Limit laws and semistability on infinite-dimensional locally compact groups

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The limit behaviour of automorphism-normalized products of independent random variables was investigated in the past and the possible limit laws, in particular stable and semistable laws are nowadays quite well understood, as long as the underlying group is a real or p-adic Lie group.

In fact, if the normalizing operators are localized on a *continuous* one-parameter group T then — without further restriction on the underlying group — the possible limit laws are concentrated on the contractible subgroup $C(T)$, in this case a closed Lie subgroup. But if the underlying group $\mathbb{G}$ is infinite-dimensional and if the normalizing automorphisms are not embedded into a continuous group then new (and unexpected) phenomena appear. There is still no general theory available but the stucture of possible limit laws can be investigated by a series of illustrative examples. As in the finite-dimensional setup the contractible subgroups $C(a)$ play an important role as the possible limit laws are concentrated on these subgroups.

The paper is organized as follows: It starts describing the role of contractible subgroups $C(a)$ showing that on metrizable locally compact groups semistable continuous convolution semigroups with trivial idempotent are representable as continuous injective homomorphic images of semistable continuous convolution semigroups on contractible completely metrizable topological groups. The investigation is continued with semistability on totally discontinuous groups including the p-adics as a detailed example.

Then, as particular examples of infinite-dimensional groups investigations of semistability on infinite products $K^{\mathbf{Z}}$ follow, including the shape of $C(a)$, marginal distributions and finally for Lie groups K, a comparision of Gaussian semistable limit laws on $K^{\mathbf{Z}}$ and on the corresponding (infinite-dimensional) Lie algebra. In fact, infinite products $\mathbb{G} = K^{\mathbf{Z}}$ of compact groups turn out to be of particular interest: The shift a defines an automorphism, a permutation of infinite order acting on the coordinates, and the existence of such automorphisms causes significant differences to the situation of finite products. We mention new features appearing in the situation $\mathbb{G} = K^{\mathbf{Z}}$:

- The intersection of the contractible parts $C(a) \cap C(a^{-1})$ is a dense subgroup.
- There exist (a, α)-semistable laws (for $\alpha \in (0,1)$) such that any projection to a finite product K^n is not semistable.

To simplify notations we shall troughout assume the underlying group $\mathbb{G}$ to be second-countable. We recall some well-known definitions. (See also [3], [14], [6], [7], [2]):

0.1. Definition. A continuous convolution semigroup $(\mu_t : t \geq 0)$ — in short $\mu_\bullet$ — is called (a, α)-*semistable* for $(a, \alpha) \in \mathrm{Aut}(\mathbb{G}) \times (0,1)$ if $a(\mu_t) = \mu_{\alpha t}, t \geq 0$.

$\mu_\bullet$ is *stable* w.r.t. a one-parameter group T iff $a_t(\mu_s) = \mu_{st}$ for $s, t > 0$, where $T = (a_t : t > 0) \subseteq \mathrm{Aut}(\mathbb{G})$ with multiplicative parametrization $a_t a_s = a_{t \cdot s}, t, s > 0$.

Note that in this definition of (semi-)stability local compactness of the underlying group is not necessary.

Continuous convolution semigroups in $\mathcal{M}^1(\mathbb{G})$ with idempotent $\mu_0 = \varepsilon_e$ are represented by *generating functionals* (cf. e.g. [9], [12]) defined on the test functions $\mathcal{D}(\mathbb{G})$

resp. on the regular functions $\mathcal{E}(\mathbb{G})$. Let $\mathcal{GF}(\mathbb{G})$ denote the cone of generating functionals.

Since (semi-)stability is closely related to the limit behaviour of automorphism-normalized convolution products we have to define domains of attraction:

0.2. Definition. $\mathrm{FDPA}(\mu_\bullet) := \{\nu \in \mathcal{M}^1(\mathbb{G}) : \exists (a_n) \subseteq \mathrm{Aut}(\mathbb{G}), k(n) \nearrow \infty,$ such that $a_n \nu^{[k(n)t]} \to \mu_t, t \geq 0\}$ (*domain of partial attraction*)
$\mathrm{FDSA}(\mu_\bullet) := \{\nu \in \mathrm{FDPA}(\mu_\bullet) : k(n)/k(n+1) \to \alpha \in (0,1)\}$ (*semi attraction*)
$\mathrm{FDA}(\mu_\bullet) := \{\nu \in \mathrm{FDPA}(\mu)\colon k(n) = n\}$ (*domain of attraction*).
If $a_n \in \{a^l : l \in \mathbb{Z}\}$ for some $a \in \mathrm{Aut}(\mathbb{G})$ (*normal attraction*) we use the notations $\mathrm{FDNPA}(\mu_\bullet)$ (if $a_n = a^{l(n)}, l(n) \nearrow \infty$), and $\mathrm{FDNSA}(\mu_\bullet)$ resp. $\mathrm{FDNA}(\mu_\bullet)$ (if $a_n = a^n$). (Cf. e.g. [5].)

