Harmonic Analysis on Negatively Curved Manifolds

- Carleson measure, Brownian motion and a gradient estimate for harmonic functions -

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This paper is mainly a summary of recent work of the author on harmonic analysis on negatively curved manifolds, and we refer the reader to [10], [6] and [7] for details.

Let (M,g) be a complete, simply connected n dimensional Riemannian manifold whose sectional curvatures K_M satisfy

$$-\infty < -\kappa_1^2 \leq K_M \leq -\kappa_2^2 < 0,$$

where κ_1 and κ_2 are positve constants. In this paper we are concerned with Hardy spaces, BMO, Carleson measure and their probabilistic aspects. Further we give a gradient estimates for harmonic functions and its application to Bloch functions on negatively curved manifolds.

Notation Throughout this paper we fix a point o in M as a reference point. The constants depending only on g, n, κ_1, κ_2 and o will usually be denoted by C or C'. But C and C' may change in value from one occurrence to the next. For two nonnegative functions f and g defined on a set U, the notation $f \lessapprox g$ indicate that $f(x) \leq Cg(x)$ for all $x \in U$, and $f \approx g$ means that $f \lessapprox g$ and $g \lessapprox f$.

Acknowledgement The author thanks to Professor Heyer, Professor Obata and Professor Hirai for their invitation to the conference on "Infinite Dimensional Harmonic Analysis".

1 Background material

Before going to the main body of this report, let us give a brief review of results obtained by Anderson and Schoen ([3]), Cifuentes and Korányi ([18]), and the author ([6], [7]).

Let $S(\infty)$ be the sphere at infinity of M, and $\overline{M}$ Eberlein and O'Neill's compactification $M \cup S(\infty)$ of M (see [23]). The following theorem plays a fundamental and important role in our work:

Theorem AS1 (Anderson and Schoen [3]; [1], [31]) (1) *The Martin compactification of M with respect to the Laplacian Δ_g on M is homoemorphic to $\overline{M}$, and the Martin boundary consists only of minimal points.*

(2) *For every $z \in M$, there exists a unique function $K_z(x, Q)$ ($Q \in S(\infty), x \in \overline{M}\backslash\{Q\}$) such that for every $Q \in S(\infty)$,*

$$K_z(\cdot, Q) \text{ is positive harmonic on } M, \tag{1}$$

$$K_z(\cdot, Q) \text{ is continuous on } \overline{M} \setminus \{Q\}, \tag{2}$$

$$K_z(Q', Q) = 0 \text{ for all } Q' \in S(\infty) \setminus \{Q\}, \text{ and} \tag{3}$$

$$K_z(z, Q) = 1. \tag{4}$$

(*This function is called the Poisson kernel normalized at z.*)

(3) *For every $z \in M$ and for every positive harmonic function u on M, there exists a unique Borel measure m_u^z on $S(\infty)$ such that*

$$u(x) = \int_{S(\infty)} K_z(x, Q) f(Q) dm_u^z(Q), \quad x \in M \tag{5}$$

(*The measure m_u^z is called the Martin representing measure relative to u and z.*)

Throughout this paper, we write $K(x, Q) = K_o(x, Q)$, and denote by ω^x the Martin representing measure relative to the constant function 1 and $x \in M$. It is called the harmonic measure relative to x. In particular, let $\omega = \omega^o$. Note that $\omega^x(S(\infty)) = 1$ and $d\omega^x(Q) = K(x, Q) d\omega(Q)$, for all $x \in M$.

For notational simplicity, we denote

$$\tilde{f}(x) = \int_{S(\infty)} K(x, Q) f(Q) d\omega(Q), \quad x \in M,$$

for every $f \in L^1(S(\infty), \omega)$.

