processes on different hypergroup structures on the same ground space, where the associated convolutions are related by some deformation.

1 Introduction

In this paper we present some Girsanov formula for Lévy processes on commutative hypergroups. We first illustrate the main result with Bessel processes on $[0,\infty[$, as these processes may be regarded as Lévy processes on the so-called Bessel-Kingman hypergroups; the understanding of this example requires no knowledge about hypergroups.

We start with an n-dimensional Brownian motion $(B_t)_{t\geq 0}$ defined on the Wiener space $(\Omega, \mathcal{F}, P)$ with

$$\Omega = C(\mathbb{R}^n) := \{f : [0,\infty[\to \mathbb{R}^n,\ f \text{ continuous}\},$$

which carries the right-continuous, complete induced filtration $(\mathcal{F}_t)_{t\geq 0}$ as usually with $\mathcal{F} = \sigma(\mathcal{F}_t : t \geq 0)$. The classical formula of Girsanov then in particular implies that for any drift vector $c \in \mathbb{R}^n$ there is a unique probability measure Q_c on $(\Omega, \mathcal{F})$ with

$$Q_c|_{\mathcal{F}_t} = e^{<c,B_t>-t\|c\|_2^2/2} P|_{\mathcal{F}_t} \qquad \text{for} \quad t \geq 0,$$

and with respect to Q_c, the process $(B_t)_{t\geq 0}$ is a Brownian motion on $\mathbb{R}^n$ with drift c. Moreover, for

$$\Phi : \mathbb{R}^n \longrightarrow [0,\infty[, \quad x \longmapsto |x| = (x_1^2 + \ldots + x_n^2)^{1/2},$$

the process $(\Phi(B_t))_{t\geq 0}$ is a Bessel process of dimension n; see [RY] for details. This process may be regarded as coordinate process $(X_t)_{t\geq 0}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ with

$$\tilde{\Omega} := \{f : [0,\infty[\to [0,\infty[,\ f \text{ continuous}\},$$

with the canonical σ-algebras, and with $\tilde{P}$ as image of P under the projection $\Psi : \Omega \to \tilde{\Omega}$ which is uniquely determined by

$$\Psi(\omega)_t = \Phi(\omega_t) \qquad \text{for} \quad t \geq 0.$$

Using the rotation invariance of $(B_t)_{t\geq 0}$ and the integral representation

$$j_{n/2-1}(x) := \int_{S^{n-1}} e^{i<x,y>}\, dU_{n-1}(y) \quad (x \in \mathbb{C})$$

of the spherical Bessel function $j_{n/2-1}$ (with U_{n-1} the uniform distribution on the unit sphere $S^{n-1} \subset \mathbb{R}^n$; see 9.1.20 of [AS]), we obtain for any drift $c \in \mathbb{R}^n$ that the distribution $\tilde{Q}_c := \Psi(Q_c) \in M^1(\tilde{\Omega}, \tilde{\mathcal{F}})$ satisfies

$$\tilde{Q}_c|_{\tilde{\mathcal{F}}_t} = e^{-t\|c\|_2^2/2} j_{n/2-1}(i\|c\|_2 X_t)\tilde{P}|_{\tilde{\mathcal{F}}_t} \qquad for \quad t \geq 0.$$

Moreover, as for a Brownian motion $(B_t)_{t\geq 0}$ on $\mathbb{R}^n$ with drift c the process $(\Phi(B_t))_{t\geq 0}$ is a non-central Bessel process with dimension n and non-centrality parameter $\|c\|_2$, it can be derived from the classical Girsanov formula that, with respect to $\tilde{Q}_c$, the coordinate process $(X_t)_{t\geq 0}$ is such a process. As there exist central and non-central Bessel processes also for "fractional dimensions" $n \in \mathbb{R}$, $n \geq 1$, it is natural to ask whether the change of measure above here also turns central Bessel processes into non-central ones. We shall give a positive answer in Theorem 3.8 below.

We shall show below how this result may be regarded as a special case of a Girsanov-type formula for Lévy processes on commutative hypergroups of the following kind: Let $(X_t)_{t\geq 0}$ be a Lévy process on some commutative hypergroup $(K, *)$ that is associated with some convolution semigroup $(\mu_t)_{t\geq 0}$. Then, for any positive semicharacter α of $(K, *)$, the hypergroup convolution $*$ can be deformed into some new hypergroup convolution, say $\bullet$ (see [BH, V1, V2]). We shall show that under some growth condition, $(\mu_t)_{t\geq 0}$ can be transformed into some convolution semigroup $(\tilde{\mu}_t)_{t\geq 0}$ on $(K, \bullet)$, and that some Girsanov-type change of measure transforms $(X_t)_{t\geq 0}$ into a Lévy process on $(K, *)$ associated with $(\tilde{\mu}_t)_{t\geq 0}$. The proof of this result will be based on a martingale characterization of Lévy processes in terms of hypergroup characters; see [RV]. This main result will be discussed in Section 2 of this paper. Section 3 will be devoted to several examples and includes, in particular, a discussion of Bessel processes.

We finally mention that the results of this paper are completely disjoint to Girsanov formulas for Brownian motions on Lie groups (see [I, Kar]), as groups do not admit nontrivial positive semicharacters and hypergroup deformations. On the other hand, we hope that martingale characterizations of Lévy processes on locally compact groups in [V3, V4] in terms of group representations may be used to generalize the results of [Kar].

2 Renormalization of commutative hypergroups and a Girsanov-type formula

We first recapitulate some notations and facts about Lévy processes on commutative hypergroups. For details on hypergroups we refer to the monograph [BH] and to [J].

2.1. Commutative hypergroups. A commutative hypergroup $(K,*)$ consists of a locally compact space K together with a commutative, weakly continuous, probability preserving convolution $*$ on the Banach space $M_b(K)$ of all bounded regular Borel measures on K satisfying certain axioms which are well known from convolutions of measures on locally compact abelian groups. We denote the identity of $(K,*)$ by e, and the hypergroup involution by $.^-$. It is well known (see [S]) that each commutative hypergroup $(K,*)$ admits a Haar measure $\omega_{(K,*)}$ which is unique up to some multiplicative constant. The dual space

$$\widehat{K}^* := \{\alpha \in C_b(K):\ \alpha \not\equiv 0,\ \int \alpha\, d(\delta_x * \delta_{\bar{y}}) = \alpha(x)\overline{\alpha(y)} \quad \text{for all } x,y \in K\}$$

is a locally compact space w.r.t. the topology of compact-uniform convergence. Elements of $\widehat{K}^*$ are called characters.

The Fourier transforms of $f \in L^1(K,\omega_{(K,*)})$ and $\mu \in M_b(K)$ are given by

$$\widehat{f}^*(\alpha) = \int_K \overline{\alpha(x)}\, f(x)\, d\omega_{(K,*)}(x) \qquad \text{and} \qquad \widehat{\mu}^*(\alpha) = \int_K \overline{\alpha}(x)\, d\mu(x) \qquad (\alpha \in \widehat{K}^*)$$

respectively. It is also well-known (Jewett [J]) that $\widehat{K}^*$ carries a unique Plancherel measure $\pi_{(K,*)}$ such that the Fourier transform on $L^1(K,\omega_{(K,*)}) \cap L^2(K,\omega_{(K,*)})$ extends uniquely to an isometric isomorphism between $L^2(K,\omega_{(K,*)})$ and $L^2(\widehat{K},\pi_{(K,*)})$. Notice that $supp\,\pi_{(K,*)}$ may be a proper subset of $\widehat{K}^*$. We here notice that the Fourier transform

$$\widehat{\cdot}^* : M_b(K) \longrightarrow C_b(supp\,\pi_{(K,*)}), \quad \mu \longmapsto \widehat{\mu}^*|_{supp\,\pi_{(K,*)}}$$

is injective (see Theorem 2.2.4 of [BH]).

2.2. Convolution semigroups and Lévy processes. A family $(\mu_t)_{t\ge0} \subset M^1(K)$ of probability measures on a commutative hypergroup $(K,*)$ is called a convolution semigroup, if

$\mu_s * \mu_t = \mu_{s+t}$ for all $s,t \geq 0$ with $\mu_0 = \delta_e$, and if $[0,\infty[\to M^1(K)$, $t \mapsto \mu_t$ is weakly continuous.

Let $(\mu_t)_{t\geq 0}$ be a convolution semigroup on $(K,*)$. A K-valued Markov process $X = (X_t)_{t\geq 0}$ with filtration $(\mathcal{F}_t)_{t\geq 0}$ (and defined on some probability space $(\Omega, \mathcal{F}, P)$) is called a Lévy process on $(K,*)$ associated with $(\mu_t)_{t\geq 0}$ and $(\mathcal{F}_t)_{t\geq 0}$, if its transition probabilities satisfy

$$P(X_t \in A |\, X_s = x) = (\mu_{t-s} * \delta_x)(A) \quad (0 \leq s \leq t,\ x \in K,\ A \subset K \text{ a Borel set}).$$

If the process X above is defined on a time interval $[0,T]$ only and has the properties above there, then it is called a restriction of a Lévy process on $(K,*)$ associated with $(\mu_t)_{t\geq 0}$ and $(\mathcal{F}_t)_{t\geq 0}$.

It can be easily checked that all (restricted) Lévy processes on $(K,*)$ are Feller processes and hence admit càdlàg versions; see [RV]. Moreover, one can construct martingales from Lévy processes on $(K,*)$ by using hypergroup characters. The following version of a martingale characterization of Lévy processes on commutative hypergroups was derived in [RV]; it is closely related with other versions for general (homogeneous) Markov processes as discussed, for instance, in Ch. 4 of [EK].

2.3. Lemma. *Let $(\mu_t)_{t\geq 0}$ be a convolution semigroup on the commutative hypergroup $(K,*)$. Then for each stochastic process X on K, which is adapted w.r.t. some filtration $(\mathcal{F}_t)_{t\geq 0}$, the following statements are equivalent:*

(1) X is a Lévy process on $(K,*)$ associated with $(\mu_t)_{t\geq 0}$ and $(\mathcal{F}_t)_{t\geq 0}$.

(2) For each $\alpha \in \widehat{K}^*$, the $\mathbb{C}$-valued process $(\widehat{\mu}_t^*(\overline{\alpha})^{-1} \cdot \alpha(X_t))_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-martingale.

(3) For each $\alpha \in supp\, \pi$, the process $(\widehat{\mu}_t(\overline{\alpha})^{-1} \cdot \alpha(X_t))_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-martingale.

An inspection of the proof of this lemma in [RV] shows that a corresponding result also holds for restricted Lévy processes.

2.4. Renormalization of commutative hypergroups. For commutative hypergroups $(K,*)$, the support $supp\, \pi_{(K,*)}$ of the Plancherel measure may be a proper subset of $\widehat{K}^*$. It was observed in [V1] that this property is closely related with the fact that commutative hypergroups $(K,*)$ may admit positive semicharacters, i.e., positive functions $\alpha_0 \in C(K)$ that admit all properties of characters except that they may be unbounded. It was shown in [V1] that each positive semicharacter α_0 on a commutatice hypergroup $(K,*)$ induces a new hypergroup structure $(K,\bullet)$ (where, by convention, the underlying positive semicharacter α_0 as index will be suppressed); the convolution $\bullet$ is determined uniquely by the convolution of point measures:

$$\delta_x \bullet \delta_y = \frac{1}{\int \alpha_0 \, d(\delta_x * \delta_y)} \cdot \alpha_0 \cdot (\delta_x * \delta_y) \qquad (x, y \in K).$$

Identity and involution of $(K, \bullet)$ are the same as of $(K, *)$. We next give a list of further connections between the data of the hypergroups $(K, *)$ and $(K, \bullet)$; for details see [V1]:

(1) If $\mu, \nu \in M_b(K)$ satisfy $\alpha_0\mu, \alpha_0\nu \in M_b(K)$, then $\alpha_0\mu \bullet \alpha_0\nu = \alpha_0(\mu * \nu)$.

(2) $\omega_{(K,\bullet)} := \alpha_0^2 \omega_{(K,*)}$ is "the" Haar measure of $(K, \bullet)$.

(3) The dual space of $(K, \bullet)$ is given by

$$\widehat{K}^\bullet := \{\alpha/\alpha_0 : \alpha \text{ a semicharacter of } (K, *) \text{ with } |\alpha| \le \alpha_0\}.$$

(4) If $\pi_{(K,\bullet)}$ denotes the Plancherel measure of $(K, \bullet)$ on $\widehat{K}^\bullet$, then the mapping $\widehat{K}^* \longrightarrow \widehat{K}^\bullet$, $\alpha \longmapsto \alpha/\alpha_0$ is a homeomorphism that maps $\pi_{(K,*)}$ into $\pi_{(K,\bullet)}$.

(5) The hypergroups $(K, *)$ and $(K, \bullet)$ may be interchanged above by using the fact that $1/\alpha_0$ is a positive semicharacter of $(K, \bullet)$, and that the associated renormalized hypergroup structure is just the original hypergroup $(K, *)$.

Let α_0 be a positive semicharacter on a commutative hypergroup $(K, *)$. We now show how convolution semigroups on $(K, *)$ can be transformed into convolution semigroups on $(K, \bullet)$. For this we say that a convolution semigroup $(\mu_t)_{t \ge 0}$ on $(K, *)$ is α_0-continuous whenever

$$[0, \infty[\to [0, \infty[, \quad t \longmapsto h(t) := \int_K \alpha_0 \, d\mu_t$$

is finite and continuous. If $\alpha_0 \in \widehat{K}^*$ is a positive character, then clearly each convolution semigroup on $(K, *)$ is α_0-continuous.

2.5. Lemma. *Let α_0 be a positive semicharacter and $(\mu_t)_{t \ge 0}$ an α_0-continuous convolution semigroup on $(K, *)$ Then, for all $s, t \ge 0$, $h(s) \cdot h(t) = h(s + t)$, and $\left(\mu_t^{\alpha_0} := \frac{1}{h(t)} \cdot \alpha_0 \mu_t\right)_{t \ge 0}$ is a convolution semigroup on $(K, \bullet)$.*

Proof. Clearly, $\mu_t^{\alpha_0} \in M^1(K)$ for all $t \ge 0$. Hence, for all $s, t \ge 0$, $\mu_s^{\alpha_0} \bullet \mu_t^{\alpha_0} \in M^1(K)$. Moreover, by Section 2.4,

$$\mu_s^{\alpha_0} \bullet \mu_t^{\alpha_0} = \frac{1}{h(s)h(t)} (\alpha_0 \mu_s) \bullet (\alpha_0 \mu_t) = \frac{1}{h(s)h(t)} \alpha_0 (\mu_s * \mu_t) = \frac{h(s+t)}{h(s)h(t)} \frac{1}{h(s+t)} \alpha_0 \mu_{s+t}.$$

As $\frac{1}{h(s+t)} \alpha_0 \mu_{s+t} \in M^1(K)$, it follows that $h(s) \cdot h(t) = h(s+t)$ and $\mu_s^{\alpha_0} \bullet \mu_t^{\alpha_0} = \mu_{s+t}^{\alpha_0}$. The continuity of h finally ensures that $t \mapsto \mu_t^{\alpha_0}$ is vaguely and hence weakly continuous. $\square$

The following Girsanov formula connects Lévy processes associated with $(\mu_t)_{t\geq 0}$ and $(\mu_t^{\alpha_0})_{t\geq 0}$.

2.6. Theorem. *Let α_0 be a positive semicharacter and $(\mu_t)_{t\geq 0}$ an α_0-continuous convolution semigroup on the commutative hypergroup $(K,*)$. Let $(X_t)_{t\geq 0}$ be a Lévy process on $(K,*)$ with filtration $(\mathcal{F}_t)_{t\geq 0}$ and with convolution semigroup $(\mu_t)_{t\geq 0}$ that is defined on some probability space $(\Omega,\mathcal{F},P)$. Then for each $T\geq 0$, the process $(X_t)_{t\in[0,T]}$ on the probability space $(\Omega,\mathcal{F}_T,\frac{1}{h(T)}\alpha_0(X_T)\cdot P)$ is the restriction of a Lévy process on $(K,\bullet)$ associated with $(\mu_t^{\alpha_0})_{t\geq 0}$.*

Proof. As $(X_t)_{t\geq 0}$ is a Lévy process on $(K,*)$ with filtration $(\mathcal{F}_t)_{t\geq 0}$ and with convolution semigroup $(\mu_t)_{t\geq 0}$, we see that for all $s,t\geq 0$ and P-almost all $\omega\in\Omega$,

$$E(\alpha_0(X_{s+t})|\mathcal{F}_s)(\omega)=E(\alpha_0(X_{s+t})|X_s)(\omega)=\int_K \alpha_0\,d(\mu_t*\delta_{X_s(\omega)})=h(t)\cdot\alpha_0(X_s(\omega)).$$

Using $h(s+t)=h(s)h(t)$, we obtain that $(Z_t:=\frac{1}{h(t)}\alpha_0(X_t))_{t\geq 0}$ is a positive $(\mathcal{F}_t)_{t\geq 0}$-martingale with $E(Z_t)=1$. In particular, $(Z_t\cdot P|_{\mathcal{F}_t})_{t\geq 0}$ is a family of probability measures with

$$(Z_t\cdot P|_{\mathcal{F}_t})_{\mathcal{F}_s}=Z_s\cdot P|\mathcal{F}_s \qquad \text{for} \quad s,t\geq 0.$$

Now let $\alpha\in supp\,\pi_{(K,\bullet)}$ be a character of $(K,\bullet)$ contained in the support of the Plancherel measure. Section 2.4 shows that $\tilde{\alpha}:=\alpha\cdot\alpha_0$ is a character of $(K,*)$, and, by the definition of $\mu_t^{\alpha_0}$,

$$\widehat{\mu}_t^*(\overline{\tilde{\alpha}})=\int_K \alpha(x)\alpha_0(x)\,d\mu_t(x)=h(t)\cdot(\mu_t^{\alpha_0})^{\wedge\bullet}(\overline{\alpha}) \qquad (t\geq 0)$$

where $.^{\wedge\bullet}$ denotes the Fourier transform w.r.t. $(K,\bullet)$. Lemma 2.3 now yields that

$$\left(\frac{1}{(\mu_t^{\alpha_0})^{\wedge\bullet}(\overline{\alpha})}\cdot Z_t\alpha(X_t)=\frac{1}{\widehat{\mu}^*(\overline{\tilde{\alpha}})}\tilde{\alpha}(X_t)\right)_{t\geq 0}$$

is an $(\mathcal{F}_t)_{t\geq 0}$-martingale on $(\Omega,\mathcal{F},P)$. Using the properties of $(Z_t)_{t\geq 0}$, we see that for $T>0$,

$$\left(\frac{1}{(\mu_t^{\alpha_0})^{\wedge\bullet}(\overline{\alpha})}\cdot\alpha(X_t)\right)_{t\in[0,T]}$$

is an $(\mathcal{F}_t)_{t\in[0,T]}$-martingale on the probability space $(\Omega,\mathcal{F},Z_TP)$. As this holds for all $\alpha\in supp\,\pi_{(K,\bullet)}$, Lemma 2.3 implies that the process $(X_t)_{t\in[0,T]}$ on $(\Omega,\mathcal{F},Z_TP)$ is the restriction of a Lévy process on $(K,\bullet)$ associated with $(\mu_t^{\alpha_0})_{t\geq 0}$. □

We now give an extension of the preceding result to the complete time interval $[0,\infty[$.

2.7. Theorem. *Let* α_0 *be a positive semicharacter and* $(\mu_t)_{t\geq 0}$ *an* α_0*-continuous convolution semigroup on the commutative Polish hypergroup* $(K,*)$. *Let* $(X_t)_{t\geq 0}$ *be a Lévy process on* $(K,*)$ *associated with* $(\mu_t)_{t\geq 0}$ *defined on the probability space* $(\Omega,\mathcal{F},P)$ *with*

$$\Omega = \mathcal{D}(K) := \{f : [0,\infty[\to K,\ f \text{ càdlàg}\}$$

and equipped with the right-continuous and complete induced filtration $(\mathcal{F}_t)_{t\geq 0}$. *Then there exists a unique probability measure* Q *on* $(\Omega, \sigma(\mathcal{F}_t : t\geq 0))$ *with*

$$Q|_{\mathcal{F}_t} = \frac{1}{h(t)}\alpha_0(X_t)P|_{\mathcal{F}_t} \qquad for \quad t\geq 0,$$

and with respect to Q, *the process* $(X_t)_{t\geq 0}$ *is a Lévy process on* $(K,\bullet)$ *associated with* $(\mu_t^{\alpha_0})_{t\geq 0}$.

Proof. In view of the proof of the preceding result it suffices to check existence and uniqueness of Q. Uniqueness, however, is clear, and the existence follows from Lemma 16.18 of [Kal]. □

2.8. Remark. Lemmas 2.3 and 2.5 as well as Theorems 2.6 and 2.7 can easily be adapted to the setting of time-homogeneous random walks $(X_n)_{n\geq 0}$ on commutative hypergroups.