The role of contractible subgroups

The investigations of the structure of the contractible subgroups $C(a), C_K(a)$ defined below play an important role in the theory of semistability on groups. We list some properties, pointing out in particular the additional features in case of *exponential Lie groups* (of course not to be expected in the infinite-dimensional situation).

We define (cf. [15], [6], [7], [2], [11]):

1.1. Definition. Let $a \in \mathrm{Aut}(\mathbb{G})$, let K denote a compact a-invariant subgroup. Then the contractible and K-contractible parts are defined as
$C(a) := \{x \in \mathbb{G} : a^n(x) \overset{n\to\infty}{\longrightarrow} e\}$ and $C_K(a) := \{x \in \mathbb{G} : a^n(x) \cdot K \to K\}$ respectively.
For a one-parameter group $T = (a_t : t > 0)$ we define analogously
$C(T) := \{x : a_t(x) \overset{t\to 0}{\longrightarrow} e\}$ and $C_K(T) := \{x : a_t \cdot K \overset{t\to 0}{\longrightarrow} K\}$.
More generally we define for a sequence $(a_n)_{n\in\mathbb{N}} \subseteq \mathrm{Aut}(\mathbb{G})$
$C((a_n)_{n\in\mathbb{N}}) := \{x \in \mathbb{G} : a_n(x) \to e\}$, and analogously $C_K((a_n))$ is defined.
Obviously, these contractible parts $C(a), C_K(a)$, etc. are subgroups of $\mathbb{G}$.

1.2. Remarks. The following observations are frequently used:
a) We have the following characterization: $C((a_n)_{n\geq 1}) =: C = \{x :$ for any subsequence $(n') \subseteq \mathbb{N}$ there exists a subsequence $(n'') \subseteq (n')$ with $a_n x \overset{(n'')}{\longrightarrow} e\}$.
We fix a sequence $(a_n)_{n\in\mathbb{N}}$. For a subsequence $(n') \subseteq \mathbb{N}$ put $C_{(n')} := C((a_n)_{n\in(n')})$.
b) Let d be a metric on the (second countable) group $\mathbb{G}$. Put for $\varepsilon > 0$ $C^{(\varepsilon)} := \{x : \limsup d(a_n(x), e) < \varepsilon\}$. Obviously, $C^{(\varepsilon)}$ is Borel measurable. Hence $C = \bigcap_{n\geq 1} C^{(1/n)}$ is Borel measurable, and analogously we obtain measurability of $C_{(n')}$.
c) Let $\mathbb{G}$ be an *exponential Lie group* with Lie algebra $\mathbb{V}$. For $a \in \mathrm{Aut}(\mathbb{G})$ let $\overset{o}{a}$ denote the differential, defined by $\exp(\overset{o}{a}(X)) = a(\exp(X))$, $X \in \mathbb{V}$. Let (a_n) and C as above. Define $\overset{o}{C} := \{X \in \mathbb{V} : \overset{o}{a}_n(X) \to 0\}$. Then $\overset{o}{C}$ is a subalgebra and we have $\exp(\overset{o}{C}) = C$. In particular, C and the subgroups $C_{(n')}$ defined in a) are closed connected subgroups.

For *exponential Lie groups* we observe with the notations introduced above:

1.3. Proposition. a) Assume that there exists a sequence $(a_n) \subseteq \mathrm{Aut}(\mathbb{G})$ which is contracting on $\mathbb{G}$, i.e. $a_n(x) \to e$ for all $x \in \mathbb{G}$. Then $\mathbb{G}$ is a contractible Lie group, hence nilpotent and simply connected.

b) More generally, for any sequence (a_n) the contractible part $C = C((a_n)_{n\geq 1})$ is a closed connected subgroup. If C is a_n-invariant for sufficiently large n then C is contractible, hence nilpotent. In particular if $a_n = a^n$ for some $a \in \mathrm{Aut}(\mathbb{G})$ then $C(a)$ is a contractible, nilpotent and a-invariant subgroup.