In their paper [3], Anderson and Schoen generalized to the manifold M Fatou's theorem on boundary behavior of bounded harmonic functions on the open unit disc. To describe their theorem we need some notation. For $x \in M$ and $y \in \overline{M}$ ($x \neq y$), let γ_{xy} be the unit speed geodesic with $\gamma_{xy}(0) = x$ and $\gamma_{xy}(t) = y$ for some $t \in (0, +\infty]$. Since such a number t is uniquely determined, we denote it by t_{xy}. Anderson and Schoen defined the following analogue of the classical nontangential region: For $Q \in S(\infty)$ and $d > 0$, let

$$T_d(Q) = \bigcup_{t>0} B(\gamma_{oQ}(t), d), \tag{6}$$

where $B(x, r)$ is the geodesic ball with center x and radius r.

Theorem AS2 (Anderson and Schoen [3]) *Let u be a bounded harmonic function on M. Then for ω-a.e. $Q \in S(\infty)$, the nontangential limit*

$$\lim_{x \in T_d(Q)} u(x)$$

exists for all $d > 0$.

This result was extended by Ancona [1], Mouton [38] and the author [7]: Ancona proved an analogue of Fatou-Doob theorem, Mouton verified Calderón-Stein type theorem and the author obtained an analogue of a local version of Fatou-Doob theorem.

2 Admissible maximal functions and Hardy spaces

In [6], we studied another analogue to M of the classical nontangential region. In order to describe it, let us mention some terminologies: For $p \in M$, $v \in T_pM$ and $\delta > 0$, let $C(p, v, \delta)$ be the cone about the tangent vector v of angle δ defined by

$$C(p, v, \delta) := \{x \in \overline{M} : \angle_p(v, \dot{\gamma}_{px}(0)) < \delta\},$$

where $\angle_p$ denotes the angle in T_pM and $\dot{\gamma}_{px}(t)$ is its tangent vector at t.

For $z \in M \setminus \{o\}$ and $t \in \mathbf{R}$, we denote

$$C(z,t) = C(\gamma_{oz}(t_{oz}+t), \dot{\gamma}_{oz}(t_{oz}+t), \pi/4), \text{ and } z(t) = \gamma_{oz}(t_{oz}+t),$$

and let

$$\Delta(x,t) = C(x,t) \cap S(\infty).$$

Our analogue is the following:

Definition 2.1 ([6]) *For $Q \in S(\infty)$ and $\alpha \in \mathbf{R}$, let*

$$\Gamma_\alpha(Q) = \{z \in M : Q \in \Delta(z, \alpha)\}, \tag{7}$$

and we call this set an admissible region at Q.

Using this notion, we can define an analogue of nontangential maximal function, admissible maximal functions, as follows: For a function u on M, let

$$N_\alpha(u)(Q) = \sup_{x \in \Gamma_\alpha(Q)} |u(x)|, \quad Q \in S(\infty), \quad \alpha \in \mathbf{R}.$$

Furthermore we can define Hardy type spaces in terms of our maximal functions:

$$H^p_\alpha = \left\{ f \in L^1(S(\infty), \omega) : N_\alpha(\tilde{f}) \in L^p(S(\infty, \omega) \right\}, \quad 1 \le p \le \infty$$

and we denote

$$\|f\|_{H^p_\alpha} := \|N_\alpha(\tilde{f})\|_{L^p(\omega)}.$$

It is easy to prove that $(H^p_\alpha, \|\cdot\|_{H^p_\alpha})$ is a Banach space and that for every $\alpha, \beta \in \mathbf{R}$, $H^p_\alpha = H^p_\beta$, and moreover for every $f \in H^p_\alpha = H^p_\beta$,

$$C^{-1}_{\alpha,\beta}\|f\|_{H^p_\alpha} \le \|f\|_{H^p_\beta} \le C_{\alpha,\beta}\|f\|_{H^p_\alpha},$$

where $C_{\alpha,\beta}$ is a positive constant depending only on n, κ_1, κ_2, α and β (see [10]). Therefore in this paper we deal only with H^p_0, and we denote

$$H^p = H^p_0, \quad \text{and} \quad \|\cdot\|_{H^p} = \|\cdot\|_{H^p_\alpha}.$$