2.9. Remark. Theorems 2.6 and 2.7 may be regarded as special cases of more general Girsanov-type formulas for Feller processes which satisfy certain technical restrictions. We shall present details of this generalization elsewhere and include some ideas here only:

Assume that α_0 is a positive semicharacter and $(\mu_t)_{t\geq 0}$ an α_0-continuous convolution semigroup on some commutative hypergroup $(K,*)$. The associated Lévy processes are Feller, and the generator G of the associated Feller semigroup on $C_0(K)$ is given by

$$Gf(x) = \lim_{t\to 0}\frac{1}{t}(\mu_t^- * f(x) - f(x)) \qquad (x\in K,\ f\in D(G))$$

where the domain $D(G)$ of G is $\|.\|_\infty$-dense in $C_0(K)$; see [RV]. Now consider the generator G^{α_0} of the Feller semigroup on $C_0(K)$ that is associated with the renormalized convolution semigroup $(\mu_t^{\alpha_0})_{t\geq 0}$ on $(K,\bullet)$. Then, using the notation above, we have

$$((\mu_t^{\alpha_0})^- \bullet f)(x) = \frac{1}{h(t)}((\alpha_0\mu_t)^- \bullet f)(x) = \frac{1}{h(t)\alpha_0(x)}(\mu_t * \alpha_0 f)(x)$$

(see p. 408 of [V1]). Moreover, by Lemma 2.5 we have $h(t) = e^{ct}$ for some $c\in\mathbb{R}$, and hence

$\lim_{t\to 0} \frac{1}{t}(1/h(t) - 1) = -c$. Hence,

$$\begin{aligned} G^{\alpha_0} f(x) &= \lim_{t\to 0} \frac{1}{t}(((\mu_t^{\alpha_0})^- \bullet f)(x) - f(x)) = \lim_{t\to 0} \frac{1}{t}\Big(\frac{1}{h(t)\alpha_0(x)}(\mu_t * \alpha_0 f)(x) - f(x)\Big) \\ &= \frac{1}{\alpha_0(x)} \lim_{t\to 0} \frac{1}{t}\Big(\frac{1}{h(t)}(\mu_t * \alpha_0 f)(x) - (\alpha_0 f)(x)\Big) \\ &= \frac{1}{\alpha_0(x)} G(\alpha_0 f)(x) + \frac{1}{\alpha_0(x)} \lim_{t\to 0}\Big(\frac{1}{t}(1/h(t) - 1)(\mu_t * \alpha_0 f)(x)\Big) \\ &= \frac{1}{\alpha_0(x)} G(\alpha_0 f)(x) - cf(x). \end{aligned}$$

Therefore, if M_g is the multiplication operator with some function $g \in C(K)$, then formally

$$G^{\alpha_0} = M_{1/\alpha_0} \circ G \circ M_{\alpha_0} - c \tag{2.1}$$

where α_0 is an eigenfunction of G with eigenvalue c.

We expect that Theorems 2.6 and 2.7 can be extended in this way to arbitrary generators G of Feller semigroups on locally compact spaces K and arbitrary "eigenfunctions" $\alpha_0 \in C(K)$ of G with eigenvalue c under certain restrictions concerning the domain of G. We mention that a related result for Feller processes on finite state spaces is given in Section IV.22 of [RW].

Lemma 2.5 admits the following converse statement:

2.10. Lemma. *Let α_0 be a positive semicharacter on $(K,*)$ with $\alpha_0 \geq 1$, and let $(\mu_t)_{t\geq 0}$ a convolution semigroup on $(K,*)$ with generator G. Assume that*

$$G^{\alpha_0} := M_{1/\alpha_0} \circ G \circ M_{\alpha_0} - c$$

(where c satisfies $G\alpha_0 = c\alpha_0$, and M is given as in 2.9) is the generator of a convolution semigroup $(\tilde{\mu}_t^{\alpha_0})_{t\geq 0}$ on the modified hypergroup $(K,\bullet)$. Then $(\mu_t)_{t\geq 0}$ is α_0-continuous, and $(\tilde{\mu}_t^{\alpha_0})_{t\geq 0}$ is equal to the convolution semigroup $(\mu_t^{\alpha_0})_{t\geq 0}$ of Lemma 2.5.

Proof. By our assumption, $1/\alpha_0$ is a positive character on $(K,\bullet)$. Now apply Lemma 2.5 and Remark 2.9 to $1/\alpha_0$ and the $1/\alpha_0$-continuous convolution semigroup $(\tilde{\mu}_t^{\alpha_0})_{t\geq 0}$ on $(K,\bullet)$. Then the renormalization of $\bullet$ is just $*$, and the generator of the convolution semigroup on $(K,*)$, which is the deformation of $(\tilde{\mu}_t^{\alpha_0})_{t\geq 0}$ according to 2.5, is given by G. Hence, for $t \geq 0$,

$$\mu_t = \frac{1}{\tilde{h}(t)\cdot\alpha_0}\tilde{\mu}_t^{\alpha_0} \quad \text{where} \quad t \longmapsto \tilde{h}(t) := \int_K 1/\alpha_0 \, d\tilde{\mu}_t^{\alpha_0} \quad \text{is continuous.}$$

This shows that the function h of Lemma 2.5 is equal to $1/\tilde{h}$ and hence continuous. The remaining assertions are now obvious. ☐

3 Examples

In this section we present a few examples to which the Girsanov-type formulas 2.6 and 2.7 may be applied. The most prominent examples will be Bessel processes which may be regarded as Lévy processes on the Bessel-Kingman hypergroups and their modifications. As a preparation we first discuss positive semicharacters on general Sturm-Liouville hypergroups on $[0,\infty[$.

3.1 Sturm-Liouville hypergroups on $[0,\infty[$

(1) A function $A \in C([0,\infty[) \cap C^1(]0,\infty[)$ is called admissible if $A(x) > 0$ for $x > 0$, and if there exist constants $\epsilon > 0$, $\alpha_0 \geq 0$ and $\alpha_1 \in C^\infty(]-\epsilon,\epsilon[)$ with

$$A'(x)/A(x) = \frac{\alpha_0}{x} + x \cdot \alpha_1(x) \quad \text{for all } x \in]0,\epsilon[.$$

In the singular case $\alpha_0 > 0$ we assume in addition that α_1 is even.

(2) The Sturm-Liouville operator associated with an admissible A is defined by

$$L^A f(x) := -\frac{1}{A(x)} \cdot (A(x) \cdot f'(x))' \quad \text{for } f \in C^2(]0,\infty[),\ x > 0.$$

(3) A hypergroup $([0,\infty[,*)$ is called a Sturm-Liouville hypergroup if there exists an admissible function A such that for each even $f \in C^\infty(\mathbb{R})$ the function $u_f(x,y) := \int_0^\infty f\, d(\delta_x * \delta_y)$ $(x,y \geq 0)$ satisfies $u_f \in C^2([0,\infty[^2)$ with

$$L_x^A u(x,y) - L_y^A u(x,y) = 0 \qquad \text{and} \qquad (u_f)_y(x,0) = 0 \qquad \text{for} \quad x,y \geq 0$$

where subscripts indicate variables with respect to which the operator L^A is applied.

3.1. Facts. Let $([0,\infty[,*)$ be a Sturm-Liouville hypergroup associated with some admissible function A that satisfies some further technical restriction; see [Z] and Ch. 3.5 of [BH]. Then the following statements hold:

(1) $\rho := \frac{1}{2}\lim_{x\to\infty} A'(x)/A(x)$ exists with $\rho \geq 0$; it is called the index of K.

(2) A function $\alpha \in C([0,\infty[)$ is multiplicative on K, i.e., $(\delta_x * \delta_y)(\alpha) = \alpha(x)\alpha(y)$ for all $x, y \geq 0$, if and only if $\alpha \in C^2([0,\infty[)$, and if α is the unique solution of the eigenvalue problem

$$L^A\alpha = s_\alpha \cdot \alpha \quad \text{with} \quad \alpha(0) = 1,\ \alpha'(0) = 0 \quad \text{for some } s_\alpha \in \mathbb{C}.$$

According to [BH, Z], we parametrize the eigenvalues by $\lambda_\alpha^2 + \rho^2 = s_\alpha$ with $\lambda_\alpha \in \mathbb{C}$. In this notation, the dual space $\widehat{K}$ and the support of the Plancherel measure are given by $\widehat{K} = \{\alpha \text{ multiplicative}: \lambda_\alpha \in [0,\infty[\cup i]0,\rho]\}$ and $supp\,\pi = \{\alpha \in \widehat{K} : \lambda_\alpha \in [0,\infty[\}$. Moreover, α is a positive semicharacter if and only if $\lambda_\alpha \in i\cdot[0,\infty[$ holds; see [V1, Z].

(3) If α is a positive character on $([0,\infty[,*)$ with $\lambda_\alpha \in i\cdot[0,\infty[$, then the associated modified hypergroup $([0,\infty[,\bullet)$ is the Sturm-Liouville hypergroup associated with the admissible function $A_\alpha(x) := \alpha(x)^2 A(x)$; see [V1].

3.2. Diffusions on $[0,\infty[$ as Lévy processes. It is well known (see [C,RV]) that for each Sturm-Liouville hypergroup $([0,\infty[,*)$ with admissible A, the operator $-L^A$ is the generator of a convolution semigroup $(\mu_t)_{t\geq 0}$ on $([0,\infty[,*)$. Now let α is an arbitrary positive character on $([0,\infty[,*)$ with $\lambda_\alpha \in i\cdot[0,\infty[$. We now check that the assumptions of Theorems 2.6 and 2.7 are satisfied:

3.3. Lemma. *In the above setting, $(\mu_t)_{t\geq 0}$ is α-continuous with*

$$h(t) := \int_0^\infty \alpha\, d\mu_t = e^{-t(\lambda_\alpha^2+\rho^2)} \qquad (t \geq 0)$$

Proof. The lemma is obvious for $\alpha \in \widehat{K}$, i.e., $\lambda_\alpha^2 + \rho^2 \geq 0$. Otherwise we have $\alpha \geq 1$ on $[0,\infty[$ (see [BH] or [Z]) and we may consider the modified hypergroup $([0,\infty[,\bullet)$ with $A_\alpha := \alpha^2 A$ which is associated with α. A short computation yields

$$(M_{1/\alpha} \circ (-L^A) \circ M_\alpha) + \lambda_\alpha^2 + \rho^2 = -L^{\alpha^2 A}$$

where, by our considerations above, $-L^{\alpha^2 A}$ is the generator of a convolution semigroup on $([0,\infty[,\bullet)$. The lemma now follows from Lemma 2.10. □

Theorem 2.7 now reads as follows in our present case:

3.4. Theorem. *Let $([0,\infty[,*)$ be a Sturm-Liouville hypergroup with associated function A and index ρ. Then the operator*

$$-L^A = \frac{1}{A(x)} \cdot \frac{d}{dx}(A(x) \cdot \frac{d}{dx})$$

is the generator of a convolution semigroup $(\mu_t)_{t\geq 0}$ on $([0,\infty[,*)$. Let $(X_t)_{t\geq 0}$ be an associated Lévy process $([0,\infty[,*)$, i.e., $(X_t)_{t\geq 0}$ is a diffusion with generator $-L^A$. Assume that $(X_t)_{t\geq 0}$ is defined on the probability space $(\Omega,\mathcal{F},P)$ with

$$\Omega := \{f : [0,\infty[\to [0,\infty[,\ f \text{ continuous}\}$$

and is equipped with the right-continuous, complete induced filtration $(\mathcal{F}_t)_{t\geq 0}$. Then for each positive semicharacter α on $([0,\infty[,*)$, there exists a unique probability measure Q on $(\Omega, \sigma(\mathcal{F}_t :\ t \geq 0))$ with

$$Q|_{\mathcal{F}_t} = e^{t(\lambda_\alpha^2+\rho^2)}\alpha(X_t)P|_{\mathcal{F}_t} \qquad for \quad t \geq 0,$$

and with respect to Q, the process $(X_t)_{t\geq 0}$ is a diffusion with generator $-L^{\alpha^2 A}$.

We now investigate concrete examples, namely Bessel processes which are Lévy processes on the so-called Bessel-Kingman hypergroups.

3.2 Bessel-Kingman hypergroups and Bessel processes

3.5. Bessel-Kingman hypergroups (see [BH, J, Ki, RV]). For a first motivation, fix some integer $n \geq 1$ and consider the Banach spaces

$$M_b^{rad}(\mathbb{R}^n) := \{\mu \in M_b(\mathbb{R}^n) : A(\mu) = \mu \text{ for all rotations } A \in SO(n)\} \quad \text{for } n \geq 2$$

$$\text{and} \quad M_b^{rad}(\mathbb{R}^1) := \{\mu \in M_b(\mathbb{R}) : \mu(B) = \mu(-B) \text{ for all Borel sets } B \subset \mathbb{R}\}$$

consisting of all "radial" measures on $\mathbb{R}^n$. $M_b^{rad}(\mathbb{R}^n)$ is a Banach-$*$-subalgebra of $M_b(\mathbb{R}^n)$, and the extension of the projection $\Phi : \mathbb{R}^n \longrightarrow [0,\infty[,\ x \longmapsto |x| = (x_1^2 + \ldots + x_n^2)^{1/2}$ to measures is an isometric isomorphism between the Banach-$*$-algebras $M_b^{rad}(\mathbb{R}^n)$ and $M_b([0,\infty[)$ where the second space has to carry the corresponding convolution and involution. This leads to a symmetric hypergroup $([0,\infty[,*)$, the "Bessel-Kingman hypergroup of index $\alpha = n/2 - 1$".

The Bessel-Kingman hypergroup of arbitrary index $\alpha \geq -1/2$ is defined as the Sturm-Liouville hypergroup on $[0,\infty[$ with admissible function

$$A_\alpha(x) = x^{2\alpha+1} \qquad \text{for } x \geq 0.$$

The dual space is given by $\{\varphi_\lambda^\alpha : \lambda \geq 0\}$ where the φ_λ^α satisfy $\varphi_\lambda^\alpha(x) := j_\alpha(\lambda x)$ with the normalized Bessel functions

$$j_\alpha(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(\alpha+1)}{2^{2k} k!\, \Gamma(\alpha+k+1)} z^{2k} \quad (z \in \mathbb{C}).$$

3.6. Bessel processes. The convolution semigroup $(\rho_t^\alpha)_{t\geq 0}$ on the Bessel-Kingman hypergroup of index $\alpha \geq -1/2$ with generator

$$-L^A/2 = \frac{1}{2}\frac{d^2}{dx^2} + \frac{\alpha + 1/2}{x}\frac{d}{dx}$$

is given by the Rayleigh distributions

$$d\rho_t^\alpha(x) = \frac{1}{\Gamma(\alpha+1)}\,\frac{2^\alpha}{t^{\alpha+1}}\, x^{2\alpha+1}\, e^{-x^2/(2t)}\, dx \quad \text{on } [0,\infty[\quad \text{for } t > 0; \tag{3.1}$$

see 7.3.18 of [BH]. Associated diffusions are called called Bessel processes of index α. Notice that in this notation, projections $(\Phi(B_t^n))_{t\geq 0}$ of n-dimensional Brownian motions $(B_t^n)_{t\geq 0}$ are Bessel processes of index $\alpha = n/2 - 1$.

We next consider the modification of Bessel-Kingman hypergroups.

3.7. Modified Bessel-Kingman hypergroups and non-central Bessel processes. For any $\alpha \geq -1/2$ and $\rho \geq 0$, the Bessel function $\varphi_{i\rho}^{\alpha}$ is a positive semicharacter on the Bessel-Kingman hypergroup of index α. The associated modified Sturm-Liouville hypergroup will be called modified Bessel-Kingman hypergroup of index α and non-centrality parameter ρ; the associated admissible function is

$$A_{\alpha,\rho}(x) := x^{2\alpha+1} \cdot (\varphi_{i\rho}^{\alpha}(x))^2 \quad (x \geq 0).$$

Diffusions on $[0, \infty[$ with the differential operator

$$-L^{A_{\alpha,\rho}}/2 = \frac{1}{2}\frac{d^2}{dx^2} + \Big(\frac{\alpha+1/2}{x} + \frac{\varphi_{i\rho}^{\alpha\prime}}{\varphi_{i\rho}^{\alpha}}\Big)\frac{d}{dx}$$

are called non-central Bessel processes with index α and non-centrality parameter ρ.

To motivate these notions, consider the n-dimensional Euclidean space $\mathbb{R}^n$ $(n \geq 1)$. Fix some non-centrality parameter $\rho \geq 0$ and consider the multiplicative mapping

$$h_\rho : \mathbb{R}^n \to]0, \infty[, \quad x \mapsto e^{<c_\rho, x>} \quad \text{with} \quad c_\rho := (\rho, 0, \ldots, 0) \in \mathbb{R}^n.$$

By [V2], the vector space

$$\{\mu \in M_b(\mathbb{R}^n) : \mu = h_\rho \cdot \nu,\ \nu \in M_b^{rad}(\mathbb{R}^n) \text{ with compact support}\}$$

is a subalgebra of $M_b(\mathbb{R}^n)$ whose total variation-closure $M_b^{rad,\rho}(\mathbb{R}^n)$ is a Banach subalgebra of $M_b(\mathbb{R}^n)$. Similar as in Section 3.5, the projection $\Phi : \mathbb{R}^n \to [0, \infty[$ leads to an isometric isomorphism between the Banach algebras $M_b^{rad,\rho}(\mathbb{R}^n)$ and $M_b([0, \infty[)$ where the latter has to be equipped with the corresponding "convolution". It can be easily verified (see [V2]) that $[0, \infty[$ with this convolution is the modified Bessel-Kingman hypergroup of index $\alpha = n/2 - 1$ and non-centrality parameter ρ. Moreover, if $(B_t^{n,\rho})_{t\geq 0}$ is an n-dimensional Brownian motion with drift c_ρ (i.e., $(B_t^{n,\rho} - tc_\rho)_{t\geq 0}$ is a Brownian motion), then $(\Phi(B_t^n))_{t\geq 0}$ is a non-central Bessel process with index $\alpha = n/2 - 1$ and non-centrality parameter ρ.

We now reformulate Theorem 3.4.

3.8. Theorem. *Let $(X_t)_{t\geq 0}$ be a Bessel process on $[0, \infty[$ of index $\alpha \geq -1/2$ which is defined on the probability space $(\Omega, \mathcal{F}, P)$ with*

$$\Omega := \{f : [0, \infty[\to [0, \infty[,\ f \text{ continuous}\},$$

and which is equipped with the right-continuous, complete induced filtration $(\mathcal{F}_t)_{t\geq 0}$. Then for each $\rho \geq 0$, there exists a unique probability measure Q on $(\Omega, \sigma(\mathcal{F}_t : t \geq 0))$ with

$$Q|_{\mathcal{F}_t} = e^{-t\rho^2/2}\varphi_{i\rho}^{\alpha}(X_t)P|_{\mathcal{F}_t} \qquad for \quad t \geq 0,$$

and with respect to Q, the process $(X_t)_{t\geq 0}$ is a non-central Bessel process with index α and non-centrality parameter ρ.

3.9. Remark. In this section we obtained non-central Bessel processes from central ones via hypergroup deformations. On the other hand we used some change of drift argument in the introduction for $\alpha = n/2 - 1$, $n \in \mathbb{N}$, in order to obtain the same result. Both methods are, in fact, related from a more abstract point of view via deformations of orbit hypergroups; for the background and possible further examples we refer to [V2].

References

[AS] M. Abramowitz, I. Stegun, *Handbook of Mathematical Functions.* Wiley, 1972.

[BH] W.R. Bloom, H. Heyer, *Harmonic Analysis of Probability Measures on Hypergroups.* De Gruyter, 1995.

[C] H. Chèbli, *Opérateurs de translation généralisée et semi-groupes de convolution.* In: Théorie du potentiel et analyse harmonique, Lecture Notes in Math., vol. 404, Springer-Verlag, Berlin, 1974, pp. 35-59.

[EK] S.N. Ethier, T.G. Kurtz, *Markov Processes, Characterization and Convergence.* Wiley, Chichester – New York, 1986.

[I] M. Ibero, *Intégrales stochastiques multiplicatives et construction de diffusions sur un groupe de Lie.* Bull. Soc. Math. France **100** (1976), 175–191.

[J] R.I. Jewett, *Spaces with an abstract convolution of measures.* Adv. Math. **18** (1975), 1 – 101.

[Kal] O. Kallenberg, *Foundations of Modern Probability.* Springer-Verlag, 1997.

[Kar] R.L. Karandikar, *Girsanov-type formula for a Lie group valued Brownian motion.* In: Sem. Probabilité XVII, Lecture Notes in Math. 986 (1983), pp. 198-204.

[Ki] J.F.C. Kingman, *Random walks with spherical symmetry.* Acta Math. **109** (1963), 11 – 53.

[RV] C. Rentzsch, M. Voit, *Lévy processes on commutative hypergroups.* Contemp. Math., to appear.

[RY] D. Revuz, M. Yor, *Continuous Martingales and Brownian Motion.* Springer-Verlag, 1994.

[RW] L.C.G. Rogers, D. Williams, *Diffusions, Markov Processes, and Martingales, Vol. II.* Wiley: Chichester – New York, 1987.

[S] R. Spector, *Mesures invariantes sur les hypergroupes.* Trans. Amer. Math. Soc. **239** (1978), 147 – 166.

[V1] M. Voit, *Positive characters on commutative hypergroups and some applications.* Math. Z. **198** (1988), 405 - 421.

[V2] M. Voit, *A generalization of orbital morphisms of hypergroups.* In: Probability Measures on Groups X (Proc. Conf. Oberwolfach 1990) Plenum Press 1991, pp. 425 - 434.

[V3] M. Voit, *Martingale characterizations of stochastic processes on compact Lie groups.* Probab. Math. Stat. **19** (1999), 211-227.

[V4] M. Voit, *A Lévy-characterization of Gaussian processes on matrix groups.* Preprint.

[WW] H. von Weizsäcker, G. Winkler, *Stochastic Integrals.* Vieweg 1990.

[Z] Hm. Zeuner, *Moment functions and laws of large numbers on hypergroups.* Math. Z. **211** (1992), 369 - 407.

ON THE PRODUCT OF RIESZ SETS IN DUAL OBJECTS OF COMPACT GROUPS

HIROSHI YAMAGUCHI

ABSTRACT. Let E_i be a Riesz set in the dual object of a compact group $K_i (i = 1, 2)$. We show that the product set $E_1 \times E_2$ is a Riesz set in the dual object of $K_1 \times K_2$. We also give a result on compact groups related to a result of Glicksberg and Graham concerned with "small p set".