[We have $C = C((a_n)_{n\geq 1}) = \mathbb{G}$ by assumption. Hence $\mathbf{V} = \overset{o}{C}$ (cf. 1.2.c), i.e. $\overset{o}{a}_n(X) \to 0$ for all $X \in \mathbf{V}$. Therefore we obtain $\|\overset{o}{a}_n\| \to 0$, hence $\overset{o}{a}_n$ — and therefore also a_n — is contractive for sufficiently large n, i.e. $(a_n)^m x \overset{m\to\infty}{\longrightarrow} e$, $x \in \mathbb{G}$. See also [17]. The rest assertions follow immediately.]

The connections between semistability and contractibility are illuminated by the following observations. (See e.g. [15], [6], [7], see also 1.6 below):

1.4. Proposition. a) Let $\mathbb{G}$ be a locally compact group and let $\mu_\bullet$ be an (a,α)-semistable continuous convolution semigroup with trivial idempotent $\mu_0 = \varepsilon_e$ and Lévy measure η. Then $\mu_\bullet$ is concentrated on $C(a)$, i.e.
$\mu_t(\mathbf{Cp}C(a)) = 0$ for all t, and furthermore $\eta(\mathbf{Cp}C(a)) = 0$.
b) And with the same proof we obtain for non-trivial idempotents: If $\mu_0 = \omega_K$ then all the measures μ_t are concentrated on the K-contraction group $C_K(a)$ of a.

Analogously, for stable continuous convolution semigroups we have:
Let $T = (a_t)_{t>0} \subseteq \mathrm{Aut}(\mathbb{G})$ be a subgroup (with $a_{t\cdot s} = a_t a_s$). Let $\mu_\bullet$ be T-stable. Then $\mu_\bullet$ is (a_t, t)-semistable for all $t \in (0,1)$. Hence 1.4 applies. For stable laws with *continuous* group T we obtain a stronger result ([6]):

1.5. Proposition. Let $(\mu_t)_{t\geq 0}$ be a T-stable continuous convolution semigroup on a locally compact group $\mathbb{G}$ such that $\mu_0 = \omega_K$. Then all μ_t are concentrated on the K-contraction group $C_K(T)$ of T.
(Note that in this situation we need not assume $\mathbb{G}$ to be second countable since according to [6] the subgroups $C_K(T)$ and $C(T)$ are closed in $\mathbb{G}$ and hence measurable.)

Proposition 1.5 applies in particular for $K = \{e\}$. We obtain:
If $\mu_\bullet$ is a T-stable continuous convolution semigroup with trivial idempotent and if T is continuous then $\mu_\bullet$ is concentrated on the *closed* subgroup $C(T)$, isomorphic to a contractible simply connected nilpotent Lie group on which T acts contractively.
Hence for *continuous* groups T the investigation of T-stable laws with trivial idempotents is completely reduced to contractible simply connected nilpotent Lie groups.

Not only limit laws, also the attracted laws are concentrated on contractible parts. Generalizing the proof of 1.4 we obtain:

1.6. Proposition. Assume $\mu_\bullet$ to be a continuous convolution semigroup with trivial idempotent $\mu_0 = \varepsilon_e$. Let ν FDPA$(\mu_\bullet)$, i.e. $k(n) \nearrow \infty$, $a_n \in \mathrm{Aut}(\mathbb{G})$ such that $a_n(\nu)^{[k(n)t]} \to \mu_t, t \geq 0$ and assume moreover $\limsup k(n)/k(n+1) < 1$.
Then $\quad \nu(C((a_n)_{n\geq 1}) = 1$.

[W.l.o.g. we assume $k(n)/k(n+1) \leq \kappa < 1$ for $n \geq 1$. Let $U \in \mathfrak{U}(e)$ be relatively compact Borel neighbourhoods. Let A denote the generating functional and η the Lévy measure of $\mu_\bullet$. According to a theorem of E. Siebert (cf. [13], [5], [4])

$$a_n(\nu)^{[k(n)t]} \to \mu_t,\ t \geq 0 \ \text{ iff } \ k(n) \cdot (a_n(\nu) - \varepsilon_e) \to A.$$

Hence $\sup_{n\geq 1} k(n) \cdot a_n(\nu)(\mathbf{Cp}U) \leq K(U) < \infty$. Therefore

$$\int \sum_{n\geq 1} 1_{\mathbf{Cp}U} \circ a_n d\nu = \sum_n a_n(\nu)(\mathbf{Cp}U)$$

$$= \frac{1}{k(1)} \sum_n \frac{k(1)}{k(2)} \cdots \frac{k(n-1)}{k(n)} \cdot k(n) \cdot a_n(\nu)(\mathbf{C}\mathbf{p}U) \leq \frac{K(U)}{k(1)} \cdot \sum \kappa^n < \infty .$$

Whence $1_{\mathbf{Cp}U} \circ a_n \to 0$ ν-a.e. In other words, $\{a_n(x)\}$ is relatively compact with $\mathrm{LIM}(a_n(x)) \subseteq U$ for ν-almost all x. (LIM denoting the set of accumulation points). Let $U_k \in \mathfrak{U}(e)$ with $U_k \downarrow \{e\}$. Repeating the above arguments we obtain $\nu(\bigcap_k \{x : \mathrm{LIM}(a_n x) \subseteq U_k\}) = \nu(C((a_n)_{n\in\mathbb{N}})) = 1$ as asserted.]