We study also atomic Hardy spaces in the sense of Coifman and Weiss and probabilistic versions of Hardy spaces. Let us describe them. First we are concerned with atomic Hardy spaces. For any $Q \in S(\infty)$, we define $\Delta_t(Q)$ to be the "ball" in $S(\infty)$ centered at Q of radius $\log(1/r)$,

$$\Delta_t(Q) := \Delta(\gamma_{oQ}(t), 0) \ (= C(\gamma_{oQ}(t), \dot{\gamma}_{oQ}(t), \pi/4) \cap S(\infty)),$$

It is easy to see that the function

$$\rho_0(Q,Q') := \left(\inf\{e^{-t} : Q' \in \Delta_t(Q)\} + \inf\{e^{-t} : Q \in \Delta_t(Q')\}\right)/2, \quad Q, Q' \in S(\infty)$$

is a quasi-distance in the sense of [19] such that $(S(\infty), \rho, \omega)$ is a space of homogeneous type. Therefore the abstract theory in [19] can be transplanted to our case. For instance, some covering lemmas, theorems on atomic Hardy spaces and BMO on spaces of homogeneous type hold true for $(S(\infty), \omega, \rho)$. Now let us mention the definition of atomic Hardy spaces on $S(\infty)$. In [19], atomic Hardy spaces and BMO on a space of homogeneous type are defined in terms of its quasi-distance. However in our case, we can prove that the family of balls defined by ρ is equivalent to $\{\Delta_t(Q)\}$, that is,

(8) $$\Delta_{\log(1/r)+k_1}(Q) \subset \{Q' : \rho(Q,Q') < r\} \subset \Delta_{\log(1/r)-k_2}(Q).$$

where k_1 and k_2 are positive constants depending only on M.

For this reason, one can define atomic Hardy spaces and BMO in terms of $\{\Delta_t(Q)\}$ which are equivalent to those defined by the quasi-distace ρ : a function a on $S(\infty)$ is called an atom if the support of a is contained in a "ball" $\Delta_r(Q)$, $\int_{S(\infty)} a d\omega = 0$, and $\|a\|_{L^\infty(\omega)} \le \omega(\Delta_r(Q))^{-1}$. Since $\omega(S(\infty)) = 1$, we regard also the constant function 1 as an atom. The atomic Hardy spaces H^1_{atom} is defined as the set of all functions h in $L^1(S(\infty), \omega)$ such that h has an atomic decomposition

(9) $$h = \sum_{j=1}^{\infty} \lambda_j a_j,$$

where $\lambda_j \in \mathbf{R}$, and a_j's are atoms and $\sum_{j=1}^{\infty} |\lambda_j| < \infty$. We set

$$\|h\|_{1,\text{atom}} = \inf\left\{\sum_{j=1}^{\infty} |\lambda_j| : h = \sum_{j=1}^{\infty} \lambda_j a_j,\ a_j\text{'s are atoms}\right\}$$

for $h \in H^1_{\text{atom}}$.

Let $\mathrm{BMO}(\omega)$ be the set of all functions $f \in L^1(S(\infty), \omega)$ such that

$$\|f\|_{\mathrm{BMO}} = \sup_{Q \in S(\infty), r \in \mathbf{R}} \frac{1}{\omega(\Delta_r(Q))} \int_{\Delta_r(Q)} |f - m_{\Delta_r(Q)} f| d\omega + \|f\|_{L^1(\omega)} < \infty,$$

where

$$m_{\Delta_r(Q)} f = \frac{1}{\omega(\Delta_r(Q))} \int_{\Delta_r(Q)} f d\omega.$$

Theorem CW ([19]). *The dual of H^1_{atom} is regarded as the space $\mathrm{BMO}(\omega)$ in the following sense: If $h = \sum \lambda_j a_j \in H^1_{\text{atom}}$, then for each $\ell \in \mathrm{BMO}(\omega)$*

$$\langle h, \ell \rangle := \lim_{m \to \infty} \lambda_j \int_X \ell a_j d\omega$$

is a well defined continuous linear functional and its norm is equivalent to $|\ell|_{\mathrm{BMO}}$. Moreover, every linear continuous functional on H^1_{atom} has this form.