1. INTRODUCTION

Let $\mathbb{T}$ and $\mathbb{Z}$ be the circle group and the integer group respectively. $\mathbb{Z}^+$ denotes the semigroup of nonnegative integers. By a well-known theorem of Bochner, each measure on $\mathbb{T}^2$ whose Fourier-Stieltjes transform vanishes off $\mathbb{Z}^+ \times \mathbb{Z}^+$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{T}^2$. This shows that the product set $\mathbb{Z}^+ \times \mathbb{Z}^+$ of the Riesz set $\mathbb{Z}^+$ in $\mathbb{Z}$ is a Riesz set in $\hat{\mathbb{T}}^2 \cong \mathbb{Z} \times \mathbb{Z}$. This holds for locally compact abelian (LCA) groups. For a LCA group G, let $L^1(G)$ and $M(G)$ be the usual group algebra and the Banach algebra of bounded regular measures on G respectively. For $\mu \in M(G)$, $\hat{\mu}$ stands for the Fourier-Stieltjes transform of μ. Let m_G denote the Haar measure of G.

Definition 1.1. *Let G be a LCA group with the dual group $\hat{G}$, and let $p \in \mathbb{N}$ (the natural numbers). A closed subset E of $\hat{G}$ is called a small p set if*

$$\forall \mu \in M_E(G) \Longrightarrow \mu^p = \overbrace{\mu * \cdots * \mu}^{p} \in L^1(G), \tag{1.1}$$

where $M_E(G) = \{\mu \in M(G) : \hat{\mu} = 0 \, on \, E^c\}$. In particular, a small 1 set is called a Riesz set.

Theorem 1.1 (cf. [12, Corollary], [10, Theorem 6]). *Let G_1 and G_2 be LCA groups, and let $p \in \mathbb{N}$. Let E_1 and E_2 be small p sets in $\hat{G}_1$ and $\hat{G}_2$ respectively. Then $E_1 \times E_2$ is a small p set in $\widehat{G_1 \oplus G_2}$.*

A condition for a set in the dual group of a LCA group to be a small 2 set was obtained by Glicksberg([6]) and Graham([7]).

Theorem 1.2 (cf. [7, Therem 1(b)]). *Let G be a LCA group, and let E be a closed set in $\hat{G}$ satisfying the following:*

$$\{\gamma \in \hat{G} : m_{\hat{G}}(E \cap (\gamma - E)) < \infty\} \text{ is dense in } \hat{G}. \tag{1.2}$$

Let $\mu, \nu \in M_E(G)$. Then $|\mu| * |\nu| \in L^1(G)$. In particular, E is a small 2 set.

On the other hand, the author proved that the product set of a Riesz set in the dual group of a compact abelian group and a Riesz set in the dual object of a compact group is a Riesz set ([16, Corollary 2.1]). In this paper, we shall show that results corresponding to Theorems 1.1 and 1.2 hold for (noncommutative) compact groups. In section 2, we state notation and our results. In section 3, we give the proofs of our results.

2. Notation and results

We often quote notation from the book of Hewitt and Ross ([9]). Let K be a compact group, and let Σ_K be the dual object of K, i.e., the set of equivalence classes of all continuous irreducible unitary representations of K. For a closed normal subgroup H of K, $A(\Sigma_K, H)$ denotes the annihilator of H in Σ_K (cf. [9, (28.7) Definition]). m_K stands for the Haar measure of K. Let $C(K)$ be the space of continuous functions on K and $M(K)$ the space of bounded regular measures on K. Let $L^1(K)$ be the group algebra. We identify $L^1(K)$ with the space of absolutely continuous measures in $M(K)$, by the Radon-Nikodym theorem. Set $M^+(K) = \{\mu \in M(K) : \mu \geq 0\}$. For $\mu \in M(K)$ and $f \in L^1(|\mu|)$, we often write $\mu(f)$ as $\int_K f(x)d\mu(x)$.

For $\sigma \in \Sigma_K$, $U^{(\sigma)}$ denotes a continuous irreducible unitary representation of K in σ with the representation space H_σ of dimension d_σ. For $\mu \in M(K)$, $\hat{\mu}$ denotes the Fourier transform of μ, i.e., for $\sigma \in \Sigma_K$ and $\xi, \eta \in H_\sigma$,

$$\langle \hat{\mu}(\sigma)\xi, \eta\rangle = \int_K \langle \overline{U}_x^{(\sigma)}\xi, \eta\rangle d\mu(x), \tag{2.1}$$

where $\overline{U}_x^{(\sigma)} = D_\sigma U_x^{(\sigma)} D_\sigma$ and D_σ is a conjugation on H_σ. Let $spec(\mu) = \{\sigma \in \Sigma_K : \hat{\mu}(\sigma) \neq 0\}$. Let $\overline{\sigma}$ denote the equivalence class in Σ_K that contains the representation $\overline{U}^{(\sigma)}$. For a subset E of Σ_K, set $M_E(K) = \{\mu \in M(K) : spec(\mu) \subset E\}$.

For $\sigma, \tau \in \Sigma_K, \sigma \times \tau$ is defined (cf. [9, (27.35) Definition]). $\sigma \times \tau$ is a finite subset of Σ_K. For a subset P of Σ_K, $[P]$ denotes the smallest subset of Σ_K that contains P and is closed under the operation '$\times$' and conjugation (cf. [9, (27.35) Definition]).

For $\sigma \in \Sigma_K, \mathfrak{T}_\sigma(K)$ is the linear span of all functions $x \to \langle U_x^{(\sigma)}\xi, \eta\rangle$, where $\xi, \eta \in H_\sigma$. Let $\mathfrak{T}(K)$ be the space of trigonometric polynomials on K, i.e., $\mathfrak{T}(K)$ is the set of finite linear combinations of functions $x \to \langle U_x^{(\sigma)}\xi, \eta\rangle$, where $\sigma \in \Sigma_K$ and $\xi, \eta \in H_\sigma$.

Let $\{\xi_1^{(\sigma)}, \cdots, \xi_{d_\sigma}^{(\sigma)}\}$ be a fixed orthonormal basis in H_σ, and let $u_{ij}^{(\sigma)}(1 \leq i, j \leq d_\sigma)$ be the coordinate function for $U^{(\sigma)} \in \sigma$ and $\{\xi_1^{(\sigma)}, \cdots, \xi_{d_\sigma}^{(\sigma)}\}$, i.e., $u_{ij}^{(\sigma)}(x) = \langle U_x^{(\sigma)}\xi_j^{(\sigma)}, \xi_i^{(\sigma)}\rangle$.

Definition 2.1. *Let p be a natural number and E a subset of Σ_K. E is called an s-small p set if*

$$\forall \mu_1, \cdots, \mu_p \in M_E(K) \Rightarrow \mu_1 * \cdots * \mu_p \in L^1(K). \tag{2.2}$$

In paticular, an s-small 1 set is called a Riesz set.

Remark 2.1. *When K is a compact abelian group, "s-small p set" and "small p set" are same notion (cf. [13, Lemma 1]).*

Theorem 2.1. *Let $p \in \mathbb{N}$, and let K_1 and K_2 be compact groups. Let E_1 and E_2 be s-small p sets in Σ_{K_1} and Σ_{K_2} respectively. Then $E_1 \times E_2$ is an s-small p set in $\Sigma_{K_1 \times K_2} \cong \Sigma_{K_1} \times \Sigma_{K_2}$.*

By the above theorem, we obtain the following corollary.

Corollary 2.1. *Let E_1 and E_2 be Riesz sets in Σ_{K_1} and Σ_{K_2} respectively. Then $E_1 \times E_2$ is a Riesz set in $\Sigma_{K_1 \times K_2} \cong \Sigma_{K_1} \times \Sigma_{K_2}$.*

Next we consider Theorem 1.2 for compact groups. When G is a compact abelian group, the condition (1.2) in Theorem 1.2 is equivalent to the following:

(1.2)' For any $\gamma_1, \gamma_2 \in \hat{G}$, $(\gamma_1 + S) \cap (\gamma_2 - S)$ is a finite set.

Theorem 2.2. *Let K be a compact group, and let Δ be a subset of Σ_K satisfying the following condition.*

(2.3) *For any $\sigma, \tau \in \Sigma_K, (\sigma \times \Delta) \cap (\tau \times \overline{\Delta})$ is a finite set,*

where $\overline{\Delta} = \{\overline{\omega} : \omega \in \Delta\}$ and $\sigma \times \Delta = \{\sigma \times \eta : \eta \in \Delta\}$. Let $\mu, \nu \in M_\Delta(K)$. Then $|\mu| * |\nu| \in L^1(K)$. In particular, Δ is an s-small 2 set.

The following also holds (cf. [7, Theorem 2]).

Theorem 2.3. *Let K be a compact group, and let $p, q \in \mathbb{N}$. Let Δ be a subset of Σ_K satisfying the following condition.*

(2.3)' *$(\sigma_1 \times \Delta) \cap \cdots \cap (\sigma_p \times \Delta) \cap (\tau_1 \times \overline{\Delta}) \cap \cdots \cap (\tau_q \times \overline{\Delta})$ is a finite set for any $\sigma_1, \cdots, \sigma_p, \tau_1, \cdots, \tau_q \in \Sigma_K$.*

Let μ_i and ν_j be measures in $M_\Delta(K)$ $(i = 1, 2, \cdots, p; j = 1, 2, \cdots, q)$. Then $|\mu_1| * \cdots * |\mu_p| * |\nu_1| * \cdots * |\nu_q| \in L^1(K)$. In particular, Δ is an s-small $p + q$ set.

Example 2.1. *Let $K = \mathbb{T} \times SU(2)$, and let T^ℓ $(\ell = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots)$ be as in* [9, (29.13)]. *Then $\Sigma_K \cong \{\tau_{n,m} : n \in \mathbb{Z};\ m = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots\}$, where $\tau_{n,m}(e^{i\theta}, u) = e^{in\theta} T_u^{(m)}$. Let $\alpha > 0$, and set $\Delta = \{\tau_{n,m} \in \Sigma_K : n \geq 0,\ m \leq \alpha n\}$. Then, by* [9, (29.26)] *and the fact that $T^{(\ell)}$ are self-conjugate (cf.* [9, (29.25)]*), Δ satisfies the condition (2.3) in Theorem 2.2. (In fact, Δ is a Riesz set, by* [3, 3.4 *Example* (*a*)].)

We prove Theorem 2.2 in the next section. We can prove Theorem 2.3 by an argument similar to that in the proof of Theorem 2.2.

3. Proofs of Theorems

In this section, we prove Theorems 2.1 and 2.2. In order to prove Theorem 2.1, we use the theory of disintegration of measures.

Lemma 3.1. *Let K_1 and K_2 be compact groups, and let $p \in \mathbb{N}$. Let $\eta_n \in M^+(K_2)$, and let $\{\nu_h^{(n)}\}_{h \in K_2}$ be a family of measures in $M(K_1)$ with the following property $(n = 1, 2, \cdots, p)$:*

(1) *$h \to (\nu_h^{(n)} \times \delta_h)(f)$ is η_n-measurable for each $f \in C(K_1 \times K_2)$.*

Then

(2) *$(h_1, \cdots, h_p) \to (\nu_{h_1}^{(1)} \times \delta_{h_1}) * \cdots * (\nu_{h_p}^{(p)} \times \delta_{h_p})(f)\ (= (\nu_{h_1}^{(1)} * \cdots * \nu_{h_p}^{(p)}) \times \delta_{h_1 \cdots h_p}(f))$ is $(\eta_1 \times \cdots \times \eta_p)$-measurable for each $f \in C(K_1 \times K_2)$.*

Proof. For $f_1, \cdots, f_p \in C(K_1 \times K_2)$, we define $f(z_1, \cdots, z_p) \in C((K_1 \times K_2)^p)$ by

$$f(z_1, \cdots, z_p) = f_1(z_1) \cdots f_p(z_p).$$

By (1),

(3) $(h_1, \cdots, h_p) \to (\nu_{h_1}^{(1)} \times \delta_{h_1}) \times \cdots \times (\nu_{h_p}^{(p)} \times \delta_{h_p})(f) = (\nu_{h_1}^{(1)} \times \delta_{h_1})(f_1) \cdots (\nu_{h_p}^{(p)} \times \delta_{h_p})(f_p)$ is $(\eta_1 \times \cdots \times \eta_p)$-measurable.

Since $\{\sum_{i=1}^n f_{1i}(z_1) \cdots f_{pi}(z_p) : f_{ji} \in C(K_1 \times K_2)\,(1 \leq j \leq p;\, n = 1, 2, \cdots)\}$ is dense in $C((K_1 \times K_2)^p)$, (3) implies that

$$(4) \qquad (h_1, \cdots, h_p) \to (\nu_{h_1}^{(1)} \times \delta_{h_1}) \times \cdots \times (\nu_{h_p}^{(p)} \times \delta_{h_p})(f) \text{ is } (\eta_1 \times \cdots \times \eta_p)\text{−measurable for each } f \in C((K_1 \times K_2)^p).$$

We define $\pi_p : (K_1 \times K_2)^p \to K_1 \times K_2$ by $\pi_p(z_1, \cdots, z_p) = z_1 \cdots z_p$. Then

$$(\nu_{h_1}^{(1)} \times \delta_{h_1}) * \cdots * (\nu_{h_p}^{(p)} \times \delta_{h_p})(g) = (\nu_{h_1}^{(1)} \times \delta_{h_1}) \times \cdots \times (\nu_{h_p}^{(p)} \times \delta_{h_p})(g \circ \pi_p)$$

for each $g \in C(K_1 \times K_2)$. Thus (2) follows from (4). □

Lemma 3.2. *Let K_1 and K_2 be metrizable compact groups, and let $p \in \mathbb{N}$. Let $\mu_n \in M(K_1 \times K_2)$, $\eta_n \in M^+(K_2)$, and let $\{\nu_h^{(n)}\}_{h \in K_2}$ be a family of measures in $M(K_1)$ with the following properties $(n = 1, 2, \cdots, p)$:*

(1) *$h \to (\nu_h^{(n)} \times \delta_h)(f)$ is η_n-measurable for each $f \in C(K_1 \times K_2)$,*

(2) *$\|\nu_h^{(n)}\| \leq 1$, and*

(3) *$\mu_n(f) = \int_{K_2} (\nu_h^{(n)} \times \delta_h)(f) d\eta_n(h)$ for all $f \in C(K_1 \times K_2)$.*

Let ρ be a measure in $M(K_1 \times K_2)$ defined by

(4) $\rho(f) = \int_{K_2} \cdots \int_{K_2} (\nu_{h_1}^{(1)} * \cdots * \nu_{h_p}^{(p)}) \times \delta_{h_1 \cdots h_p}(f) d\eta_1(h_1) \cdots d\eta_p(h_p)$

for $f \in C(K_1 \times K_2)$. Then $\rho = \mu_1 * \cdots * \mu_p$.

Proof. Let (σ_1, σ_2) be any element in $\Sigma_{K_1} \times \Sigma_{K_2}$. For any $\xi_i^{(\sigma_1)} \otimes \xi_k^{(\sigma_2)}, \xi_j^{(\sigma_1)} \otimes \xi_\ell^{(\sigma_2)} \in H_{\sigma_1} \otimes H_{\sigma_2}$, we have

$$\begin{aligned}
&\langle \hat{\rho}(\sigma_1, \sigma_2)(\xi_i^{(\sigma_1)} \otimes \xi_k^{(\sigma_2)}), \xi_j^{(\sigma_1)} \otimes \xi_\ell^{(\sigma_2)} \rangle \\
&= \int_{K_1 \times K_2} \langle \overline{U}_x^{(\sigma_1)} \otimes \overline{U}_y^{(\sigma_2)} (\xi_i^{(\sigma_1)} \otimes \xi_k^{(\sigma_2)}), \xi_j^{(\sigma_1)} \otimes \xi_\ell^{(\sigma_2)} \rangle d\rho(x, y) \\
(5) \quad &= \int_{K_2} \cdots \int_{K_2} (\nu_{h_1}^{(1)} * \cdots * \nu_{h_p}^{(p)}) \times \delta_{h_1 \cdots h_p} (\langle \overline{U}_x^{(\sigma_1)}(\xi_i^{(\sigma_1)}), \xi_j^{(\sigma_1)} \rangle \\
&\qquad \times \langle \overline{U}_y^{(\sigma_2)}(\xi_k^{(\sigma_2)}), \xi_\ell^{(\sigma_2)} \rangle) d\eta_1(h_1) \cdots d\eta_p(h_p) \\
&= \int_{K_2} \cdots \int_{K_2} \langle (\nu_{h_1}^{(1)} * \cdots * \nu_{h_p}^{(p)})\hat{}\,(\sigma_1)(\xi_i^{(\sigma_1)}), \xi_j^{(\sigma_1)} \rangle \\
&\qquad \times \langle \overline{U}_{h_1 \cdots h_p}^{(\sigma_2)}(\xi_k^{(\sigma_2)}), \xi_\ell^{(\sigma_2)} \rangle d\eta_1(h_1) \cdots d\eta_p(h_p).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\langle(\mu_1*\cdots*\mu_p)\hat{}\,(\sigma_1,\sigma_2)(\xi_i^{(\sigma_1)}\otimes\xi_k^{(\sigma_2)}),\xi_j^{(\sigma_1)}\otimes\xi_\ell^{(\sigma_2)}\rangle\\
&=\int_{K_1\times K_2}\langle\overline{U}_x^{(\sigma_1)}\otimes\overline{U}_y^{(\sigma_2)}(\xi_i^{(\sigma_1)}\otimes\xi_k^{(\sigma_2)}),\xi_j^{(\sigma_2)}\otimes\xi_\ell^{(\sigma_2)}\rangle d\mu_1*\cdots*\mu_p(x,y)\\
&=\int_{K_1\times K_2}\cdots\int_{K_1\times K_2}\langle\overline{U}_{x_1\cdots x_p}^{(\sigma_1)}\otimes\overline{U}_{y_1\cdots y_p}^{(\sigma_2)}(\xi_i^{(\sigma_1)}\otimes\xi_k^{(\sigma_2)}),\xi_j^{(\sigma_1)}\otimes\xi_\ell^{(\sigma_2)}\rangle\\
&\qquad\qquad d\mu_1(x_1,y_1)\cdots d\mu_p(x_p,y_p)\\
&=\int_{K_1\times K_2}\cdots\int_{K_1\times K_2}\langle\overline{U}_{x_1\cdots x_p}^{(\sigma_1)}(\xi_i^{(\sigma_1)}),\xi_j^{(\sigma_1)}\rangle\langle\overline{U}_{y_1\cdots y_p}^{(\sigma_2)}(\xi_k^{(\sigma_2)}),\xi_\ell^{(\sigma_2)}\rangle\\
&\qquad\qquad d\mu_1(x_1,y_1)\cdots d\mu_p(x_p,y_p)\\
&=\int_{K_1\times K_2}\cdots\int_{K_1\times K_2}\int_{K_2}(\nu_{h_1}^{(1)}\times\delta_{h_1})(\langle\overline{U}_{x_1\cdots x_p}^{(\sigma_1)}(\xi_i^{(\sigma)}),\xi_j^{(\sigma_1)}\rangle\langle\overline{U}_{y_1\cdots y_p}^{(\sigma_2)}(\xi_k^{(\sigma_2)},\xi_\ell^{(\sigma_2)}\rangle)\\
&\qquad\qquad d\eta_1(h_1)d\mu_2(x_2,y_2)\cdots\mu_p(x_p,y_p)
\end{aligned}$$

(6)

$$\begin{aligned}
&=\int_{K_1\times K_2}\cdots\int_{K_1\times K_2}\int_{K_2}\int_{K_1}\langle\overline{U}_{x_1\cdots x_p}^{(\sigma_1)}(\xi_i^{(\sigma_1)}),\xi_j^{(\sigma_1)}\rangle d\nu_{h_1}^{(1)}(x_1)\\
&\qquad\times\langle\overline{U}_{h_1y_2\cdots y_p}^{(\sigma_2)}(\xi_k^{(\sigma_2)}),\xi_\ell^{(\sigma_2)}\rangle d\eta_1(h_1)d\mu_2(x_2,y_2)\cdots d\mu_p(x_p,y_p)\\
&=\int_{K_1\times K_2}\cdots\int_{K_1\times K_2}\int_{K_2}\langle\hat{\nu}_{h_1}^{(1)}(\sigma_1)(\overline{U}_{x_2\cdots x_p}^{(\sigma_1)}(\xi_i^{(\sigma_1)})),\xi_j^{(\sigma_1)}\rangle\\
&\qquad\times\langle\overline{U}_{h_1y_2\cdots y_p}^{(\sigma_2)}(\xi_k^{(\sigma_2)}),\xi_\ell^{(\sigma_2)}\rangle d\eta_1(h_1)d\mu_2(x_2,y_2)\cdots d\mu_p(x_p,y_p)\\
&=\int_{K_2}\int_{K_1\times K_2}\cdots\int_{K_1\times K_2}\langle\overline{U}_{x_2\cdots x_p}^{(\sigma_1)}(\xi_i^{(\sigma_1)}),\hat{\nu}_{h_1}^{(1)}(\sigma_1)^*(\xi_j^{(\sigma_1)})\rangle\\
&\qquad\times\langle\overline{U}_{y_2\cdots y_p}^{(\sigma_2)}(\xi_k^{(\sigma_2)}),\overline{U}_{h_1}^{(\sigma_2)*}(\xi_\ell^{(\sigma_2)})\rangle d\mu_2(x_2,y_2)\cdots\mu_p(x_p,y_p)d\eta_1(h_1)
\end{aligned}$$

....

