Lévy Processes on Quantum Hypergroups

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Abstract. Quantum hypergroups are non-commutative versions of hypergroups and were introduced by Yu.A. Chapovsky and L.I. Vainerman. Assuming that the quantum hypergroup satisfies a certain positivity condition (Schoenberg's correspondence), we show that Lévy processes, like in the quantum group case, are given by solutions of quantum stochastic differential equations in the sense of R.L. Hudson and K.R. Parthasarathy. We prove that quantum hypergroups of double coset type satisfy Schoenberg's correspondence. As an example we discuss the quantum hypergroup $U\langle 2\rangle//U\langle 1\rangle$ with $U\langle n\rangle$ the non-commutative analogue of the coefficient algebra of the unitary group.

1. Intoduction

Let K be a hypergroup; see [2]. This means, among other conditions, that

- K is a (locally compact) topological space with a distinguished point $e \in K$.
- There is a binary operation, denoted by $\star$ and called *convolution*, on the space $\mathbf{M}_b$ of finite signed measures on K which turns $\mathbf{M}_b$ into an algebra.
- For probability measures μ and ν the convolution product $\mu\star\nu$ is again a probability measure.
- $\mu \star \delta_e = \delta_e \star \mu = \mu$ for all $\mu \in \mathbf{M}_b$

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where δ_x is the Dirac measure at x for $x \in K$.

For an appropriate complex-valued function f on K (for example, $f \in \mathrm{L}^\infty(K)$) we define the function Δf on $K \times K$ by

$$\Delta f(x, y) = \int_K f \, \mathrm{d}(\delta_x \star \delta_y).$$

If $f \in \mathrm{L}^\infty(K)$ then $\Delta f \in \mathrm{L}^\infty(K \times K)$. In many cases $\mathrm{L}^\infty(K \times K)$ will be (the closure of) the tensor product $\mathrm{L}^\infty(K) \otimes \mathrm{L}^\infty(K)$ and we will have the following situation. There is a $*$-algebra $\mathrm{F}(K)$ of functions on K such that Δ maps $\mathrm{F}(K)$ to the tensor product $\mathrm{F}(K) \otimes \mathrm{F}(K)$. The hypergroup can then be described by a triplet $(\mathrm{F}, \Delta, \delta)$ with the properties

- F is a complex $*$-algebra
- $\Delta : \mathrm{F} \to \mathrm{F} \otimes \mathrm{F}$ is a *positive* linear mapping satisfying $\Delta \mathbf{1} = \mathbf{1} \otimes \mathbf{1}$ and the coassociativity condition

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$$

- $\delta : \mathrm{F} \to \mathbb{C}$ is a $*$-algebra homomorphism satisfying the counit condition

$$(\delta \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \delta) \circ \Delta$$

If we allow not only commutative $*$-algebras and if we replace the positivity of Δ by *complete positivity* we arrive at the notion of a *quantum hypergroup* (see [3]) or, more generally, of what we call a *hyper-bialgebra.*

Important examples of hypergroups are given by a double coset structure. Let G be a semi-group with unit element e. In order to stay in a purely algebraic framework, we consider the $*$-algebra $\mathrm{R}(G)$ of functions which come from a finite-dimensional representation of G (i.e. $f \in \mathrm{R}(G)$ if $f(x) = \langle \xi, \gamma(x)\zeta \rangle$ for $\xi, \zeta \in \mathbb{C}^n$ and γ a $*$-representation of the elements of G as $n \times n$-matrices). $\mathrm{R}(G)$ becomes a $*$-bialgebra if we define the comultiplication by $\Delta_1(x, y) = f(xy)$ and the counit by $\delta_1(f) = f(e)$. Now let H be sub-semi-group of G equipped with a Haar measure λ. Since H is a semi-group, we can define a comultiplication Δ_2 and a counit δ_2 on $\mathrm{R}(H)$ in the same manner as for G. Denote by $\pi : \mathrm{R}(G) \to \mathrm{R}(H)$ the restriction to H. Then π is a $*$-bialgebra homomorphism. Denote by $\mathrm{R}(G)//\mathrm{R}(H)$ the space of functions in $\mathrm{R}(G)$ satisfying

$$f(xzy) = f(z) \text{ for all } x, y \in G,\ z \in H,$$

that is $\mathrm{R}(G)//\mathrm{R}(H)$ consists of functions on $G//H$, the space of double cosets of G with respect to H. We have

$$\mathrm{R}(G)//\mathrm{R}(H) = \{f \in \mathrm{R}(G) \mid (\pi \otimes \mathrm{id}) \circ \Delta_1 f = \mathbf{1} \otimes f \text{ and } (\mathrm{id} \otimes \pi) \circ \Delta_1 f = f \otimes \mathbf{1}\}.$$

It can be shown that the $*$-algebra $\mathrm{R}(G)//\mathrm{R}(H)$ is turned into a hyper-bialgebra if we set

$$\Delta f(x, y) = \int f(xzy)\, \mathrm{d}\lambda(z)$$

and

$$\delta f = \delta_1 f = f(e),$$

$f \in \mathrm{R}(G)//\mathrm{R}(H)$. Then

$$\Delta = (\mathrm{id} \otimes (\lambda \circ \pi) \otimes \mathrm{id}) \circ (\Delta_1 \otimes \mathrm{id}) \circ \Delta_1 \lceil \mathrm{R}(G)//\mathrm{R}(H)$$

and $\delta = \delta_1 \lceil \mathrm{R}(G)//\mathrm{R}(H)$. Examples of this construction are given by double coset hypergroups. Moreover, this constuction can be turned over to the non-commutative setting; see [3] and Section 3 of this paper.

We will be concerned with quantum stochastic processes on hyper-bialgebras, in particular, with quantum Lévy processes. These are defined in analogy to Lévy processes on $*$-bialgebras: $*$-homomorphisms are replaced by completely positive mappings; cf. also [12]. We prove that Lévy processes on hyper-bialgebras can be realized as solutions of quantum stochastic differential equations on Bose-Fock space, thus generalizing the result for bialgebras, *under the condition* that the hyper-bialgebra fulfills the principle of *Schoenberg's correspondence* (Section 2). We were not able to prove Schoenberg's correspondence in the general case of a hyper-bialgebra but only for hyper-bialgebras of double coset type with the additional assumption that the Haar measure is faithful (Section 3). In Section 4 we introduce the example of the double coset hyper-bialgebra $\mathrm{U}\langle 2\rangle//\mathrm{U}\langle 1\rangle$ with $\mathrm{U}\langle n\rangle$ denoting the non-commutative analogue of the coefficient algebra of the unitary group U_n. In Section 5 we consider a class of Brownian motions on $\mathrm{U}\langle 2\rangle//\mathrm{U}\langle 1\rangle$ for which we analyze the corresponding quantum stochastic differential equations in Section 6.

Vector spaces will be over the field of complex numbers. Algebras are always assumed to be associative, complex and unital. For a vector space $\mathcal{V}$ we denote by $\mathcal{V}'$ the vector space of linear functionals on $\mathcal{V}$. For a coalgebra $(\mathcal{C}, \Delta, \delta)$ we define the n-times comultiplication $\Delta^{(n)} : \mathcal{C} \to \mathcal{C}^{\otimes n}$, $n = 0, 1, 2, \ldots$, inductively by $\Delta^{(0)} = \delta$ and $\Delta^{(n+1)} = (\mathrm{id} \otimes \Delta^{(n)}) \circ \Delta$. Note that $\Delta^{(1)} = \mathrm{id}$ and $\Delta^{(2)} = \Delta$.