In this paper we will also deal with probabilistic analogues of Hardy spaces. To define them, we need to recall some facts on Brownian motion on M and its Markov properties: Let W be the set of all continuous maps from $[0, \infty)$ to M, and let $Z_t(w) = w(t)$, $w \in W$. Since by Yau [47] the life time of Brownian motion on M is equal to $+\infty$, so there exists a system of probability measures $\{P_x\}_{x \in M}$ on W such that (P_x, Z_t) is a Brownian motion starting at x. From Sullivan [43] or Kifer [31] it follows the following facts:

(I) There exists a limit $Z_\infty(w) := \lim_{t \to \infty} Z_t(w)$ for almost sure $w \in W$ with respect to P_x, $x \in M$. Moreover, $Z_\infty(w) \in S(\infty)$ for P_x-a.s. $w \in W$.

(II) For every $x \in M$ and for every Borel subset F of $S(\infty)$,

$$\omega^x(F) = P_x(\{w \in W : Z_\infty(w) \in F\}).$$

For every $f \in L^1(\omega)$, $\tilde{f}(x) = E_x[f(Z_\infty)]$ for all $x \in M$ and $\lim_{t \to \infty} \tilde{f}(Z_t) = f(Z_\infty)$ P_x-a.s., where $E_x[\ ]$ denotes the expectation with respect to P_x ($x \in M$). We denote $P = P_o$ and $E[\ ] = E_o[\ ]$. Let

$$H^p_{\text{prob}} := \left\{ f \in L^p(\omega) : \|f\|_{H^p_{\text{prob}}} = E\left[\sup_{0 \le t < \infty} \left|\tilde{f}(Z_t)\right|^p\right]^{1/p} < \infty \right\}, \quad 1 \le p < \infty.$$

Let $\mathcal{B}$ (resp. $\mathcal{B}_t$) be the smallest σ-field for which all random variables Z_s, $s \geq 0$ (resp. Z_s, $0 \leq s \leq t$) are measurable. For a probability Borel measure μ on M, let $P_\mu(A) = \int_{S(\infty)} P_x(A)d\mu(x)$, $A \subset W$. We denote by $(W, \mathcal{F}^\mu, \mathcal{F}_t^\mu, P_\mu)$ the usual P_μ augmentation of $(W, \mathcal{B}, \mathcal{B}_t, P_\mu)$ in the sense of [41, III 9]. In particular, $(W, \mathcal{F}^x, \mathcal{F}_t^x, P_x)$ denotes the P_x-augumentation of $(W, \mathcal{B}, \mathcal{B}_t, P_\mu)$. Put $\tilde{\mathcal{F}} := \bigcap \mathcal{F}^\mu$ and $\tilde{F}_t := \bigcap \mathcal{F}_t^\mu$, where the intersection is taken over all probability Borel measures μ on M. Then $(Z_t, W, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_t, P_x : x \in M)$ is a strong Markov process. If fact, considering that M is diffeomorphic to $\mathbf{R}^n$, it is a honest FD diffusion in the sense of [41, III 3, III 13].