1.7. Corollary. a) Assume (as in the case of stable $\mu_\bullet$) that $k(n)/k(n+1) \to 1$. Then for any $\alpha \in (0,1)$ there exists a subsequence (n') with $k(n)/k(n+1) \overset{(n')}{\longrightarrow} \alpha$. And according to 1.6 we conclude $\nu(C((a_n)_{n\in(n')})) = 1$.
b) (*Domains of normal (semi-)attraction*). Let $a_n = a^n$ for some $a \in \mathrm{Aut}(\mathbb{G})$, $k(n)/k(n+1) \to \alpha \in (0,1)$ and $a^n \nu^{[k(n)t]} \to \mu_t, t \geq 0$. Then $\nu(C(a)) = 1$.

Let $a_n = a^{l(n)}$ with $l(n) \nearrow \infty$. Let $C := C((a_n))$. Then $\nu(C) = 1$, but in general $C \neq C(a)$ is possible. However, for *exponential Lie groups* we observe
1.8. Proposition. Let $\mathbb{G}$ be an exponential Lie group, let $a \in \mathrm{Aut}(\mathbb{G}), l(n) \nearrow \infty$. Then $C := C((a^{l(n)})_{n\in\mathbb{N}}) = C(a)$.
[Let $\mathbf{V}$ denote the Lie algebra of $\mathbb{G}$, let as above $\overset{o}{a} \in \mathrm{GL}(\mathbf{V})$ denote the differential of a defined by $\exp(\overset{o}{a} X) = a(\exp X), X \in \mathbf{V}$. For $x \in \mathbb{G}$ let $X = \exp^{-1}(x) \in \mathbf{V}$. $\mathbb{G}$ being exponential, $a^{l(n)}x \to e$ iff $\overset{o}{a}^{l(n)} X \to 0$. As easily seen, this is the case iff X belongs to the contractible $\overset{o}{a}$-invariant subspace $\bigcup_{|z|<1} \{Y : (\overset{o}{a} - zI)^k Y = 0$ for some $k \in \mathbb{N}\} = C(\overset{o}{a})$. Therefore $\overset{o}{a}^n X \overset{n\to\infty}{\longrightarrow} 0$; whence $a^n x \to e$ follows.]

The relevance of the description of $C((a_n)_{n\geq 1})$ in 1.1.a) is shown by the following
1.9. Proposition. Let $\mathbb{G}$ be a group in which the subgroups $C((a_n))$ are closed, e.g. an exponential Lie group (1.2.c)). Let (a_n) be a sequence in $\mathrm{Aut}(\mathbb{G})$ and let $\nu \in \mathcal{M}^1(\mathbb{G})$, such that $a_n(\nu) \to \varepsilon_e$ (infinitesimality). Then $\mathrm{supp}(\nu) \subseteq C((a_n))$.
[Let $\nu \in \mathcal{M}^1(\mathbb{G})$ and assume $a_n \nu \to \varepsilon_e$, for some sequence $(a_n) \subseteq \mathrm{Aut}(\mathbb{G})$. Consider the probability space $(\mathbb{G}, \mathcal{B}, \nu), \mathcal{B}$ denoting the Borel sets. Consider $(a_n = a_n(\cdot))_{n\in\mathbb{N}}$ as a sequence of $\mathbb{G}$-valued random variables on the probability space $(\mathbb{G}, \mathcal{B}, \nu)$. By assumption, $a_n(\nu) \to \varepsilon_e$, hence $a_n(\cdot)$ converge to e in distribution, equivalently in probability. Therefore for any subsequence $(n') \subseteq \mathbb{N}$ there exists a subsequence $(n'') \subseteq (n')$ with $a_n(\cdot) \overset{(n'')}{\longrightarrow} e$ ν-a.e. I.e., we have $\nu(C_{(n'')}) = 1$, with the notations from above.
$C_{(n'')}$ being closed, $\mathrm{supp}(\nu) \subseteq C_{(n'')}$ follows. Therefore, $\mathrm{supp}(\nu) \subseteq \bigcap_{(n')} C_{(n'')} = C$.]