A $*$-algebra is an algebra $\mathcal{A}$ equipped with an involution, i.e. an antilinear mapping $a \mapsto a^*$ satisfying $(ab)^* = b^* a^*$ and $(a^*)^* = a$. An element of a $*$-algebra is called positive if it is a finite sum of elements of the form $a^* a$. A linear mapping Φ from a $*$-algebra $\mathcal{A}$ to a $*$-algebra $\mathcal{B}$ is called positive if $\Phi(a^* a)$ is a positive element in $\mathcal{B}$ for all $a \in \mathcal{A}$, i.e. if Φ maps positive elements to positive elements. We call Φ completely positive (c.p.) if $\Phi \mathbf{1} = \mathbf{1}$ and if $\Phi \otimes \mathrm{id} : \mathcal{A} \otimes \mathcal{M}_n(\mathbb{C}) \to \mathcal{B} \otimes \mathcal{M}_n(\mathbb{C})$ is positive for all $n \in \mathbb{N}$ where $\mathcal{M}_n(\mathbb{C})$ denotes the $*$-algebra of $n \times n$-matrices. The tensor product of two c.p. mappings is again c.p.

2. Lévy processes on hyper-bialgebras

A *quantum probability space* is a pair $(\mathcal{A}, \Phi)$ consisting of a $*$-algebra $\mathcal{A}$ and a state Φ on $\mathcal{A}$, see [1] and also [9, 8, 4, 13]. For a complex vector space $\mathcal{V}$ a linear mapping $j : \mathcal{V} \to \mathcal{A}$ is called a quantum random variable (q.r.v.). The unital sub-$*$-algebras $\mathcal{A}_1, \ldots, \mathcal{A}_n$ of $\mathcal{A}$ are called (tensor) *independent* if $[\mathcal{A}_k, \mathcal{A}_l] = 0$ for $k \neq l$, and if $\Phi(a_1 \ldots a_n) = \Phi(a_1) \ldots \Phi(a_n)$ for all $a_k \in \mathcal{A}_k$, $k = 1, \ldots, n$. The q.r.v. $j_1, \ldots, j_n$, $j_k : \mathcal{V}_k \to \mathcal{A}$, are said to be independent if the $*$-algebras $*\text{-}\underline{\mathrm{alg}}\,(j_k(\mathcal{V}_k))$, $k = 1, \ldots, n$, are independent where $*\text{-}\underline{\mathrm{alg}}$ means 'unital $*$-algebra generated by'.

The $*$-tensor algebra $\mathcal{T}(\mathcal{V})$ over a vector space $\mathcal{V}$ is defined to be the free $*$-algebra generated by $\mathcal{V}$. This space can be realized as the vector space

$$\bigoplus_{n=0}^{\infty} (\mathcal{V} \oplus \overline{\mathcal{V}})^{\otimes n}$$

with $\overline{\mathcal{V}}$ a complex conjugate copy of $\mathcal{V}$ and the $*$-algebra structure given by

$$(v_1 \otimes \ldots v_n)\, v = v_1 \otimes \ldots v_n \otimes v; \;\; v^* = \bar{v}.$$

For a q.r.v. j we denote by $\mathcal{T}(j)$ the unique extension of j to $\mathcal{T}(\mathcal{V})$ as a $*$-algebra homomorphism. The *distribution* of j is the state $\Phi \circ \mathcal{T}(j)$ on $\mathcal{T}(\mathcal{V})$.

A *Lévy process* on a coalgebra $\mathcal{C}$ is a family of q.r.v. (j_{st}) over the same quantum probability space, indexed by pairs (s,t) of real numbers with $0 \leq s \leq t$, and satisfying

- $j_{rs} \star j_{st} = j_{rt}, \; 0 \leq r \leq s \leq t$
- $j_{tt} = \delta \,\mathrm{id}$
- $j_{t_1 t_2}, \ldots, j_{t_n t_{n+1}}$ independent for $0 \leq t_1 \leq t_2 \leq \cdots \leq t_{n+1}$
- the distribution of j_{st} only depends on $t-s$ (we write Φ_t for the distribution of j_{0t})
- $\lim_{t \to 0+} \Phi_t(c_1 \otimes \ldots \otimes c_n) = \delta(c_1) \ldots \delta(c_n)$ (where we put $\delta(\bar{c}) = \overline{\delta(c)}$)

Notice that $(\mathcal{T}(\mathcal{C}), \mathcal{T}(\Delta), \mathcal{T}(\delta))$ is a $*$-bialgebra. The above definition of a Lévy processes says that $\mathcal{T}(j_{st})$ is a Lévy process on the $*$-bialgebra $\mathcal{T}(\mathcal{C})$ in the sense of [11]. Therefore, the theory of Lévy processes developed in [11] applies and we obtain a realization of our Lévy process on a Bose-Fock-space as the solution to a quantum stochastic differential equation in the sense of Hudson and Parthasarathy [7].

We describe the situation more precisely. Let D be a pre-Hilbert space. We denote by $\mathrm{L}(D)$ the $*$-algebra formed by all linear operators $R: D \to D$ which possess an adjoint R^* on D (i.e. there exists a linear operator $R^*: D \to D$ such that $\langle \xi, R\zeta \rangle = \langle R^*\xi, \zeta \rangle$ for all $\xi, \zeta \in D$.) Suppose that we are given

- a linear mapping $\mathrm{r}: \mathcal{C} \to \mathrm{L}(D)$
- a linear mapping $\mathrm{e}: \mathcal{C} \oplus \overline{\mathcal{C}} \to D$
- a linear mapping $\psi: \mathcal{C} \to \mathbb{C}$.

We put $\mathrm{r}(\bar{c}) = \mathrm{r}(c)^*$ and $\psi(\bar{c}) = \overline{\psi(c)}$ and we will always assume that the set $\{\mathrm{r}(b_1) \ldots \mathrm{r}(b_n)\mathrm{e}(b) \mid b, b_1, \ldots, b_n \in \mathcal{C} \oplus \overline{\mathcal{C}}\}$ is total in D.

Consider the quantum stochastic differential equation

$$\mathrm{d}j_{st} = j_{st} \star \mathrm{d}I_t \; ; \quad j_{ss} = \delta$$

with

$$I_t(c) = A_t^*(\mathrm{e}(c)) + \Lambda_t(\mathrm{r}(c) - \delta(c)\mathrm{id}) + A_t(\mathrm{e}(\bar{c})) + \psi(c)t; \;\; c \in \mathcal{C} \oplus \overline{\mathcal{C}}$$

in the sense of [11], Theorem 2.5.1. Then the solution to these equations is a Lévy process on $\mathcal{C}$ whose generator $\Psi : \mathrm{T}(\mathcal{C}) \to \mathbb{C}$ is given by

$$\begin{aligned}
\Psi(c) &= \psi(c) \text{ for } c \in \mathcal{C} \oplus \overline{\mathcal{C}} \\
\Psi(c_1 \otimes c_2) &= \langle \mathrm{e}(\bar{c}_1), \mathrm{e}(c_2) \rangle \text{ for } c_1, c_2 \in \mathcal{C} \oplus \overline{\mathcal{C}} \\
\Psi(c_1 \otimes \ldots \otimes c_n) &= \langle \mathrm{e}(\bar{c}_1), \mathrm{r}(c_2) \ldots \mathrm{r}(c_{n-1}) \mathrm{e}(c_n) \rangle \text{ for } c_1, \ldots, c_n \in \mathcal{C} \oplus \overline{\mathcal{C}}, \; n \geq 3
\end{aligned}$$

Conversely, starting from a Lévy process, by applying the GNS construction to its generator, one obtains D, e, r, ψ such that the above quantum stochastic differential equation yields a version of the process. The quantum probability space underlying our Fock-representation of the Lévy process is given by the $*$-algebra $\mathrm{L}(\mathcal{E}_D)$ and the vacuum state. Here

$$\mathcal{E}_D = \bigcap_{\alpha \geq 0} \operatorname{dom} \alpha^{\mathrm{N}} \cap \bigcup_E \Gamma(E)$$

with N the number operator, $\Gamma(E)$ the Bose-Fock-space over $\mathrm{L}^2(\mathbb{R}+) \otimes E$, and where the union is taken over all finite dimensional subspaces E of D.

