CHARACTERIZATION OF HIDA MEASURES IN WHITE NOISE ANALYSIS

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1. INTRODUCTION

In the recent paper [3] by Asai et al., the growth order of holomorphic functions on a nuclear space has been considered. For this purpose, certain classes of growth functions u are introduced and many properties of Legendre transform of such functions are investigated. In [4], applying Legendre transform of u under the conditions (U0), (U2) and (U3) (see §2), the Gel'fand triple

$$[\mathcal{E}]_u \subset (L^2) \subset [\mathcal{E}]_u^*$$

associated with a growth function u is constructed.

The main purpose of this work is to prove Theorem 4.4, so-called, the characterization theorem of Hida measures (generalized measures). As examples of such measures, we shall present the Poisson noise measure and the Grey noise measure in Example 4.5 and 4.6, respectively.

The present paper is organized as follows. In §2, we give a quick review of some fundamental results in white noise analysis and introduce the notion of Legendre transform utilized by Asai et al. in [3],[4]. In §3, we simply cite some useful properties of the Legendre transform from [3]. In §4, we discuss the characterization of Hida measures (generalized measures).

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Acknowledgements. N. Asai wants to thank the Daiko Foundation and the Kamiyama Foundation for research support.

2. Preliminaries

In this section, we will summarize well-known results in white noise analysis [9],[20],[22] and notions from Asai et al.[1],[2],[3],[4]. Complete details and further developments will be appeared in [5]. Some similar results have been obtained independently by Gannoun et al. [8].

Let $\mathcal{E}_0$ be a real separale Hilbert space with the norm $|\cdot|_0$. Suppose $\{|\cdot|_p\}_{p=0}^{\infty}$ is a sequence of densely defined inner product norms on $\mathcal{E}_0$. Let $\mathcal{E}_p$ be the completion of $\mathcal{E}$ with respect to the norm $|\cdot|_p$. In addition we assume

(a) There exists a constant $0 < \rho < 1$ such that $|\cdot|_0 \le \rho|\cdot|_1 \le \cdots \le \rho^p|\cdot|_p \le \cdots$.
(b) For any $p \ge 0$, there exists $q \ge p$ such that the inclusion $i_{q,p} : \mathcal{E}_q \hookrightarrow \mathcal{E}_p$ is a Hilbert-Schmidt operator.

Let $\mathcal{E}'$ and $\mathcal{E}_p'$ denote the dual spaces of $\mathcal{E}$ and $\mathcal{E}_p$, respectively. We can use the Riesz representation theorem to identify $\mathcal{E}_0$ with its dual space $\mathcal{E}_0'$. Let $\mathcal{E}$ be the projective limit of $\{\mathcal{E}_p;\ p \ge 0\}$. Then we get the following continuous inclusions:

$$\mathcal{E} \subset \mathcal{E}_p \subset \mathcal{E}_0 \subset \mathcal{E}_p' \subset \mathcal{E}', \qquad p \ge 0.$$

The above condition (b) says that $\mathcal{E}$ is a nuclear space and so $\mathcal{E} \subset \mathcal{E}_0 \subset \mathcal{E}'$ is a Gel'fand triple.

Let μ be the standard Gaussian measure on $\mathcal{E}'$ with the characteristic function given by

$$\int_{\mathcal{E}'} e^{i\langle x,\xi\rangle}\, d\mu(x) = e^{-\frac{1}{2}|\xi|_0^2}, \qquad \xi \in \mathcal{E}.$$

The probability space $(\mathcal{E}', \mu)$ is called a *white noise space* or *Gaussian space*. For simplicity, we will use (L^2) to denote the Hilbert space of μ-square integrable functions on $\mathcal{E}'$. By the Wiener-Itô theorem, each $\varphi \in (L^2)$ can be uniquely expressed as

$$\varphi(x) = \sum_{n=0}^{\infty} I_n(f_n)(x) = \sum_{n=0}^{\infty} \langle :x^{\otimes n}:, f_n\rangle, \qquad f_n \in \mathcal{E}_0^{\widehat{\otimes} n}, \tag{2.1}$$

where I_n is the multiple Wiener integral of order n and $:x^{\otimes n}:$ is the Wick tensor of $x \in \mathcal{E}'$ (see [20].) Moreover, the (L^2)-norm of φ is given by

$$\|\varphi\|_0 = \left(\sum_{n=0}^{\infty} n!|f_n|_0^2\right)^{1/2}. \tag{2.2}$$

Let $u \in C_{+,\frac{1}{2}}$ be the set of all positive continuous functions on $[0,\infty)$ satisfying

$$\lim_{r\to\infty} \frac{\log u(r)}{\sqrt{r}} = \infty.$$

In addition, we introduce conditions:

(U0) $\inf_{r\geq 0} u(r) = 1$.

(U1) u is increasing and $u(0) = 1$.

(U2) $\lim_{r\to\infty} r^{-1} \log u(r) < \infty$.

(U3) $\log u(x^2)$ is convex on $[0,\infty)$.

Obviously, (U1) is a stronger condition than (U0).