It is known that the usual P_x-augumentation $(W, \mathcal{F}^x, \mathcal{F}_t^x, P_x)$ satisfies the so-called usual condition (see [41, III 9]). Moreover, for every harmonic function u on M, the process $u(Z_t)$ is a continuous local $(P_x, \mathcal{F}_t^x)$-martingale. Denote by $(W, \mathcal{F}, \mathcal{F}_t, P)$ the usual P_o-augumentation $(W, \mathcal{F}^o, \mathcal{F}_t^o, P_o)$. As usual, Hardy spaces of martingales are defined as follows:

$$\mathcal{M}^p := \left\{ X \in L^1(W, \mathcal{W}, P) : \|X\|_{\mathcal{M}^p} := E\left[\sup_{0 \leq t < \infty} |E[X|\mathcal{F}_t]|^p\right]^{1/p} < \infty \right\},$$

$(1 \leq p < \infty)$, where and always $E[\cdot|\mathcal{C}]$ denotes the conditional expectation with respect to P and a sub σ-field $\mathcal{C}$ of $\mathcal{F}$. Note that Meyer's previsibility theorem ([41, VI 15, Theorem 15.4]) implies that for every $X \in L^1(W, P)$, the process $(E[X/\mathcal{F}_t])_{t \geq 0}$ is an $(\mathcal{F}_t)$-continuous martingale.

For $X \in L^1(W, \mathcal{F}, P)$, let $\mathcal{N}'(X) := E[X|\sigma(Z_\infty)]$, where $\sigma(Z_\infty)$ is the sub σ-field of $\mathcal{F}$ generated by the random variable Z_∞. Then by (I) there exists a unique element $f \in L^1(\omega)$ such that $\mathcal{N}'(X) = f(Z_\infty)$, P-a.s. Denote the function f by $\mathcal{N}X$.

Now we can mention another probabilistic analogue of Hardy spaces:

$$H^p_{\text{mart}} := \{\mathcal{N}(X) : X \in \mathcal{M}^p\}, \quad 1 \leq p < \infty,$$

and as a norm on H^p_{mart}, we consider $\|\mathcal{N}(X)\|_{H^p_{\text{mart}}} := \|X\|_{\mathcal{M}^p}$.

For two normed spaces $(A, \| \|_A)$ and $(B, \| \|_B)$, we denote by $A \preceq B$ that $A \subset B$ and $\|x\|_B \leq C\|x\|_A$ for every $x \in A$, where C is a constant independent of x. Further we set $A \simeq B$ if $A \preceq B$ and $B \preceq A$.

In 1987, we announced in [6] the following Theorems 2.1 and 2.2 (see [10] for detailed proofs). :

Theorem 2.1 ([6]; see also [10])

$$H^1(\omega) \preceq H^1_{\text{prob}} \preceq H^1_{\text{mart}} \preceq H^1_{\text{atom}}(\omega)$$

Let k be a constant such that for every $Q_1, Q_2 \in S(\infty)$ and $r \in \mathbf{R}$, $\Delta_r(Q_1) \cap \Delta_r(Q_2) \neq \emptyset$ implies $\Delta_r(Q_2) \subset \Delta_{r-k}(Q_1)$. (This constant always exsits.)

Theorem 2.2 ([6]; see also [10]) *Consider the following geometric condition:*

(β) *For every $Q \in S(\infty)$, $t > k$ and $z \in C(\gamma_{oQ}(t), 0)$,*

$$\Delta_t(\gamma_{\zeta z}(+\infty)) \bigcap \Delta_t(Q) \neq \emptyset.$$

If our manifold M satisfies the condition (β), we have $H^1_{\text{atom}}(\omega) \preceq H^1(\omega)$.

When M is rotationally symmetric at o or the dimension of M is two, the condition (β) is satisfied. However recently, Cifuentes and Korányi proved the following

Theorem CK2 (Cifuentes and Korányi [18]) *The manifold M satisfies always the condition (β).*

Therefore combining our Theorems 2.1 and 2.2 with Theorem CK2, the following theorem is obtained:

Theorem 2.3 (Arai [6], Cifuentes and Korányi [18])

$$H^1(\omega) \simeq H^1_{\text{atom}}(\omega) \simeq H^1_{\text{prob}} \simeq H^1_{\text{mart}}$$