$$\begin{aligned}
&=\overbrace{\int_{K_2}\cdots\int_{K_2}}^{r}\overbrace{\int_{K_1\times K_2}\cdots\int_{K_1\times K_2}}^{p-r}\langle\overline{U}_{x_{r+1}\cdots x_p}^{(\sigma_1)}(\xi_i^{(\sigma_1)}),(\nu_{h_1}^{(1)}*\cdots*\nu_{h_r}^{(r)})\hat{}\,(\sigma_1)^*(\xi_j^{(\sigma_1)})\rangle\\
&\times\langle\overline{U}_{y_{r+1}\cdots y_p}^{(\sigma_2)}(\xi_k^{(\sigma_2)}),\overline{U}_{h_1\cdots h_r}^{(\sigma_2)*}(\xi_\ell^{(\sigma_2)})\rangle d\mu_{r+1}(x_{r+1},y_{r+1})\cdots d\mu_p(x_p,y_p)d\eta_1(h_1)\cdots d\eta_r(h_r)
\end{aligned}$$

....

$$
\begin{aligned}
&= \int_{K_2} \cdots \int_{K_2} \langle \xi_i^{(\sigma_1)}, (\nu_{h_1}^{(1)} * \cdots * \nu_{h_p}^{(p)})\hat{}(\sigma_1)^*(\xi_j^{(\sigma_1)})\rangle \\
&\qquad\qquad \times \langle \xi_k^{(\sigma_2)}, \overline{U}_{h_1\cdots h_p}^{(\sigma_1)*}(\xi_\ell^{(\sigma_2)})\rangle d\eta_1(h_1)\cdots d\eta_p(h_p) \\
&= \int_{K_2} \cdots \int_{K_2} \langle (\nu_{h_1}^{(1)} * \cdots * \nu_{h_p}^{(p)})\hat{}(\sigma_1)(\xi_i^{(\sigma_1)}), \xi_j^{(\sigma_1)}\rangle \\
&\qquad\qquad \times \langle \overline{U}_{h_1\cdots h_p}^{(\sigma_2)}(\xi_k^{(\sigma_2)}), \xi_\ell^{(\sigma_2)}\rangle d\eta_1(h_1)\cdots d\eta_p(h_p),
\end{aligned}
$$

where $\hat{\nu}_{h_1}^{(1)}(\sigma_1)^*$ and $\overline{U}_{h_1\cdots h_r}^{(\sigma_2)*}$ are the adjoints of $\hat{\nu}_{h_1}^{(1)}(\sigma_1)$ and $\overline{U}_{h_1\cdots h_r}^{(\sigma_2)}$ respectively. By (5) and (6), we have

$$
\begin{aligned}
&\langle \hat{\rho}(\sigma_1, \sigma_2)(\xi_i^{(\sigma_1)} \otimes \xi_k^{(\sigma_2)}), \xi_j^{(\sigma_1)} \otimes \xi_\ell^{(\sigma_2)}\rangle \\
&\quad = \langle (\mu_1 * \cdots * \mu_p)\hat{}(\sigma_1, \sigma_2)(\xi_i^{(\sigma_1)} \otimes \xi_k^{(\sigma_2)}), \xi_j^{(\sigma_1)} \otimes \xi_\ell^{(\sigma_2)}\rangle
\end{aligned}
$$

for any $(\sigma_1, \sigma_2) \in \Sigma_{K_1} \times \Sigma_{K_2}$ and $\xi_i^{(\sigma_1)} \otimes \xi_k^{(\sigma_2)}$, $\xi_j^{(\sigma_1)} \otimes \xi_\ell^{(\sigma_2)} \in H_{\sigma_1} \otimes H_{\sigma_2}$. This yields $\rho = \mu_1 * \cdots * \mu_p$. □

Proposition 3.1. *Let K_1 and K_2 be metrizable compact groups, and let $p \in \mathbb{N}$. Let E_1 be an s-small p set in Σ_{K_1}, and let $\mu_1, \cdots, \mu_p \in M_{E_1 \times \Sigma_{K_2}}(K_1 \times K_2)$. Then $\lim_{x \to e_1} \|\delta_{(x,e_2)} * \mu_1 * \cdots * \mu_p - \mu_1 * \cdots * \mu_p\| = 0$, where e_i is the unit element of K_i $(i = 1, 2)$.*

Proof. Let $\pi : K_1 \times K_2 \to K_2$ be the projection, and let $\eta_n = \pi(|\mu_n|)$ $(n = 1, 2, \cdots, p)$. Then, by the theory of disintegration of measures (cf. [1] or [14, Corollary 1.6]), there exists a family $\{\lambda_h^{(n)}\}_{h \in K_2}$ of measures in $M(K_1 \times K_2)$ with the following properties:

(1) $\quad h \to \lambda_h^{(n)}(f)$ is η_n–measurable for each $f \in C(K_1 \times K_2)$,

(2) $\quad \|\lambda_h^{(n)}\| \leq 1$,

(3) $\quad supp(\lambda_h^{(n)}) \subset \pi^{-1}(h)$, and

(4) $\quad \mu_n(f) = \int_{K_2} \lambda_h^{(n)}(f) d\eta_n(h)$ for all $f \in C(K_1 \times K_2)$.

By (2) and (3), there exists a measure $\nu_h^{(n)} \in M(K_1)$, with $\|\nu_h^{(n)}\| \leq 1$, such that

(5) $$\lambda_h^{(n)} = \nu_h^{(n)} \times \delta_h.$$

Let $\sigma_1 \notin E_1$. Let σ_2 be any element in Σ_{K_2}. For any $\xi_i^{(\sigma_1)} \otimes \xi_k^{(\sigma_2)}$, $\xi_j^{(\sigma_1)} \otimes \xi_\ell^{(\sigma_2)} \in H_{\sigma_1} \otimes H_{\sigma_2}$, we have

$$\begin{aligned}
0 = & \langle \hat{\mu}_n(\sigma_1, \sigma_2)(\xi_i^{(\sigma_1)} \otimes \xi_k^{(\sigma_2)}), \xi_j^{(\sigma_1)} \otimes \xi_\ell^{(\sigma_2)} \rangle \\
= & \int_{K_1 \times K_2} \langle \overline{U}_x^{(\sigma_1)} \otimes \overline{U}_y^{(\sigma_2)} (\xi_i^{(\sigma_1)} \otimes \xi_k^{(\sigma_2)}), \xi_j^{(\sigma_1)} \otimes \xi_\ell^{(\sigma_2)} \rangle d\mu_n(x, y) \\
= & \int_{K_1 \times K_2} \langle \overline{U}_x^{(\sigma_1)} (\xi_i^{(\sigma_1)}), \xi_j^{(\sigma_1)} \rangle \langle \overline{U}_y^{(\sigma_2)} (\xi_k^{(\sigma_2)}), \xi_\ell^{(\sigma_2)} \rangle d\mu_n(x, y) \\
= & \int_{K_2} \int_{K_1} \langle \overline{U}_x^{(\sigma_1)} (\xi_i^{(\sigma_1)}), \xi_j^{(\sigma_1)} \rangle d\nu_h^{(n)}(x) \\
& \qquad \times \langle \overline{U}_h^{(\sigma_2)} (\xi_k^{(\sigma_2)}), \xi_\ell^{(\sigma_2)} \rangle d\eta_n(h) \qquad \text{(by (4) and (5))} \\
= & \int_{K_2} \langle \hat{\nu}_h^{(n)}(\sigma_1) \xi_i^{(\sigma_1)}, \xi_j^{(\sigma_1)} \rangle \overline{u}_{\ell k}^{(\sigma_2)}(h) d\eta_n(h),
\end{aligned}$$

which yields

$$\int_{K_2} \langle \hat{\nu}_h^{(n)}(\sigma_1) \xi_i^{(\sigma_1)}, \xi_j^{(\sigma_1)} \rangle p(h) d\eta_n(h) = 0$$

for all $p \in \mathfrak{T}(K_2)$. Hence

$$\langle \hat{\nu}_h^{(n)}(\sigma_1) \xi_i^{(\sigma_1)}, \xi_j^{(\sigma_1)} \rangle = 0 \qquad \eta_n\text{-a.a. } h \in K_2 \quad (1 \leq \forall i, j \leq d_{\sigma_1}).$$

Thus

$$\hat{\nu}_h^{(n)}(\sigma_1) = 0 \qquad \eta_n\text{-a.a. } h \in K_2.$$

Since Σ_{K_1} is countable, we have

(6) $$\hat{\nu}_h^{(n)}(\sigma_1) = 0 \quad \text{for all } \sigma_1 \in \Sigma_{K_1} \setminus E_1 \quad \eta_n\text{-a.a. } h \in K_2.$$

Since E_1 is an s-small p set, we have

(7) $$\nu_{h_1}^{(1)} * \cdots * \nu_{h_p}^{(p)} \in L^1(K_1) \quad (\eta_1 \times \cdots \times \eta_p)\text{-a.a. } (h_1, \cdots, h_p) \in K_2^p.$$

It follows from Lemmas 3.1 and 3.2 that $(h_1, \cdots, h_p) \to (\nu_{h_1}^{(1)} * \cdots * \nu_{h_p}^{(p)}) \times \delta_{h_1 \cdots h_p}(f)$ is $(\eta_1 \times \cdots \times \eta_p)$-measurable for each $f \in C(K_1 \times K_2)$ and

(8) $$\begin{aligned}
& \mu_1 * \cdots * \mu_p(f) \\
& = \int_{K_2} \cdots \int_{K_2} (\nu_{h_1}^{(1)} * \cdots * \nu_{h_p}^{(p)}) \times \delta_{h_1 \cdots h_p}(f) d\eta_1(h_1) \cdots d\eta_p(h_p)
\end{aligned}$$

for all $f \in C(K_1 \times K_2)$. For $x \in K_1$, we note that $(h_1, \cdots, h_p) \to (\delta_x * \nu_{h_1}^{(1)} * \cdots * \nu_{h_p}^{(p)}) \times \delta_{h_1 \cdots h_p}(f)$ is $(\eta_1 \times \cdots \times \eta_p)$-measurable for each $f \in C(K_1 \times K_2)$. It follows from (8) that

(9) $$\begin{aligned}
& \delta_{(x, e_2)} * \mu_1 * \cdots * \mu_p(f) \\
& = \int_{K_2} \cdots \int_{K_2} (\delta_x * \nu_{h_1}^{(1)} * \cdots * \nu_{h_p}^{(p)}) \times \delta_{h_1 \cdots h_p}(f) d\eta_1(h_1) \cdots d\eta_p(h_p)
\end{aligned}$$

for all $f \in C(K_1 \times K_2)$. Let $\mathcal{A} = \{f_n\}$ be a countable dense set in $C(K_1 \times K_2)$. Since

$$\begin{aligned}
&\|\delta_x * \nu_{h_1}^{(1)} * \cdots * \nu_{h_p}^{(p)} - \nu_{h_1}^{(1)} * \cdots * \nu_{h_p}^{(p)}\| \\
&= \sup_{\substack{f_n \in \mathcal{A} \\ \|f_n\|_\infty \leq 1}} |\{(\delta_x * \nu_{h_1}^{(1)} * \cdots * \nu_{h_p}^{(p)}) \times \delta_{h_1 \cdots h_p} - (\nu_{h_1}^{(1)} * \cdots * \nu_{h_p}^{(p)}) \times \delta_{h_1 \cdots h_p}\}(f_n)|,
\end{aligned}$$

we note that

$$(h_1, \cdots, h_p) \to \|\delta_x * \nu_{h_1}^{(1)} * \cdots * \nu_{h_p}^{(p)} - \nu_{h_1}^{(1)} * \cdots * \nu_{h_p}^{(p)}\|$$

is $(\eta_1 \times \cdots \times \eta_p)$-measurable. Let $\{s_n\}$ be a sequence in K_1 such that $\lim_{n\to\infty} s_n = e_1$. Then, by (7),

$$\lim_{n\to\infty} \|\delta_{s_n} * \nu_{h_1}^{(1)} * \cdots * \nu_{h_p}^{(p)} - \nu_{h_1}^{(1)} * \cdots * \nu_{h_p}^{(p)}\| = 0 \qquad (\eta_1 \times \cdots \times \eta_p)\text{–a.a. } (h_1, \cdots, h_p) \in K_2^p,$$

which, together with (8) and (9), yields

$$\begin{aligned}
&\lim_{n\to\infty} \|\delta_{(s_n,e_2)} * \mu_1 * \cdots * \mu_p - \mu_1 * \cdots * \mu_p\| \\
&= \lim_{n\to\infty} \sup_{\substack{f \in \mathcal{A} \\ \|f\|_\infty \leq 1}} |\ \delta_{(s_n,e_2)} * \mu_1 * \cdots * \mu_p(f) - \mu_1 * \cdots * \mu_p(f)\ | \\
&= \lim_{n\to\infty} \sup_{\substack{f \in \mathcal{A} \\ \|f\|_\infty \leq 1}} \left| \int_{K_2} \cdots \int_{K_2} \{(\delta_{s_n} * \nu_{h_1}^{(1)} * \cdots * \nu_{h_p}^{(p)}) \times \delta_{h_1 \cdots h_p} - \right. \\
&\qquad\qquad \left. (\nu_{h_1}^{(1)} * \cdots * \nu_{h_p}^{(p)}) \times \delta_{h_1 \cdots h_p}\}(f) d\eta_1(h_1) \cdots d\eta_p(h_p) \right| \\
&\leq \lim_{n\to\infty} \int_{K_2} \cdots \int_{K_2} \|\delta_{s_n} * \nu_{h_1}^{(1)} * \cdots * \nu_{h_p}^{(p)} - \nu_{h_1}^{(1)} * \cdots * \nu_{h_p}^{(p)}\| d\eta_1(h_1) \cdots d\eta_p(h_p) \\
&= 0. \qquad \text{(by the Lebesgue convergence theorem)}
\end{aligned}$$

Since K_1 is metrizable, the proposition is obtained. □

Similarly we get the following proposition.

Proposition 3.2. *Let K_1 and K_2 be metrizable compact groups, and let $p \in \mathbb{N}$. Let E_2 be an s-small p set in Σ_{K_2}, and let $\mu_1, \cdots, \mu_p \in M_{\Sigma_{K_1} \times E_2}(K_1 \times K_2)$. Then $\lim_{y\to e_2} \|\mu_1 * \cdots * \mu_p - \delta_{(e_1,y)} * \mu_1 * \cdots * \mu_p\| = 0$.*

Proposition 3.3. *Let K_1 and K_2 be metrizable compact groups, and let $p \in \mathbb{N}$. Let E_1 and E_2 be s-small p sets in Σ_{K_1} and Σ_{K_2} respectively. Then $E_1 \times E_2$ is an s-small p set in $\Sigma_{K_1 \times K_2} \cong \Sigma_{K_1} \times \Sigma_{K_2}$.*

Proof. Let $\mu_n \in M_{E_1 \times E_2}(K_1 \times K_2)$ $(n = 1, 2, \cdots, p)$. It follows from Propositions 3.1 and 3.2 that

(1) $$\lim_{x\to e_1} \|\mu_1 * \cdots * \mu_p - \delta_{(x,e_2)} * \mu_1 * \cdots * \mu_p\| = 0, \quad \text{and}$$

(2) $$\lim_{y\to e_2} \|\mu_1 * \cdots * \mu_p - \delta_{(e_1,y)} * \mu_1 * \cdots * \mu_p\| = 0.$$

Thus we have

$$\begin{aligned}\lim_{(x,y)\to(e_1,e_2)} &\|\mu_1 * \cdots * \mu_p - \delta_{(x,y)} * \mu_1 * \cdots * \mu_p\| \\ &\le \lim_{(x,y)\to(e_1,e_2)} \{\|\mu_1 * \cdots * \mu_p - \delta_{(x,e_2)} * \mu_1 * \cdots * \mu_p\| \\ &\qquad\qquad + \|\delta_{(x,e_2)} * \mu_1 * \cdots * \mu_p - \delta_{(x,y)} * \mu_1 * \cdots * \mu_p\|\} \\ &= 0,\end{aligned}$$

which implies $\mu_1 * \cdots * \mu_p \in L^1(K_1 \times K_2)$. This completes the proof. □

Lemma 3.3. *Let K be a compact group, and let H be a closed normal subgroup of K. Let $\nu \in M(K/H)$, and let $\pi : K \to K/H$ be the canonical map. Then there exists a measure $\mu \in M(K)$ with the following :*

(1) $\pi(\mu) = \nu$,

(2) $\hat{\mu}(\sigma) = 0$ *for* $\sigma \in \Sigma_K \setminus A(\Sigma_K, H)$, *and*

(3) $\{\sigma \in A(\Sigma_K, H) : \hat{\mu}(\sigma) \neq 0\} = \{\sigma \in A(\Sigma_K, H) : \hat{\nu}(\sigma) \neq 0\}$.

Proof. Let $\nu \in M(K/H)$. For $f \in C(K)$, let $[f]$ be a continuous function in $C(K/H)$ defined by

$$[f](\dot{x}) = \int_H f(xy)dm_H(y),$$

and we define $\mu \in M(K)$ by

$$\mu(f) = \int_{K/H} [f](\dot{x})d\nu(\dot{x})$$

for $f \in C(K)$. It is easy to verify that

(4) $\pi(\mu) = \nu$.

Claim 1. $\hat{\mu}(\sigma) = 0$ for $\sigma \in \Sigma_K \setminus A(\Sigma_K, H)$.

Let $\sigma \in \Sigma_K \setminus A(\Sigma_K, H)$. For $\xi, \eta \in H_\sigma$, we have

$$\begin{aligned}\langle \hat{\mu}(\sigma)\xi, \eta\rangle &= \int_K \langle \overline{U}_x^{(\sigma)}\xi, \eta\rangle d\mu(x) \\ &= \int_{K/H}\int_H \langle \overline{U}_{xy}^{(\sigma)}\xi, \eta\rangle dm_H(y)d\nu(\dot{x}) \\ &= \int_{K/H}\int_H \langle \overline{U}_y^{(\sigma)}\xi, \overline{U}_x^{(\sigma)*}\eta\rangle dm_H(y)d\nu(\dot{x}) \\ &= \int_{K/H} \langle \hat{m}_H(\sigma)\xi, \overline{U}_x^{(\sigma)*}\eta\rangle d\nu(\dot{x}) \\ &= 0. \qquad \text{(by [9, 28.72(g), p.112])}\end{aligned}$$

This shows that $\hat{\mu}(\sigma) = 0$.

Claim 2. Let $\sigma \in A(\Sigma_K, H)$. Then $\hat{\mu}(\sigma) \neq 0$ if and only if $\hat{\nu}(\sigma) \neq 0$.

For $\xi, \eta \in H_\sigma$, we have, by the fact that $\sigma \in A(\Sigma_K, H)$,

$$\begin{aligned}\langle \hat{\mu}(\sigma)\xi, \eta\rangle &= \int_{K/H}\int_H \langle \overline{U}^{(\sigma)}_{xy}\xi, \eta\rangle dm_H(y)d\nu(\dot{x}) \\ &= \int_{K/H} \langle \overline{U}^{(\sigma)}_{xH}\xi, \eta\rangle d\nu(\dot{x}) \\ &= \langle \hat{\nu}(\sigma)\xi, \eta\rangle.\end{aligned}$$

Thus Claim 2 follows. By (4) and Claims 1 and 2, the lemma is obtained. □

Lemma 3.4. *Let K be a compact group, and let H be a closed normal subgroup of K. Let $p \in \mathbb{N}$. If E is an s-small p set in Σ_K, then $E \cap A(\Sigma_K, H)$ is an s-small p set in $\Sigma_{K/H} \cong A(\Sigma_K, H)$.*

Proof. We note that $\Sigma_{K/H} \cong A(\Sigma_K, H)$ (cf. [9, (28.10) Corollary]).
Let $\nu_n \in M_{E\cap A(\Sigma_K,H)}(K/H)\,(n = 1, 2, \cdots, p)$, and let $\pi : K \to K/H$ be the canonical map. It follows from Lemma 3.3 that there exists $\mu_n \in M(K)$ such that

(1) $\qquad \pi(\mu_n) = \nu_n,$

(2) $\qquad \hat{\mu}_n(\sigma) = 0 \quad \text{for } \sigma \in \Sigma_K \setminus A(\Sigma_K, H), \quad \text{and}$

(3) $\qquad \{\sigma \in A(\Sigma_K, H) : \hat{\mu}_n(\sigma) \neq 0\} = \{\sigma \in A(\Sigma_K, H) : \hat{\nu}_n(\sigma) \neq 0\}.$

Then

$$\{\sigma \in \Sigma_K : \hat{\mu}_n(\sigma) \neq 0\} \subset E \cap A(\Sigma_K, H).$$

Since E is an s-small p set, $\mu_1 * \cdots * \mu_p$ belongs to $L^1(K)$, which yields that $\nu_1 * \cdots * \nu_p = \pi(\mu_1 * \cdots * \mu_p) \in L^1(K/H)$. This completes the proof. □

The following lemma is due to [16]. For a subset P of Σ_K, $A(K, P)$ denotes the annihilator of P in K.

Lemma 3.5 (cf. [16, Lemma 3.3]). *Let K be a compact group. Let μ_0 be a nonzero measure in $M(K)$, and let μ and ν be mutually singular positive measures in $M(K)$. Let σ_0 be an element in Σ_K such that $\hat{\mu}_0(\sigma_0) \neq 0$. Then there exists a countable subset P of Σ_K, with $[P] = P$, such that*

(i) $\quad \sigma_0 \in P,$

(ii) $\quad \pi(\mu_0)\hat{}(\sigma_0) \neq 0,$ *and*

(iii) $\quad \pi(\mu) \perp \pi(\nu),$

where $H = A(K, P)$ and $\pi : K \to K/H$ is the canonical map. Moreover, for any $P' \supset P$ with $[P'] = P'$, we have

(iv) $\quad \pi'(\mu) \perp \pi'(\nu),$

where $H' = A(K, P')$ and $\pi' : K \to K/H'$ is the canonical map.