Let $C_{+,\log}$ denote the set of all positive continuous functions u on $[0,\infty)$ satisfying the condition:

$$\lim_{r\to\infty} \frac{\log u(r)}{\log r} = \infty.$$

It is easy to see $C_{+,\frac{1}{2}} \subset C_{+,\log}$.

The *Legendre transform* ℓ_u of $u \in C_{+,\log}$ is defined to be the function

$$\ell_u(t) = \inf_{r>0} \frac{u(r)}{r^t}, \qquad t \in [0,\infty).$$

Some useful properties of the Legendre transform will be refered in section 3.

¿From now on, we take a function $u \in C_{+,\frac{1}{2}}$ satisfying (U0) (U2) (U3). We shall constract a Gel'fand triple associated with u. For $\varphi \in (L^2)$ being represented by Equation (2.1) and $p \geq 0$, define

$$\|\varphi\|_{p,u} = \left(\sum_{n=0}^{\infty} \frac{1}{\ell_u(n)} |f_n|_p^2 \right)^{1/2}. \tag{2.3}$$

Let $[\mathcal{E}_p]_u = \{\varphi \in (L^2);\ \|\varphi\|_{p,u} < \infty\}$. Define the space $[\mathcal{E}]_u$ of *test functions* on $\mathcal{E}'$ to be the projective limit of $\{[\mathcal{E}_p]_u;\ p \geq 0\}$. The dual space $[\mathcal{E}]_u^*$ of $[\mathcal{E}]_u$ is called the space of *generalized functions* on $\mathcal{E}'$.

Choose an appropriate p_0 such that $c\rho^{2p_0}\sqrt{2} \leq 1$ for some c. Then two conditions (a) and (U2) imply that $[\mathcal{E}_p]_u \subset (L^2)$ for all $p \geq p_0$. Hence $[\mathcal{E}]_u \subset (L^2)$ holds. By identifying (L^2) with its dual space we get the following continuous inclusions:

$$[\mathcal{E}]_u \subset [\mathcal{E}_p]_u \subset (L^2) \subset [\mathcal{E}_p]_u^* \subset [\mathcal{E}]_u^*, \qquad p \geq p_0,$$

where $[\mathcal{E}_p]_u^*$ is the dual space of $[\mathcal{E}_p]_u$. Moreover, $[\mathcal{E}]_u$ is a nuclear space and so $[\mathcal{E}]_u \subset (L^2) \subset [\mathcal{E}]_u^*$ is a Gel'fand triple. Note that $[\mathcal{E}]_u^* = \cup_{p\geq 0}[\mathcal{E}_p]_u^*$ and for $p \geq p_0$, $[\mathcal{E}_p]_u^*$ is the completion of (L^2) with respect to the norm

$$\|\varphi\|_{-p,(u)} = \left(\sum_{n=0}^{\infty} (n!)^2 \ell_u(n)\, |f_n|_{-p}^2 \right)^{1/2}. \tag{2.4}$$

For ξ belonging to the complexification $\mathcal{E}_c$ of $\mathcal{E}$, the renormalized exponential function $:e^{\langle\cdot,\xi\rangle}:$ is defined by

$$:e^{\langle\cdot,\xi\rangle}:=\sum_{n=0}^{\infty}\frac{1}{n!}\langle:\cdot^{\otimes n}:,\xi^{\otimes n}\rangle.$$

Then we have the norm estimate,

$$\|:e^{\langle\cdot,\xi\rangle}:\|_{-q,(u)}^2=\sum_{n=0}^{\infty}\ell_u(n)|\xi|_{-q}^{2n}=:\mathcal{L}_u(|\xi|_{-q}^2). \tag{2.5}$$

For later uses, let us define the notion of *equivalent functions* here.

Definition 2.1. Two positive functions f and g on $[0,\infty)$ are called *equivalent* if there exist constants $c_1, c_2, a_1, a_2 > 0$ such that

$$c_1 f(a_1 r)\le g(r)\le c_2 f(a_2 r),\qquad \forall\, r\in[0,\infty).$$

Example 2.2.

$$g_k(r)=\exp\left[2\sqrt{r\log_{k-1}\sqrt{r}}\right], \tag{2.6}$$

where $\log_k(r)$ is given by

$$\log_1(r)=\log(\max\{e,r\}),\quad \log_k(r)=\log_1(\log_{k-1}(r)),\quad k\ge 2.$$

Then the function g_k belongs to $C_{+,1/2}$ and satisfies conditions (U1) (U2) (U3). In the sense of Definition 2.1, the function g_k is equivalent to the function given by

$$u_k(r)=\sum_{n=0}^{\infty}\frac{1}{b_k(n)n!}r^n$$

where $b_k(n)$ is the *k-th order Bell number.* Hence we get the Gel'fand triple,

$$[\mathcal{E}]_{g_k}\subset(L^2)\subset[\mathcal{E}]_{g_k}^*$$

known as the *CKS-space associated with* g_k, which is the same as the one defined by the k-th order Bell number $b_k(n)$. See more details in [1],[2],[3], [4],[5],[6],[15],[16].

