Samples of algebraic central limit theorems based on $Z/2Z$

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1. Introduction

Random walks associated with subgroups of an infinitly many free product G of $\mathbf{Z}/2\mathbf{Z}$ bring us various samples of algebraic central limit theorems. Let $\mathbf{F}_i, \sigma_i$ be a copy of $\mathbf{Z}/2\mathbf{Z}$ and its generator σ. Taking the left regular representation of G on $l^2(G)$, a pair $(\mathcal{A}, \phi)$ of a group $*$-algebra $\mathcal{A}$ of G and a tracial state $\phi(\cdot) := \langle \cdot \delta_e, \delta_e \rangle$ is considered an algebraic probability space, where δ_e is a characteristic function of the unit e of G.

It is well-known fact that the limit distribution under ϕ associated with a discrete Laplacian

$$\frac{\sigma_1 + \sigma_2 + \cdots + \sigma_N}{\sqrt{N}}$$

converges to the Wigner semi-circle law $\frac{1}{2\pi}\chi_{[-2,2]}\sqrt{4-x^2}dx$, of which limit process has a free Fock representation

$$\lim_{N\to\infty} \phi\left((\frac{\sigma_1 + \sigma_2 + \cdots + \sigma_N}{\sqrt{N}})^m \right) = \langle (A^\dagger + A)^m \mathbf{1}, \mathbf{1} \rangle,$$

where $A^\dagger$ and A are canonical creation and annihilation operators actiong on an 1-mode free Fock space $\Gamma(\mathbf{C})$ with a cyclic element $\mathbf{1}$.

Let us take a sequence $\{w_{ij} := \sigma_i \sigma_j \mid i \neq j\}$. The assymptotic behavior of a Laplacian

$$\frac{1}{\sqrt{N(N-1)}} \sum_{1 \le i \ne j \le N} w_{ij}$$

under ϕ is grasped as a special case $\lambda = 1$ of a Fock representation

$$\langle (A^\dagger + A + \lambda P)^m \mathbf{1}, \mathbf{1} \rangle,$$

where P is a projection orthogonal to the vacuum $\mathbf{1}$, that coinsides with a representation obtained in the studies of Haagerup state [14] and [2], [3] where the concept of the singleton independence was investigated. Starting with a partial sum

$$S_2(\gamma, N) := \frac{1}{\sqrt{v}} \sum_{\substack{1 \le i < j \le N \\ i \le \max\{\gamma N, 1\}}} (w_{ij} + w_{ji})$$

where v is a constant so that $\phi(S_2(\gamma, N)) = 1$, the limit process has a representation, for instance, if γ equals to a constant $0 \le \alpha \le 1$,

$$\lim_{N\to\infty} \phi\Big(S_2(\gamma,N)^m\Big) = \left\langle \left(\sqrt{\frac{\alpha}{2-\alpha}}(A^\dagger + A + P) + \sqrt{\frac{1-\alpha}{2-\alpha}}(X^\dagger + X + Y^\dagger + Y + Q + R) \right)^m \mathbf{1}, \mathbf{1} \right\rangle$$

on a 4-mode Fock space Γ, a free product of four 1-mode Fock spaces, where $A, A^\dagger, X, X^\dagger, Y, Y^\dagger$ are canonical creations and annihilations and P, Q, R are projections orthogonal to $\mathbf{1}$ with certain mutual relations (section 4).

Considering sequences such as $\{w_{ijk} = \sigma_i \sigma_j \sigma_k \mid i, j, k : \text{diffrent each other}\}$ drives us into another generalization. The asymptotic behavior of

$$\frac{1}{\sqrt{N(N-1)(N-2)}} \sum_{\substack{1 \le i,j,k \le N \\ i,j,k\,:\,\text{different each other}}} w_{ijk}$$

has a representation

$$\langle ((A^\dagger)^3 + B^\dagger + B + A^3)^m \mathbf{1}, \mathbf{1} \rangle$$

on a 1-mode Fock space, where $A^\dagger$ and A are canonical creation and annihilation operators, $B^\dagger$ and B are 'conditional' creation and annihilation ones, which kill the vacuum $\mathbf{1}$, acting on the subspace orthogonal to $\mathbf{1}$ where $A^\dagger = B^\dagger$ and $A = B$ hold. (The term 'conditional' is borrowed from the significant paper [7].)

[†]This work is supported by JSPS Research Fellow.

Throughout this study, the lattice path counting works effectively, which gives exact solutions to moment problems associated with some of these limit processes, with the help of the reflection method (e.g. [16]) and residue calculi. In the case of the last sample, a residue

$$f(t) := Res_{z=0} \frac{1-z^6}{(1-t(z^3+z+\frac{1}{z}+\frac{1}{z^3}))z}$$

gives the moment generating function $F(t) = 1/(1-t^2 f(t))$.

The aim of this study is to collect samples of algebraic central limit theorems for detecting new concepts of independences in the sense of the algebraic probability theory, in a category of 'non-free' algebra. Such researchs on relations between the independences and algebraic relations will bring us interpolative concepts from the classical independence to the free independence. It is an important work to interpret these samples in terms of interacting Fock spaces [1], giving us a united understanding of algebraic central limit theorems.

2. The Wigner semi-circle law on $*\mathbf{Z}/2\mathbf{Z}$

Let F_i and σ_i be a copy of $\mathbf{Z}/2\mathbf{Z}$ and its generator respectively. Taking the left regular representation π of $G = *F_i$, an infinitly many product of F_i's, a pair $(\mathcal{A}, \phi)$ of a group $*$-algebra $\mathcal{A}$ of G and a tracial state $\phi(\cdot) := \langle \cdot\delta_e, \delta_e\rangle$ is considered an algebraic probability space, where δ_e is a characteristic function of the unit e of G.

To obtain the algebraic central limit theorem with respect to freely independent elements σ_i's,

$$S_1(N) := \frac{\sigma_1+\sigma_2+\cdots+\sigma_N}{\sqrt{N}},$$

let us observe the action of each terms $\pi(\sigma_{i_1})\pi(\sigma_{i_2})\cdots\pi(\sigma_{i_m})/(\sqrt{N})^m$ on δ_e, in an expansion of

$$\left(\frac{\pi(\sigma_1)+\pi(\sigma_2)+\cdots+\pi(\sigma_N)}{\sqrt{N}}\right)^m$$