Now we prove Theorem 2.1. Suppose there exist measures $\mu_n \in M_{E_1\times E_2}(K_1 \times K_2)\,(n = 1, 2, \cdots, p)$ such that $\mu_1 * \cdots * \mu_p$ does not belong to $L^1(K_1 \times K_2)$. Let

$$\mu_1 * \cdots * \mu_p = \mu_a + \mu_s$$

be the Lebesgue decomposition of $\mu_1 * \cdots * \mu_p$ with respect to $m_{K_1 \times K_2}$. Then $\mu_s \neq 0$. Thus there exists $\sigma_0 = (\sigma_1, \sigma_2) \in \Sigma_{K_1} \times \Sigma_{K_2}$ such that $\hat{\mu}_s(\sigma_0) \neq 0$. It follows from Lemma 3.5 that there exists a countable subset P of $\Sigma_{K_1 \times K_2}$, with $[P] = P$, such that

$$\sigma_0 = (\sigma_1, \sigma_2) \in P, \tag{3.1}$$

$$\pi(\mu_s)\hat{}(\sigma_0) \neq 0, \quad \text{and} \tag{3.2}$$

$$\pi(|\mu_s|) \perp \pi(m_{K_1 \times K_2}), \tag{3.3}$$

where $\pi : K_1 \times K_2 \to K_1 \times K_2/A(K_1 \times K_2, P)$ is the canonical map. Moreover, P can be chosen so that, for any $P' \supset P$ with $[P'] = P'$,

$$\pi'(|\mu_s|) \perp \pi'(m_{K_1 \oplus K_2}), \tag{3.4}$$

where $\pi' : K_1 \times K_2 \to K_1 \times K_2/A(K_1 \times K_2, P')$ is the canonical map. Let $\tau_i : \Sigma_{K_1} \times \Sigma_{K_2} (\cong \Sigma_{K_1 \times K_2}) \to \Sigma_{K_i}$ be the projection (i = 1, 2), and let P_i be a countable subset of Σ_{K_i} such that $\tau_i(P) \subset P_i$ and $[P_i] = P_i$ $(i = 1, 2)$. Set $H_i = A(K_i, P_i)$, and put $H = H_1 \times H_2$. Then H_i and H are closed normal subgroups of K_i and $K_1 \times K_2$ respectively. Let $\pi_H : K_1 \times K_2 \to K_1 \times K_2/H \cong K_1/H_1 \times K_2/H_2$ be the natural map. Since $P \subset P_1 \times P_2$, we have, by (3.4),

$$\pi_H(|\mu_s|) \perp \pi_H(m_{K_1 \times K_2}). \tag{3.5}$$

Since $\sigma_0 = (\sigma_1, \sigma_2) \in P_1 \times P_2$ and $\hat{\mu}_s(\sigma_0) \neq 0$, we note that

$$\pi_H(\mu_s)\hat{}(\sigma_0) \neq 0 \tag{3.6}$$

(cf. the proof of Lemma 3.3 in [16]). It follows from Lemma 3.4 that $E_i \cap A(\Sigma_{K_i}, H_i)$ is an s-small p set. Since P_i is countable, K_i/H_i is a metrizable compact group. Hence $(E_1 \cap A(\Sigma_{K_1}, H_1)) \times (E_2 \cap A(\Sigma_{K_2}, H_2))$ is an s-small p set in $\Sigma_{K_1 \times K_2/H} \cong A(\Sigma_{K_1}, H_1) \times A(\Sigma_{K_2}, H_2) (\cong P_1 \times P_2)$, by Proposition 3.3. Since $spec(\pi_H(\mu_n)) \subset (E_1 \cap A(\Sigma_{K_1}, H_1)) \times (E_2 \cap A(\Sigma_{K_2}, H_2))$, we have

$$\pi_H(\mu_1 * \cdots * \mu_p) = \pi_H(\mu_1) * \cdots * \pi_H(\mu_p) \in L^1(K_1 \times K_2/H). \tag{3.7}$$

On the other hand, (3.5) shows that $\pi_H(\mu_1 * \cdots * \mu_p) = \pi_H(\mu_a) + \pi_H(\mu_s)$ is the Lebesgue decomposition of $\pi_H(\mu_1 * \cdots * \mu_p)$ with respect to $\pi_H(m_{K_1 \times K_2})$. By (3.6), we have $\pi_H(\mu_s) \neq 0$, which contradicts (3.7). This shows that $E_1 \times E_2$ is an s-small p set in $\Sigma_{K_1 \times K_2}$, and the proof is complete.

Next we prove Theorem 2.2. We need several lemmas.
For $\mu \in M(K)$, define $\overline{\mu} \in M(K)$ by

$$\overline{\mu}(B) = \overline{\mu(B)} \tag{3.8}$$

for Borel sets B on K. Let $\sigma \in \Sigma_K$. We denote by $B(H_\sigma)$ the space of all bounded linear operators on H_σ. For $\mu \in M(K)$, we define $T_\mu \in B(H_\sigma)$ by

$$\langle T_\mu \xi, \eta \rangle = \int_K \langle D_\sigma \overline{U}_x^{(\sigma)} D_\sigma \xi, \eta \rangle d\mu(x) \tag{3.9}$$

for $\xi, \eta \in H_\sigma$. The following can be found in the proof of [9, (28.44) Theorem].

Lemma 3.6. *There exists an onto linear isometry $C : H_{\overline{\sigma}} \to H_\sigma$ such that $\hat{\mu}(\overline{\sigma}) = C^{-1}T_\mu C$.*

Lemma 3.7. *Let $\mu \in M(K)$ and $\sigma \in \Sigma_K$. Then $\hat{\overline{\mu}}(\sigma) = D_\sigma T_\mu D_\sigma$.*

Proof. For ξ, $\eta \in H_\sigma$, we have

$$\begin{aligned}\langle \hat{\overline{\mu}}(\sigma)\xi, \eta\rangle &= \int_K \langle \overline{U}_x^{(\sigma)}\xi, \eta\rangle d\overline{\mu}(x) = \overline{\int_K \overline{\langle \overline{U}_x^{(\sigma)}\xi, \eta\rangle} d\mu(x)} \\ &= \overline{\int_K \langle D_\sigma \overline{U}_x^{(\sigma)}\xi, D_\sigma\eta\rangle d\mu(x)} = \overline{\int_K \langle D_\sigma \overline{U}_x^{(\sigma)} D_\sigma D_\sigma \xi, D_\sigma\eta\rangle d\mu(x)} \\ &= \overline{\langle T_\mu D_\sigma\xi, D_\sigma\eta\rangle} = \langle D_\sigma T_\mu D_\sigma \xi, \eta\rangle.\end{aligned}$$

This completes the proof. □

Remark 3.1. *Let $\mu \in M(K)$ and $\sigma \in \Sigma_K$. It follows from Lemmas 3.6 and 3.7 that the following are equivalent.*

(i) $\hat{\mu}(\overline{\sigma}) \neq 0$.

(ii) $\hat{\overline{\mu}}(\sigma) \neq 0$.

Corollary 3.1. *Let $\mu \in M(K)$. Then $spec(\overline{\mu}) = spec(\mu)^-$, where $spec(\mu)^- = \{\overline{\sigma} : \sigma \in spec(\mu)\}$.*

Proof. For $\sigma \in \Sigma_K$, we note that $\overline{\overline{\sigma}} = \sigma$. Thus the corollary follows from Remark 3.1. □

The following lemma is due to [15].

Lemma 3.8 (cf. [15, Lemma 3.3]). *Let $\sigma \in \Sigma_K$ and $\Delta \subset \Sigma_K$. For $f \in \mathfrak{T}_\sigma(K)$ and $\mu \in M(K)$ with $spec(\mu) \subset \Delta$, we have $spec(f\mu) \subset \sigma \times \Delta$.*

Now we prove Theorem 2.2. Let $\mu, \nu \in M_\Delta(K)$. Then

$$(u_{ij}^{(\sigma)}\mu) * (u_{k\ell}^{(\tau)}\overline{\nu}) \in L^1(K) \tag{3.10}$$

for all $\sigma, \tau \in \Sigma_K$; $u_{ij}^{(\sigma)} \in \mathfrak{T}_\sigma(K), u_{k\ell}^{(\tau)} \in \mathfrak{T}_\tau(K)$. In fact, since $spec(\mu) \subset \Delta$, we have, by Lemma 3.8,

$$spec(u_{ij}^{(\sigma)}\mu) \subset \sigma \times \Delta.$$

Similary Corollary 3.1, together with the previous lemma, yields

$$spec(u_{k\ell}^{(\tau)}\overline{\nu}) \subset \tau \times \overline{\Delta}.$$

Hence we have

$$spec((u_{ij}^{(\sigma)}\mu) * (u_{k\ell}^{(\tau)}\overline{\nu})) \subset (\sigma \times \Delta) \cap (\tau \times \overline{\Delta}),$$

which implies (3.10), since $(\sigma \times \Delta) \cap (\tau \times \overline{\Delta})$ is finite by the hypothesis (2.3). It follows from (3.10) that

$$(f\mu) * (h\overline{\nu}) \in L^1(K) \quad \text{for any } f, h \in \mathfrak{T}(K). \tag{3.11}$$

On the other hand, there exist sequences $\{f_n\}$ and $\{h_n\}$ in $\mathfrak{T}(K)$ such that $\lim_{n\to\infty} \|f_n\mu - |\mu|\| = 0$ and $\lim_{n\to\infty} \|h_n\overline{\nu} - |\overline{\nu}|\| = 0$. Since $\lim_{n\to\infty} \|(f_n\mu) * (h_n\overline{\nu}) - |\mu| * |\overline{\nu}|\| = 0$, (3.11) yields $|\mu| * |\nu| = |\mu| * |\overline{\nu}| \in L^1(K)$. This completes the proof.

REFERENCES

[1] N. Bourbaki, Intégration, Élements de Mathématique, Livre VI, ch 6, Paris, Herman, 1959.

[2] R.G.M. Brummelhuis, An F. and M. Riesz theorem for bounded symmetric domains, Ann. Inst. Fourier **37** (1987), 139-150.

[3] R.G.M. Brummelhuis, A note on Riesz sets and lacunary sets, J. Austral. Math. soc. **48** (1990), 57-65.

[4] G.A. Edgar, Disintegration of measures and vector-valued Radon-Nikodym theorem, Duke Math. J. **42** (1975), 447-450.

[5] G.B. Folland, Real Analysis: Modern Techniques and Their Applications, Wiley-Interscience, New York, 1984.

[6] I. Glicksberg, Fourier-Stieltjes transforms with small supports, Illinois J. Math. **9** (1965), 418-427.

[7] C.C. Graham, Fourier-Stieltjes transforms with small supports, Illinois J. Math. **18** (1974), 532-534.

[8] E. Hewitt and K.A. Ross, Abstract Harmonic Analysis, Vol. I, New York-Heidelberg-Berlin, Springer-Verlag, 1963.

[9] E. Hewitt and K.A. Ross, Abstract Harmonic Analysis, Vol. II, New York-Heidelberg-Berlin, Springer-Verlag, 1970.

[10] H.A. MacLean, Riesz sets and a theorem of Bochner, Pacific J. Math. **113** (1984), 115-135.

[11] W. Rudin, Fourier Analysis on Groups, Interscience, New York, 1962.

[12] Y. Takahashi and H. Yamaguchi, On measures which are continuous by certain translation, Hokkaido Math. J. **13** (1984), 109-117.

[13] H. Yamaguchi, On the product of a Riesz set and a small p set, Proc. Amer. Math. Soc. **81** (1981), 273-278.

[14] H. Yamaguchi, Idempotent multipliers on the space of analytic singular measures, Hokkaido Math. J. **14** (1985), 49-73.

[15] H. Yamaguchi, Absolute continuity of measures on compact transformation groups, Hokkaido Math. J. **22** (1993), 25-33.

[16] H. Yamaguchi, An F. and M. Riesz theorem on compact groups, Transactions of German-Jpanese Symposium on 'Infinite-Dimensional Harmonic Analysis' held in Tübingen, 249-273, Verlag D.+M. Gräbner, Bamberg, 1996.

Department of mathematics
Josai University
Sakado, Saitama
Japan

TWO DUAL PAIR METHODS IN THE STUDY OF GENERALIZED WHITTAKER MODELS FOR IRREDUCIBLE HIGHEST WEIGHT MODULES

HIROSHI YAMASHITA

INTRODUCTION

Let G be a connected simple linear Lie group of Hermitian type, and let K be a maximal compact subgroup of G. The Lie algebras of G and K are denoted by $\mathfrak{g}_0$ and $\mathfrak{k}_0$ respectively. The purpose of this note is to make an overview of our algebraic and geometric approach to the study of generalized Whittaker models for irreducible admissible representations of G with highest weights. We employ two kinds of dual pair methods in the course of our study.

To be more precise, we write $G_{\mathbf{C}}, K_{\mathbf{C}}$ (resp. $\mathfrak{g}, \mathfrak{k}$) for the complexifications of G, K (resp. $\mathfrak{g}_0, \mathfrak{k}_0$) respectively. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a complexified Cartan decomposition of $\mathfrak{g}$. The G-invariant complex structure on $K\backslash G$ gives a triangular decomposition $\mathfrak{g} = \mathfrak{p}_+ + \mathfrak{k} + \mathfrak{p}_-$ of $\mathfrak{g}$. It is well-known that $\mathfrak{p}_+$ admits precisely $r+1$ number of $K_{\mathbf{C}}$-orbits $\mathcal{O}_m$ $(m = 0, 1, \ldots, r)$ arranged as $\dim \mathcal{O}_0 = 0 < \dim \mathcal{O}_1 < \cdots < \dim \mathcal{O}_r = \dim \mathfrak{p}_+$, where r denotes the real rank of G.

These nilpotent $K_{\mathbf{C}}$-orbits $\mathcal{O}_m$ are essentially related to the highest weight representations. In reality, the Harish-Chandra module of an irreducible admissible G-representation with highest weight is isomorphic to the unique simple quotient $L(\tau)$ of generalized Verma module $M(\tau)$ attached to an irreducible representation (τ, V_τ) of K. Then, the associated variety (i.e., the support) $\mathcal{V}(L(\tau))$ of $L(\tau)$ coincides with the closure of a single $K_{\mathbf{C}}$-orbit $\mathcal{O}_{m(\tau)}$ in $\mathfrak{p}_+$, where $m(\tau)$ depends on τ. On the other hand, following the recipe by Kawanaka [12] (see also [23]), one can construct a generalized Gelfand-Graev representation $\Gamma_m = \mathrm{Ind}_{\mathfrak{n}(m)}^{G}(\eta_m)$ (GGGR for short; see Definition 4.1) attached to the nilpotent G-orbit $\mathcal{O}'_m$ in $\mathfrak{g}_0$ corresponding to each $K_{\mathbf{C}}$-orbit $\mathcal{O}_m$ through the Kostant-Sekiguchi bijection. The GGGR Γ_m is induced from certain one-dimensional representation η_m of a nilpotent Lie subalgebra $\mathfrak{n}(m)$ of $\mathfrak{g}$, and it is far from irreducible.

In this note, we are concerned with the following problem.

Problem. Describe the $(\mathfrak{g}, K)$-embeddings, i.e., the generalized Whittaker models, of $L(\tau)$ into these GGGRs Γ_m.

As for $L(\tau)$'s isomorphic to the irreducible generalized Verma modules $M(\tau)$, we already have a complete answer in [24, Part II]. Hence our main interest is in the case where the corresponding $M(\tau)$ is reducible.

In order to specify the embeddings, we use the invariant differential operator $\mathcal{D}_{\tau^*}$ on $K\backslash G$ of gradient type associated to the K-representation τ^* dual to τ (Definition 2.2). This operator $\mathcal{D}_{\tau^*}$ is due to Enright, Davidson and Stanke ([2], [3], [4]), and the K-finite kernel of $\mathcal{D}_{\tau^*}$ realizes the dual lowest weight module $L(\tau)^*$. Our first dual pair method, which comes essentially from a duality of Peter-Weyl type for irreducible $(\mathfrak{g}, K)$-modules,

Date: January 25, 2000, Revised February 23, 2000.
1991 *Mathematics Subject Classification*. Primary: 22E46; Secondary: 17B10.

tells us that the space $\mathcal{Y}(\tau, m)$ of η_m-covariant solutions F of differential equation $\mathcal{D}_{\tau^*}F = 0$ is isomorphic to the space of $(\mathfrak{g}, K)$-homomorphisms in question. The space $\mathcal{Y}(\tau, m)$ can be intrinsically analyzed by an algebraic method, thanks to the Cayley transform on $G_{\mathbf{C}}$ which carries the bounded realization of $K\backslash G$ to the unbounded one.

As consequences, it is shown that $L(\tau)$ embeds into the GGGR Γ_m with nonzero and finite multiplicity if and only if the corresponding $\mathcal{O}_m$ is the unique open $K_{\mathbf{C}}$-orbit $\mathcal{O}_{m(\tau)}$ in the associated variety $\mathcal{V}(L(\tau))$. If $L(\tau)$ is unitarizable, we can specify the space $\mathcal{Y}(\tau) := \mathcal{Y}(\tau, m(\tau))$ in terms of the principal symbol at the origin Ke of the differential operator $\mathcal{D}_{\tau^*}$. This reveals a natural action on $\mathcal{Y}(\tau)$ of the isotropy subgroup $K_{\mathbf{C}}(X)$ of $K_{\mathbf{C}}$ at a certain point $X \in \mathcal{O}_{m(\tau)}$. Furthermore, we find that the dimension of $\mathcal{Y}(\tau)$ coincides with the multiplicity of $L(\tau)$ at the defining ideal of $\mathcal{V}(L(\tau))$. See Theorems 5.1 and 5.2.

If G is one of the classical groups $G = SU(p,q)$, $Sp(2n,\mathbb{R})$ and $SO^*(2n)$, the theory of reductive dual pair gives realizations of unitarizable highest weight modules $L(\tau)$ (cf. [11], [7], [3]). The generalized Whittaker models for such an $L(\tau)$ can be described more explicitly by using the oscillator representation of the pair (G, G') with a compact group G' dual to G. This is our second dual pair method. The case $SU(p,q)$ has been studied by Tagawa [20] motivated by author's observation in 1997 for the case $Sp(n,\mathbb{R})$. In this note we focus our attention on the remaining case $SO^*(2n)$.

The full detail of this overview will appear elsewhere (see [27]).

We organize this note as follows.

Section 1 concerns our first dual pair method. Namely, we provide with a kernel theorem (Theorem 1.2) which will be utilized for describing the generalized Whittaker models in later sections. We introduce in Section 2 the differential operator $\mathcal{D}_{\tau^*}$ on $K\backslash G$ of gradient type associated to τ^*, after [4]. Section 3 is devoted to characterizing the associated variety and multiplicity of irreducible highest weight module $L(\tau)$ by means of the principal symbol of $\mathcal{D}_{\tau^*}$ (Theorem 3.3). After introducing the GGGRs Γ_m in Section 4, we state our main results (Theorems 5.1 and 5.2) in Section 5. Also, we discuss the case of classical group $SO^*(2n)$ more explicitly in 5.2, through our second dual pair method.

1. The first dual pair method – Kernel theorem

In this section, let G be any connected semisimple Lie group with finite center. We employ the same notation as in Introduction. Conventionally, the complexification in $\mathfrak{g}$ of any real vector subspace $\mathfrak{s}_0$ of $\mathfrak{g}_0$ will be denoted by $\mathfrak{s}$ by dropping the subscript 0. We write $U(\mathfrak{m})$ (resp. $S(\mathfrak{v})$) for the universal enveloping algebra of a Lie algebra $\mathfrak{m}$ (resp. the symmetric algebra of a vector space $\mathfrak{v}$). A $U(\mathfrak{g})$-module X is called a $(\mathfrak{g}, K)$-module if the subalgebra $U(\mathfrak{k})$ acts on X locally finitely, and if the $\mathfrak{k}_0$-action gives rise to a representation of K on X through exponential map.

The group G acts on the space $C^\infty(G)$ of all smooth functions on G by left translation L and by right translation R as follows:

$$(1.1) \qquad g^L f(x) := f(g^{-1}x), \quad g^R f(x) := f(xg) \quad (g \in G, x \in G; f \in C^\infty(G)).$$

Through differentiation one gets two $U(\mathfrak{g})$-representations on $C^\infty(G)$ denoted again by L and R respectively. Let $C^\infty_K(G)$ be the space of all functions in $C^\infty(G)$ which are left K-finite and also right K-finite. Then $C^\infty_K(G)$ becomes a $(\mathfrak{g}, K)$-module through L or R.

The following well-known lemma says that a duality of Peter-Weyl type holds for irreducible $(\mathfrak{g}, K)$ modules.

Lemma 1.1. *Let X be an irreducible $(\mathfrak{g}, K)$-module, and let f be in $C^\infty_K(G)$. Then the $(\mathfrak{g}, K)$-module $U(\mathfrak{g})^L f$ generated by f through L is isomorphic to X if and only if the* *corresponding $U(\mathfrak{g})^R f$ through R is isomorphic to the dual $(\mathfrak{g}, K)$-module $\boldsymbol{X}^*$ consisting of all K-finite linear forms on $\boldsymbol{X}$.*

For an irreducible $(\mathfrak{g}, K)$-module $\boldsymbol{X}$, we fix once and for all an irreducible finite-dimensional representation (τ, V_τ) of K which occurs in $\boldsymbol{X}$, and fix an embedding $i_\tau : V_\tau \hookrightarrow \boldsymbol{X}$ as K-modules. Then the adjoint operator i_τ^* of i_τ gives a surjective K-homomorphism from $\boldsymbol{X}^*$ to V_τ^*, where (τ^*, V_τ^*) denotes the representation of K contragredient to τ.