We pose the following question. Let the coalgebra $\mathcal{B}$ also carry the structure of a $*$-algebra such that the following are satisfied

- the comultiplication $\Delta : \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}$ is completely positive (c.p.)
- the counit $\delta : \mathcal{B} \to \mathbb{C}$ is a $*$-algebra homomorphism

We call such an object a *hyper-bialgebra*; see [3] where the concept of a quantum hypergroup was introduced. What are the conditions on the coefficients e, r and ψ such that the corresponding Lévy process consists of c.p. mappings?

We equip $\mathcal{T}(\mathcal{B})$ with an other multiplication, denoted by $\cdot$, by setting

$$(b_1 \otimes \ldots \otimes b_2) \cdot (c_1 \otimes \ldots \otimes c_m) = b_1 \otimes \ldots \otimes b_{n-1} \otimes (b_n c_1) \otimes c_2 \otimes \ldots \otimes c_m.$$

which turns $\mathcal{T}(\mathcal{B})$ into a hyper-bialgebra. This new hyper-bialgebra has another interpretation. Consider the free product

$$\mathcal{B} \sqcup_1 \mathbb{C}(p)$$

of unital hyper-bialgebras $\mathcal{B}$ and $\mathbb{C}(p)$ (cf. [12]) where $\mathbb{C}(p)$ denotes the $*$-bialgebra generated by a single projection p (i.e. an indeterminate satisfying $p^2 = p = p^*$). Then

$$\mathcal{B} \sqcup_1 \mathbb{C}(p) = (\operatorname{kern} \mathcal{B}) \sqcup \operatorname{kern} \mathbb{C}(p) \oplus \mathbb{C}\mathbf{1}$$

(here $\sqcup$ is the free product of algebras) and the $*$-bialgebra $\mathcal{T}(\mathcal{B})$ can be recovered in $\mathcal{B} \sqcup_1 \mathbb{C}(p)$ if we identify $b_1 \otimes \ldots \otimes b_n$ with $p^\perp b_1 p^\perp \ldots p^\perp b_n p^\perp$. Moreover, the hyper-bialgebra $\mathcal{T}(\mathcal{B})$ is also a sub-hyper-bialgebra of $\mathcal{B} \sqcup_1 \mathbb{C}(p)$ if we send $b_1 \otimes \ldots \otimes b_n$ to $b_1 p^\perp \ldots p^\perp b_n$.

We say that a hyper-bialgebra $\mathcal{B}$ satisfies *Schoenberg's correspondence* if for a linear functional Ψ on $\mathcal{T}(\mathcal{B})$ the following are equivalent:

(i) $\Psi(\mathbf{1}) = 0$, $\Psi(B^*) = \overline{\Psi(B)}$ for all $B \in \mathcal{T}(\mathcal{B})$ and $\Psi(B^* \cdot B) \geq 0$ for all $B \in \operatorname{kern} \mathcal{T}(\delta)$

(ii) $\exp_*(t\Psi)(\mathbf{1}) = 1$ and $\exp_*(t\Psi)(B^* \cdot B) \geq 0$ for all $B \in \mathcal{T}(\mathcal{B})$

where the convolution in (ii) is with respect to the comultiplication $\mathcal{T}(\Delta)$.

A $*$-representation of an algebra $\mathcal{A}$ on a pre-Hilbert space D is a $*$-algebra homomorphism from $\mathcal{A}$ to $\mathrm{L}(D)$. For a $*$-representation ρ of the $*$-algebra $\mathcal{A}$ on a pre-Hilbert space D and for a $*$-homomorphism $\delta : \mathcal{A} \to \mathbb{C}$ the pre-Hilbert space D becomes a two-sided $\mathcal{A}$-module if we put

$$a.\xi.b = \rho(a)\xi\delta(b) \text{ for } a, b \in \mathcal{A} \text{ and } \xi \in D.$$

We speak of (ρ, δ)-cocycles and -coboundaries of the Hochschildt cohomology associated with this bimodule structure of D.

Theorem 2.1 *Let $\mathcal{B}$ be a hyper-bialgebra which we suppose to satisfy Schoenberg's correspondence. Let j_{st} be a Lévy process on $\mathcal{B}$ with coefficients D,* e, r *and ψ. Then the q.r.v. j_{st} are c.p. if and only if there exist*

- *a pre-Hilbert space E and an isometry $V : D \to E$*
- *a $*$-representation ρ of $\mathcal{B}$ on E*
- *a (ρ, δ)-1-cocycle $\eta : \mathcal{B} \to E$*

such that

- $e = V^* \circ \eta$
- $r(b) = V^* \circ \rho(b) \circ V$
- $-\langle \eta(b^*), \eta(c) \rangle$ is the (δ, δ)-coboundary of ψ .

Proof: Using Schoenberg's correspondence, it is not difficult to see that a Lévy process on $\mathcal{B}$ is c.p. if and only if its generator satisfies the condition (i) above. However, then we can apply Corollary 2.5 of [12].$\diamondsuit$

Let j_{st} be a Lévy process with the extra property that the isometry V appearing in the canonical construction of j_{st} is unitary. We call such a process *basic*. In this case $\eta = e$, $r = \rho$ and $\Psi(b_1 \otimes \ldots \otimes b_n) = \psi(b_1 \ldots b_n)$. Therefore, a basic Lévy process is given by a conditionally positive, hermitian linear functional ψ with $\psi(\mathbf{1}) = 0$ on $\mathcal{B}$. In fact, there is a 1-1-correspondence between such functionals and basic Lévy processes.

3. Double coset hyper-bialgebras

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be $*$-bialgebras. Suppose that we are given a *Haar measure* λ on $\mathcal{B}_2$ that is $\lambda : \mathcal{B}_2 \to \mathbb{C}$ is a state satisfying

$$(\mathrm{id} \otimes \lambda) \circ \Delta_2 = \lambda \mathbf{1} = (\lambda \otimes \mathrm{id}) \circ \Delta_2.$$

We will also assume that λ is *faithful*, a condition needed for the proof of Theorem 3.1 below. Let $\pi : \mathcal{B}_1 \to \mathcal{B}_2$ be a $*$-bialgebra epimorphism. We put

$$\begin{aligned} \mathcal{B}_1/\mathcal{B}_2 &= \{ b \in \mathcal{B}_1 \mid (\mathrm{id} \otimes \pi) \circ \Delta_1(b) = b \otimes \mathbf{1} \} \\ \mathcal{B}_2 \backslash \mathcal{B}_1 &= \{ b \in \mathcal{B}_1 \mid (\pi \otimes \mathrm{id}) \circ \Delta_1(b) = \mathbf{1} \otimes b \} \\ \mathcal{B} &= \mathcal{B}_1/\mathcal{B}_2 \cap \mathcal{B}_2 \backslash \mathcal{B}_1 \end{aligned}$$

Next we define

$$\tilde{\Delta} : \mathcal{B}_1 \to \mathcal{B}_1 \otimes \mathcal{B}_1$$

by

$$\tilde{\Delta} b = (\mathrm{id} \otimes (\lambda \circ \pi) \otimes \mathrm{id}) \circ \Delta_1^{(3)}.$$

It is not difficult to check that $(\mathcal{B}, \Delta, \delta)$ with $\Delta = \tilde{\Delta} \lceil \mathcal{B}$ and $\delta = \delta_1 \lceil \mathcal{B}$ is an example of a hyper-bialgebra; see [3]. We sometimes write $\mathcal{B} = \mathcal{B}_1 // \mathcal{B}_2$ and call $\mathcal{B}$ a *double coset hyper-bialgebra.*

Theorem 3.1 *Double coset hyper-bialgebras satisfy Schoenberg's correspondence.*

The proof will be given at the end of this section.