(abbreviate π, the rest). Since σ_i's are algebraic free, only the terms with the subindices forming a non-crossing pair partition survive in the limit $N \to \infty$. For a term $\sigma_{i_1}\cdots\sigma_{i_m}$, the rule

$$\begin{array}{llll} \sigma_{i_m} & \longleftrightarrow & \nwarrow, & \\ \sigma_{i_k} & \longleftrightarrow & \nwarrow, & \text{if } |\sigma_{i_k}\sigma_{i_{k+1}}\cdots\sigma_{i_m}| > |\sigma_{i_{k+1}}\cdots\sigma_{i_m}| \quad \text{and} \\ \sigma_{i_k} & \longleftrightarrow & \swarrow, & \text{if } |\sigma_{i_k}\sigma_{i_{k+1}}\cdots\sigma_{i_m}| < |\sigma_{i_{k+1}}\cdots\sigma_{i_m}|, \end{array}$$

gives a correspondence of the terms $\sigma_{i_1}\cdots\sigma_{i_m}$ to sequences $\swarrow \cdots \nwarrow$ of up-down arrows, where $|\sigma_{i_1}\cdots\sigma_{i_m}|$ denotes the reduced length of the product. Such a sequence $\epsilon_1\cdots\epsilon_m$ of arrows $\epsilon_i = \nwarrow$ or $\swarrow$ satisfies

$$\begin{array}{ll} \#\{i \mid \epsilon_i = \nwarrow, k \le i \le m\} \ge \#\{i \mid \epsilon_i = \swarrow, k \le i \le m\}, & \text{for } k > 1 \quad \text{and} \\ \#\{i \mid \epsilon_i = \nwarrow, 1 \le i \le m\} = \#\{i \mid \epsilon_i = \swarrow, 1 \le i \le m\}, & \end{array}$$

which is called *a sequence of Catalan type* here. $\eta_1(\epsilon_1\cdots\epsilon_m)$ denotes *the height* of $\epsilon_1\cdots\epsilon_m$ defined as $\eta_1(\epsilon_1\cdots\epsilon_m) = \eta_1(\epsilon_1)+\cdots+\eta_1(\epsilon_m)$ where $\eta_1(\nwarrow) = +1$ and $\eta_1(\swarrow) = -1$. Then, a sequence $\epsilon_1\cdots\epsilon_m$ is of Catalan type if and only if $\eta_1(\epsilon_k\cdots\epsilon_m) \ge 0$ $(k > 1)$ and $\eta_1(\epsilon_1\cdots\epsilon_m) = 0$ hold. The number of terms of corresponding to a sequence $\epsilon_1\cdots\epsilon_m$ of Catalan type is

$$N(N-1)\cdots(N-\frac{m}{2}+1);$$

of order $O((\sqrt{N})^m)$, allowing an expression

$$M_m := \lim_{N\to\infty} \phi\left(\left(\frac{\sigma_1+\sigma_2+\cdots+\sigma_N}{\sqrt{N}}\right)^m\right) = \#\{\text{sequence } \epsilon_1\cdots\epsilon_m \text{ of up-down arrows of Catalan type}\}.$$

Taking $\nwarrow$ for a creation and $\swarrow$ for an annihilation, the right hand side coinsides with a Fock representation

$$\langle (A^\dagger + A)^m \mathbf{1}, \mathbf{1}\rangle,$$

where $A^\dagger$ and A are canonical creation and annihilation operators respectively actiong on an 1-mode free Fock space $\Gamma(\mathbf{C})$ with a cyclic element $\mathbf{1}$.

A sequence $\epsilon_1\cdots\epsilon_{2m}$ of up-down arrows of Catalan type corresponds to *a Catalan path*: a minimal path on a lattice $\mathbf{Z}^2$ from $(0,0)$ to (m,m) laying under the diagonal line $y = x+1$. The reflection method (*cf.* [16][22]) shows that the number of Catalan paths with length $2m$ equals to

$$\#\{\text{ minimal path from } (0,0) \text{ to } (m,m)\ \} - \#\{\text{ minimal path from } (-1,1) \text{ to } (m,m)\ \},$$

which is eqivalent to

$$\begin{aligned}
&[z^0]\left(z+\frac{1}{z}\right)^m - [z^2]\left(z+\frac{1}{z}\right)^m \\
&= [z^0]\left(z+\frac{1}{z}\right)^m - [z^{-2}]\left(z+\frac{1}{z}\right)^m \\
&= \text{ constant term in } (1-z^2)\left(z+\frac{1}{z}\right)^m \\
&= Res_{z=0}\left\{\left(\frac{1-z^2}{z}\right)\left(z+\frac{1}{z}\right)^m\right\},
\end{aligned}$$

where $[z^k]f(z)$ denotes a coefficient of z^k in a Laurent series $f(z)$. Then a residue calculus gives the moment generating function

$$\begin{aligned}
f(t) &= \sum_{m=0}^{\infty} M_m t^m \\
&= Res_{z=0}\frac{1-z^2}{\left(1-t(z+\frac{1}{z})\right)z} \\
&= \frac{1-\sqrt{1-4t^2}}{2t^2}.
\end{aligned}$$

As the Cauchy transform of the limit distribution μ associated with $S_1(N)$ equals to

$$\frac{1}{t}f\left(\frac{1}{t}\right) = \frac{t-\sqrt{t^2-4}}{2},$$

the Stieltjes inversion formula (*cf.*[5]) yields the Wigner law

$$d\mu = \frac{1}{2\pi}\chi_{[-2,2]}\sqrt{4-x^2}dx.$$

3. Folding of free elements I

Let us consider elements $w_{ij} := \sigma_i\sigma_j$ $(i \neq j)$, which are not free each other. A noticeable difference from the previous section is that, in some cases, a muliplication by w_{ij} fixes the reduced length of a product, e.g., $|w_{12}w_{23}| = |\sigma_1\sigma_3| = 2 = |w_{23}|$. Thus, an observation of the action of a product $w_{i_1j_1}\cdots w_{i_mj_m}$ on δ_e allows a correspondens of such a product to a sequence of symbols $\nwarrow$, $\swarrow$ and $\smile$ by way of the rule

$$\begin{aligned}
&w_{i_mj_m} \longleftrightarrow \nwarrow, \\
&w_{i_kj_k} \longleftrightarrow \nwarrow, \quad \text{if } |w_{i_kj_k}w_{i_{k+1}j_{k+1}}\cdots w_{i_mj_m}| > |w_{i_{k+1}j_{k+1}}\cdots w_{i_mj_m}|, \\
&w_{i_kj_k} \longleftrightarrow \smile, \quad \text{if } |w_{i_kj_k}w_{i_{k+1}j_{k+1}}\cdots w_{i_mj_m}| = |w_{i_{k+1}j_{k+1}}\cdots w_{i_mj_m}|, \quad \text{and} \\
&w_{i_kj_k} \longleftrightarrow \swarrow, \quad \text{if } |w_{i_kj_k}w_{i_{k+1}j_{k+1}}\cdots w_{i_mj_m}| < |w_{i_{k+1}j_{k+1}}\cdots w_{i_mj_m}|.
\end{aligned}$$

By definition, for a product $w_{i_1j_1}\cdots w_{i_mj_m}$,

$$\phi(w_{i_1j_1}\cdots w_{i_mj_m}) = 1$$

holds provided that the sequence $i_1j_1\cdots i_mj_m$ of subindices forms a non-crossing pair partition with $i_k \neq j_k$ $(k = 1,\ldots,m)$, and as seen in the previous section, only such products survive in the limit $N \to \infty$. Those products correspond to sequences $\epsilon_1\cdots\epsilon_m$ of symbols $\nwarrow$, $\swarrow$ and $\smile$ of *Catalan type with inner singletons* [2]:

Definition 3.1. A sequence $\epsilon_1\cdots\epsilon_m$ of symbols $\nwarrow$, $\swarrow$ and $\smile$ is called *Catalan type with inner singletons* provided that

(i) the rest sequence $\epsilon_{i_1}\cdots\epsilon_{i_k}$ removed all $\smile$'s from $\epsilon_1\cdots\epsilon_m$ is of Catalan type.