We now consider the C^∞-induced representation $\mathrm{Ind}_K^G(\tau^*)$ acting on the space

$$C_{\tau^*}^\infty(G) := \{\Phi : G \xrightarrow{C^\infty} V_\tau^* \mid \Phi(kg) = \tau^*(k)\Phi(g) \ (g \in G,\ k \in K)\}, \tag{1.2}$$

endowed with G- and $U(\mathfrak{g})$-module structures through right translation R. Equip $C_{\tau^*}^\infty(G)$ with a Fréchet space topology of compact uniform convergence of functions on G and each of their derivatives. Then the G-action on $C_{\tau^*}^\infty(G)$ is smooth. By the Frobenius reciprocity, there corresponds (to i_τ^*) a unique $(\mathfrak{g}, K)$-embedding A_{τ^*} from $\boldsymbol{X}^*$ into $C_{\tau^*}^\infty(G)$ through

$$A_{\tau^*}(\varphi)(g) = \tilde{i}_\tau^*(\pi^*(g)\varphi) \quad (g \in G;\ \varphi \in \boldsymbol{X}^*). \tag{1.3}$$

Here $\tilde{i}_\tau^*$ denotes the unique continuous extension of $i_\tau^* : \boldsymbol{X}^* \to V_\tau^*$ to any irreducible admissible G-module $\boldsymbol{H}^*$ with K-finite part $\boldsymbol{X}^*$.

Let $\mathrm{Hom}_{\mathfrak{g},K}(\boldsymbol{X},\ C^\infty(G))$ be the space of $(\mathfrak{g}, K)$-homomorphisms from $\boldsymbol{X}$ into $C^\infty(G)$ (under the action L). The right action R on $C^\infty(G)$ naturally gives a G-module structure on this space of $(\mathfrak{g}, K)$-homomorphisms. For each element W in $\mathrm{Hom}_{\mathfrak{g},K}(\boldsymbol{X},\ C^\infty(G))$, one can define $F \in C_{\tau^*}^\infty(G)$ by

$$\langle F(g), v\rangle = ((W \circ i_\tau)(v))(g) \quad (g \in G,\ v \in V_\tau). \tag{1.4}$$

Here $\langle\ \cdot\ ,\cdot\ \rangle$ stands for the dual pairing on $V_\tau^* \times V_\tau$. Then it is easily seen that the assignment $W \mapsto F$ sets up a G-embedding

$$\mathrm{Hom}_{\mathfrak{g},K}(\boldsymbol{X},\ C^\infty(G)) \hookrightarrow C_{\tau^*}^\infty(G). \tag{1.5}$$

Lemma 1.1 together with our argument in [25, I, §2] allows us to prove the following kernel theorem.

Theorem 1.2. *Under the above notation, if $\mathcal{D}$ is any continuous G-homomorphism from $C_{\tau^*}^\infty(G)$ to a smooth Fréchet G-module $\boldsymbol{M}$ such that*

$$A_{\tau^*}(\boldsymbol{X}^*) = \{F \in C_{\tau^*}^\infty(G) \mid F \text{ is right } K\text{-finite and } \mathcal{D}F = 0\}, \tag{1.6}$$

then the full kernel space $\mathrm{Ker}\,\mathcal{D}$ *of $\mathcal{D}$ in $C_{\tau^*}^\infty(G)$ coincides with the image of the G-embedding* (1.5). *Hence one gets*

$$\mathrm{Hom}_{\mathfrak{g},K}(\boldsymbol{X},\ C^\infty(G)) \simeq \mathrm{Ker}\,\mathcal{D} \quad \text{as } G\text{-modules}. \tag{1.7}$$

This claim can be deduced also from the work of Kashiwara and Schmid (cf. [10] and [19]) on the maximal globalization of Harish-Chandra modules, by noting that $\mathrm{Ker}\,\mathcal{D}$ gives the maximal globalization of the irreducible $(\mathfrak{g}, K)$-module $\boldsymbol{X}^*$.

Example 1.3. We mention that an operator $\mathcal{D}$ satisfying the requirement in Theorem 1.2 has been constructed when $\boldsymbol{X}^*$ is the $(\mathfrak{g}, K)$-module associated with: (a) discrete series ([18], [9]) and more generally Zuckerman cohomologically induced module ([22], [1]), with parameter "far from the walls", or (b) highest weight representation ([2], [4]; see also Theorem 2.5). In each of these cases, $\mathcal{D}$ is given as a G-invariant differential operator of gradient type acting on $C_{\tau^*}^\infty(G)$, where τ^* is the unique extreme K-type of $\boldsymbol{X}^*$.

We will apply the above kernel theorem later in order to describe the generalized Whittaker models for irreducible admissible highest weight representations.

2. Differential operators of gradient type

From now on, let us assume that G is of Hermitian type as in Introduction. We consider the irreducible highest weight $(\mathfrak{g}, K)$-modules $L(\tau)$ with extreme K-types τ. In this section we construct, following [4], the differential operators $\mathcal{D}_{\tau^*}$ of gradient type on $K\backslash G$ whose K-finite kernels realize the dual lowest weight $(\mathfrak{g}, K)$-modules $L(\tau)^*$ (Theorem 2.5).

2.1. **Generalized Verma modules.** First, we fix some notation concerning simple Lie algebras of Hermitian type (cf. [24, Part I, §5] and [8, 3.3]). Take the complexification $G_{\mathbf{C}}$ of G, and the analytic subgroup $K_{\mathbf{C}}$ of $G_{\mathbf{C}}$ with Lie algebra $\mathfrak{k} = \mathfrak{k}_0 \otimes_{\mathbf{R}} \mathbb{C}$. Then there exists a unique (up to sign) central element Z_0 of $\mathfrak{k}_0$ such that $\operatorname{ad} Z_0$ restricted to $\mathfrak{p}_0$ gives an $\operatorname{Ad}(K)$-invariant complex structure on $\mathfrak{p}_0$. One gets a triangular decomposition of $\mathfrak{g}$ as follows:

$$\begin{gathered} \mathfrak{g} = \mathfrak{p}_- \oplus \mathfrak{k} \oplus \mathfrak{p}_+ \quad \text{such that} \\ [\mathfrak{k}, \mathfrak{p}_\pm] \subset \mathfrak{p}_\pm, \quad [\mathfrak{p}_+, \mathfrak{p}_-] \subset \mathfrak{k}, \quad [\mathfrak{p}_+, \mathfrak{p}_+] = [\mathfrak{p}_-, \mathfrak{p}_-] = \{0\}, \end{gathered} \tag{2.1}$$

where $\mathfrak{p}_\pm$ denotes the eigenspace of $\operatorname{ad} Z_0$ on $\mathfrak{g}$ with eigenvalue $\pm\sqrt{-1}$ respectively.

Let $\mathfrak{t}_0$ be a compact Cartan subalgebra of $\mathfrak{g}_0$ contained in $\mathfrak{k}_0$. We write Δ for the root system of $\mathfrak{g}$ with respect to $\mathfrak{t}$. For each $\gamma \in \Delta$, the corresponding root subspace of $\mathfrak{g}$ will be denoted by $\mathfrak{g}(\mathfrak{t};\gamma)$. We choose root vectors $X_\gamma \in \mathfrak{g}(\mathfrak{t};\gamma)$ $(\gamma \in \Delta)$ such that

$$X_\gamma - X_{-\gamma},\ \sqrt{-1}(X_\gamma + X_{-\gamma}) \in \mathfrak{k}_0 + \sqrt{-1}\mathfrak{p}_0, \quad [X_\gamma, X_{-\gamma}] = H_\gamma, \tag{2.2}$$

where H_γ is the element of $\sqrt{-1}\mathfrak{t}_0$ corresponding the coroot $\gamma^\vee := 2\gamma/(\gamma,\gamma)$ through the identification $\mathfrak{t}^* = \mathfrak{t}$ by the Killing form B of $\mathfrak{g}$. Let Δ_c (resp. Δ_n) denote the subset of all compact (resp. noncompact) roots in Δ.

Take a positive system Δ^+ of Δ compatible with the decomposition (2.1):

$$\mathfrak{p}_\pm = \bigoplus_{\gamma \in \Delta_n^+} \mathfrak{g}(\mathfrak{t};\pm\gamma) \quad \text{with} \quad \Delta_n^+ := \Delta^+ \cap \Delta_n, \tag{2.3}$$

and fix a lexicographic order on $\sqrt{-1}\mathfrak{t}_0^*$ which yields Δ^+. Using this order we define a fundamental sequence $(\gamma_1, \gamma_2, \ldots, \gamma_r)$ of strongly orthogonal (i.e., $\gamma_i \pm \gamma_j \notin \Delta \cup \{0\}$ for $i \neq j$) noncompact positive roots in such a way that γ_k is the maximal element of Δ^+, which is strongly orthogonal to $\gamma_{k+1}, \ldots, \gamma_r$. Then r is equal to the real rank of G.

Let (τ, V_τ) be any irreducible finite-dimensional representation of K with Δ_c^+-highest weight $\lambda = \lambda(\tau)$. We consider the *generalized Verma* $U(\mathfrak{g})$*-module* induced from τ:

$$M(\tau) := U(\mathfrak{g}) \otimes_{U(\mathfrak{k}+\mathfrak{p}_+)} V_\tau. \tag{2.4}$$

Here τ is extended to a representation of the maximal parabolic subalgebra $\mathfrak{k} + \mathfrak{p}_+$ by the null $\mathfrak{p}_+$-action on V_τ. $M(\tau)$ admits a natural $(\mathfrak{g}, K)$-module structure. Let $N(\tau)$ be the unique maximal proper $(\mathfrak{g}, K)$-submodule of $M(\tau)$. Then the quotient $L(\tau) := M(\tau)/N(\tau)$ gives an irreducible $(\mathfrak{g}, K)$-module with Δ^+-highest weight λ.

Note that $M(\tau) = U(\mathfrak{p}_-)V_\tau$ is canonically isomorphic to the tensor product $S(\mathfrak{p}_-) \otimes V_\tau = S(\mathfrak{p}_-) \otimes_{\mathbf{C}} V_\tau$ as a K-module, where $S(\mathfrak{p}_-)$ ($\simeq U(\mathfrak{p}_-)$, since $\mathfrak{p}_-$ is abelian) denotes the symmetric algebra of $\mathfrak{p}_-$ looked upon as a K-module by the adjoint action. This isomorphism yields a gradation of the K-module $M(\tau)$:

$$M(\tau) = \bigoplus_{j=0}^{\infty} M_j(\tau) \quad \text{with} \quad M_j(\tau) := S^j(\mathfrak{p}_-)V_\tau \simeq S^j(\mathfrak{p}_-) \otimes V_\tau. \tag{2.5}$$

Here we write $S^j(\mathfrak{p}_-)$ for the K-submodule of $S(\mathfrak{p}_-)$ consisting of all homogeneous elements of $S(\mathfrak{p}_-)$ of degree j. Observe that the submodule $N(\tau)$ is graded:

$$N(\tau) = \bigoplus_{j=0}^{\infty} N_j(\tau) \quad \text{with} \quad N_j(\tau) := N(\tau) \cap M_j(\tau). \tag{2.6}$$

Since $M(\tau) = S(\mathfrak{p}_-)V_\tau$ is finitely generated over the Noetherian ring $S(\mathfrak{p}_-)$, so is the submodule $N(\tau)$, too. This implies that, if $N(\tau) \neq \{0\}$, there exist finitely many irreducible K-submodules $W_1, \ldots, W_q$ of $N(\tau)$ such that

$$N(\tau) = \sum_{u=1}^{q} S(\mathfrak{p}_-)W_u \quad \text{with} \quad W_u \subset S^{i_u}(\mathfrak{p}_-)V_\tau \simeq S^{i_u}(\mathfrak{p}_-) \otimes V_\tau \tag{2.7}$$

for some positive integers i_u $(u = 1, \ldots, q)$ arranged as

$$i(\tau) := i_1 \leq i_2 \leq \cdots \leq i_q. \tag{2.8}$$

We call $i(\tau)$ *the level of reduction* of $M(\tau)$.

For unitarizable $L(\tau)$'s, Joseph [5] gives a simple description of the maximal submodule $N(\tau)$ as follows. Assume that $L(\tau)$ is unitarizable and that $N(\tau) \neq \{0\}$. Then the level $i(\tau)$ of reduction of $M(\tau)$ turns to be an integer such that $1 \leq i(\tau) \leq r$, where r is the real rank of G. Let $Q_{i(\tau)}$ be the irreducible K-submodule of $S^{i(\tau)}(\mathfrak{p}_-)$ with lowest weight $-\gamma_r - \ldots - \gamma_{r-i(\tau)+1}$. Then the tensor product $Q_{i(\tau)} \otimes V_\tau$ has a unique irreducible K-submodule W_1, called the PRV(Parthasarathy, Rao and Varadarajan)-component, with extreme weight $\lambda - \gamma_r - \ldots - \gamma_{r-i(\tau)+1}$. We regard W_1 as a K-submodule of $M_{i(\tau)}(\tau)$.

Theorem 2.1 ([5, 5.2, 6.5 and 8.3], see also [3, 3.1]). *Under the above assumption and notation, the maximal submodule $N(\tau)$ of $M(\tau)$ is a highest weight $(\mathfrak{g}, K)$-module generated over $S(\mathfrak{p}_-)$ by the PRV-component W_1.*

2.2. **A realization of the dual lowest weight module $L(\tau)^*$.** For each irreducible representation (τ, V_τ) of K, let $L(\tau)^*$ be the irreducible lowest weight $(\mathfrak{g}, K)$-module which is dual to $L(\tau)$. Since $L(\tau)^*$ contains the extreme K-type (τ^*, V_τ^*) with multiplicity one, there exists a unique (up to constant multiple) $(\mathfrak{g}, K)$-embedding A_{τ^*} form $L(\tau)^*$ into $C^\infty_{\tau^*}(G)$. We are going to introduce a differential operator of gradient type whose K-finite kernel coincides with the image $A_{\tau^*}(L(\tau)^*)$.

For this, we take a basis $X_1, \ldots, X_s$ of the $\mathbb{C}$-vector space $\mathfrak{p}_+$ such that $B(X_j, \overline{X}_k) = \delta_{jk}$ (Kronecker's δ), where $\overline{X}_i \in \mathfrak{p}_-$ denotes the complex conjugate of $X_i \in \mathfrak{p}_+$ with respect to the real form $\mathfrak{g}_0$. Set

$$X^\alpha := X_1^{\alpha_1} \cdots X_s^{\alpha_s} \in U(\mathfrak{p}_+) \quad \text{and} \quad \overline{X}^\alpha := \overline{X}_1^{\alpha_1} \cdots \overline{X}_s^{\alpha_s} \in U(\mathfrak{p}_-) \tag{2.9}$$

for every multi-index $\alpha = (\alpha_1, \ldots, \alpha_s)$ of nonnegative integers $\alpha_1, \ldots, \alpha_s$. We call $|\alpha| := \alpha_1 + \cdots + \alpha_s$ the length of α. For each positive integer n we define the gradients ∇^n and $\overline{\nabla}^n$ of order n on $C^\infty_{\tau^*}(G)$ as follows.

$$\nabla^n F(x) := \sum_{|\alpha|=n} \overline{X}^\alpha \otimes (X^\alpha)^L F(x), \tag{2.10}$$

$$\overline{\nabla}^n F(x) := \sum_{|\alpha|=n} X^\alpha \otimes (\overline{X}^\alpha)^L F(x), \tag{2.11}$$

for $x \in G$ and $F \in C^\infty_{\tau^*}(G)$. It is then easy to see that $\nabla^n F$ and $\overline{\nabla}^n F$ are independent of the choice of a basis $X_1, \ldots, X_s$, and that the operators ∇^n and $\overline{\nabla}^n$ give continuous G-homomorphisms

$$\nabla^n : C^\infty_{\tau^*}(G) \to C^\infty_{\tau^*(-n)}(G), \quad \overline{\nabla}^n : C^\infty_{\tau^*}(G) \to C^\infty_{\tau^*(+n)}(G). \tag{2.12}$$

Here $\tau^*(\pm n)$ denotes the K-representation on the tensor product $S^n(\mathfrak{p}_\pm) \otimes V_\tau^*$ respectively.

Let W_u ($u = 1, \ldots, q$) be, as in (2.7), the irreducible K-submodules of $S^{i_u}(\mathfrak{p}_-)V_\tau \subset N(\tau)$ which generate $N(\tau)$ over $S(\mathfrak{p}_-)$ when $N(\tau) \neq \{0\}$. For each u, the adjoint operator P_u of the embedding

$$W_u \hookrightarrow S^{i_u}(\mathfrak{p}_-)V_\tau \simeq S^{i_u}(\mathfrak{p}_-) \otimes V_\tau \tag{2.13}$$

gives a surjective K-homomorphism:

$$P_u : S^{i_u}(\mathfrak{p}_+) \otimes V_\tau^* \simeq (S^{i_u}(\mathfrak{p}_-) \otimes V_\tau)^* \longrightarrow W_u^*, \tag{2.14}$$

where $\mathfrak{p}_+$ is identified with the dual space of $\mathfrak{p}_-$ through the Killing form B, which is nondegenerate on $\mathfrak{p}_+ \times \mathfrak{p}_-$.

Definition 2.2. Keep the above notation.

(1) Let $\mathcal{D}_{\tau^*}$ be a continuous G-homomorphism from $C^\infty_{\tau^*}(G)$ to $C^\infty_\rho(G)$ defined by

$$\mathcal{D}_{\tau^*}F(x) := \nabla^1 F(x) \oplus (\oplus_{u=1}^q P_u(\overline{\nabla}^{i_u} F(x))) \tag{2.15}$$

for $x \in G$ and $F \in C^\infty_{\tau^*}(G)$. Here we write $\rho = \rho(\tau^*)$ for the representation of K on

$$(\mathfrak{p}_- \otimes V_\tau^*) \oplus (\oplus_{u=1}^q W_u^*), \tag{2.16}$$

and $\mathcal{D}_{\tau^*}$ should be understood as $\mathcal{D}_{\tau^*} = \nabla^1$ if $N(\tau) = \{0\}$, or equivalently $M(\tau) = L(\tau)$. We call $\mathcal{D}_{\tau^*}$ the *differential operator of gradient type* associated to τ^*.

(2) Put for $X \in \mathfrak{p}_+$ and $v^* \in V_\tau^*$,

$$\sigma(X, v^*) := \sum_{u=1}^q P_u(X^{i_u} \otimes v^*) \in W^* := \oplus_{u=1}^q W_u^*. \tag{2.17}$$

We call σ the *principal symbol* of $\mathcal{D}_{\tau^*}$ at the origin. Here σ should be understood as $\sigma(X, v^*) = 0$ for every $X \in \mathfrak{p}_+$ and every $v^* \in V_\tau^*$, when $\mathcal{D}_{\tau^*} = \nabla^1$.

Remark 2.3. A function $F \in C^\infty_{\tau^*}(G)$ gives an anti-holomorphic section of the vector bundle on $K\backslash G$ associated to τ^* if and only if $\nabla^1 F = 0$. Hence the elements of $\operatorname{Ker} \mathcal{D}_{\tau^*}$ are necessarily anti-holomorphic. The converse is true when $N(\tau) = \{0\}$.

Remark 2.4. If $L(\tau)$ is unitarizable, one sees from Theorem 2.1 that

$$\mathcal{D}_{\tau^*} = \nabla^1 \oplus (P_1 \circ \overline{\nabla}^{i(\tau)}). \tag{2.18}$$

Here $i(\tau)$ is the level of reduction of $M(\tau)$, and the K-homomorphism P_1 is defined through the PRV-component $W_1 \subset S^{i(\tau)}(\mathfrak{p}_-) \otimes V_\tau$.

The following theorem, equivalent to [4, Prop.7.6] due to Davidson and Stanke, realizes the lowest weight module $L(\tau)^*$ by means of $\mathcal{D}_{\tau^*}$.

Theorem 2.5. *The image of the $(\mathfrak{g}, K)$-embedding A_{τ^*} from $L(\tau)^*$ into $C^\infty_{\tau^*}(G)$ coincides with the K-finite kernel of the differential operator $\mathcal{D}_{\tau^*}$ of gradient type.*

3. Associated variety and principal symbol

This section concerns the relationship between the associated variety (with multiplicity) of $L(\tau)$ and the principal symbol σ of the differential operator $\mathcal{D}_{\tau^*}$ of gradient type. The result is summarized as Theorem 3.3.

For every integer m such that $0 \le m \le r = \mathbb{R}\text{-rank}\, G$, we set

$$\mathcal{O}_m := \mathrm{Ad}(K_{\mathbb{C}})X(m) \quad \text{with} \quad X(m) := \sum_{k=r-m+1}^{r} X_{\gamma_k} \quad (\text{see } (2.2)). \tag{3.1}$$

where $X(0)$ should be understood as 0. The following proposition is well-known.

Proposition 3.1. *The subspace $\mathfrak{p}_+$ splits into a disjoint union of $r+1$ number of $K_{\mathbb{C}}$-orbits $\mathcal{O}_m$ $(0 \le m \le r)$: $\mathfrak{p}_+ = \coprod_{0 \le m \le r} \mathcal{O}_m$, and the closure $\overline{\mathcal{O}_m}$ of each orbit $\mathcal{O}_m$ is equal to $\cup_{k \le m} \mathcal{O}_k$ for every m.*

Let $L(\tau)$ be the irreducible highest weight $(\mathfrak{g}, K)$-module with extreme K-type (τ, V_τ). The annihilator $\mathrm{Ann}_{S(\mathfrak{p}_-)} L(\tau)$ of $L(\tau)$ in $S(\mathfrak{p}_-) = U(\mathfrak{p}_-)$ defines an affine algebraic variety

$$\mathcal{V}(L(\tau)) := \{X \in \mathfrak{p}_+ \mid D(X) = 0 \quad \text{for all } D \in \mathrm{Ann}_{S(\mathfrak{p}_-)} L(\tau)\} \subset \mathfrak{p}_+, \tag{3.2}$$

which is called the *associated variety* of the $(\mathfrak{g}, K)$-module $L(\tau)$. Here $S(\mathfrak{p}_-)$ is identified with the ring of polynomial functions on $\mathfrak{p}_+$ through the Killing form B of $\mathfrak{g}$. By noting that the ideal $\mathrm{Ann}_{S(\mathfrak{p}_-)} L(\tau)$ is stable under $\mathrm{Ad}(K_{\mathbb{C}})$, we see from Proposition 3.1 that there exists a unique integer $m(\tau)$ $(0 \le m(\tau) \le r)$ such that

$$\mathcal{V}(L(\tau)) = \overline{\mathcal{O}_{m(\tau)}}. \tag{3.3}$$

In particular, the variety $\mathcal{V}(L(\tau))$ is irreducible.