To analyse the situation consider first a convolution semi-group φ_t on $\mathcal{B} = \mathcal{B}_1//\mathcal{B}_2$. We know that φ_t is the convolution exponential of $\psi = \frac{\mathrm{d}}{\mathrm{d}t}\varphi_t|_{t=0}$, the pointwise derivative at 0 of φ_t, i.e.

$$\varphi_t = \exp_\star(t\psi)$$

which is defined pointwise as the series

$$\sum_{n=0}^{\infty} \frac{\psi^{\star n}}{n!} t^n = \delta + \psi t + \frac{\psi^{\star 2}}{2!} t^2 + \dots,$$

see [10]. Now a linear functional β on $\mathcal{B} \subset \mathcal{B}_1$ can be extended to $\mathcal{B}_1$ by setting

$$\tilde{\beta} = \beta \circ \big((\lambda \circ \pi) \otimes \mathrm{id} \otimes (\lambda \circ \pi)\big) \circ \Delta^{(3)}$$

because $\big((\lambda \circ \pi) \otimes \mathrm{id} \otimes (\lambda \circ \pi)\big) \circ \Delta^{(3)}$ maps $\mathcal{B}_1$ to $\mathcal{B}$. Moreover, the restriction of $\tilde{\beta}$ to $\mathcal{B}$ gives back β. We may write

$$\tilde{\beta}\lceil\mathcal{B} = \beta \text{ and } \tilde{\beta} = (\lambda \circ \pi) \star \beta \star (\lambda \circ \pi).$$

The convolution semi-group φ_t is mapped to $\tilde{\varphi}_t$ with the properties

$$\begin{aligned} \tilde{\varphi}_{s+t} &= \tilde{\varphi}_s \star \tilde{\varphi}_t \text{ (with respect to } \Delta_1) \\ \tilde{\varphi}_t &\to \lambda \circ \pi = \tilde{\varphi}_0 \text{ for } t \to 0+ \end{aligned}$$

Thus $\tilde{\varphi}_t$ is a continuous convolution semi-group on $\mathcal{B}_1$ which does not start at the counit δ_1 but at $\lambda \circ \pi$!

This leads to the following general consideration. Let $\mathcal{B}$ be a $*$-bialgebra and suppose that we are given linear functionals φ_t satisfying

$$\begin{aligned} \varphi_{s+t} &= \varphi_s \star \varphi_t \\ \varphi_t &\to \varphi_0 \end{aligned}$$

Can we differentiate φ_t at 0? Let us look at matrices first. Let $A_t \in \mathcal{M}_d(\mathbb{C})$ with $A_{s+t} = A_s A_t$ and $A_t \to A_0$. Since $A_0^2 = A_0$ we can find a basis of $\mathbb{C}^d$ such that A_0 is of the form

$$\begin{pmatrix} \mathrm{I} & 0 \\ 0 & 0 \end{pmatrix}$$

with I the $n \times n$-unit matrix, $n \leq d$. We have $A_0 A_t = A_t = A_t A_0$ which means that A_t has the form

$$\begin{pmatrix} B_t & 0 \\ 0 & 0 \end{pmatrix}$$

with $B_t \in \mathcal{M}_n(\mathbb{C})$ and

$$B_{s+t} = B_s B_t, \quad B_t \to \mathrm{I}.$$

We know that $B_t = \mathrm{e}^{tG}$ with $G = \frac{\mathrm{d}}{\mathrm{d}t} B_t|_{t=0}$ and therefore

$$A_t = \begin{pmatrix} \mathrm{e}^{tG} & 0 \\ 0 & 0 \end{pmatrix} = A_0\, \mathrm{e}^{t\tilde{G}} \tag{1}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} A_t|_{t=0} = \tilde{G}$$

where we put $\tilde{G} = \begin{pmatrix} G & 0 \\ 0 & 0 \end{pmatrix}$. In the case of a general coalgebra $\mathcal{C}$ and $\varphi_t \in \mathcal{C}'$, $\varphi_{s+t} = \varphi_s \star \varphi_t$, $\varphi_t \to \varphi_0$, we use for a given element b in $\mathcal{C}$ the fundamental theorem on coalgebras (see [14]) to find a finite-dimensional sub-coalgebra $\mathcal{C}_b$ of $\mathcal{C}$ containing b. For $\mathrm{T}_t : \mathcal{C}_b \to \mathcal{C}_b$, $\mathrm{T}_t(c) = (\mathrm{id} \otimes \varphi_t) \circ \Delta(c)$, $c \in \mathcal{C}_b$, we have $\mathrm{T}_{s+t} = \mathrm{T}_s \mathrm{T}_t$, $\mathrm{T}_t \to \mathrm{T}_0$. By what we saw for matrices it follows $\mathrm{T}_t = \mathrm{T}_0\, \mathrm{e}^{t\tilde{G}}$ and

$$\varphi_t(c) = \delta \circ \mathrm{T}_t(c) = \varphi_0 \star \mathrm{e}_\star^{t\psi}(c) \text{ for } c \in \mathcal{C}_b$$

with $\psi = \delta \circ \tilde{G}$. We also have

$$\frac{\mathrm{d}}{\mathrm{d}t} \varphi_t(c)|_{t=0} = (\varphi_0 \star \psi)(c) = (\psi \star \varphi_0)(c) = \psi(c)$$

for $c \in \mathcal{C}_b$. Since the intersection of two sub-coalgebras is a sub-coalgebra, ψ can be defined on the whole of $\mathcal{B}$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} \varphi_t|_{t=0} = \psi; \; \varphi_t = \varphi_0 \star \mathrm{e}_\star^{t\psi}.$$

A sesqui-linear form L on a vector space $\mathcal{V}$ is called *positive* if $L(v, v) \geq 0$ for all $v \in \mathcal{V}$. In order to prove a Schoenberg type result for convolution semi-groups (on $*$-bialgebras) which do not start at the counit, we proceed like in [10] by showing

Lemma 3.2 *Let C be a coalgebra. We form the tensor product $(\bar{C}\otimes C, \Lambda, \bar{\delta}\otimes\delta)$ of the coalgebras $(\bar{C}, \bar{\Delta}, \bar{\delta})$ and (C, Δ, δ) where $\bar{C}$ denotes the complex conjugate coalgebra of C. Let L_t be linear functionals on $\bar{C}\otimes C$ (that is the L_t are sesqui-linear forms on C) satisfying*

- $L_{s+t} = L_s \star L_t$ *(with respect to Λ)*
- $L_t \to L_0$ *pointwise for $t \to 0+$*

Then for

$$K = \frac{\mathrm{d}}{\mathrm{d}t} L_t|_{t=0}$$

the following conditions are equivalent:

(i) *L_0 is positive and*
$K(c, c) \geq 0$ for all $c \in C$ with $L_0(c, c) = 0$, and $K(c, d) = \overline{K(d, c)}$ for all $c, d \in C$

(ii) *L_t are positive for all $t \in \mathbb{R}_+$*

Proof: The proof is similar to the counit case. We give it here in a version adapted to our situation.— (ii) $\Longrightarrow$ (i) is proved by differentiating. For the proof of (i) $\Longrightarrow$ (ii) it suffices to show that $L_0 \star \mathrm{e}_\star^K$ is positive. Thanks to the fundamental theorem on coalgebras we may restrict ourselves to a finite-dimensional C.

We choose a scalar product S in C. We begin by showing that to each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$L_0(c, c) \leq \delta \text{ and } \| c \| = 1 \implies K(c, c) > -\epsilon.$$

(Notice that by assumtion $K(c, c)$ is real.) To see this we form the sets

$$A_{n,\epsilon} = \{c \in C \mid \| c \| = 1 \text{ and } L_0(c, c) \leq \frac{1}{n} \text{ but } K(c, c) \leq -\epsilon\}.$$

The $A_{n,\epsilon}$ are closed with $\bigcap_n A_{n,\epsilon} = \emptyset$. The latter follows from the fact that $K(c,c) \geq 0$ if $L_0(c,c) = 0$. By compactness there is n_0 such that $A_{n_0,\epsilon} = \emptyset$. Put $\delta = \frac{1}{n_0}$.