(ii) $\eta_2(\epsilon_{k+1}\cdots\epsilon_m) > 0$ holds if $\epsilon_k = \smile$, where $\eta_2(\epsilon_1\cdots\epsilon_m)$ denotes *the height* of $\epsilon_1\cdots\epsilon_m$ defined as $\eta_2(\epsilon_1\cdots\epsilon_m) = \eta_2(\epsilon_1)+\cdots+\eta_2(\epsilon_m)$, $\eta_2(\nwarrow) = +2$, $\eta_2(\swarrow) = -2$ and $\eta_2(\smile) = 0$. $\smile$ is called *an inner singleton* here.

Since the number of terms in an expansion of

$$S_2(N)^m := \left(\frac{1}{\sqrt{N(N-1)}}\sum_{1\leq i\neq j\leq N} w_{ij}\right)^m$$

corresponding to the same sequence $\epsilon_1 \cdots \epsilon_m$ of Catalan type with inner singletons, which is equivalent to nothing but the number of sequences $i_1 j_1 \cdots i_m j_m$ of subindices forming non-crossing pair partitions with $i_k \neq j_k$ $(k = 1, \ldots, m)$, equals to

$$m! \binom{N}{m} = O(N^m),$$

the m-th moment has an expression

$$\lim_{N\to\infty} \phi(S_2(N)^m) = \#\{\text{sequence } \epsilon_1 \cdots \epsilon_m \text{ of Catalan type with inner singletons}\}.$$

$A^\dagger$, A and P denote a creation, an annihilation and a projection othogonal to the vacuum $\mathbf{1}$ respecticely, acting on an 1-mode free Fock space $\Gamma(\mathbf{C})$. Then, taking $\nwarrow$, $\swarrow$ and $\smile$ for $A^\dagger$, A and P respectively yields a Fock representation for assymptotic behavior of $S_2(N)$:

Theorem 3.2.

$$\lim_{N\to\infty} \phi\left(\left(\frac{1}{\sqrt{N(N-1)}} \sum_{1\leq i\neq j\leq N} w_{ij}\right)^m\right) = \left\langle (A^\dagger + A + P)^m \mathbf{1}, \mathbf{1}\right\rangle.$$

In the investigation of the Haagerup state [2], a general representation

$$\left\langle (A^\dagger + A + \lambda P)^m \mathbf{1}, \mathbf{1}\right\rangle$$

with a parameter λ. A description

$$\left\langle (A^\dagger + A + \lambda P)^m \mathbf{1}, \mathbf{1}\right\rangle = \sum_{k=0}^{m-2} \#\{\epsilon_1 \cdots \epsilon_m \ : \text{ of Catalan type with } k \text{ inner singletons }\} \cdot \lambda^k$$

is connected with a lattce path counting on $\mathbf{Z}^2$ by way of the rule

$$\begin{array}{lll} \nwarrow & \longleftrightarrow & \Omega_+ : (x,y) \to (x+1,y) \to (x+2,y), \\ \swarrow & \longleftrightarrow & \Omega_- : (x,y) \to (x,y+1) \to (x,y+2) \qquad \text{and} \\ \smile & \longleftrightarrow & \Omega_0 : (x,y) \to (x,y+1) \to (x+1,y+1). \end{array}$$

A sequence $\epsilon_1 \cdots \epsilon_m$ of Catalan type with inner singletons corresponds to a lattice path $\omega_1 \cdots \omega_m$ from $(0,0)$ to (m,m) which consist of moves Ω_+, Ω_- and Ω_0, walking under the line $y = x + 1$ without accrossing the diagonal $y = x$. Let l be the largest number that $\eta_2(\epsilon_l \cdots \epsilon_m) = 0$ holds, then by definition, $\epsilon_m = \nwarrow$, $\epsilon_l = \swarrow$ and $2 \leq l \leq m$. In the part $\epsilon_{l+1} \cdots \epsilon_{m-1}$, $\smile$'s occur with no restrictions: only **Definition 3.1** (i) holds, named *of Catalan type with singletons*. The corresponding path $\omega_{l+1} \cdots \omega_{m-1}$ lays under the line $y = x$ without accrossing the line $y = x - 1$, connecting $(2,0)$ with $(m - l + 1, m - l - 1)$. Putting

$$\begin{aligned} F_m &:= \sum_{k=0}^{m-2} \#\{\epsilon_1 \cdots \epsilon_m \ : \text{ of Catalan type with } k \text{ inner singletons }\} \cdot \lambda^k \qquad \text{and} \\ f_m &:= \sum_{k=0}^{m} \#\{\epsilon_1 \cdots \epsilon_m \ : \text{ of Catalan type with } k \text{ singletons }\} \cdot \lambda^k, \end{aligned}$$

the decomposition

$$\epsilon_1 \cdots \epsilon_m = \epsilon_1 \cdots \epsilon_{l-1} \cdot \swarrow \epsilon_{l+1} \cdots \epsilon_{m-1} \nwarrow$$

implies a recurrence formula

$$F_m = \sum_{l=0}^{m-2} F_{l-1} f_{m-l-1}, \tag{3.1}$$

which is nothing but a conditional moment-cumulant formula [7] with a cumulant $R_2(\swarrow, \nwarrow) = 1$. Since $\smile$'s have no restrictions in the sequence $\epsilon_1 \cdots \epsilon_m$ of Catalan type **with singletons**, it follows that

$$\begin{aligned} &\#\{\epsilon_1 \cdots \epsilon_m \ : \text{ of Catalan type with } k \text{ singletons }\} \\ &= \binom{m}{k} \#\{\epsilon_1 \cdots \epsilon_{m-k} \ : \text{ of Catalan type}\} \\ &= \binom{m}{k} \cdot \text{ constant term in } (1 - z^2)\left(z + \frac{1}{z}\right)^{m-k}. \end{aligned}$$