3 Carleson measure

In this section we study a condition on a measure μ on M in order that the Martin integral operator,

$$K[f](z) = \int_{S(\infty)} K(z, Q) f(Q) d\omega(Q) \; (= \tilde{f}(z)), \; z \in M,$$

is bounded from $L^p(\omega)$ to $L^p(M, \mu)$. This problem was studied by L. Carleson in the classical Euclidean case, and he found a necessary and sufficient condition called now "Carleson condition". We study a version to M of "Carleson condition":

Definition 3.1 *For a set $A \subset S(\infty)$ and $r > 0$, let*

$$S_r[A] := \{z \in M \setminus B(o, r) : \Delta(z, 0) \subset A\}.$$

A given complex Borel measure μ *on* M *is said to be a Carleson measure on* M *if for every* $r > 0$,

$$\|\mu\|_{c,r} := \sup_{Q\in S(\infty),t>1} \frac{|\mu|(S_r[\Delta_t(Q)])}{\omega(\Delta_t(Q))} + |\mu|(M) < \infty,$$

where $|\mu|$ *is the total variation of* μ. *We wirte* $\|\mu\|_c = \|\mu\|_{c,1}$.

As an analogue of the classical Carleson-Hörmander's theorem, we obtain the following

Theorem 3.1 ([10]) *Let* μ *be a complex Borel measure on* M. *Then the following are equivalent:*

(i) μ *is a Carleson measure on* M.

(ii) $\|\mu\|_{c,r} < \infty$ *for some* $r > 0$.

(iii) *For every* $1 \leq p < \infty$, *the Martin integral operator* K *is bounded from* $H^p(\omega)$ *to* $L^p(M, |\mu|)$.

(iv) *For every* $1 < p < \infty$, *the operator* K *is bounded from* $L^p(\omega)$ *to* $L^p(M, |\mu|)$.

(v) *For some* $1 < p < \infty$, *the operator* K *is bounded from* $L^p(\omega)$ *to* $L^p(M, |\mu|)$.

Furthermore, for every $r > 0$, *there is a constant* C'_r *depending only on* M, o *and* r *such that*

$$C_r'^{-1}\|\mu\|_{c,r} \leq \|\mu\|_c \leq C'_r\|\mu\|_{c,r}.$$

We give also a kind of an analytic characterization of Carleson measures. Let $G(x, y)$ be Green's function on M (see [3] or [4]). For a Borel measure μ on M, the function

$$G[\mu](x) = \int_M G(x, y)d\mu(y), \quad x \in M$$

is called the Green potential of μ. In this section we study boundary behavior of the Green potentials of the following weighted measures: for a nonnegative Borel measure μ on M, let

$$\mu_0(A) = \int_A \frac{1}{G(o, w)}d\mu(w), \quad A \subset M.$$

A nonnegative function f on M is said to be asymptotically bounded if there exists a positive constant $R > 0$ such that $\sup_{x\in M\setminus B(o,R)} f(x) < \infty$. Then we have the following

Theorem 3.2 ([10]) *Let* μ *be a nonnegative Borel measure on* M. *Suppose that* $\mu(H) < \infty$ *for every compact set* H *in* M. *Then the following statements are equivalent:*

(i) $G[\mu_0]$ *is asymptotically bounded on* M.

(ii) μ *is a Carleson measure and satisfies the following condition* (F)*:*

(F) *There exist positive constants* r *and* C *such that*

$$(10) \qquad \int_{B(z,1)} G(z,w)d\mu(w) \leq CG(o,z) \quad \text{for every } z \in M \setminus B(o,r).$$

For $f \in L^1(\omega)$, let

$$d\mu_f(w) = G(o,w)|\nabla \tilde{f}(w)|^2 dV(w),$$

where dV is the volume measure with respect to the metric g, and $|\nabla \tilde{f}(w)|$ is the norm of the gradient of $\tilde{f}$ with respect to g, that is, in a local coordinate neighborhood,

$$|\nabla \tilde{f}(w)|^2 = \sum_{ij} g^{ij}(w) \frac{\partial f(w)}{\partial x_i} \frac{\partial f(w)}{\partial x_j},$$

where $(g^{ij}(w))$ is the inverse matrix of the metric $(g_{ij}(w))$. This is an analogue to M of the classical Littlewood-Paley measure.