Next we show that to each $\epsilon > 0$ there exists n_ϵ such that

$$L_0 + \frac{K + \epsilon S}{n}$$

is positive for all $n \geq n_\epsilon$. By the first part there is a $\delta > 0$ such that for $\| c \| = 1$

$$K(c,c) + \epsilon \geq 0 \text{ if } L_0(c,c) \leq \delta.$$

This means

$$(L_0 + \frac{K + \epsilon S}{n})(c,c) \geq 0$$

for all $c \in \mathcal{C}$ with $\| c \| = 1$ and $L_0(c,c) \leq \delta$. For $c \in \mathcal{C}$ with $\| c \| = 1$ and $L_0(c,c) > \delta$ we find n_ϵ such that

$$\left| \frac{K(c,c) + \epsilon}{n} \right| \leq \frac{\| K \| + \epsilon}{n} \leq \delta$$

for all $n \geq n_\epsilon$. Then

$$\left(L_0 + \frac{K + \epsilon S}{n} \right) (c,c) \geq 0$$

for all $c \in \mathcal{C}$, $\| c \| = 1$, $L_0(c,c) > \delta$, $n \geq n_\epsilon$. Thus

$$L_0 + \frac{K + \epsilon S}{n}$$

is positive for all $n \geq n_\epsilon$. Since the convolution product of two positive forms on $\mathcal{C}$ is positive we have that

$$L_0 \star (L_0 + \frac{K + \epsilon S}{n}) \star L_0 = L_0 \star (\bar{\delta} \otimes \delta + \frac{K + \epsilon\, L_0 \star S \star L_0}{n}) \geq 0$$

for all $n \geq n_\epsilon$ and

$$0 \leq L_0 \star \left(L_0 + \frac{K + \epsilon\, L_0 \star S \star L_0}{n} \right)^{\star n} = L_0 \star \left(\bar{\delta} \otimes \delta + \frac{K + \epsilon\, L_0 \star S \star L_0}{n} \right)^{\star n}$$

converges pointwise to the form $L_0 \star \mathrm{e}_\star^{K + \epsilon\, L_0 \star S \star L_0}$ which, therefore, must be positive. By lettting ϵ tend to 0, we arrive at the desired result.$\diamondsuit$

As a direct consequence we have

Theorem 3.3 *Let $\mathcal{B}$ be a $*$-bialgebra and let $\varphi_t \in \mathcal{B}'$, $t \in \mathbb{R}_+$, satisfy*

- $\varphi_{s+t} = \varphi_s \star \varphi_t$
- $\varphi_t \to \varphi_0$

Then for $\psi = \frac{\mathrm{d}}{\mathrm{d}t}\varphi_t|_{t=0}$ *the following conditions are equivalent:*

(i) φ_0 *is positive and*

$\psi(b^*b) \geq 0$ *for all* $b \in \mathcal{B}$ *with* $\varphi_0(b^*b) = 0$, *and* $\varphi(b^*) = \overline{\varphi(b)}$ *for all* $b \in \mathcal{B}$

(ii) φ_t *is positive for all* $t \in \mathbb{R}_+$

Proof: We observe that, by applying the mapping

$$\mathcal{F} : \mathcal{B}' \to (\bar{\mathcal{B}} \otimes \mathcal{B})'$$

given by

$$\mathcal{F}(\varphi)(c, d) = \varphi(c^*d),$$

we can reduce everything to the situation of the preceeding lemma.$\diamondsuit$

Proof of Theorem 3.1: Let $\mathcal{B} = \mathcal{B}_1//\mathcal{B}_2$ be a double coset hyper-bialgebra. Then we define the homomorphism $\tilde{\pi}$ from $(\mathcal{T}(\mathcal{B}_1), \cdot)$ to $\mathcal{B}_2$ by

$$\tilde{\pi}(b_1 \otimes \ldots \otimes b_n) = b_1 \ldots b_n.$$

It is straightforward to check that $\mathcal{T}(\mathcal{B})$ equals $\mathcal{T}(\mathcal{B}_1)//\mathcal{B}_2$, so that $\mathcal{T}(\mathcal{B})$ is again a double coset hyper-bialgebra. Thus it is sufficient to prove that for a linear functional ψ on a given double coset hyper-bialgebra we have

$$\psi \text{ conditionally positive and hermitian} \implies \varphi_t = \mathrm{e}_\star^{t\psi} \text{ positive}$$

However, $\tilde{\varphi}_t$ and $\tilde{\psi}$ satisfy the conditions of Theorem 3.3 with $\tilde{\varphi}_0 = \lambda \circ \pi$. To see that $\tilde{\psi}$ satisfies (i) of Theorem 3.3 we remark first that $(\lambda \circ \pi)(b^*b) = 0$ if and only if $b \in \operatorname{kern} \pi$ since λ is faithful. Then, using the fact that $\operatorname{kern} \pi$ is a bi-ideal, one shows that, for $b \in \operatorname{kern} \pi$, $\big((\lambda \circ \pi) \otimes \mathrm{id} \otimes (\lambda \circ \pi)\big) \circ \Delta_1^{(3)}\, b^*b$ is of the form $\sum c_i^* c_i$ with $c_i \in \operatorname{kern} \delta$. An application of Theorem 3.3 yields the positivity of $\tilde{\varphi}_t$ and of φ_t. $\diamondsuit$

4. The hyper-bialgebra $\mathrm{U}\langle 2\rangle//\mathrm{U}\langle 1\rangle$

For $d \in \mathbb{N}$ we denote by $\mathrm{U}\langle d\rangle$ the free (non-commutative!) $*$-algebra generated by indeterminates x_{kl}, $k, l = 1, \ldots, d$, with the unitarity relations

$$\sum_{n=1}^{d} x_{kn} x_{ln}^{*} = \delta_{kl} \tag{2}$$

$$\sum_{n=1}^{d} x_{kn}^{*} x_{nl} = \delta_{kl} \tag{3}$$

The $*$-algebra $\mathrm{U}\langle d\rangle$ is turned into a $*$-bialgebra if we put

$$\begin{aligned} \Delta_1 x_{kl} &= \sum_{n=1}^{d} x_{kn} \otimes x_{nl} \\ \delta_1 x_{kl} &= \delta_{kl}. \end{aligned}$$

This $*$-bialgebra has been investigated by P. Glockner und W. von Waldenfels [6]. If we assume that the generators x_{kl}, x_{kl}^{*} commute we obtain the coefficient algebra of the unitary group U_d. This is why $\mathrm{U}\langle d\rangle$ was sometimes called the *non-commutative analogue of the coefficient algebra of the unitary group*. It is equal to the $*$-algebra generated by mappings

$$\xi_{kl} : \mathcal{U}(\mathbb{C}^d \otimes \mathcal{H}) \to \mathcal{B}(\mathcal{H})$$

with

$$\xi_{kl}(U) = U_{kl}, \; U \in \mathcal{U}(\mathbb{C}^d \otimes \mathcal{H}) \subset \mathrm{M}_d(\mathcal{B}(\mathcal{H}))$$

where $\mathcal{H}$ is an infinite-dimensional Hilbert space and $\mathcal{U}(\mathbb{C}^d \otimes \mathcal{H})$ denotes the group of unitary operators on $\mathbb{C}^d \otimes \mathcal{H}$. Moreover, $\mathcal{B}(\mathcal{H})$ is the $*$-algebra of bounded operators on $\mathcal{H}$ and $\mathrm{M}_d(\mathcal{B}(\mathcal{H}))$ denotes the $*$-algebra of $d \times d$-matrices with elements from $\mathcal{B}(\mathcal{H})$.