Hence

$$\begin{aligned} f_m &= \sum_{k=0}^{m} \binom{m}{k} \lambda^k \cdot \text{ constant term in } (1-z^2)\left(z+\frac{1}{z}\right)^{m-k} \\ &= \text{constant term in } (1-z^2)\left(z+\frac{1}{z}+\lambda\right)^m \\ &= Res_{z=0}\left\{\left(\frac{1-z^2}{z}\right)\left(z+\frac{1}{z}+\lambda\right)^m\right\}. \end{aligned}$$

Then the generating function

$$f(t) := \sum_{m=0}^{\infty} f_m t^m$$

is given by

$$\begin{aligned} f(t) &= \sum_{m=0}^{\infty} Res_{z=0}\left\{\left(\frac{1-z^2}{z}\right)\left(z+\frac{1}{z}+\lambda\right)^m\right\} t^m \\ &= Res_{z=0}\left\{\frac{1-z^2}{\left(1-t\left(z+\frac{1}{z}+\lambda\right)\right)z}\right\} \\ &= \frac{1-\lambda t-\sqrt{((\lambda+2)t-1)((\lambda-2)t-1)}}{2t^2}. \end{aligned}$$

In view of (3.1), the generating function

$$F(t) := \sum_{m=0}^{\infty} F_m t^m$$

has a functional equation

$$F(t)-1 = t^2 f(t) F(t),$$

and hence

$$F(t) = \frac{1+\lambda t-\sqrt{((\lambda+2)t-1)((\lambda-2)t-1)}}{2(\lambda+t)t}.$$

The Cauchy transform $G(t)$ of the distribution μ_λ associated with the operator $A^\dagger + A + \lambda P$ under the tracial state $\langle \cdot \mathbf{1}, \mathbf{1}\rangle$ is given by

$$\begin{aligned} G(t) &= \frac{1}{t} F\left(\frac{1}{t}\right) \\ &= \frac{t+\lambda-\sqrt{(\lambda+2-t)(\lambda-2-t)}}{2(1+\lambda t)}. \end{aligned} \tag{3.2}$$

Again, the Stieltjes inversion formula yields a non-symmetric deformation of the semi-circle law:

Theorem 3.3. The distribution μ_λ associated with the operator $A^\dagger + A + \lambda P$ under the tracial state $\langle \cdot \mathbf{1}, \mathbf{1}\rangle$ is given by

$$\mu_\lambda = \begin{cases} \tilde{\mu}_\lambda, & \lambda^2 \le 1, \\ \left(1-\frac{1}{\lambda^2}\right)\delta_{-1/\lambda} + \tilde{\mu}_\lambda, & \lambda^2 \ge 1, \end{cases}$$

where

$$d\tilde{\mu}_\lambda = \frac{1}{2\pi}\chi_{[\lambda-2,\lambda+2]}(x)\frac{\sqrt{(\lambda+2-x)(x-\lambda+2)}}{1+\lambda x}dx \tag{3.3}$$

for any $\lambda \in \mathbf{R}$.

Remark. In the study of Haagerup state [15], the same distribution (3.3) is obtained only for $-1 \le \lambda \le 0$. Moreover, a coordinate exchange

$$t = 1 + \lambda x \quad \text{and} \quad \beta = \lambda^2$$

give a connection with the free Poisson distribution (*cf.* [7])

$$\pi_{\beta,\beta} = \begin{cases} (1-\beta)\delta_0 + \tilde{\pi}_{\beta,\beta}, & 0 \le \beta \le 1, \\ \tilde{\pi}_{\beta,\beta}, & 1 \le \beta, \end{cases}$$

where

$$\begin{aligned} d\tilde{\pi}_{\beta,\beta} &= \frac{1}{2\pi}\chi_{[(1-\sqrt{\beta})^2,(1+\sqrt{\beta})^2]}(t)\frac{\sqrt{4\beta-(t-1-\beta)^2}}{t}dt \\ &= \lambda^2 d\tilde{\mu}_\lambda. \end{aligned}$$

According to a relation between the Cauchy transform of a distribution and its orthogonal polynomials (*cf.*[32]), a continued fractional expression

$$g(t) = \cfrac{1}{t - b_1 - \cfrac{c_2}{t - b_2 - \cfrac{c_3}{t - b_3 - \cdots}}}$$

of the Cauchy transform of a measure induces recurrence relations among its *monic* orthogonal polynomials $\{p_n(t)\}$,

$$\begin{aligned} &p_0(t) = 1, \qquad p_1(t) = t - b_1, \\ &p_n(t) = (t - b_n)p_{n-1}(t) - c_n p_{n-2}(t) \qquad (n \ge 2). \end{aligned}$$

In the case of $G(t)$ in (3.2), a direct calculation gives an unfavorable expression (*cf.*[7])

$$G(t) = \cfrac{1}{t + \lambda - \cfrac{1 + \lambda t}{t + \lambda - \cfrac{1 + \lambda t}{t + \lambda - \cdots}}},$$

however, a small trick removes the difficulty. Note that $G(t)$ is a solution of a quadratic equation in G,

$$(t + \lambda - (1 + \lambda t)G)G = 1. \tag{3.4}$$

Put $(1 + \lambda t)G(t) = \alpha g(t) + \beta$ where α and β are constants, and suppose that $g(t)$ is a solution of

$$(t - b - cg)g = 1 \tag{3.5}$$

which implies $g(t)$ has a suitable continued fractional expression

$$g(t) = \cfrac{1}{t - b - \cfrac{c}{t - b - \cfrac{c}{t - b - \cdots}}}.$$

Substitution of g into (3.4) and comparison with (3.5) give the solution

$$\alpha = 1, \quad \beta = \lambda, \quad b = \lambda \text{ and } c = 1,$$

hence

$$g(t) = \frac{1}{t - \lambda - g(t)} = \cfrac{1}{t - \lambda - \cfrac{1}{t - \lambda - \cfrac{1}{t - \lambda - \cdots}}},$$

$$G(t) = \frac{1}{t - g(t)} = \cfrac{1}{t - \cfrac{1}{t - \lambda - \cfrac{1}{t - \lambda - \cdots}}}.$$

Thus, the monic orthogonal polynomials associated with $d\mu_\lambda$ are determined by

$$\begin{aligned} &p_0(t) = 1, \qquad p_1(t) = t, \\ &p_n(t) = (t - \lambda)p_{n-1}(t) - p_{n-2}(t) \qquad (n \ge 2), \end{aligned}$$

with the Jacobi parameters [1]

$$\alpha_1 = 0, \qquad \alpha_n = \lambda \qquad (n \geq 2), \qquad \omega_n = 1 \qquad (n \geq 1), \tag{3.6}$$

which declares that **Theorem 3.2** gives nothing but an interacting Fock representation with the Jacobi parameters (3.6).