Example 2.3. For $0\le\beta<1$, let u be the function defined by

$$u(r)=\exp\left[(1+\beta)r^{\frac{1}{1+\beta}}\right].$$

It is easy to check that u belongs to $C_{+,1/2}$ and satisfies conditions (U1) (U2) (U3). Hence this Gel'fand triple,

$$(\mathcal{E})_\beta\subset(L^2)\subset(\mathcal{E})_\beta^*$$

which is well-known as the *Hida-Kubo-Takenaka space* for $\beta=0$ [9],[10],[17], [18],[22] and the *Kondratiev-Streit space* for a general β [12], [20]. For $\beta=1$ case, see [11],[13],[14].

Remark. We have the following chain of Gel'fand triples:

$(\mathcal{E})_1 \subset [\mathcal{E}]_{g_k} \subset [\mathcal{E}]_{g_l} \subset (\mathcal{E})_\beta \subset (\mathcal{E})_\gamma \subset (L^2) \subset (\mathcal{E})_\gamma^* \subset (\mathcal{E})_\beta^* \subset [\mathcal{E}]_{g_l}^* \subset [\mathcal{E}]_{g_k}^* \subset (\mathcal{E})_1^*$
where $0 \le \gamma \le \beta < 1$ and $2 \le l \le k$.

3. Properties of Legendre transforms

First we mention the following notions of concave and convex functions which will be used frequently.

Definition 3.1. A positive function f on $[0, \infty)$ is called

(1) *log-concave* if the function $\log f$ is concave on $[0, \infty)$;
(2) *log-convex* if the function $\log f$ is convex on $[0, \infty)$;
(3) *(log, exp)-convex* if the function $\log f(e^x)$ is convex on $\mathbb{R}$;
(4) *(log, x^2)-convex* if the function $\log f(x^2)$ is convex on $[0, \infty)$.

We will need the fact that if f is log-concave, then the sequence $\{f(n)\}_{n=0}^\infty$ is log-concave. To check this fact, note that for any $t_1, t_2 \ge 0$ and $0 \le \lambda \le 1$,

$$f(\lambda t_1 + (1-\lambda)t_2) \ge f(t_1)^\lambda f(t_2)^{1-\lambda}.$$

In particular, take $t_1 = n, t_2 = n+2$, and $\lambda = 1/2$ to get

$$f(n)f(n+2) \le f(n+1)^2, \qquad \forall\, n \ge 0.$$

Hence the sequence $\{f(n)\}_{n=0}^\infty$ is log-concave.

The next theorem is from Lemma 3.4 in [3].

Theorem 3.2. *Let $u \in C_{+,\log}$. Then the Legendre transform ℓ_u is log-concave. (Hence ℓ_u is continuous on $[0,\infty)$ and the sequence $\{\ell_u(n)\}_{n=0}^\infty$ is log-concave.)*

¿From Theorem 2 (b) in [1] we have the fact: If $\{\alpha(n)/n!\}_{n=0}^\infty$ is log-concave and $\alpha(0) = 1$, then

$$\alpha(n+m) \le \binom{n+m}{n} \alpha(n)\alpha(m), \qquad \forall\, n, m \ge 0.$$

By Theorem 3.2 the sequence $\{\ell_u(n)\}$ is log-concave. Hence we can apply the above fact to the sequence $\alpha(n) = n!\ell_u(n)/\ell_u(0)$ to get the next theorem.

Theorem 3.3. *Let $u \in C_{+,\log}$. Then for all integers $n, m \ge 0$, we have*

$$\ell_u(0)\ell_u(n+m) \le \ell_u(n)\ell_u(m).$$

In the next theorem we state some properties of the Legendre transform ℓ_u of a (log, exp)-convex function u in $C_{+,\log}$. It is from Lemmas 3.6 and 3.7 in [3].

Theorem 3.4. *Let $u \in C_{+,\log}$ be (log, exp)-convex. Then*

(1) *$\ell_u(t)$ is decreasing for large t,*
(2) $\lim_{t\to\infty} \ell_u(t)^{1/t} = 0$,

(3) $u(r) = \sup_{t\geq 0} \ell_u(t) r^t$ *for all* $r \geq 0$.

On the other hand, for a (log, x^2)-convex function u in $C_{+,\log}$, its Legendre transform ℓ_u has the properties in the next theorem from Lemmas 3.9 and 3.10 in [3]. If in addition u is increasing, then u is also (log, exp)-convex and hence ℓ_u has the properties in the above Theorem 3.4.

Theorem 3.5. *Let* $u \in C_{+,\log}$. *We have the assertions:*

(1) u *is (log,* x^2*)-convex if and only if* $\ell_u(t)t^{2t}$ *is log-convex.*

(2) *If* u *is (log,* x^2*)-convex, then for any integers* $n, m \geq 0$,

$$\ell_u(n)\ell_u(m) \leq \ell_u(0) 2^{2(n+m)} \ell_u(n+m).$$

Now, suppose $u \in C_{+,\log}$ and assume that $\lim_{n\to\infty} \ell_u(n)^{1/n} = 0$. We define the *L-function* $\mathcal{L}_u$ of u by

$$\mathcal{L}_u(r) = \sum_{n=0}^{\infty} \ell_u(n) r^n. \tag{3.1}$$

Note that $\mathcal{L}_u$ is an entire function. By Theorem 3.4 (2), ℓ_u is defined for any (log, exp)-convex function u in $\in C_{+,\log}$. Moreover, we have the next theorem from Theorem 3.13 in [3].