It is easy to see that for $f \in L^1(\omega)$, $\mu_f(M) < \infty$ if and only if $f \in L^2(\omega)$.

As a corollary of Theorem 3.2 we obtain the following characterization of BMO functions in terms of Carleson measures and Green potentials:

Theorem 3.3 ([10]) *Let* $f \in L^2(\omega)$. *Then the following are equivalent:*

(i) $f \in \mathrm{BMO}(\omega)$

(ii) μ_f *is a Carleson measure on* M.

(iii) *The Green potential*

$$G_f(x) := \int_M G(x,w)|\nabla \tilde{f}(w)|^2 dV(w)$$

is asymptotically bounded.

(iv) *The potential* G_f *defined in* (iii) *is bounded on* M.

Remark. As known, in the classical Euclidean case, the part "(i)$\Longleftrightarrow$ (ii)" was obtained by Fefferman and Stein [24]. In the case of the Bergman ball in $\mathbf{C}^n$, analogous results to Theorem 3.3 were proved in Jevtić [27]. See also [8] and [9].

4 A gradient estimate for harmonic functions and Bloch functions.

In this section we will apply Theorem 3.3 to Bloch function theory on Riemannian manifolds.

Classicaly Bloch functions were defined on the open unit disc D in $\mathbf{C}$ as follows: a holomorphic function f on D is said to be a Bloch function on D if

$$\sup_{z \in D}(1 - |z|)|f'(z)| < \infty. \tag{11}$$

This means that f is a Bloch function if and only if the norm of gradient $|\nabla f|$ with respect to the Poincaré metric is bounded. Now the notion of Bloch functions is naturally extended to Riemannian manifold $(\mathcal{R}, h)$:

Definition 4.1 *Let f be a harmonic function on $\mathcal{R}$. Then f is said to be a harmonic Bloch function on $\mathcal{M}$ if*

$$\|f\|_B := \sup_{x \in \mathcal{R}} |\nabla f(x)| < \infty,$$

where $|\nabla f|$ is the norm of gradient of f with respect to the metric h, i.e. $|\nabla f(x)|^2 = \sum_{i,j} h^{ij}(x)(\partial f(x)/\partial x_i)(\partial f(x)/\partial x_j)$, where $(h^{ij}(x))$ is the inverse matrix of the Riemannian metric $(h_{ij}(x))$.

In particular, if $(\mathcal{R}, h)$ is a Kähler manifold, then a function u is said to be a holomorphic Bloch function on M if u is a harmonic Bloch function and holomorphic on $\mathcal{R}$.

In [32], Krantz and Ma defined Bloch functions on a bounded strongly pseudoconvex domain with smooth boundary. See Timoney [44] for Bloch functions on symmetric domains. If $(\mathcal{R}, h)$ is a bounded smoothly strongly pseudoconvex domain endowed with the Bergman metric, it is easy to see that our definition of Bloch functions is equivalent to one by Krantz and Ma.

If the Ricci curvature of $\mathcal{R}$ is nonnegative, then from Yau and Chen's results it follows that the class of Bloch functions is equal to the class of harmonic functions with linear order growth (see [34] and [30]).

Theorem 4.1 ([10]) *Suppose $f \in BMO(\omega)$. Then $\tilde{f}$ is a harmonic Bloch function on M. Indeed*

$$\sup_{x \in M} \|\nabla \tilde{f}(x)\| \leq C\|f\|_{\mathrm{BMO}}, \tag{12}$$

where C is a positive constant depending only on M and o.

In particular, there exists a unbounded harmonic Bloch function on M.