4. Folding of free elements II

Let us start with a partial sum of $S_2(N)$,

$$S_2(\gamma, N) := \frac{1}{\sqrt{v}} \sum_{\substack{1 \leq i < j \leq N \\ i \leq \max\{\gamma N, 1\}}} (w_{ij} + w_{ji})$$

where v denotes the variance $v = \gamma N((2-\gamma)N - 1)$ so that $\phi(S_2(\gamma, N)^2) = 1$. Contrast to the previous section, the asymmetricity on the subindices causes more rich phenomena, depending on the growth rate of γ to N. We observe the three cases:

(A) $\gamma N \equiv 1$,
(B) $\gamma N \to \infty$ and $\gamma \to 0$ as $N \to \infty$,
(C) γ equals to a constant $0 \leq \alpha \leq 1$.

A product $w_{i_1 j_1} \cdots w_{i_m j_m}$ is connected with a sequence $\epsilon_1 \cdots \epsilon_m$ of symbols ${}_{\circ}{\nwarrow}^{\bullet}$, ${}_{\bullet}{\nwarrow}^{\circ}$, ${}_{\bullet}{\nwarrow}^{\bullet}$, ${}^{\circ}{\smile}^{\bullet}$, ${}^{\bullet}{\smile}^{\circ}$, ${}^{\bullet}{\smile}^{\bullet}$, ${}^{\circ}{\swarrow}_{\bullet}$, ${}^{\bullet}{\swarrow}_{\circ}$ and ${}^{\bullet}{\swarrow}_{\bullet}$ by way of the following rule:

$$\begin{aligned} w_{i_m j_m} &\longleftrightarrow {}_{\bullet}{\nwarrow}^{\circ}, \text{ if } i_m \leq \gamma N < j_m, \\ w_{i_m j_m} &\longleftrightarrow {}_{\circ}{\nwarrow}^{\bullet}, \text{ if } j_m \leq \gamma N < i_m, \\ w_{i_m j_m} &\longleftrightarrow {}_{\bullet}{\nwarrow}^{\bullet}, \text{ if } i_m, j_m \leq \gamma N, \end{aligned}$$

in the case of $i_k \leq \gamma N < j_k$,

$$\begin{aligned} w_{i_k j_k} &\longleftrightarrow {}_{\bullet}{\nwarrow}^{\circ}, && \text{if } |w_{i_k j_k} w_{i_{k+1} j_{k+1}} \cdots w_{i_m j_m}| > |w_{i_{k+1} j_{k+1}} \cdots w_{i_m j_m}|, \\ w_{i_k j_k} &\longleftrightarrow {}^{\bullet}{\smile}^{\circ}, && \text{if } |w_{i_k j_k} w_{i_{k+1} j_{k+1}} \cdots w_{i_m j_m}| = |w_{i_{k+1} j_{k+1}} \cdots w_{i_m j_m}|, \qquad \text{and} \\ w_{i_k j_k} &\longleftrightarrow {}^{\bullet}{\swarrow}_{\circ}, && \text{if } |w_{i_k j_k} w_{i_{k+1} j_{k+1}} \cdots w_{i_m j_m}| < |w_{i_{k+1} j_{k+1}} \cdots w_{i_m j_m}|, \end{aligned}$$

in the case of $j_k \leq \gamma N < i_k$,

$$\begin{aligned} w_{i_k j_k} &\longleftrightarrow {}_{\circ}{\nwarrow}^{\bullet}, && \text{if } |w_{i_k j_k} w_{i_{k+1} j_{k+1}} \cdots w_{i_m j_m}| > |w_{i_{k+1} j_{k+1}} \cdots w_{i_m j_m}|, \\ w_{i_k j_k} &\longleftrightarrow {}^{\circ}{\smile}^{\bullet}, && \text{if } |w_{i_k j_k} w_{i_{k+1} j_{k+1}} \cdots w_{i_m j_m}| = |w_{i_{k+1} j_{k+1}} \cdots w_{i_m j_m}|, \qquad \text{and} \\ w_{i_k j_k} &\longleftrightarrow {}^{\circ}{\swarrow}_{\bullet}, && \text{if } |w_{i_k j_k} w_{i_{k+1} j_{k+1}} \cdots w_{i_m j_m}| < |w_{i_{k+1} j_{k+1}} \cdots w_{i_m j_m}|, \end{aligned}$$

in the case of $i_k, j_k \leq \gamma N$,

$$\begin{aligned} w_{i_k j_k} &\longleftrightarrow {}_{\bullet}{\nwarrow}^{\bullet}, && \text{if } |w_{i_k j_k} w_{i_{k+1} j_{k+1}} \cdots w_{i_m j_m}| > |w_{i_{k+1} j_{k+1}} \cdots w_{i_m j_m}|, \\ w_{i_k j_k} &\longleftrightarrow {}^{\bullet}{\smile}^{\bullet}, && \text{if } |w_{i_k j_k} w_{i_{k+1} j_{k+1}} \cdots w_{i_m j_m}| = |w_{i_{k+1} j_{k+1}} \cdots w_{i_m j_m}|, \qquad \text{and} \\ w_{i_k j_k} &\longleftrightarrow {}^{\bullet}{\swarrow}_{\bullet}, && \text{if } |w_{i_k j_k} w_{i_{k+1} j_{k+1}} \cdots w_{i_m j_m}| < |w_{i_{k+1} j_{k+1}} \cdots w_{i_m j_m}|. \end{aligned}$$

For instance, the product $w_{1a} w_{a2} w_{2b} w_{b1}$ $(a, b > \gamma N)$ corresponds to ${}^{\bullet}{\swarrow}_{\circ}\ {}^{\circ}{\smile}^{\bullet}\ {}^{\bullet}{\smile}^{\circ}\ {}_{\circ}{\nwarrow}^{\bullet}$.

Consider an action of the symbols ${}_{\circ}\!\nwarrow^{\bullet}$, ${}_{\bullet}\!\nwarrow^{\circ}$, ${}_{\bullet}\!\nwarrow^{\bullet}$, ${}^{\prime}\!\smile^{\circ}$, ${}^{\circ}\!\smile^{\bullet}$, ${}^{\bullet}\!\smile^{\bullet}$, ${}^{\circ}\!\swarrow_{\bullet}$, ${}^{\bullet}\!\swarrow_{\circ}$ and ${}^{\bullet}\!\swarrow_{\bullet}$ on a sequence $\kappa = \kappa_1 \cdots \kappa_m$ of o's and •'s given by

$$
{}_{\bullet}\!\nwarrow^{\circ}\,\kappa = \bullet\circ\kappa, \qquad {}_{\circ}\!\nwarrow^{\bullet}\,\kappa = \circ\bullet\kappa, \qquad {}_{\bullet}\!\nwarrow^{\bullet}\,\kappa = \bullet\bullet\kappa,
$$