Theorem 3.6. *(1) Let* $u \in C_{+,\log}$ *be (log, exp)-convex. Then its L-function* $\mathcal{L}_u$ *is also (log, exp)-convex and for any* $a > 1$,

$$\mathcal{L}_u(r) \leq \frac{ea}{\log a} u(ar), \qquad \forall r \geq 0.$$

(2) Let $u \in C_{+,\log}$ *be increasing and (log,* x^2*)-convex. Then there exists a constant* C *such that*

$$u(r) \leq C\mathcal{L}_u(2^2 r), \qquad \forall r \geq 0.$$

Recall from Proposition 2.3 (3) in [3]: If f is increasing and (log, x^2)-convex for some $k > 0$, then f is (log, exp)-convex. Hence the above Theorem 3.6 yields the next theorem.

Theorem 3.7. *Let* $u \in C_{+,\log}$ *be increasing and (log,* x^2*)-convex. Then the functions* u *and* $\mathcal{L}_u$ *are equivalent.*

In the next section 4, we will consider the characterization of Hida measures (generalized measures). We prepare two lemmas for this purpose. The proof of Lemma 3.8 is simple application of Theorem 3.5 so that we just state it without proof.

Lemma 3.8. *Suppose* $u \in C_{+,\log}$ *is (log,* x^2*)-convex. Then*

$$\mathcal{L}_u(r)^2 \leq \ell_u(0)\mathcal{L}_u(2^3 r), \qquad \forall r \in [0, \infty). \tag{3.2}$$

Remark. Note that $\mathcal{L}_u(r) \geq \ell_u(0)$ for all $r \geq 0$. Hence we have

$$\ell_u(0)\mathcal{L}_u(r) \leq \mathcal{L}_u(r)^2 \leq \ell_u(0)\mathcal{L}_u(2^3 r), \qquad \forall r \in [0, \infty).$$

Thus $\mathcal{L}_u$ and $\mathcal{L}_u^2$ are equivalent for any (log, x^2)-convex function $u \in C_{+,\log}$. If, in addition, u is increasing, then u and $\mathcal{L}_u$ are equivalent by Theorem 3.7. It follows that u and u^2 are equivalent for such a function u.

The next Lemma 3.9 can be obtained from Theorem 3.8 and Lemma 3.6.

Lemma 3.9. *Suppose $u \in C_{+,\log}$ is increasing and (log, x^2)-convex. Then for any $a > 1$, we have*

$$\mathcal{L}_u(r) \leq \sqrt{\ell_u(0)\frac{ea}{\log a}}\, u\big(a2^3 r\big)^{1/2}. \tag{3.3}$$

4. Characterization of Hida Measures

Before going to the main theorem, we need to introduce another equivalent family of norms on $[\mathcal{E}]_u$, i.e., $\{\|\cdot\|_{\mathcal{A}_{p,u}}; p \geq 0\}$. This family of norms is intrinsic in the sense that $\|\varphi\|_{\mathcal{A}_{p,u}}$ is defined directly in terms of the analyticity and growth condition of φ.

First, it is well-known that each test function φ in $[\mathcal{E}]_u$ has a unique analytic extension (see §6.3 of [20]) given by

$$\varphi(x) = \langle\!\langle :e^{\langle\cdot,x\rangle}:, \Theta\varphi\rangle\!\rangle, \qquad x \in \mathcal{E}_c', \tag{4.1}$$

where Θ is the unique linear operator taking $e^{\langle\cdot,\xi\rangle}$ into $:e^{\langle\cdot,\xi\rangle}:$ for all $\xi \in \mathcal{E}_c$. By Theorem 6.2 in [20] with minor modifications, Θ is shown to be a continuous linear operator from $[\mathcal{E}]_u$ into itself. Note that we still assume conditions (U0), (U2) and (U3) on u given in section 2.

Now, let $p \geq 0$ be any fixed number. Choose $p_1 > p$ such that $2\rho^{2(p_1-p)} \leq 1$. Then use Equations (4.1), (2.5) and Theorem 3.6 to get

$$|\varphi(x)| \leq \|\Theta\varphi\|_{p_1,u}\, \|:e^{\langle\cdot,x\rangle}:\|_{-p_1,(u)} \leq \|\Theta\varphi\|_{p_1,u} \sqrt{\frac{2e}{\log 2}}\, u\big(2|x|_{-p_1}^2\big)^{1/2}.$$

Note that $2|x|_{-p_1}^2 \leq 2\rho^{2(p_1-p)}|x|_{-p}^2 \leq |x|_{-p}^2$ by the above choice of p_1. Since u is an increasing function, we see that

$$|\varphi(x)| \leq \|\Theta\varphi\|_{p_1,u} \sqrt{\frac{2e}{\log 2}}\, u\big(|x|_{-p}^2\big)^{1/2}.$$

But Θ is a continuous linear operator from $[\mathcal{E}]_u$ into itself. Hence there exist positive constants q and $K_{p,q}$ such that $\|\Theta\varphi\|_{p_1,u} \leq K_{p,q}\|\varphi\|_{q,u}$. Therefore,

$$|\varphi(x)| \leq C_{p,q}\|\varphi\|_{q,u}\, u\big(|x|_{-p}^2\big)^{1/2}, \qquad x \in \mathcal{E}_{p,c}', \tag{4.2}$$

where $C_{p,q} = K_{p,q}\sqrt{2e/\log 2}$. This is the growth condition for test functions.