Let $\mathbf{T}$ be the unit circle. Denote by BMOA($\mathbf{T}$) the set of all functions f in BMO($\mathbf{T}$) such that the Poisson integral of f is holomorphic in the open unit disc D. Then it is known that if $f \in$ BMOA($\mathbf{T}$), then its Poisson integral is a holomorphic Bloch function on D (cf. [40]). Krantz and Ma [32] extended this fact to bounded strongly pseudoconvex domains with smooth boundaries. Our proof of Theorem 4.1 is different from their proofs.

It should be noted that the inequality (12) is closely related to Jerison and Kenig [28, Lemma 9.9] for harmonic functions with respect to the Euclidean Laplacian.

Let $u(z) = \sum_{k=m}^{\infty} z^{15^k}$ $(z \in D)$. Then u is a holomorphic Bloch function, and for large m,

$$\limsup_{r\to 1} \frac{|u(re^{i\theta})|}{\sqrt{\log(1-r)^{-1}\log\log\log(1-r)^{-1}}} > 0.685\|u\|_B \text{ a.e. } \theta \in [0, 2\pi)$$

(see [40, p.194]).

In 1985, Makarov proved the following

Theorem M (Makarov [36]; see also Pommerenke [40, p.186]) *Let u be a holomorphic Bloch function on D. Then for almost every $\theta \in [0, 2\pi)$,*

$$\limsup_{r\to 1} \frac{|u(re^{i\theta})|}{\sqrt{\log(1-r)^{-1}\log\log\log(1-r)^{-1}}} \le \|u\|_B.$$

Also a probabilistic version of Theorem M was obtained by Lyons [35]:

Theorem L (Lyons [35]) Let u be a holomorphic Bloch function on D. Let X_t be hyperbolic Brownian motion on D. Then

$$\limsup_{t\to\infty} \frac{|u(X_t)|}{\sqrt{\log(1-|X_t|)^{-1}\log\log\log(1-|X_t|)^{-1}}} \le \|u\|_B.$$

We will generalize Theorem L to our manifold M. We begin with characterizing Bloch functions in terms of Brownian motion:

Theorem 4.2 ([10]) *For a harmonic function u on M, the following* (i) *and* (ii) *are equivalent:*

(i) *u is a harmonic Bloch function on M.*

(ii) *The stochastic process $\{u(Z_t)\}_t$ satisfies that*

$$\|u\|^2_{B,\mathrm{prob}} := \sup_{x\in M} \left\{ \frac{E_x[|u(Z_T) - u(Z_0)|^2]}{E_x[T]} : T \in \mathcal{T}_x, E_x[T] > 0 \right\} < \infty,$$

where $\mathcal{T}_x$ is the set of all $(\mathcal{F}_t^x)$-stopping times. Furthermore, $\|u\|_B \le \|u\|_{B,\mathrm{prob}} \le \sqrt{2}\|u\|_B$.

In the case of the open unit disc in $\mathbf{C}$, a martingale characterization of holomorphic Bloch functions was given in Muramoto [39]. We will prove Theorem 4.2 by simplifying and exploiting the method in [39] by combining an idea in Lyons [35].

Now we describe on our genealization of Thoerem L:

Theorem 4.3 ([10]) *Let u be a harmonic Bloch functions on M. Then*

$$\limsup_{t\to\infty} \frac{|u(Z_t)|}{\sqrt{d(o,Z_t)\log\log d(o,Z_t)}} \leq C\|u\|_B \quad P\text{-a.s.}$$

As an immediate consequence of Theorem 4.3 we have the following

Corollary 4.4 ([10]) *Let $M = \{x \in \mathbf{R}^n : |x| < 1\}$ and let g be the hyperbolic metric on M. Then for a harmonic Bloch function u on (M, g),*

$$\limsup_{t\to\infty} \frac{|u(Z_t) - u(o)|}{\sqrt{\log(1-|Z_t|)^{-1}\log\log\log(1-|Z_t|)^{-1}}} \leq C\|u\|_B \quad a.s.P^o$$

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