$$
{}_{\bullet}\!\nwarrow^{\circ}\,\mathbf{1} = \bullet\circ, \qquad {}_{\circ}\!\nwarrow^{\bullet}\,\mathbf{1} = \circ\bullet, \qquad {}_{\bullet}\!\nwarrow^{\bullet}\,\mathbf{1} = \bullet\bullet, \qquad {}_{\bullet}\!\nwarrow^{\circ}\,\mathbf{0} = {}_{\circ}\!\nwarrow^{\bullet}\,\mathbf{0} = {}_{\bullet}\!\nwarrow^{\bullet}\,\mathbf{0} = \mathbf{0},
$$

$$
{}^{\prime}\!\smile^{\circ}\,\kappa = \begin{cases} \bullet\kappa_2\cdots\kappa_m, & \text{if } \kappa_1 = \circ, \\ \mathbf{0}, & \text{otherwise}, \end{cases} \qquad {}^{\bullet}\!\swarrow_{\circ}\,\kappa = \begin{cases} \kappa_3\cdots\kappa_m, & \text{if } \kappa_1\kappa_2 = \circ\bullet, \\ \mathbf{0}, & \text{otherwise}, \end{cases}
$$

$$
{}^{\circ}\!\smile^{\bullet}\,\kappa = \begin{cases} \circ\kappa_2\cdots\kappa_m, & \text{if } \kappa_1 = \bullet, \\ \mathbf{0}, & \text{otherwise}, \end{cases} \qquad {}^{\circ}\!\swarrow_{\bullet}\,\kappa = \begin{cases} \kappa_3\cdots\kappa_m, & \text{if } \kappa_1\kappa_2 = \bullet\circ, \\ \mathbf{0}, & \text{otherwise}, \end{cases}
$$

$$
{}^{\bullet}\!\smile^{\bullet}\,\kappa = \begin{cases} \kappa, & \text{if } \kappa_1 = \bullet, \\ \mathbf{0}, & \text{otherwise}, \end{cases} \qquad {}^{\bullet}\!\swarrow_{\bullet}\,\kappa = \begin{cases} \kappa_3\cdots\kappa_m, & \text{if } \kappa_1\kappa_2 = \bullet\bullet, \\ \mathbf{0}, & \text{otherwise}, \end{cases}
$$

where $\mathbf{0}$ is a fixed point of all symbols and $\mathbf{1}$ an initial point. The reduction rule among w_{ij}'s, such as $w_{1a}w_{a2} = \sigma_1\sigma_2$, is reflected faithfully in the above rule. The equation $w_{i_1j_1}\cdots w_{i_mj_m} = e$ corresponds to $\epsilon_1\cdots\epsilon_m\mathbf{1} = \mathbf{1}$ particularly. $\eta_2(\epsilon_1\cdots\epsilon_m)$ denotes the height of $\epsilon_1\cdots\epsilon_m$ given as the length of the sequence $\epsilon_1\cdots\epsilon_m\mathbf{1}$ of o's and •'s, putting the length of $\mathbf{1} = 0$ and that of $\mathbf{0} = -\infty$.

The action of the symbols produces a direct combinatorial expression on a free Fock space. Let $\Gamma = \Gamma(a,b,x,y)$ be a unital algebra over $\mathbf{C}$ freely generated by a,b,x,y with the unit $\mathbf{1}$, taken for a free product of four 1-mode Fock spaces, $\Gamma = \Gamma(\mathbf{C}a) * \Gamma(\mathbf{C}b) * \Gamma(\mathbf{C}x) * \Gamma(\mathbf{C}y)$, equipped with a canonical inner product. An interpretation

$$
\bullet\bullet \leftrightarrow a, \qquad \circ\circ \leftrightarrow b, \qquad \bullet\circ \leftrightarrow x, \qquad \circ\bullet \leftrightarrow y,
$$

induces operators $A^\dagger$,A,P,$X^\dagger$,X,$Y^\dagger$,Y,Q,R corresponding to ${}_{\bullet}\!\nwarrow^{\bullet}$, ${}^{\bullet}\!\swarrow_{\bullet}$, ${}^{\bullet}\!\smile^{\bullet}$, ${}_{\bullet}\!\nwarrow^{\circ}$, ${}^{\circ}\!\swarrow_{\bullet}$, ${}_{\circ}\!\nwarrow^{\bullet}$, ${}^{\bullet}\!\swarrow_{\circ}$, ${}^{\circ}\!\smile^{\bullet}$ and ${}^{\bullet}\!\smile^{\circ}$ respectively, acting on Γ, under the rule defined below: for $u \in \Gamma$,

$$
A^\dagger u = au, \qquad Au = \begin{cases} u', & \text{if } u = au', \quad u' \in \Gamma, \\ 0, & \text{otherwise,} \end{cases}
$$

$$
X^\dagger u = xu, \qquad Xu = \begin{cases} u', & \text{if } u = xu', \quad u' \in \Gamma, \\ 0, & \text{otherwise,} \end{cases}
$$

$$
Y^\dagger u = yu, \qquad Yu = \begin{cases} u', & \text{if } u = yu', \quad u' \in \Gamma, \\ 0, & \text{otherwise,} \end{cases}
$$

$$
\begin{array}{lllll}
Pau = au, & Pbu = 0, & Pxu = xu, & Pyu = 0, & P\mathbf{1} = 0, \\
Qxu = bu, & Qyu = 0, & Qau = yu, & Qbu = 0, & Q\mathbf{1} = 0, \\
Rxu = 0, & Ryu = au, & Rau = 0, & Rbu = xu, & R\mathbf{1} = 0.
\end{array}
$$

4.1. The case of (A): $\gamma N \equiv 1$.

Since a morphism $\omega_{1i} \to g_i$ (and then, $\omega_{i1} \to g_i^{-1}$) yields an isomorhism from the subgroup of $G = *\mathbf{Z}/2\mathbf{Z}$ generated by $\{w_{1i}\}$ to a group freely generated by $\{g_i\}$, $S_2(1/N,N)$ induces the free central limit theorem. A 1-mode Fock representation is given by

$$
\lim_{N\to\infty} \phi\Big(S_2\left(\frac{1}{N},N\right)^m\Big) = \big\langle (A^\dagger + A)^m \mathbf{1}, \mathbf{1}\big\rangle,
$$

4.2. The case of (B): $\gamma N > 1$ and $\gamma \to 0$ as $N \to \infty$.

An effect of folding free elements appears, however, the asymmetricity on the subindices causes a difference from the previous section. Consider a product $w_{xa}w_{ab}w_{bx} = e$ with $a,b \le \gamma N$ and $x \le N$. This type of products have no contribution to the limit distribution, as the number of such indices (a,b,x) has smaller order than $\sqrt{v}^3$. This observation shows that a product $w_{i_1j_1}\cdots w_{i_mj_m}$ containing a factor $w_{i_kj_k}$ with $i_k, j_k \le \gamma N$ has no contribution in the limit $N \to \infty$, exactly,

Lemma 4.1. For a equation $\epsilon_1\cdots\epsilon_m\mathbf{1} = \mathbf{1}$, let T_N be the number of products $w_{i_1j_1}\cdots w_{i_mj_m} = e$ of w_{ij}'s $(1 \le i \ne j \le N)$ corresponding to $\epsilon_1\cdots\epsilon_m$. Then,

$$
\lim_{N\to\infty} \frac{T_N}{(\sqrt{v})^m} = \begin{cases} 0, & \text{if } k > 0, \\ \left(\frac{1}{\sqrt{2}}\right)^m, & \text{if } k = 0, \end{cases}
$$

where k denotes the total number of ⟍'s, ◡'s and ⟋'s appear in $\epsilon_1 \cdots \epsilon_m$.