Being motivated by Equation (4.2), we define

$$\|\varphi\|_{\mathcal{A}_{p,u}} = \sup_{x\in\mathcal{E}'_{p,c}} |\varphi(x)|\, u\big(|x|_{-p}^2\big)^{-1/2}. \tag{4.3}$$

Obviously, $\|\cdot\|_{\mathcal{A}_{p,u}}$ is a norm on $[\mathcal{E}]_u$ for each $p \geq 0$.

Theorem 4.1. *Suppose $u \in C_{+,1/2}$ satisfies conditions (U1) (U2) (U3). Then the families of norms $\{\|\cdot\|_{\mathcal{A}_{p,u}}; p \geq 0\}$ and $\{\|\cdot\|_{p,u}; p \geq 0\}$ are equivalent, i.e., they generate the same topology on $[\mathcal{E}]_u$.*

Remark. This theorem gives an alternative construction of test functions. This idea is due to Lee [21], see also §15.2 of [20]. For $p \geq 0$, let $\mathcal{A}_{p,u}$ consist of all functions φ on $\mathcal{E}'_c$ satisfying the conditions:

(a) φ is an analytic function on $\mathcal{E}'_{p,c}$.
(b) There exists a constant $C \geq 0$ such that

$$|\varphi(x)| \leq Cu\big(|x|_{-p}^2\big)^{1/2}, \qquad \forall x \in \mathcal{E}'_{p,c}.$$

For each $\varphi \in \mathcal{A}_{p,u}$, define $\|\varphi\|_{\mathcal{A}_{p,u}}$ by Equation (4.3). Then $\mathcal{A}_{p,u}$ is a Banach space with norm $\|\cdot\|_{\mathcal{A}_{p,u}}$. Let $\mathcal{A}_u$ be the projective limit of $\{\mathcal{A}_{p,u}; p \geq 0\}$. We can use the above theorem to conclude that $\mathcal{A}_u = [\mathcal{E}]_u$ as vector spaces with the same topology. Here the equality $\mathcal{A}_u = [\mathcal{E}]_u$ requires the use of analytic extensions of test functions in $[\mathcal{E}]_u$, which exists in view of Equation (4.1).

Proof. Let $p \geq 0$ be any given number. We have already shown that there exist constants $q > p$ and $C_{p,q} \geq 0$ such that Equation (4.2) holds. It follows that

$$\|\varphi\|_{\mathcal{A}_{p,u}} = \sup_{x\in\mathcal{E}'_{p,c}} |\varphi(x)|\, u\big(|x|_{-p}^2\big)^{-1/2} \leq C_{p,q}\|\varphi\|_{q,u}.$$

Hence for any $p \geq 0$, there exist constants $q > p$ and $C_{p,q} \geq 0$ such that

$$\|\varphi\|_{\mathcal{A}_{p,u}} \leq C_{p,q}\|\varphi\|_{q,u}, \qquad \forall \varphi \in [\mathcal{E}]_u. \tag{4.4}$$

To show the converse, first note that by condition (U2) there exist constants $c_1, c_2 > 0$ such that $u(r) \leq c_1 e^{c_2 r}$, $r \geq 0$. Next note that by Fernique's theorem (see [7], [19], [20]) we have

$$\int_{\mathcal{E}'} e^{2c_2|x|_{-\lambda}^2}\, d\mu(x) < \infty \qquad \text{for all large } \lambda.$$

Now, let $p \geq 0$ be any given number. Choose $q > p$ large enough such that

$$4e^2\|i_{q,p}\|_{HS}^2 < 1, \qquad \int_{\mathcal{E}'} e^{2c_2|x|_{-q}^2}\, d\mu(x) < \infty. \tag{4.5}$$

With this choice of q we will show below that

$$\|\varphi\|_{p,u} \leq L_{p,q}\|\varphi\|_{\mathcal{A}_{q,u}}, \qquad \forall \varphi \in [\mathcal{E}]_u, \tag{4.6}$$

where $L_{p,q}$ is the constant given by

$$L_{p,q} = \sqrt{c_1}\left(1 - 4e^2\|i_{q,p}\|_{HS}^2\right)^{-1/2} \int_{\mathcal{E}'} e^{2c_2|x|_{-q}^2}\, d\mu(x). \tag{4.7}$$

Observe that the theorem follows from Equations (4.4) and (4.6).