Proposition 4.1
(a) *On* $\mathrm{U}\langle 1\rangle$ *a faithful Haar measure is given by* $\lambda(x^n) = \delta_{0,n}$, $n \in \mathbb{Z}$.
(b) *On* $\mathrm{U}\langle 1\rangle$ *an antipode is given by setting* $Sx = x^*$ *and extending* S *as a* $*$*-algebra homomorphism.*
(c) *For* $d > 1$ *the* $*$*-bialgebra* $\mathrm{U}\langle d\rangle$ *does not posses an antipode.*

Proof: Only (c) requires a proof. Let us suppose that we are given an antipode S on $\mathrm{U}\langle d\rangle$, $d > 1$. Then

$$\begin{aligned}
\sum_{m=1}^{d} \sum_{n=1}^{d} S(x_{kn}) x_{nl} x_{lm}^{*} &= x_{lk}^{*} \\
&= \sum_{n=1}^{d} S(x_{kl}) \sum_{m=1}^{d} x_{nl} x_{lm}^{*} \\
&= \sum_{n=1}^{d} S(x_{kn}) \delta_{nl}. \\
&= S(x_{kl})
\end{aligned}$$

Similarly, one proves that $S(x_{kl}^{*}) = x_{lk}$. Since S is an antipode it has to be an algebra anti-homomorphism. Therefore,

$$\begin{aligned}
S\Big(\sum_{n=1}^{d} x_{kn} x_{ln}^{*}\Big) &= \sum_{n=1}^{d} S(x_{ln}^{*}) S(x_{kn}) \\
&= \sum_{n=1}^{d} x_{nl} x_{nk}^{*}
\end{aligned}$$

which is not equal to δ_{kl} if $d > 1$.$\Diamond$

Using the result of Glockner and von Waldenfels, we can describe the coalgebra structure of $\mathrm{U}\langle d\rangle$ as follows. Define a mapping

$$\tilde{\Delta}_1 : \mathrm{U}\langle d\rangle \to \underline{\mathrm{Map}}\left(\mathcal{U}(\mathbb{C}^d \otimes \mathcal{H}) \times \mathcal{U}(\mathbb{C}^d \otimes \mathcal{H}), \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H})\right)$$

by setting

$$\tilde{\Delta}_1 \, \xi_{kl} \, (U, V) = \sum_{n=1}^{d} U_{kn} \otimes V_{nl}.$$

An emdedding ι of $\mathrm{U}\langle d\rangle \otimes \mathrm{U}\langle d\rangle$ into $\underline{\mathrm{Map}}\left(\mathcal{U}(\mathbb{C}^d \otimes \mathcal{H}) \times \mathcal{U}(\mathbb{C}^d \otimes \mathcal{H}), \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H})\right)$ is given by

$$\iota(b \otimes c)(U, V) = a(U) \otimes b(V)$$

and we have

$$\tilde{\Delta}_1 \mathrm{U}\langle d\rangle \subset \iota(\mathrm{U}\langle d\rangle \otimes \mathrm{U}\langle d\rangle) \text{ with } \Delta_1 = \iota^{-1} \circ \tilde{\Delta}_1.$$

Let us now apply the construction in the beginning of this paragraph to the situation

$$\mathcal{B}_1 = \mathcal{U}\langle 2\rangle;\ \mathcal{B}_2 = \mathcal{U}\langle 1\rangle = \mathbb{C}\langle x, x^*\rangle / xx^* = \mathbf{1} = x^*x$$

and

$$\begin{pmatrix} \pi(x_{11}) & \pi(x_{12}) \\ \pi(x_{21}) & \pi(x_{22}) \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & \mathbf{1} \end{pmatrix}$$

In order to describe $\mathcal{B}$ in this case, we introduce two gradings l and g on $\mathrm{U}\langle d\rangle$ by setting

$$\begin{aligned} \mathrm{l}\,(x_{kl}^{(\epsilon)}) &= \begin{cases} 1 & \text{if} \quad k=1 \text{ and } \epsilon = 0 \\ -1 & \text{if} \quad k=1 \text{ and } \epsilon = 1 \\ 0 & \text{if} \quad k=2 \text{ and } \epsilon = 1 \end{cases} \\ \mathrm{g}\,(x_{kl}^{(\epsilon)}) &= \mathrm{l}\,(x_{lk}^{(\epsilon)}) \end{aligned}$$

where we use the notation $x_{kl}^{(0)} = x_{kl}$ and $x_{kl}^{(1)} = x_{kl}^*$. Since (2) and (3) are homogeneous elements of the free $*$-algebra generated by x_{kl}, the gradings l and g are well-defined. Denote by $\mathcal{B}_1^{(0)}$ and $\mathcal{B}_{1,(0)}$ the space of homogeneous elements of degree 0 in $\mathrm{U}\langle d\rangle$ in the l- and g-grading repectively.

Proposition 4.2

$$\begin{aligned} \mathcal{B}_1^{(0)} &= \{b \in \mathrm{U}\langle 2\rangle \mid (\pi \otimes \mathrm{id}) \circ \Delta_1 = \mathbf{1} \otimes b\} \\ \mathcal{B}_{1,(0)} &= \{b \in \mathrm{U}\langle 2\rangle \mid (\mathbf{1} \otimes \pi) \circ \Delta_1 = b \otimes \mathbf{1}\} \\ \mathcal{B} &= \mathcal{B}_1^{(0)} \cap \mathcal{B}_{1,(0)} \end{aligned}$$

Proof: We prove the first identity. If we consider $(\pi \otimes \mathrm{id}) \circ \Delta_1 b$ as an element of $\underline{\mathrm{Map}}\big(\mathcal{U}(\mathcal{H}) \times \mathcal{U}(\mathbb{C}^d \otimes \mathcal{H}), \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H})\big)$ we have for a monomial $b = \xi_{k_1 l_1}^{(\epsilon_1)} \cdots \xi_{k_n l_n}^{(\epsilon_n)}$

$$\begin{aligned} (\pi \otimes \mathrm{id}) \circ \Delta_1 b\Big(u, \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}\Big) &= \xi_{k_1 l_1}^{(\epsilon_1)} \cdots \xi_{k_n l_n}^{(\epsilon_n)} \begin{pmatrix} u \otimes U_{11} & U \otimes U_{12} \\ \mathbf{1} \otimes U_{21} & \mathbf{1} \otimes U_{22} \end{pmatrix} \\ &= u^{\mathrm{l}(b)} \otimes \xi_{k_1 l_1}^{(\epsilon_1)} \cdots \xi_{k_n l_n}^{(\epsilon_n)} \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}. \end{aligned}$$

For an arbitrary element

$$b = \sum_{n \in \mathbb{Z}} b^{(n)},\ b^{(n)} \in \mathcal{B}_1^{(n)}$$

in $\mathrm{U}\langle d\rangle$ we have

$$b\begin{pmatrix} u\otimes U_{11} & U\otimes U_{12} \\ \mathbf{1}\otimes U_{21} & \mathbf{1}\otimes U_{22}\end{pmatrix} = \sum_{n\in\mathbf{Z}} u^n \otimes b^{(n)} \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22}\end{pmatrix}$$

which is equal to

$$\sum_{n\in\mathbf{Z}} \mathbf{1}\otimes b^{(n)} \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22}\end{pmatrix}$$

for all $u \in \mathcal{U}(\mathcal{H})$ and all $U \in \mathcal{U}(\mathbb{C}^d \otimes \mathcal{H})$ if and only if $b^{(n)} = 0$ for $n \neq 0$.$\diamondsuit$

$\mathcal{B}$ is not a $*$-bialgebra. We have $\Delta x_{22} = x_{22} \otimes x_{22}$ but

$$\Delta x_{22}x_{22}^* = x_{22}x_{22}^* \otimes x_{22}x_{22}^* + (\mathbf{1} - x_{22}x_{22}^*) \otimes (1 - x_{11}x_{11}^*).$$

Notice that $\mathrm{U}_2//\mathrm{U}_1$ is the unit sphere S^1, so, in this sense, $\mathrm{U}\langle 2\rangle//\mathrm{U}\langle 1\rangle$ might be regarded as a non-commutative version of S^1.