Proof. By definitions, the number of choice of subindices $i_s j_s$'s assymptotically equals to

$$(\gamma N)^{\frac{m}{2}}((1-\gamma)N)^{\frac{m-k}{2}}(\gamma N)^{\frac{k}{2}},$$

hence the assertion. □

As a result, a Fock representation on $\Gamma(a,b,x,y)$ is obtained.

Theorem 4.2. The assymptotic behavior of $S_2(\gamma, N)$ with $\gamma N > 1$ and $\gamma \to 0$ as $N \to \infty$ has a representation on the Fock space $\Gamma(a,b,x,y)$,

$$\lim_{N\to\infty} \phi\Big(S_2(\gamma,N)^m\Big) = \left\langle \left(\frac{1}{\sqrt{2}}(X^\dagger + X + Y^\dagger + Y + Q + R)\right)^m \mathbf{1}, \mathbf{1}\right\rangle.$$

Suppose that $\epsilon_1 \cdots \epsilon_m \mathbf{1} = \mathbf{1}$ holds. Like the innner singletons, ◡'s and ◡'s occur only at the height > 0, however, by definition, ◡ and ◡ should appear pairwise at the same height, which brings us another combinatorial description. Let us consider the Fock space $\Gamma(a,b,x,y)$ defined above. Putting $z = (x+y)/\sqrt{2}$ and $c = (a+b)/\sqrt{2}$, the action of $Z^\dagger = X^\dagger + Y^\dagger$, $Z = X + Y$ and $O = Q + R$ is given by

$$Z^\dagger u = \sqrt{2} z u, \ Zzu = \sqrt{2} u, \ Ozu = cu, \ Ocu = zu \qquad (u \in \Gamma(a,b,x,y)).$$

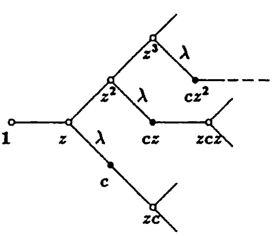
Hence we have

$$\left\langle \left(\frac{1}{\sqrt{2}}(X^\dagger + X + Y^\dagger + Y + Q + R)\right)^m \mathbf{1}, \mathbf{1}\right\rangle = \left\langle (Z^\dagger + Z + \frac{1}{\sqrt{2}} O)^m \mathbf{1}.\mathbf{1}\right\rangle$$

Let us consider more general situation

$$\langle (Z^\dagger + Z + \lambda O)^m \mathbf{1}, \mathbf{1}\rangle$$

with a parameter λ, which is connected with the weighted walks, starting the origin $\mathbf{1}$ and returning there after m-step, on an induced subgraph of the binary tree. (The weights are given in the figuer below.)



Let F_m be the number of m-step walks leaving and returning to $\mathbf{1}$, allowed reaching $\mathbf{1}$ several times in the middle of the walks. Samely let f_m be the number of m-step walks leaving and returning to z without reaching $\mathbf{1}$, allowed reaching z several times in the middle of the walks. By the self-similarity of the graph, one has for $m \geq 2$,

$$\begin{aligned} f_m &= \sum_{k=0}^{m-2} (f_k + \lambda F_k) f_{m-k-2}, \\ F_m &= \sum_{k=0}^{m-2} f_k F_{m-k-2}, \end{aligned}$$

where $f_0 = F_0 = 1$. Putting the moment functions, $F(t) = \sum_m F_m t^m$ and $f(t) = \sum_m f_m t^m$, one has

$$\begin{aligned} f(t) - 1 &= t^2(f(t) + \lambda F(t)) f(t), \\ F(t) - 1 &= t^2 F(t)^2. \end{aligned}$$

Hence

$$\lambda^2 t^2 F(t)^3 + (1-\lambda^2) t^2 F(t)^2 - F(t) + 1 = 0,$$

and the Cauchy transform $G(t)$ of the distribution $d\mu_\lambda$ associated with **Theorem 4.2** is given as a solution of

$$\lambda^2 tG(t)^3 + (1-\lambda^2)G(t)^2 - tG(t) + 1 = 0.$$

Remark. Putting $\lambda^2 = 1/2$, $d\mu_\lambda$ coinside with the distribution in *Examples 1.5* (1.16) and (1.17) of [23], up to the variance, where the anti-commutation $ab+ba$ of semi-circle elements a, b which are free each other is observed. Indeed what we have done in the case of (B) is a calculation of the anti-commutation of semi-circle elements. Intuitively, this is because, in the limit we have

$$S_2(\gamma, N) \sim \left(\frac{\sigma_1 + \cdots + \sigma_{\gamma N}}{\sqrt{\gamma N}}\right)\left(\frac{\sigma_{\gamma N+1} + \cdots + \sigma_N}{\sqrt{N}}\right) + \left(\frac{\sigma_{\gamma N+1} + \cdots + \sigma_N}{\sqrt{N}}\right)\left(\frac{\sigma_1 + \cdots + \sigma_{\gamma N}}{\sqrt{\gamma N}}\right),$$

which is noting but the anti-commutation of semi-circle elements that are free each other.

4.3. The case of (C): γ equals to a constant $0 \le \alpha \le 1$.

In this case, such a product $w_{xa}w_{ab}w_{by}$ with $a, b \le \gamma N$ and $\gamma N < x, y \le N$ contributes to the limit distribution; the symbols •↖, ⌣ and ↙• appear.

Lemma 4.3. For a equation $\epsilon_1 \cdots \epsilon_m \mathbf{1} = \mathbf{1}$, let T_N be the number of products $w_{i_1 j_1} \cdots w_{i_m j_m} = e$ of w_{ij}'s $(1 \le i \ne j \le N)$ corresponding to $\epsilon_1 \cdots \epsilon_m$. Then,

$$\lim_{N\to\infty} \frac{T_N}{(\sqrt{v})^m} = \left(\frac{\alpha}{2-\alpha}\right)^{\frac{k}{2}} \left(\frac{1-\alpha}{2-\alpha}\right)^{\frac{m-k}{2}}$$

where k denotes the total number of •↖'s, ⌣'s and ↙•'s appear in $\epsilon_1 \cdots \epsilon_m$.

Proof. Just repeat the proof of **Lemma 4.1** in the case of (C). □

Then, again a Fock representation on $\Gamma(a, b, x, y)$ is in hand, which interpolates the distributions in **Theorem 3.2** and **Theorem 4.2**.