Now let I_m be the prime ideal of $S(\mathfrak{p}_-)$ that defines the irreducible variety $\overline{\mathcal{O}_m}$ $(0 \le m \le r)$. If M is a finitely generated $S(\mathfrak{p}_-)$-module, the *multiplicity* $\mathrm{mult}_{I_m}(M)$ of M at I_m is defined to be the length of the localization M_{I_m} as an $S(\mathfrak{p}_-)_{I_m}$-module. The associated variety $\mathcal{V}(L(\tau))$ with the multiplicity $\mathrm{mult}_{I_{m(\tau)}}(L(\tau))$ is called the *associated cycle* of $L(\tau)$.

For each $X \in \mathfrak{p}_+$, let $\mathfrak{m}(X)$ be the maximal ideal of $S(\mathfrak{p}_-)$ which defines the variety $\{X\}$ of a single element X. We set

$$\mathcal{W}(X, \tau) := L(\tau)/\mathfrak{m}(X)L(\tau). \tag{3.4}$$

Then we see that $\dim \mathcal{W}(X, \tau) < \infty$, and that the isotropy group $K_{\mathbb{C}}(X)$ of $K_{\mathbb{C}}$ at X acts on $\mathcal{W}(X, \tau)$ naturally. Let σ be the principal symbol of $\mathcal{D}_{\tau^*}$ as in Definition 2.2. The map $v^* \mapsto \sigma(X, v^*)$ gives a $K_{\mathbb{C}}(X)$-homomorphism $\sigma(X, \cdot\,)$ from V_τ^* to W^*. Hence $\mathrm{Ker}\, \sigma(X, \cdot\,)$ is a $K_{\mathbb{C}}(X)$-submodule of V_τ^*.

The following lemma relates the above kernel of σ with the $K_{\mathbb{C}}(X)$-module $\mathcal{W}(X, \tau)$.

Lemma 3.2. *For each $X \in \mathfrak{p}_+$, the natural map*

$$V_\tau \hookrightarrow M(\tau) \to L(\tau) = M(\tau)/N(\tau) \to \mathcal{W}(X, \tau) = L(\tau)/\mathfrak{m}(X)L(\tau) \tag{3.5}$$

from V_τ onto $\mathcal{W}(X, \tau)$ induces a $K_{\mathbb{C}}(X)$-isomorphism

$$\mathcal{W}(X, \tau)^* \simeq \mathrm{Ker}\, \sigma(X, \cdot\,) \subset V_\tau^* \tag{3.6}$$

through the contravariant functor $\mathrm{Hom}_{\mathbb{C}}(\cdot\,, \mathbb{C})$.

By applying the argument of Vogan in [21, Section 2] in view of Lemma 3.2, we can deduce the following theorem.

Theorem 3.3. *Let $L(\tau)$ be any irreducible highest weight $(\mathfrak{g}, K)$-module with extreme K-type τ, and let $\sigma : \mathfrak{p}_+ \times V_\tau^* \to W^*$ be the principal symbol of the differential operator $\mathcal{D}_{\tau^*}$ of gradient type associated to τ^*. Then it holds that*

$$\mathcal{V}(L(\tau)) = \{X \in \mathfrak{p}_+ \mid \operatorname{Ker} \sigma(X, \cdot) \neq \{0\}\}. \tag{3.7}$$

Moreover, if X is an element of the unique open $K_{\mathbf{C}}$-orbit $\mathcal{O}_{m(\tau)}$ of $\mathcal{V}(L(\tau))$, the dimension of $\operatorname{Ker} \sigma(X, \cdot)$ is equal to the multiplicity of $S(\mathfrak{p}_-)$-module $L(\tau)/I_{m(\tau)}L(\tau)$ at the prime ideal $I_{m(\tau)}$.

As for the unitarizable highest weight modules $L(\tau)$, some results of Joseph [15, Lem.2.4 and Th.5.6] (due to Davidson, Enright and Stanke [3] for $\mathfrak{g}$ classical) assure that the prime ideal $I_{m(\tau)}$ annihilates $L(\tau)$. Thus we obtain

Corollary 3.4. *One has* $\operatorname{mult}_{I_{m(\tau)}}(L(\tau)) = \dim \mathcal{W}(X, \tau)$ $(X \in \mathcal{O}_{m(\tau)})$ *for every irreducible unitarizable highest weight module $L(\tau)$.*

Remark 3.5. We can get the same kind of characterization of the associated cycle also for irreducible $(\mathfrak{g}, K)$-modules of discrete series, by using the results of [9] and [26]. We will discuss it elsewhere.

Remark 3.6. For classical groups $Sp(2n, \mathbb{R})$, $U(p,q)$ and $O^*(2p)$, Nishiyama, Ochiai and Taniguchi [17, Th.7.18 and Th.9.1] have described the associated cycle and the Bernstein degree of unitarizable highest weight module $L(\tau)$ by using the theory of reductive dual pairs (G, G') with compact G'. They deal with the case where the dual pair (G, G') is in the stable range with smaller G', through detailed study of K-types of $L(\tau)$. On the other hand, the above corollary gives another simple method for describing the multiplicity $\operatorname{mult}_{I_{m(\tau)}}(L(\tau))$ by means of the $K_{\mathbf{C}}(X)$-module $\mathcal{W}(X, \tau)$ (cf. 5.2).

4. Cayley transform and generalized Gelfand-Graev representations

In this section, we introduce the generalized Gelfand-Graev representations of G attached to the Cayley transforms of nilpotent $K_{\mathbf{C}}$-orbits $\mathcal{O}_m = \operatorname{Ad}(K_{\mathbf{C}})X(m)$ $(m = 0, \ldots, r)$ in $\mathfrak{p}_+$.

For this, we consider an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$:

$$X(m) = \sum_{k=r-m+1}^{r} X_{\gamma_k}, \quad H(m) := \sum_{k=r-m+1}^{r} H_{\gamma_k}, \quad Y(m) := \sum_{k=r-m+1}^{r} X_{-\gamma_k}, \tag{4.1}$$

and the *Cayley transform* $\mathbf{c} = \operatorname{Ad}(c)$ on $\mathfrak{g}$ defined by the element

$$c := \exp\left(\frac{\pi}{4} \cdot \sum_{k=1}^{r} (X_{\gamma_k} - X_{-\gamma_k})\right) \in G_{\mathbf{C}}. \tag{4.2}$$

We put

$$\begin{cases} X'(m) := -\sqrt{-1}\mathbf{c}^{-1}(X(m)) = \dfrac{\sqrt{-1}}{2}(H(m) - X(m) + Y(m)), \\ H'(m) := \mathbf{c}^{-1}(H(m)) = X(m) + Y(m), \\ Y'(m) := \sqrt{-1}\mathbf{c}^{-1}(Y(m)) = -\dfrac{\sqrt{-1}}{2}(H(m) + X(m) - Y(m)). \end{cases} \tag{4.3}$$

Then $(X'(m), H'(m), Y'(m))$ forms an $\mathfrak{sl}_2$-triple in the real form $\mathfrak{g}_0$ of $\mathfrak{g}$. Set $\mathcal{O}'_m := \operatorname{Ad}(G)X'(m)$. We note that the nilpotent G-orbit $\mathcal{O}'_m$ in $\mathfrak{g}_0$ corresponds to the $K_{\mathbf{C}}$-orbit $\mathcal{O}_m$ in $\mathfrak{p}_+ \subset \mathfrak{p}$ through the Kostant-Sekiguchi correspondence (cf. [8, Th.3.1]).

Now, let η_m be the one-dimensional representation (i.e., character) of abelian Lie subalgebra $\mathfrak{n}(m) := \mathbf{c}([\mathfrak{k}, Y(m)])$ defined by

$$\eta_m(U) := -\sqrt{-1}B(U, Y'(m)) = -B(c^{-1}U, X(m)) \quad \text{for} \quad U \in \mathfrak{n}(m). \tag{4.4}$$

Then, we can form a C^∞-induced G- and $(\mathfrak{g}, K)$-representation Γ_m acting on the space

$$C^\infty(G; \eta_m) := \{f \in C^\infty(G) \mid U^R f = -\eta_m(U)f \quad (U \in \mathfrak{n}(m))\} \tag{4.5}$$

by left translation L. Note that

$$C^\infty(G; \eta_r) \subset C^\infty(G; \eta_{r-1}) \subset \cdots \subset C^\infty(G; \eta_0) = C^\infty(G), \tag{4.6}$$

since one sees $\mathfrak{n}(m) \subset \mathfrak{n}(m')$ and $\eta_{m'}|\mathfrak{n}(m) = \eta_m$ for $m \leq m'$.

Definition 4.1. We call $(\Gamma_m, C^\infty(G; \eta_m))$ the *generalized Gelfand-Graev representation* (GGGR for short) of G attached to the nilpotent G-orbit $\mathcal{O}'_m = \mathrm{Ad}(G)X'(m)$ in $\mathfrak{g}_0$.

Remark 4.2. The GGGRs attached to arbitrary nilpotent orbits have been constructed in full generality by Kawanaka [12] for reductive algebraic groups. See also [23] for the GGGRs of real semisimple Lie groups.

In order to describe the generalized Whittaker models for $L(\tau)$, we need the bounded and unbounded realizations of Hermitian symmetric space $K\backslash G$. To be more precise, let $P_\pm := \exp \mathfrak{p}_\pm$ be the connected Lie subgroups of $G_{\mathbf{C}}$ with Lie algebras $\mathfrak{p}_\pm$, respectively. Note that the exponential map gives holomorphic diffeomorphisms from $\mathfrak{p}_\pm$ onto $P_\pm$. Consider an open dense subset $P_+K_{\mathbf{C}}P_-$ of $G_{\mathbf{C}}$, which is holomorphically diffeomorphic to the direct product $P_+ \times K_{\mathbf{C}} \times P_-$ through multiplication. For each $x \in P_+K_{\mathbf{C}}P_-$, let $p_+(x)$, $k_{\mathbf{C}}(x)$, and $p_-(x)$ denote respectively the elements of P_+, $K_{\mathbf{C}}$, and P_- such that $x = p_+(x)k_{\mathbf{C}}(x)p_-(x)$. Set $\xi(x) := \log p_-(x) \in \mathfrak{p}_-$.

Proposition 4.3 (cf. [13, Chapter VII]). (1) *One has $Gc \cup G \subset P_+K_{\mathbf{C}}P_-$, where c is the Cayley element of $G_{\mathbf{C}}$ in* (4.2).

(2) *The assignment $x \mapsto \xi(x)$ $(x \in G)$ sets up an anti-holomorphic diffeomorphism from $K\backslash G$ onto a bounded domain $\{\xi(x) \mid x \in G\}$ in $\mathfrak{p}_-$.*

(3) *Similarly, $x \mapsto \xi(xc)$ $(x \in G)$ induces an anti-holomorphic diffeomorphism from $K\backslash G$ onto an unbounded domain $\{\xi(xc) \mid x \in G\}$ in $\mathfrak{p}_-$.*

5. Generalized Whittaker models

For any irreducible finite-dimensional K-module (τ, V_τ), let $L(\tau) = M(\tau)/N(\tau)$ (see 2.1) be the irreducible highest weight $(\mathfrak{g}, K)$-module with extreme K-type τ. Consider the GGGRs $(\Gamma_m, C^\infty(G; \eta_m))$ $(m = 0, \ldots, r)$ induced from the characters $\eta_m : \mathfrak{n}(m) \to \mathbb{C}$. We say that $L(\tau)$ has a *generalized Whittaker model* of type η_m if $L(\tau)$ is isomorphic to a $(\mathfrak{g}, K)$-submodule of $C^\infty(G; \eta_m)$. In this section, we give an answer to the problem posed in Introduction.

5.1. **Main results.** We are going to describe the generalized Whittaker models for $L(\tau)$ by specifying the vector space of $(\mathfrak{g}, K)$-homomorphisms from $L(\tau)$ into $C^\infty(G;\, \eta_m)$. To do this, let $\mathcal{D}_{\tau^*} : C^\infty_{\tau^*}(G) \to C^\infty_\rho(G)$ be, as in Definition 2.2, the G-invariant differential operator of gradient type whose kernel realizes the maximal globarization of lowest weight module $L(\tau)^*$. We set

$$\mathcal{Y}(\tau, m) := \{F \in C^\infty_{\tau^*}(G) \mid \mathcal{D}_{\tau^*}F = 0,\;\; U^R F = -\eta_m(U)F \;(U \in \mathfrak{n}(m))\}. \tag{5.1}$$

Then the kernel theorem (Theorem 1.2) gives a linear isomorphism

$$\mathrm{Hom}_{\mathfrak{g},K}(L(\tau), C^\infty(G;\eta_m)) \simeq \mathcal{Y}(\tau, m) \tag{5.2}$$

through the correspondence (1.4). Thus our task amounts to specifying the space $\mathcal{Y}(\tau, m)$ for each τ and m.

Let $\mathcal{O}_{m(\tau)}$ be the unique open $K_{\mathbb{C}}$-orbit in the associated variety $\mathcal{V}(L(\tau))$ of $L(\tau)$. Among the generalized Whittaker models for $L(\tau)$, those of type $\eta_{m(\tau)}$ are most important. We obtain the following result on the corresponding linear space $\mathcal{Y}(\tau, m)$ with $m = m(\tau)$.

Theorem 5.1. *Let (τ, V_τ) be an irreducible finite-dimensional representation of K. Set $m = m(\tau)$ and $\mathcal{Y}(\tau) := \mathcal{Y}(\tau, m)$ for short. Then,*

(1) $\mathcal{Y}(\tau)$ is a nonzero, finite-dimensional vector space.

(2) For any $F \in \mathcal{Y}(\tau)$, there exists a unique polynomial function φ on $\mathfrak{p}_-$ with values in V_τ^* such that

$$F(x) = \exp B(X(m), \xi(xc))\tau^*(k_{\mathbb{C}}(xc))\varphi(\xi(xc)) \quad (x \in G). \tag{5.3}$$

(3) Let $\sigma : \mathfrak{p}_+ \times V_\tau^* \to W^*$ be the principal symbol of the differential operator $\mathcal{D}_{\tau^*}$ of gradient type, defined by (2.17). For $v^* \in V_\tau^*$, we write F_{v^*} for the function in (5.3) corresponding to the constant polynomial $\varphi : \mathfrak{p}_- \ni Z \mapsto v^* \in V_\tau^*$. Then the assignment $v^* \longmapsto \chi_\tau(v^*) := F_{v^*}$ ($v^* \in \mathrm{Ker}\,\sigma(X(m), \cdot\,)$) yields an injective linear map

$$\chi_\tau : \mathrm{Ker}\,\sigma(X(m), \cdot\,) \hookrightarrow \mathcal{Y}(\tau). \tag{5.4}$$

(4) Assume that $L(\tau)$ is unitarizable. Then the linear embedding χ_τ in (3) is surjective. Hence one gets

$$\mathrm{Hom}_{\mathfrak{g},K}(L(\tau), C^\infty(G;\eta_m)) \simeq \mathcal{Y}(\tau) \simeq \mathrm{Ker}\,\sigma(X(m), \cdot\,) \simeq \mathcal{W}(X(m), \tau) \tag{5.5}$$

as vector spaces, where $\mathcal{W}(X(m), \tau) = L(\tau)/\mathfrak{m}(X(m))L(\tau)$ is as in (3.4). Moreover, the dimension of the vector spaces in (5.5) equals the multiplicity $\mathrm{mult}_{I_m}(L(\tau))$ of the $S(\mathfrak{p}_-)$-module $L(\tau)$ at the unique associated prime I_m, by Corollary 3.4.

As for $\mathcal{Y}(\tau, m')$ with $m' \neq m(\tau)$, we can deduce the following

Theorem 5.2. *The linear space $\mathcal{Y}(\tau, m')$ vanishes (resp. is infinite-dimensional) if $m' > m(\tau)$ (resp. $m' < m(\tau)$).*

These two theorems are the main results of this note.

Remark 5.3. (1) Theorem 5.1 (4) recovers, to a great extent, our earlier work [24, Part II] on the generalized Whittaker models for the holomorphic discrete series.

(2) The vanishing of $\mathcal{Y}(\tau, m')$ ($m' > m(\tau)$) in Theorem 5.2 follows also from a general result of Matumoto [16, Th.1].

5.2. **The second dual pair method: case of $SO^*(2n)$.** Let G be the group $SO^*(2n)$ consisting of all matrices in $SL(2n, \mathbb{C})$ satisfying

$$g\begin{pmatrix} I_n & O \\ O & -I_n \end{pmatrix} {}^t\overline{g} = \begin{pmatrix} I_n & O \\ O & -I_n \end{pmatrix} \quad \text{and} \quad {}^tg\begin{pmatrix} O & I_n \\ I_n & O \end{pmatrix} g = \begin{pmatrix} O & I_n \\ I_n & O \end{pmatrix},$$

where I_n denotes the identity matrix of size n. The totality of unitary matrices in G forms a maximal compact subgroup K. In this subsection, we describe the space $\mathcal{W}(X(m), \tau)$ in (5.5) by using the oscillator representation of the pair (G, G') with $G' = Sp(k)$.

5.2.1. First, we note that, under a natural identification, $K_{\mathbb{C}} = GL(n,\mathbb{C})$ acts on the space $\mathfrak{p}_+ = \mathrm{Alt}_n$ of all complex alternating matrices of size n by

$$g \cdot X = gX{}^tg, \quad g \in GL(n,\mathbb{C}),\ X \in \mathrm{Alt}_n. \tag{5.6}$$

For every positive integer k, we realize the compact group $G' = Sp(k)$ as

$$G' = \{g \in U(2k) \mid {}^tgJ_kg = J_k\} \quad \text{with } J_k = \begin{pmatrix} O & I_k \\ -I_k & O \end{pmatrix}. \tag{5.7}$$

The group $K_{\mathbb{C}} \times G'_{\mathbb{C}}$ acts on the vector space $M := M_{n,2k}$ by

$$(g, g') \cdot Z := gZg'^{-1}, \quad (g,g') \in K_{\mathbb{C}} \times G'_{\mathbb{C}},\ Z \in M, \tag{5.8}$$

where $G'_{\mathbb{C}} = Sp(k,\mathbb{C})$ is the complexification of G', and $M_{p,q}$ denotes the space of all complex matrices of size $p \times q$.

We set $\psi(Z) := \frac{1}{2}ZJ_k{}^tZ$ for $Z \in M$. Note that $\psi : M \to \mathfrak{p}_+$ is a $K_{\mathbb{C}} \times G'_{\mathbb{C}}$-equivariant polynomial map of degree two, where the $G'_{\mathbb{C}}$-action on $\mathfrak{p}_+$ is trivial. For each $Y \in \mathfrak{p}_-$, let h_Y be a polynomial on M defined by

$$h_Y(Z) := B(\psi(Z), Y) \quad (B \text{ the Killing form of } \mathfrak{g}). \tag{5.9}$$

Let $\mathbb{C}[M]$ denote the ring of polynomial functions on the complex vector space M. One can define a $(\mathfrak{g}, K)$-representation ω on $\mathbb{C}[M]$ in the following fashion. First, the $\mathfrak{p}_-$ action on $\mathbb{C}[M]$ is given by multiplication:

$$\omega(Y)f(Z) := h_Y(Z)f(Z), \quad Y \in \mathfrak{p}_-, \tag{5.10}$$

for $f \in \mathbb{C}[M]$. Second, $\mathfrak{p}_+$ acts by differentiation:

$$\omega(X)f(Z) := \kappa(h_{\overline{X}}(\partial)f)(Z), \quad X \in \mathfrak{p}_+. \tag{5.11}$$

Here $h_{\overline{X}}(\partial)$ stands for the constant coefficient differential operator on M defined by the polynomial $h_{\overline{X}}$, and the constant κ depends only on the Lie algebra $\mathfrak{g}_0$ of G. Third, the complexification $K_{\mathbb{C}}$ acts on $\mathbb{C}[M]$ holomorphically as

$$\omega(g)f(Z) := (\det g)^{-k} f((g^{-1}, e) \cdot Z), \quad g \in K_{\mathbb{C}}. \tag{5.12}$$

On the other hand, $\mathbb{C}[M]$ has a natural $G'_{\mathbb{C}}$-module structure through

$$R(g')f(Z) := f((e, g'^{-1}) \cdot Z), \quad g' \in G'_{\mathbb{C}}. \tag{5.13}$$

Then it is easily seen that these two representations ω and R commute with each other. The resulting $(\mathfrak{g}, K) \times G'_{\mathbb{C}}$-representation (ω, R) on $\mathbb{C}[M]$ will be called the Fock model of the (infinitesimal) *oscillator representation* of the pair (G, G') (cf. [3, §7]).