Following [11], Section 5, a basic Brownian motion on $\mathcal{B}$ is a basic Lévy process on $\mathcal{B}$ whose generator ψ satisfies

$$\psi(bc) = \psi(b)\delta(c) + \delta(b)\psi(c) + \overline{\mathrm{d}(c^*)}\mathrm{d}(b)$$

where d is a derivation on $\mathcal{B}$, i.e. a linear functional on $\mathcal{B}$ with

$$\mathrm{d}(bc) = \mathrm{d}(b)\delta(c) + \delta(b)\mathrm{d}(c), \ \ b, c \in \mathcal{B}.$$

5. Examples of generators on $\mathrm{U}\langle 2\rangle//\mathrm{U}\langle 1\rangle$

We will now consider a class of basic Lévy processes on $\mathcal{B} = \mathrm{U}\langle 2\rangle//\mathrm{U}\langle 1\rangle$. Let $B = (b_{ij})$ a hermitian 2×2-matrix and let A_{ij}, $1 \leq i,j \leq 2$, be four complex matrices. Define ρ, η, and ψ on the generators of $\mathrm{U}\langle 2\rangle$ by

$$\begin{aligned}
\rho(x_{ij}) &= \delta_1(x_{ij})\mathrm{id}_{\mathcal{M}_d}, && 1\leq i,j\leq 2,\\
\rho(x_{ij}^*) &= \delta_1(x_{ij})\mathrm{id}_{\mathcal{M}_d}, && 1\leq i,j\leq 2,\\
\eta(x_{ij}) &= A_{ij}, && 1\leq i,j\leq 2,\\
\eta(x_{ij}^*) &= -A_{ji}, && 1\leq i,j\leq 2,\\
\psi(x_{ij}) &= ib_{ij} - \frac{1}{2}\sum_{k=1}^{2}\langle A_{kj}, A_{ki}\rangle, && 1\leq i,j\leq 2,\\
\psi(x_{ij}^*) &= -ib_{ji} - \frac{1}{2}\sum_{k=1}^{2}\langle A_{ki}, A_{kj}\rangle, && 1\leq i,j\leq 2,
\end{aligned}$$

where $\langle A, A'\rangle = \sum \overline{a_{ij}} a'_{ij}$ for $A = (a_{ij})$, $A' = (a'_{ij}) \in \mathcal{M}_d$ is a scalar product on $\mathcal{M}_d$. These maps extend to a unique triple on $\mathrm{U}\langle 2\rangle$ in the sense of Definition 2.3 of [5]. Actually, this is the form of a general Gaussian triple on $\mathrm{U}\langle 2\rangle$, cf. [11], Section 5 and [5]. The restrictions of ρ, η and ψ to $\mathcal{B}$ define a triple on $\mathcal{B}$ and therefore the quantum differential equation

$$\mathrm{d}j_{st} = j_{st} \star \mathrm{d}I_t \ ; \quad j_{ss} = \delta$$

yields a basic Lévy process on $\mathcal{B}$.
It is instructive to compare this process with the process $\tilde{\jmath}$ on $\mathrm{U}\langle 2\rangle$ obtained by solving the quantum stochastic equation

$$\mathrm{d}\tilde{\jmath}_{st} = \tilde{\jmath}_{st} \star_1 \mathrm{d}I_t \ ; \quad \tilde{\jmath}_{ss} = \delta_1$$

where $\star_1$ denotes the convolution w.r.t. the coproduct Δ_1 of $\mathrm{U}\langle 2\rangle$. Even though the differentials appearing in these two quantum differential equations coincide, in general they are different because they come from different coproducts. Therefore one expects the processes to be different, too. This is the case. It can be checked by computing the expectation values or by verifying that j_{st} is not a $*$-homomorphism (wheras $\tilde{\jmath}$ is a Lévy process and therefore always a $*$-homomorphism).

Let us study the first few moments of $\tilde{\jmath}_{0t}$: We have

$$\begin{aligned}
\psi(x_{ij}x_{kl}) &= -\langle A_{ji}, A_{kl}\rangle + \delta_{ij}\psi(x_{kl}) + \delta_{kl}\psi(x_{ij}),\\
\psi(x^*_{ij}x_{kl}) &= \langle A_{ij}, A_{kl}\rangle + \delta_{ij}\psi(x_{kl}) + \delta_{kl}\overline{\psi(x_{ij})},\\
\psi(x_{ij}x^*_{kl}) &= \langle A_{ji}, A_{lk}\rangle + \delta_{ij}\overline{\psi(x_{kl})} + \delta_{kl}\psi(x_{ij}),\\
\psi(x^*_{ij}x^*_{kl}) &= -\langle A_{ij}, A_{lk}\rangle + \delta_{ij}\overline{\psi(x_{kl})} + \delta_{kl}\overline{\psi(x_{ij})}
\end{aligned}$$

for the values of ψ on products of the generators. In particular, we have $\psi(x_{11}x^*_{11}) = ||A_{11}||^2 + ib_{11} - \frac{||A_{11}||^2 - ||A_{21}||^2}{2} - ib_{11} - \frac{||A_{11}||^2 - ||A_{21}||^2}{2} = -||A_{21}||^2$, $\psi(x_{22}x^*_{22}) = -||A_{12}||^2$.

Due to the form of the coproduct Δ_1 on $\mathrm{U}\langle 2\rangle$, we get

$$\begin{aligned}
E(\tilde{\jmath}_{0t}(x_{ij})) &= \left(e^{t(\psi(x_{kl}))_{1\le k,l\le 2}}\right)_{ij},\\
E(\tilde{\jmath}_{0t}(x^*_{ij})) &= \left(e^{t\left(\overline{\psi(x_{kl})}\right)_{1\le k,l\le 2}}\right)_{ij},
\end{aligned}$$

and similar formulae for the second-order elements, i.e. write $\psi(x_{ij}^{(\epsilon)}x_{kl}^{(\epsilon)})$ as a matrix, with the elements ordered in the following way,

$$\psi\begin{pmatrix} x_{11}^{(\epsilon)}x_{11}^{(\epsilon)} & x_{11}^{(\epsilon)}x_{12}^{(\epsilon)} & x_{12}^{(\epsilon)}x_{11}^{(\epsilon)} & x_{12}^{(\epsilon)}x_{12}^{(\epsilon)} \\ x_{11}^{(\epsilon)}x_{21}^{(\epsilon)} & x_{11}^{(\epsilon)}x_{22}^{(\epsilon)} & x_{12}^{(\epsilon)}x_{21}^{(\epsilon)} & x_{12}^{(\epsilon)}x_{22}^{(\epsilon)} \\ x_{21}^{(\epsilon)}x_{11}^{(\epsilon)} & x_{21}^{(\epsilon)}x_{12}^{(\epsilon)} & x_{22}^{(\epsilon)}x_{11}^{(\epsilon)} & x_{22}^{(\epsilon)}x_{12}^{(\epsilon)} \\ x_{21}^{(\epsilon)}x_{21}^{(\epsilon)} & x_{21}^{(\epsilon)}x_{22}^{(\epsilon)} & x_{22}^{(\epsilon)}x_{21}^{(\epsilon)} & x_{22}^{(\epsilon)}x_{22}^{(\epsilon)} \end{pmatrix}$$

and then exponentiate this matrix.
For the moments of j_{0t} we get

$$\begin{aligned} E(j_{0t}(x_{22})) &= e^{t\psi(x_{22})} = \exp\left(itb_{22} - t\frac{||A_{12}||^2 + ||A_{22}||^2}{2}\right), \\ E(j_{0t}(x_{22}^*)) &= e^{t\overline{\psi(x_{22})}} = \exp\left(-itb_{22} - t\frac{||A_{12}||^2 + ||A_{22}||^2}{2}\right), \end{aligned}$$

and

$$\begin{aligned} E(j_{0t}(x_{22}x_{22})) &= e^{t\psi(x_{22}x_{22})} = \exp\left(2itb_{22} - t||A_{12}||^2 - 2t||A_{22}||^2\right), \\ E(j_{0t}(x_{22}^*x_{22}^*)) &= e^{t\overline{\psi(x_{22}x_{22}^*)}} = \exp\left(-2itb_{22} - t||A_{12}||^2 - 2t||A_{22}||^2\right). \end{aligned}$$