Finally, we prove Equation (4.6). Let $\varphi \in [\mathcal{E}]_u$. Then we can use an integral form of S-transform (see [20]) given by

$$F(\xi) = S\varphi(\xi) = \int_{\mathcal{E}'} \varphi(x+\xi)\, d\mu(x), \qquad \xi \in \mathcal{E}_c.$$

Hence for the above choice of q, we have

$$\begin{aligned}|F(\xi)| &\le \int_{\mathcal{E}'} |\varphi(x+\xi)|\, d\mu(x) \\ &\le \int_{\mathcal{E}'} \left(|\varphi(x+\xi)|\, u\big(|x+\xi|_{-q}^2\big)^{-1/2}\right) u\big(|x+\xi|_{-q}^2\big)^{1/2}\, d\mu(x) \\ &\le \|\varphi\|_{\mathcal{A}_{q,u}} \int_{\mathcal{E}'} u\big(|x+\xi|_{-q}^2\big)^{1/2}\, d\mu(x).\end{aligned}$$

Here by condition (U1), we have $u(r)^{1/2} \le u(r)$ for all $r \ge 0$. Therefore,

$$|F(\xi)| \le \|\varphi\|_{\mathcal{A}_{q,u}} \int_{\mathcal{E}'} u\big(|x+\xi|_{-q}^2\big)\, d\mu(x). \tag{4.8}$$

By condition (U3), we have

$$u\left(\left(\tfrac{1}{2}r_1 + \tfrac{1}{2}r_2\right)^2\right) \le u\big(r_1^2\big)^{1/2} u\big(r_2^2\big)^{1/2}, \qquad \forall r_1, r_2 \ge 0.$$

Put $r_1 = 2|x|_{-q}$ and $r_2 = 2|\xi|_{-q}$ to get

$$\begin{aligned}u\big(|x+\xi|_{-q}^2\big) &\le u\left(\left(\tfrac{1}{2}2|x|_{-q} + \tfrac{1}{2}2|\xi|_{-q}\right)^2\right) \\ &\le u\big(4|x|_{-q}^2\big)^{1/2} u\big(4|\xi|_{-q}^2\big)^{1/2}.\end{aligned}$$

Then integrate over $\mathcal{E}'$ to obtain the inequality:

$$\int_{\mathcal{E}'} u\big(|x+\xi|_{-q}^2\big)\, d\mu(x) \le u\big(4|\xi|_{-q}^2\big)^{1/2} \int_{\mathcal{E}'} u\big(4|x|_{-q}^2\big)^{1/2}\, d\mu(x). \tag{4.9}$$

Put Equation (4.9) into Equation (4.8) to get

$$|F(\xi)| \le \|\varphi\|_{\mathcal{A}_{q,u}} u\big(4|\xi|_{-q}^2\big)^{1/2} \int_{\mathcal{E}'} u\big(4|x|_{-q}^2\big)^{1/2}\, d\mu(x). \tag{4.10}$$

Now, by the inequality $u(r) \le c_1 e^{c_2 r}$, we have

$$\int_{\mathcal{E}'} u\big(4|x|_{-q}^2\big)^{1/2}\, d\mu(x) \le \sqrt{c_1} \int_{\mathcal{E}'} e^{2c_2|x|_{-q}^2}\, d\mu(x), \tag{4.11}$$

which is finite by the choice of q in Equation (4.5).

¿From Equations (4.10) and (4.11), we see that

$$|F(\xi)| \le \|\varphi\|_{\mathcal{A}_{q,u}} \sqrt{c_1} \left(\int_{\mathcal{E}'} e^{2c_2|x|_{-q}^2} \, d\mu(x) \right) u\big(4|\xi|_{-q}^2\big)^{1/2}, \qquad \xi \in \mathcal{E}_c.$$

With this inequality and the choice of q in Equation (4.5) we can apply Lemma 4.2 (see below) and Equation (2.3) to show that for any $\varphi \in [\mathcal{E}]_u$,

$$\|\varphi\|_{q,u} \le L_{p,q} \|\varphi\|_{\mathcal{A}_{q,u}},$$

where $L_{p,q}$ is given by Equation(4.7). Thus the inequality in Equation (4.6) holds and so the proof is completed. □

In the proof of the prevous theorem, we have used the next lemma from [3].

Lemma 4.2 ([3]). *Suppose $u \in C_{+,1/2}$ satisfies conditions (U1) (U2) (U3). Let F be a complex-valued function on $\mathcal{E}_c$ satisfying the conditions:*

(1) *For any $\xi, \eta \in \mathcal{E}_c$, the function $F(z\xi + \eta)$ is an entire function of $z \in \mathbb{C}$.*
(2) *There exist constants $K, a, p \ge 0$ such that*

$$|F(\xi)| \le K u\big(a|\xi|_{-p}^2\big)^{1/2}, \qquad \xi \in \mathcal{E}_c.$$

Let $q \in [0, p)$ be a number such that $ae^2\|i_{p,q}\|_{HS}^2 < 1$. Then there exist functions $f_n \in \mathcal{E}_{q,\mathbb{C}}^{\widehat{\otimes}}$ such that $F(\xi) = \sum_{n=0}^{\infty} \langle f_n, \xi^{\widehat{\otimes}} \rangle$ and

$$|f_n|_q^2 \le K(ae^2\|i_{p,q}\|_{HS}^2)^n \ell_u(n). \tag{4.12}$$

Definition 4.3. A measure ν on $\mathcal{E}'$ is called a *Hida measure* associated with u if $[\mathcal{E}]_u \subset L^1(\nu)$ and the linear functional $\varphi \mapsto \int_{\mathcal{E}'} \varphi(x)\, d\nu(x)$ is continuous on $[\mathcal{E}]_u$.