Theorem 4.4. The assymptotic behavior of $S_2(\gamma, N)$ with γ =constant α $(0 \le \alpha \le 1)$ has a representation on the Fock space $\Gamma(a, b, x, y)$,

$$\lim_{N\to\infty} \phi\Big((S_2(\gamma, N))^m\Big) = \left\langle \left(\sqrt{\frac{\alpha}{2-\alpha}}(A^\dagger + A + P) + \sqrt{\frac{1-\alpha}{2-\alpha}}(X^\dagger + X + Y^\dagger + Y + Q + R)\right)^m \mathbf{1}, \mathbf{1} \right\rangle.$$

5. Multi-folding of free elements

In the previous sections, we saw that the double folding of free elements gives samples for conditionally free central limit theorems. However multi-folding of free elements suggests more general concept of independence. For instance, let us consider elements $w_{ijk} := \sigma_i\sigma_j\sigma_k$ $(i \ne j \ne k \ne i)$. Note that the difference of reduced length of $w_{i_1j_1k_1}w_{i_2j_2k_2}\cdots w_{i_mj_mk_m}$ and $w_{i_2j_2k_2}\cdots w_{i_mj_mk_m}$ equals to ± 3 or ± 1. Then, for a product $w_{i_1j_1k_1}\cdots w_{i_mj_mk_m}$, one associate a sequence of symbols $A^\dagger, A, B^\dagger, B$'s by way of the rule

$$\begin{aligned}
w_{i_mj_mk_m} &\longleftrightarrow A^\dagger, \\
w_{i_sj_sk_s} &\longleftrightarrow A^\dagger, \text{ if } |w_{i_sj_sk_s}w_{i_{s+1}j_{s+1}k_{s+1}}\cdots w_{i_mj_mk_m}| - |w_{i_{s+1}j_{s+1}k_{s+1}}\cdots w_{i_mj_mk_m}| = +3, \\
w_{i_sj_sk_s} &\longleftrightarrow B^\dagger, \text{ if } |w_{i_sj_sk_s}w_{i_{s+1}j_{s+1}k_{s+1}}\cdots w_{i_mj_mk_m}| - |w_{i_{s+1}j_{s+1}k_{s+1}}\cdots w_{i_mj_mk_m}| = +1, \\
w_{i_sj_sk_s} &\longleftrightarrow B, \text{ if } |w_{i_sj_sk_s}w_{i_{s+1}j_{s+1}k_{s+1}}\cdots w_{i_mj_mk_m}| - |w_{i_{s+1}j_{s+1}k_{s+1}}\cdots w_{i_mj_mk_m}| = -1, \text{ and} \\
w_{i_sj_sk_s} &\longleftrightarrow A, \text{ if } |w_{i_sj_sk_s}w_{i_{s+1}j_{s+1}k_{s+1}}\cdots w_{i_mj_mk_m}| - |w_{i_{s+1}j_{s+1}k_{s+1}}\cdots w_{i_mj_mk_m}| = -3.
\end{aligned}$$

Suppose that $w_{i_1j_1k_1}\cdots w_{i_mj_mk_m} = e$, that is the sequence of sub indices $i_1j_1k_1\cdots i_mj_mk_m$ forms a non crossing pair partition, which implies m is to be an even number. Let $\epsilon_1 \cdots \epsilon_m$ be the corresponding sequence of $A^\dagger, A, B^\dagger, B$ defined above. By definitions, such a sequence $\epsilon_1 \cdots \epsilon_m$ corresponds to a restricted Catalan path on $\mathbf{Z}^2$ from $(0,0)$ to $(3\lfloor m/2\rfloor, 3\lfloor m/2\rfloor)$ in the following way: each symbol ϵ_s is taken for a three step walk,

$$\begin{aligned}
A^\dagger &\longleftrightarrow \Omega_{+3} : (x,y) \to (x+1,y) \to (x+2,y) \to (x+3,y), \\
B^\dagger &\longleftrightarrow \Omega_{+1} : (x,y) \to (x,y+1) \to (x+1,y+1) \to (x+2,y+1), \\
B &\longleftrightarrow \Omega_{-1} : (x,y) \to (x,y+1) \to (x,y+2) \to (x+1,y+2) \text{ and} \\
A &\longleftrightarrow \Omega_{-3} : (x,y) \to (x,y+1) \to (x,y+2) \to (x,y+3),
\end{aligned}$$

and the corresponding lattice path consists of the walks $\Omega_{\pm 3}$ and $\Omega_{\pm 1}$, walking under the line $y = x + 1$ with out accrossing the diagonal $y = x$. Note that the walks Ω_{+1} and Ω_{-1} may start only from the trianguler areas under the line $y = x - 1$ and $y = x - 2$ respectively.

Let us observe the assymptotic behavior of

$$S_3(N) := \frac{1}{\sqrt{N(N-1)(N-2)}} \sum_{1\leq i\neq j\neq k\neq i\leq N} w_{ijk}.$$

From the argument above, it is easily seen that all odd moments vanish and the $2m$-th moment has an expression

$$\lim_{N\to\infty} \phi(S_3(N)^{2m}) = \#\{\text{Catalan path on } \mathbf{Z}^2 \text{ from } (0,0) \text{ to } (3m,3m) \text{ consisting of } \Omega_{\pm 3}, \Omega_{\pm 1}\}.$$

Summing up, we have an combinatorial description.

Theorem 5.1. Let $A^\dagger$ and A be canonical creation and annihilation operators on a 1-mode Fock space $\Gamma(\mathbf{C})$, and $B^\dagger$ and B be operators killing the vacuum $\mathbf{1}$, acting on the subspace orthogonal to $\mathbf{1}$ where $A^\dagger = B^\dagger$ and $A = B$ holds. Then the assymptotic behavior of $S_3(N)$ has a combinatorial description

$$\lim_{N\to\infty} \phi(S_3(N)^m) = \left\langle ((A^\dagger)^3 + B^\dagger + B + A^3)^m \mathbf{1}, \mathbf{1} \right\rangle_{\Gamma(\mathbf{C})}.$$

Remark. According to [7], Jacobi parameters associated with conditionally free central limit distributions are of the form

$$\omega_1 = p,\ \omega_n = q\ (n \geq 2),\ \alpha_n = 0\ (n \geq 0).$$

Contrast to the conditionally free case, above example has aperiodic Jacobi parameters,

$$\omega_1 = 1, \quad \omega_2 = 3,\ \omega_3 = 6,\ \omega_4 = 8/3,\ \omega_5 = 217/48,\ \ldots,$$
$$\alpha_n = 0 \quad (n \geq 0).$$

Acknowledgement. The auther expresses his gratitude to Professor R. Speicher for pointing out mistakes in the case of (B) of section 4, and for many valuable instructions.

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