We have

$$\begin{aligned} \Delta x_{11}x_{11}^* &= x_{11}x_{11}^* \otimes x_{11}x_{11}^* + (1 - x_{11}x_{11}^*) \otimes (1 - x_{22}x_{22}^*), \\ \Delta x_{22}x_{22}^* &= x_{22}x_{22}^* \otimes x_{22}x_{22}^* + (1 - x_{22}x_{22}^*) \otimes (1 - x_{11}x_{11}^*), \end{aligned}$$

and therefore

$$\begin{aligned} \dot{\varphi}_1 &= \varphi_1\psi(x_{11}x_{11}^*) - (1 - \varphi_1)\psi(x_{22}x_{22}^*), \\ \dot{\varphi}_2 &= \varphi_2\psi(x_{22}x_{22}^*) - (1 - \varphi_2)\psi(x_{11}x_{11}^*), \end{aligned}$$

for $\varphi_i(t) = E(j_{0t}(x_{ii}x_{ii}^*))$, $i = 1, 2$. Since $\varphi_i(0) = \delta(x_{ii}x_{ii}^*) = 1$, we get

$$\begin{aligned} E(j_{0t}(x_{11}x_{11}^*)) &= \varphi_1(t) = \frac{\psi(x_{11}x_{11}^*)e^{t(\psi(x_{11}x_{11}^*)+\psi(x_{22}x_{22}^*))} + \psi(x_{22}x_{22}^*)}{\psi(x_{11}x_{11}^*) + \psi(x_{22}x_{22}^*)}, \\ E(j_{0t}(x_{22}x_{22}^*)) &= \varphi_2(t) = \frac{\psi(x_{22}x_{22}^*)e^{t(\psi(x_{11}x_{11}^*)+\psi(x_{22}x_{22}^*))} + \psi(x_{11}x_{11}^*)}{\psi(x_{11}x_{11}^*) + \psi(x_{22}x_{22}^*)}, \end{aligned}$$

6. Quantum stochastic differential equations

On U⟨2⟩ we have

$$\mathrm{d}\tilde{j}_{st}(x_{ij}) = \sum_{k=1}^{2} \tilde{j}_{st}(x_{ik})\mathrm{d}I_t(x_{kj}),$$

where

$$I_t(x_{kj}) = A_t^*(A_{kj}) - A_t(A_{jk}) + \psi(x_{kj})t$$

On $\mathcal{B}$ we have, e.g.,

$$\mathrm{d}j_{st}(x_{22}) = j_{st}(x_{22})\mathrm{d}I_t(x_{22}), \qquad \mathrm{d}j_{st}(x_{22}^*) = j_{st}(x_{22}^*)\mathrm{d}I_t(x_{22}^*),$$

and

$$\begin{aligned} \mathrm{d}j_{st}(x_{22}x_{22}^*) &= j_{st}(x_{22}x_{22}^*)\mathrm{d}I_t(x_{22}x_{22}^*) + (j_{st}(x_{22}x_{22}^*) - \mathrm{id})\mathrm{d}I_t(x_{11}x_{11}^*) \\ &= j_{st}(x_{22}x_{22}^*)(\mathrm{d}I_t(x_{22}x_{22}^*) + \mathrm{d}I_t(x_{11}x_{11}^*)) - \mathrm{d}I_t(x_{11}x_{11}^*). \quad (4) \end{aligned}$$

For $j_{st}(x_{22})j_{st}(x_{22}^*)$, on the other hand, we get

$$\begin{aligned} \mathrm{d}(j_{st}(x_{22})j_{st}(x_{22}^*)) &= j_{st}(x_{22})\mathrm{d}j_{st}(x_{22}^*) + \mathrm{d}j_{st}(x_{22})j_{st}(x_{22}^*) + \mathrm{d}j_{st}(x_{22}) \bullet \mathrm{d}j_{st}(x_{22}^*) \\ &= j_{st}(x_{22})j_{st}(x_{22}^*)\mathrm{d}I_t(x_{22}^*) + j_{st}(x_{22})j_{st}(x_{22}^*)\mathrm{d}I_t(x_{22}) \\ &\quad + j_{st}(x_{22})j_{st}(x_{22}^*)\mathrm{d}I_t(x_{22}) \bullet \mathrm{d}I_t(x_{22}^*) \\ &= j_{st}(x_{22})j_{st}(x_{22}^*)(\mathrm{d}I_t(x_{22}^*) + \mathrm{d}I_t(x_{22}) + \mathrm{d}I_t(x_{22}) \bullet \mathrm{d}I_t(x_{22}^*)) \end{aligned}$$

But since $\mathrm{d}I_t$ is a $*$-homomorphism on ker δ_1 and $\mathrm{d}I_t(1) = 0$, we get

$$\begin{aligned} & \mathrm{d}I_t(x_{22}^*) + \mathrm{d}I_t(x_{22}) + \mathrm{d}I_t(x_{22}) \bullet \mathrm{d}I_t(x_{22}^*) \\ = \; & \mathrm{d}I_t(x_{22}^* - 1) + \mathrm{d}I_t(x_{22} - 1) + \mathrm{d}I_t(x_{22} - 1) \bullet \mathrm{d}I_t(x_{22}^* - 1) \\ = \; & \mathrm{d}I_t(x_{22} - 1 + x_{22}^* - 1 + (x_{22} - 1)(x_{22}^* - 1)) \\ = \; & \mathrm{d}I_t(x_{22}x_{22}^* - 1) = \mathrm{d}I_t(x_{22}x_{22}^*), \end{aligned}$$

and therefore

$$\mathrm{d}(j_{st}(x_{22})j_{st}(x_{22}^*)) = j_{st}(x_{22})j_{st}(x_{22}^*)\mathrm{d}I_t(x_{22}x_{22}^*) \quad (5)$$

We see that the quantum stochastic differential equations (4) and (5) differ if $dI_t(x_{11}x_{11}^*) \neq 0$, and therefore we get

$$j_{st}(x_{22})j_{st}(x_{22}^*) \neq j_{st}(x_{22}x_{22}^*)$$

in that case, i.e., j_{st} is not a homomorphism. Note that $I_t(x_{11}x_{11}^*)$ and $I_t(x_{22}x_{22}^*)$ are of the form

$$\begin{aligned} I_t(x_{11}x_{11}^*) &= -2A_t^*(A_{11}) - 2A_t(A_{11}) - ||A_{21}||^2 t, \\ I_t(x_{22}x_{22}^*) &= -2A_t^*(A_{22}) - 2A_t(A_{22}) - ||A_{12}||^2 t \end{aligned}$$

since $\psi(x_{11}x_{11}^*) = -||A_{21}||^2$, $\psi(x_{22}x_{22}^*) = -||A_{12}||^2$ and

$$\eta(x_{ii}x_{ii}^*) = \rho(x_{ii})\eta(x_{ii}^*) - \eta(x_{ii})\delta_1(x_{ii}^*) = -2A_{ii},$$

$i = 1, 2$, so that it is not difficult to give the explicite solutions of (4) and (5).

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