In this case, ν induces a generalized function, denoted by $\widetilde{\nu}$, in $[\mathcal{E}]_u^*$ such that

$$\langle\langle \widetilde{\nu}, \varphi \rangle\rangle = \int_{\mathcal{E}'} \varphi(x)\, d\nu(x), \qquad \varphi \in [\mathcal{E}]_u. \tag{4.13}$$

Theorem 4.4. *Suppose $u \in C_{+,1/2}$ satisfies conditions (U1) (U2) (U3). Then a measure ν on $\mathcal{E}'$ is a Hida measure with $\widetilde{\nu} \in [\mathcal{E}]_u^*$ if and only if ν is supported by $\mathcal{E}_p'$ for some $p \ge 0$ and*

$$\int_{\mathcal{E}_p'} u\big(|x|_{-p}^2\big)^{1/2} \, d\nu(x) < \infty. \tag{4.14}$$

Remarks. (a) The integrability condition in the theorem can be replaced by

$$\int_{\mathcal{E}_p'} u\big(|x|_{-p}^2\big) \, d\nu(x) < \infty.$$

To verify this fact, just note that u and u^2 are equivalent (from the Remark of Lemma 3.8) and $|x|_{-q} \le \rho^{q-p}|x|_{-p}$ for $0 \le p \le q$ and $x \in \mathcal{E}_p'$.

(b) This theorem is due to Lee [21] for the case $u(r) = e^r$. See §15.2 of the book [20] for the case $u(r) = \exp\left[(1+\beta)r^{\frac{1}{1+\beta}}\right]$, $0 \leq \beta < 1$. In the case of $\beta = 1$, we need special treatment since our Legendre transform method should be modified. In order to take care of $\beta = 1$ case, we have to remove the assumption

$$\lim_{r\to\infty} \frac{\log u(r)}{\sqrt{r}} = \infty$$

on u introduced in §2, for example. It will be discussed in the future. On the other hand, there are references [13],[14] discussed this case with a diffrent way from our point of view.

Proof. To prove the sufficiency, suppose ν is supported by $\mathcal{E}'_p$ for some $p \geq 0$ and Equation (4.14) holds. Then for any $\varphi \in [\mathcal{E}]_u$,

$$\begin{aligned}
\int_{\mathcal{E}'} |\varphi(x)|\, d\nu(x) &= \int_{\mathcal{E}'_p} |\varphi(x)|\, d\nu(x) \\
&= \int_{\mathcal{E}'_p} \left(|\varphi(x)| u\left(|x|_{-p}^2\right)^{-1/2}\right) u\left(|x|_{-p}^2\right)^{1/2} d\nu(x) \\
&\leq \|\varphi\|_{\mathcal{A}_{p,u}} \int_{\mathcal{E}'_p} u\left(|x|_{-p}^2\right)^{1/2} d\nu(x).
\end{aligned} \tag{4.15}$$

By Theorem 4.1, $\{\|\cdot\|_{\mathcal{A}_{p,u}}; p \geq 0\}$ and $\{\|\cdot\|_{p,u}; p \geq 0\}$ are equivalent. Hence Equation (4.15) implies that $[\mathcal{E}]_u \subset L^1(\nu)$ and the linear functional

$$\varphi \longmapsto \int_{\mathcal{E}'} \varphi(x)\, d\nu(x), \qquad \varphi \in [\mathcal{E}]_u,$$

is continuous on $[\mathcal{E}]_u$. Thus ν is a Hida measure with $\widetilde{\nu}$ in $[\mathcal{E}]_u^*$.

To prove the necessity, suppose ν is a Hida measure inducing a generalized function $\widetilde{\nu} \in [\mathcal{E}]_u^*$. Then for all $\varphi \in [\mathcal{E}]_u$,

$$\langle\!\langle \widetilde{\nu}, \varphi \rangle\!\rangle = \int_{\mathcal{E}'} \varphi(x)\, d\nu(x). \tag{4.16}$$

Since $\{\|\cdot\|_{\mathcal{A}_{p,u}}; p \geq 0\}$ and $\{\|\cdot\|_{p,u}; p \geq 0\}$ are equivalent, the linear functional $\varphi \mapsto \langle\!\langle \widetilde{\nu}, \varphi \rangle\!\rangle$ is continuous with respect to $\{\|\cdot\|_{\mathcal{A}_{p,u}}; p \geq 0\}$. Hence there exist constants $K, q \geq 0$ such that for all $\varphi \in [\mathcal{E}]_u$,

$$\left|\langle\!\langle \widetilde{\nu}, \varphi \rangle\!\rangle\right| \leq K \|\varphi\|_{\mathcal{A}_{q,u}}. \tag{4.17}$$

Note that by continuity, Equations (4.16) and (4.17) also hold for all $\varphi \in \mathcal{A}_{q,u}$ defined in the Remark of Theorem 4.1.