5.2.2. Let (σ, V_σ) be an irreducible finite-dimensional representation of the compact group G'. Extend σ to a holomorphic representation of $G'_{\mathbb{C}}$ in the canonical way. We set

$$L[\sigma] := \mathrm{Hom}_{G'_{\mathbb{C}}}(V_\sigma, \mathbb{C}[M]), \tag{5.14}$$

which turns to be a $(\mathfrak{g}, K)$-module through the representation ω on $\mathbb{C}[M]$. Let $\Sigma(k)$ denote the totality of equivalence classes of irreducible finite-dimensional representations σ of G' such that $L[\sigma] \neq \{0\}$. Then one gets a natural isomorphism

$$\mathbb{C}[M] \simeq \bigoplus_{\sigma \in \Sigma(k)} L[\sigma] \otimes V_\sigma \quad \text{as } (\mathfrak{g}, K) \times G'_{\mathbb{C}}\text{-modules}. \tag{5.15}$$

The following theorem states the theta correspondence associated to (G, G').

Theorem 5.4 ([11], [6], [7]; cf. [3]). (1) *$L[\sigma]$ is an irreducible unitarizable highest weight $(\mathfrak{g}, K)$-module for every $\sigma \in \Sigma(k)$. In particular,* (5.15) *gives the irreducible decomposition of the $(\mathfrak{g}, K) \times G'_{\mathbb{C}}$-module $\mathbb{C}[M]$.*

(2) *Let $\sigma_1, \sigma_2 \in \Sigma(k)$. Then, $V_{\sigma_1} \simeq V_{\sigma_2}$ as $G'_{\mathbb{C}}$-modules if and only if $L[\sigma_1] \simeq L[\sigma_2]$ as $(\mathfrak{g}, K)$-modules.*

Let $\tau[\sigma]$ denote the extreme K-type of highest weight $(\mathfrak{g}, K)$-module $L[\sigma]$, i.e., $L[\sigma] = L(\tau[\sigma])$. We note that the correspondence $\sigma \leftrightarrow \tau[\sigma]$ can be explicitly described in terms of their highest weights. For this, see the articles cited in the above theorem.

For each $m = 0, \dots, r = [n/2]$, the $K_{\mathbb{C}}$-orbit $\mathcal{O}_m$ in $\mathfrak{p}_+$ consists of all the matrices in $\mathfrak{p}_+ = \mathrm{Alt}_n$ of rank $2m$. Let $E_{s,t}(i,j)$ denote the (i,j)-matrix unit of size $s \times t$ whose (k,l)-matrix entry e_{kl} is equal to 1 if $(k,l) = (i,j)$; $e_{kl} = 0$ otherwise. We take an element $X(m) \in \mathcal{O}_m$ explicitly as

$$X(m) := \sum_{i=1}^{m} (E_{n,n}(i, m+i) - E_{n,n}(m+i, i))/2. \tag{5.16}$$

It is easily verified that the image $\psi(M)$ of the $K_{\mathbb{C}} \times G'_{\mathbb{C}}$-equivariant map $\psi : M \to \mathfrak{p}_+$ is a $K_{\mathbb{C}}$-stable, irreducible algebraic variety described as

$$\psi(M) = \overline{\mathcal{O}_{m_k}} \quad \text{with} \quad m_k := \min(k, r), \tag{5.17}$$

where M and ψ depend on k. By (5.10) and (5.15), we find that, for any $\sigma \in \Sigma(k)$, the associated variety of $L[\sigma]$ is equal to the closure of the $K_{\mathbb{C}}$-orbit $\mathcal{O}_{m_k} = \mathrm{Ad}(K_{\mathbb{C}})X(m_k)$.

5.2.3. We consider the maximal ideal:

$$\mathfrak{m} := \mathfrak{m}(X(m_k)) = \sum_{Y \in \mathfrak{p}_-} (Y - B(X(m_k), Y))S(\mathfrak{p}_-) \subset S(\mathfrak{p}_-) \quad (\text{cf. } (3.4)), \tag{5.18}$$

for each positive integer k. For $m = 0, \dots, r$, let $K_{\mathbb{C}}(m) := K_{\mathbb{C}}(X(m))$ be the isotropy subgroup of $K_{\mathbb{C}}$ at $X(m) \in \mathcal{O}_m$. We want to describe the $K_{\mathbb{C}}(m_k)$-modules

$$\mathcal{W}[\sigma] := \mathcal{W}(X(m_k), \tau[\sigma]) = L[\sigma]/\mathfrak{m}L[\sigma] \simeq \mathrm{Hom}_{G'_{\mathbb{C}}}(V_\sigma, \mathbb{C}[M]/\omega(\mathfrak{m})\mathbb{C}[M]). \tag{5.19}$$

Namely, our task is to decompose the quotient $K_{\mathbb{C}}(m_k) \times G'_{\mathbb{C}}$-module $\mathbb{C}[M]/\omega(\mathfrak{m})\mathbb{C}[M]$.

To do this, we note that $\omega(\mathfrak{m})\mathbb{C}[M]$ is equal to the ideal of $\mathbb{C}[M]$ generated by all matrix entries of the following polynomial function of degree two:

$$M \ni Z \longmapsto \psi(Z) - X(m_k) \in \mathfrak{p}_+. \tag{5.20}$$

We write $\mathcal{V}_k$ for the corresponding affine algebraic variety of M:

$$\mathcal{V}_k := \{Z \in M \mid \psi(Z) = X(m_k)\} = \psi^{-1}(X(m_k)). \tag{5.21}$$

Clearly, $\mathcal{V}_k$ is stable under the action of $K_{\mathbb{C}}(m_k) \times G'_{\mathbb{C}}$.

We define a subgroup $G'_{\mathbb{C}}(k-r)$ of $G'_{\mathbb{C}}$ by

$$G'_{\mathbb{C}}(k-r) := \begin{cases} \{I_{2k}\} \ (\text{the unit group}) & \text{if } k \le r, \\ \left\{ \begin{pmatrix} I_k & O & O & O \\ O & h_{11} & O & h_{12} \\ O & O & I_k & O \\ O & h_{21} & O & h_{22} \end{pmatrix} \in G'_{\mathbb{C}} \;\middle|\; h_{ij} \in M_{k-r,k-r} \right\} & \text{if } k > r. \end{cases} \tag{5.22}$$

Note that if $k > r$, the group $G'_{\mathbb{C}}(k-r)$ is naturally isomorphic to $Sp(k-r, \mathbb{C})$.

Lemma 5.5. (1) *If $k \le r$, one has*

$$\mathcal{V}_k = G'_{\mathbf{C}} \cdot I_{n,2k}(2k) \simeq G'_{\mathbf{C}} \quad \text{as } G'_{\mathbf{C}}\text{-sets}, \tag{5.23}$$

where $I_{s,t}(l) := \sum_{i=1}^{l} E_{s,t}(i,i) \in M_{s,t} \quad (l = 0, \dots, \min(s,t))$.

(2) *If $k > r = n/2$ with even integer n, the variety $\mathcal{V}_k$ is described as*

$$\mathcal{V}_k = G'_{\mathbf{C}} \cdot \begin{pmatrix} I_{r,k}(r) & O \\ O & I_{r,k}(r) \end{pmatrix} \simeq G'_{\mathbf{C}}/G'_{\mathbf{C}}(k-r), \tag{5.24}$$

where $G'_{\mathbf{C}}(k-r) \simeq Sp(k-r, \mathbf{C})$ (cf. (5.22)) coincides with the isotropy subgroup of $G'_{\mathbf{C}}$ at the matrix $\begin{pmatrix} I_{r,k}(r) & O \\ O & I_{r,k}(r) \end{pmatrix}$ in $M = M_{2r,2k}$.

(3) *If $k > r = (n-1)/2$ with odd integer n, $\mathcal{V}_k$ consists of two $G'_{\mathbf{C}}$-orbits. In fact, we set*

$$(z_1, z_2)^{\sim} := \begin{pmatrix} I_r & O & O & O \\ O & O & I_r & O \\ o & z_1 & o & z_2 \end{pmatrix} \quad \text{for} \quad (z_1, z_2) \in M_{1,2(k-r)} = M_{1,k-r} \times M_{1,k-r}. \tag{5.25}$$

Then $\mathcal{V}_k$ decomposes as

$$\mathcal{V}_k = G'_{\mathbf{C}} \cdot \tilde{M}_{1,2(k-r)} = G'_{\mathbf{C}} \cdot (0 \dots 0, 0 \dots 0)^{\sim} \coprod G'_{\mathbf{C}} \cdot (1\, 0 \dots 0, 0 \dots 0)^{\sim}, \tag{5.26}$$

where $\tilde{M}_{1,2(k-r)} := \{(z_1, z_2)^{\sim} \mid z_1, z_2 \in M_{1,k-r}\}$.

The above lemma implies in particular that the affine variety $\mathcal{V}_k$ is irreducible. This allows us to deduce the following proposition by applying [14, Lemma 4].

Proposition 5.6. *The ideal $\omega(\mathfrak{m})\mathbf{C}[M]$ of $\mathbf{C}[M]$ coincides with the defining ideal of $\mathcal{V}_k$ in $\mathbf{C}[M]$. Hence one gets a natural isomorphism*

$$\mathbf{C}[M]/\omega(\mathfrak{m})\mathbf{C}[M] \simeq \mathbf{C}[\mathcal{V}_k] \quad \text{as } K_{\mathbf{C}}(\mathfrak{m}_k) \times G'_{\mathbf{C}}\text{-modules}, \tag{5.27}$$

where $\mathbf{C}[\mathcal{V}_k]$ denotes the affine coordinate ring of $\mathcal{V}_k$.

5.2.4. We are now in a position to specify the $K_{\mathbf{C}}(\mathfrak{m}_k)$-modules $\mathcal{W}[\sigma]$ for every $\sigma \in \Sigma(k)$ $(k = 1, 2, \dots)$. Let us introduce a $G'_{\mathbf{C}}(k-r)$-stable subvariety $\mathcal{U}_k$ of $\mathcal{V}_k$ as

$$\mathcal{U}_k := \begin{cases} \{I_{n,2k}(2k)\} & (k \le r = [n/2]) \\ \left\{ \begin{pmatrix} I_{r,k}(r) & O \\ O & I_{r,k}(r) \end{pmatrix} \right\} & (k > r = n/2 \text{ with } n \text{ even}), \\ \tilde{M}_{1,2(k-r)} & (k > r = (n-1)/2 \text{ with } n \text{ odd}). \end{cases} \tag{5.28}$$

Then it follows from Lemma 5.5 that $\mathcal{V}_k = G'_{\mathbf{C}} \cdot \mathcal{U}_k$, and that the $G'_{\mathbf{C}}$-orbits $\mathcal{X}$ in $\mathcal{V}_k$ are in one-one correspondence with the $G'_{\mathbf{C}}(k-r)$-orbits $\mathcal{X} \cap \mathcal{U}_k$ in $\mathcal{U}_k$.

Now Proposition 5.6 together with (5.19) allows us to deduce the following

Proposition 5.7. *Under the above notation, let $\mathbf{C}[\mathcal{U}_k]$ be the coordinate ring of $G'_{\mathbf{C}}(k-r)$-stable variety $\mathcal{U}_k$ viewed as a $G'_{\mathbf{C}}(k-r)$-module in the canonical way. Then one has a linear isomorphism*

$$\mathcal{W}[\sigma] \simeq \mathrm{Hom}_{G'_{\mathbf{C}}(k-r)}(V_\sigma, \mathbf{C}[\mathcal{U}_k]) \simeq (V_\sigma^* \otimes \mathbf{C}[\mathcal{U}_k])^{G'_{\mathbf{C}}(k-r)} \quad (\sigma \in \Sigma(k)). \tag{5.29}$$

In particular, it holds that

$$\mathcal{W}[\sigma] \simeq \begin{cases} (V_\sigma^*)^{G'_{\mathbf{C}}(k-r)} & \text{if } n \text{ is even and } k > r, \\ V_\sigma^* & \text{if } k \le r. \end{cases} \tag{5.30}$$

Here $(V_\sigma^* \otimes \mathbb{C}[\mathcal{U}_k])^{G'_{\mathbb{C}}(k-r)}$ denotes the subspace of $V_\sigma^* \otimes \mathbb{C}[\mathcal{U}_k]$ of $G'_{\mathbb{C}}(k-r)$-fixed vectors.

Remark 5.8. For the case $k > r$ with odd n, $\mathbb{C}[\mathcal{U}_k]$ decomposes into a direct sum of the irreducible representations $V(l)$ $(l = 0, 1, \dots)$ of $G'_{\mathbb{C}}(k-r) = Sp(k-r, \mathbb{C})$ with highest weights $(l, 0, \dots, 0)$: $\mathbb{C}[\mathcal{U}_k] \simeq \oplus_{l \geq 0} V(l)$.

At the end, we are going to clarify how the isotropy subgroup $K_{\mathbb{C}}(m_k)$ acts on the space $\mathcal{W}[\sigma] \simeq \mathrm{Hom}_{G'_{\mathbb{C}}(k-r)}(V_\sigma, \mathbb{C}[\mathcal{U}_k])$. To do this, we note that the elements g of the subgroup $K_{\mathbb{C}}(m)$ $(0 \leq m \leq r)$ of $K_{\mathbb{C}}$ are written as follows.

$$g = \begin{pmatrix} g_{11} & g_{12} \\ O & g_{22} \end{pmatrix} \in K_{\mathbb{C}} = GL(n, \mathbb{C}) \quad \text{with} \quad g_{11} \in Sp(m, \mathbb{C}). \tag{5.31}$$

Define a group homomorphism

$$\alpha : K_{\mathbb{C}}(m_k) \to G'_{\mathbb{C}}, \quad g \mapsto \alpha(g), \tag{5.32}$$

by putting

$$\alpha(g) := \begin{pmatrix} p_{11} & O & p_{12} & O \\ O & I_{k-r} & O & O \\ p_{21} & O & p_{22} & O \\ O & O & O & I_{k-r} \end{pmatrix} \quad \text{with} \quad g_{11} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}. \tag{5.33}$$

Here p_{ij} is a matrix of size k, and $\alpha(g)$ should be understood as g_{11} if $k \leq r$. Note that the elements of $\alpha(K_{\mathbb{C}}(m_k))$ commute with those of the subgroup $G'_{\mathbb{C}}(k-r)$.

Now we can deduce

Theorem 5.9. *If n is even or $k \leq r$, it holds that*

$$\mathcal{W}[\sigma] \simeq (\det(\,\cdot\,)^{-k} \otimes (\sigma^* \circ \alpha), \ \ (V_\sigma^*)^{G'_{\mathbb{C}}(k-r)}) \quad \text{as } K_{\mathbb{C}}(m_k)\text{-modules.} \tag{5.34}$$

In particular, $\mathcal{W}[\sigma]$ is an irreducible $K_{\mathbb{C}}(m_k)$-module if $k \leq r$.

Next we consider the remaining case: $k > r$ with odd n. Then, $\beta(g) := g_{22}$ $(g \in K_{\mathbb{C}}(r))$ defines a group homomorphism β from $K_{\mathbb{C}}(r)$ to $GL(1, \mathbb{C}) = \mathbb{C}^\times$. The group $K_{\mathbb{C}}(r)$ acts on $\mathbb{C}[\mathcal{U}_k] \simeq \mathbb{C}[M_{1,2(k-r)}]$ naturally through the left multiplication composed with β. We denote by ν the resulting representation of $K_{\mathbb{C}}(r)$ on $\mathbb{C}[\mathcal{U}_k]$. Note that ν as well as $\sigma^* \circ \alpha$ commutes with the $G'_{\mathbb{C}}(k-r)$-action.

Theorem 5.10. *If $k > r$ with odd n, the reductive part of $K_{\mathbb{C}}(r)$ acts on $\mathcal{W}[\sigma] \simeq (V_\sigma^* \otimes \mathbb{C}[\mathcal{U}_k])^{G'_{\mathbb{C}}(k-r)}$ by the representation* $\det(\,\cdot\,)^{-k} \otimes (\sigma^* \circ \alpha) \otimes \nu$.

Similar descriptions of $\mathcal{W}[\sigma]$ can be obtained for the groups $G = SU(p,q)$ and $Sp(n, \mathbb{R})$ also. For this we refer to [20] and [27, Section 5].

References

[1] L. Barchini, Szegö kernels associated with Zuckerman modules, J.Funct.Anal., **131** (1995), 145–182.

[2] M.G. Davidson, T.J. Enright and R.J. Stanke, Covariant differential operators, Math.Ann., **288** (1990), 731–739.

[3] M.G. Davidson, T.J. Enright and R.J. Stanke, Differential operators and highest weight representations, Mem.Amer.Math.Soc. No. **455**, American Mathematical Society, Rrovidence, R.I., 1991.

[4] M.G. Davidson and T.J. Stanke, Szegö maps and highest weight representations, Pacific J.Math., **158** (1993), 67–91.

[5] T.J. Enright and A. Joseph, An intrinsic analysis of unitarizable highest weight modules, Math.Ann., **288** (1990), 571–594.

[6] T.J Enright and R. Parthasarathy, A proof of a conjecture of Kashiwara and Vergne. *in* "Noncommutative harmonic analysis and Lie groups (Marseille, 1980)", Lecture Notes in Math., 880, Springer, Berlin-New York, 1981, pp. 74–90.

[7] T.J. Enright, R. Howe, and N.R. Wallach, A classification of unitary highest weight modules, *in* "Representation theory of reductive groups (Park City, Utah, 1982; P.C.Trombi ed.)", Progress in Math., Vol. **40**, Birkhäuser, 1983, pp.97–143.

[8] A. Gyoja and H. Yamashita, Associated variety, Kostant-Sekiguchi correspondence, and locally free $U(\mathfrak{n})$-action on Harish-Chandra modules, J. Math. Soc. Japan, **51** (1999), 129–149.

[9] R. Hotta and R. Parthasarathy, Multiplicity formulae for discrete series, Invent.Math., **26** (1974), 133–178.

[10] M. Kashiwara and W. Schmid, Quasi-equivariant $\mathcal{D}$-modules, equivariant derived category, and representations of reductive Lie groups, *in* "Lie theory and geometry (J.L.Brylinski et al. eds.)", Birkhäuser, 1994, pp.457–488.

[11] M. Kashiwara and M. Vergne, On the Segal-Shale-Weil representations and harmonic polynomials, Invent.Math., **44** (1978), 1–47.

[12] N. Kawanaka, Generalized Gelfand-Graev representation and Ennola duality, *in* "Algebraic groups and related topics", Advanced Studies in Pure Math. **6** (1985), 175–206.

[13] A.W. Knapp, *Lie groups beyond an introduction*, Progress in Mathematics Vol. **140**, Birkhäuser, Boston-Besel-Stuttgart, 1996.

[14] B. Kostant, Lie group representations on polynomial rings, Amer. J. Math., **85** (1963), 327–404.

[15] A. Joseph, Annihilators and associated varieties of unitary highest weight modules, Ann.Sci. Éc.Norm.Sup., **25** (1992), 1–45.

[16] H. Matumoto, Whittaker vectors and associated varieties, Invent.Math., **89** (1987), 219–224.

[17] K. Nishiyama, H. Ochiai and K. Taniguchi, Bernstein degree and associated cycles of Harish-Chandra modules – Hermitian symmetric case, Preprint (1999), Kyushu University Preprint Series in Mathematics, 1999-16.

[18] W. Schmid, Homogeneous complex manifolds and representations of semisimple Lie groups, Dissertation, University of California, Berkeley, 1967; reprinted *in* "Representation theory and harmonic analysis on semisimple Lie groups (P.L Sally and D.A Vogan eds.)", Mathematical Surveys and Monograph Vol. **31**, Amer.Math.Soc., 1989, pp.223–286.

[19] W. Schmid, Boundary value problems for group invariant differential equations, *in* "Elie Cartan et les Mathématiques d'Aujourd'hui", Astérisque, Numéro hors série, 1985, pp.311–321.

[20] M. Tagawa, Generalized Whittaker models for unitarizable highest weight representations (in Japanese), Master Thesis, Kyoto University, 1998.

[21] D.A. Vogan, Associated varieties and unipotent representations, *in* "Harmonic Analysis on Reductive Groups (W.Barker and P.Sally eds.)," Progress in Math., Vol. **101**, Birkhäuser, 1991, pp.315-388.

[22] H.W. Wong, Dolbeault cohomological realization of Zuckerman modules associated with finite rank representations, J.Funct.Anal., **129** (1995), 428–454.

[23] H Yamashita, Finite multiplicity theorems for induced representations of semisimple Lie groups II: Applications to generalized Gelfand-Graev representations, J.Math.Kyoto.Univ., **28** (1988), 383–444.

[24] H. Yamashita, Multiplicity one theorems for generalized Gelfand-Graev representations of semisimple Lie groups and Whittaker models for the discrete series, Advanced Studies in Pure Math. **14** (1988), 31–121.

[25] H. Yamashita, Embeddings of discrete series into induced representations of semisimple Lie groups, I: General theory and the case of $SU(2,2)$, Japan.J.Math.,**16** (1990), 31-95; II: generalized Whittaker models for $SU(2,2)$, J.Math.Kyoto Univ., **31** (1991), 543–571.

[26] H. Yamashita, Description of the associated varieties for the discrete series representations of a semisimple Lie group: An elementary proof by means of differential operators of gradient type, Comment. Math. Univ. St. Paul., **47** (1998), 35–52.

[27] H. Yamashita, Cayley transform and generalized Whittaker models for irreducible highest weight modules, Preprint (1999), 41 pages.

Division of Mathematics, Graduate School of Science, Hokkaido University, Sapporo, 060-0810 JAPAN

E-mail address: yamasita@math.sci.hokudai.ac.jp

無限次元の調和解析

報告書

日独シンポジウム

開催日　１９９９年９月２０～２４日

於　　京都大学

編集

ハーバート・ハイヤー　チュービンゲン大学数学研究所

平井　武　　　　　　　京都大学理学部

尾畑　伸明　　　　　　名古屋大学大学院多元数理科学研究科

京都，１９９９