Now, with this q, we define a function θ on $\mathcal{E}'_{q,c}$ by

$$\theta(x) = \mathcal{L}_u\left(2^{-4}\langle x, x\rangle_{-q}\right), \qquad x \in \mathcal{E}'_{q,c},$$

where $\langle\cdot,\cdot\rangle_{-q}$ is the bilinear pairing on $\mathcal{E}'_{q,c}$. Obviously, θ is analytic on $\mathcal{E}'_{q,c}$. On the other hand, apply Lemma 3.9 with $a = k = 2$ to get

$$|\theta(x)| \leq \mathcal{L}_u\big(2^{-4}|x|_{-q}^2\big) \leq \sqrt{\frac{2e}{\log 2}}\, u\big(|x|_{-q}^2\big)^{1/2}, \qquad \forall x \in \mathcal{E}'_{q,c}.$$

This shows that $\theta \in \mathcal{A}_{q,u}$ and we have

$$\|\theta\|_{\mathcal{A}_{q,u}} \leq \sqrt{\frac{2e}{\log 2}}.$$

Then apply Equation (4.17) to the function θ,

$$\big|\langle\!\langle \widetilde{\nu}, \theta\rangle\!\rangle\big| \leq K\|\theta\|_{\mathcal{A}_{q,u}} \leq K\sqrt{\frac{2e}{\log 2}}.$$

Therefore, from Equation (4.16) with $\varphi = \theta$ we conclude that

$$\left|\int_{\mathcal{E}'} \theta(x)\, d\nu(x)\right| \leq K\sqrt{\frac{2e}{\log 2}}. \tag{4.18}$$

Note that $\theta(x) = \mathcal{L}_u\big(2^{-4}|x|_{-q}^2\big)$ for $x \in \mathcal{E}'$. Hence Equation (4.18) implies that

$$\int_{\mathcal{E}'} \mathcal{L}_u\big(2^{-4}|x|_{-q}^2\big)\, d\nu(x) < \infty.$$

But $u(r) \leq C\mathcal{L}_u(4r)$ from Theorem 3.6 (2) with $k = 2$. Therefore,

$$\int_{\mathcal{E}'} u\big(2^{-6}|x|_{-q}^2\big)\, d\nu(x) < \infty.$$

Now, choose $p > q$ large enough such that $\rho^{2(p-q)} \leq 2^{-6}$. Then $|x|_{-p}^2 \leq 2^{-6}|x|_{-q}^2$. Recall that u is increasing. Hence

$$\int_{\mathcal{E}'} u\big(|x|_{-p}^2\big)\, d\nu(x) < \infty.$$

Note that $u(r) \geq 1$ and so $u(r)^{1/2}(r) \leq u(r)$. Thus we conclude that

$$\int_{\mathcal{E}'} u\big(|x|_{-p}^2\big)^{1/2}\, d\nu(x) < \infty.$$

This inequality implies that ν is supported by $\mathcal{E}'_p$ and Equation (4.14) holds. □

Example 4.5. (Poisson noise measure)
Let $\mathcal{P}$ be the Poisson measure on $\mathcal{E}^*$ given by

$$\exp\Big(\int_{\mathbb{R}} (e^{i\xi(t)} - 1)dt\Big) = \int_{\mathcal{E}^*} e^{i\langle x,\xi\rangle}\mathcal{P}(dx), \quad \xi \in \mathcal{E}^*.$$

It has been shown [6] that the Poisson noise measure induces a generalized function in $[\mathcal{E}]_{g_2}^*$. Thus by Theorem 4.4 and Example 2.2 we have the integrability condition

$$\int_{\mathcal{E}_p^*} \exp\Big(|x|_{-p}\sqrt{\log|x|_{-p}}\Big)\mathcal{P}(dx) < \infty$$

for some p.

Example 4.6. (Grey noise measure)
Let $0 < \lambda \leq 1$. The grey noise measure on $\mathcal{E}^*$ is the measure ν_λ having the characteristic function

$$L_\lambda(|\xi|_0^2) = \int_{\mathcal{E}^*} e^{i\langle x,\xi\rangle}\nu_\lambda(dx), \quad \xi \in \mathcal{E},$$

where $L_\lambda(t)$ is the Mittag-Leffler function with parameter λ;

$$L_\lambda(t) = \sum_{n=0}^{\infty} \frac{(-t)^n}{\Gamma(1+\lambda n)}.$$

Here Γ is the Gamma function. This measure was introduced by Schneider [23]. It is shown in [20] that ν_λ is a Hida measure which induces a generalized function Φ_{ν_λ} in $(\mathcal{E})^*_{1-\lambda}$. Therefore by Theorem 4.4 and Example 2.3 the grey noise measure ν_λ satisfies

$$\int_{\mathcal{E}_p^*} \exp\big(\frac{1}{2}(2-\lambda)|x|_{-p}^{\frac{2}{2-\lambda}}\big)\nu_\lambda(dx) < \infty$$

for some p.

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