Retopologisation of $C(a)$: Intrinsic topologies

We recall the following results from E. Siebert's investigations ([16]):
Let $a \in \mathrm{Aut}(\mathbb{G})$. Then there exists a unique topology $\mathcal{O}_\tau$ turning $C(a)$ into a topological Hausdorff group $\tilde{C}(a)$ (not necessarily locally compact), furthermore there exist $\tilde{a} \in \mathrm{Aut}(\tilde{C}(a))$ and a continuous injective homomorphism $\varphi : \tilde{C}(a) \to \mathbb{G}$ such that $\varphi \circ \tilde{a} = a \circ \varphi$ (hence $\varphi(\tilde{C}(a) = C(a))$.

2.1. Properties. a) If $\mathbb{G}$ is complete and metrizable and if $a \in \mathrm{Aut}(\mathbb{G})$ is contractive then we have $\tilde{C}(a) = C(a) = \mathbb{G}$.

b) $\mathcal{O}_\tau$ is stronger than the relative topology of $C(a)$ (as a subspace of $\mathbb{G}$).
c) If $\mathbb{G}$ is metrizable then $\tilde{C}(a)$ is metrizable too.
d) If $\mathbb{G}$ has a countable basis then $\tilde{C}(a)$ has a countable basis too.
e) If $\mathbb{G}$ is complete then $\tilde{C}(a)$ is complete too.
f) If $\mathbb{G}$ is totally disconnected then $\tilde{C}(a)$ is totaly disconnected too.

Let $\mu_\bullet$ be a continuous (a,α)-semistable convolution semigroup with $\mu_0 = \varepsilon_e$. According to 2.1 there exists a contractible completely metrizable group $\mathbb{H} := \tilde{C}(a)$ with contractive automorphism $\tilde{a} \in \mathrm{Aut}(\mathbb{H})$ and an injective continuous homomorphism $\varphi : \mathbb{H} \hookrightarrow \mathbb{G}$ such that $\varphi(\mathbb{H}) = C(a)$ and $\varphi \circ \tilde{a} = a \circ \varphi$.
Then φ^{-1} is a Borel isomorphism $C(a) \to \mathbb{H}$. Hence φ induces a bijection $\mathcal{M}^1(\mathbb{H}) \longleftrightarrow \{\nu \in \mathcal{M}^1(\mathbb{G}) : \nu(C(a)) = 1\}, \nu \mapsto \varphi(\nu) =: \mu$. In fact, a continuous affine bijective convolution homomorphism. But φ^{-1} need not be continuous.

Nevertheless any continuous convolution semigroup $\mu_\bullet$ concentrated on $C(a)$ generates a continuous convolution semigroup $\varphi^{-1}(\mu_\bullet) =: \nu_\bullet$ on $\mathbb{H}$:
2.2. Proposition. Let $\mathbb{G}$ and $\mathbb{H}$ be completely metrizable topological groups and $\varphi : \mathbb{H} \hookrightarrow \mathbb{G}$ be an injective continuous homomorphism. Put $\mathbb{L} := \varphi(\mathbb{H})$. If $\mathbb{H}$ is σ-compact then $\mathbb{L}$ is measurable.
a) If $\nu_\bullet \subseteq \mathcal{M}^1(\mathbb{H})$ is a continuous convolution semigroup then $\bigcup_{t>0} \mathrm{supp}(\nu_t)$ generates a (closed) σ-compact subgroup $\mathbb{H}_1$. Hence $\varphi(\nu_\bullet) = \mu_\bullet$ defines a continuous convolution semigroup in $\mathcal{M}^1(\mathbb{G})$ concentrated on the measurable subgroup $\mathbb{L}_1 := \varphi(\mathbb{H}_1) \subseteq \mathbb{G}$.
b) Conversely, assume $\mathbb{L} = \varphi(\mathbb{H})$ to be a measurable subgroup $\subseteq \mathbb{G}$. Let $\mu_\bullet$ be a continuous convolution semigroup in $\mathcal{M}^1(\mathbb{G})$ with $\mu_t(\mathbb{L}) = 1$ for $t \geq 0$. Then $\nu_\bullet = \varphi^{-1}(\mu_\bullet) \subseteq \mathcal{M}^1(\mathbb{H})$ is a continuous convolution semigroup (with $\varphi(\nu_\bullet) = \mu_\bullet$).
Proof: a) is obvious by continuity of φ.
To prove b) note first that $\nu_\bullet$ is uniquely defined by $\mu_\bullet$ and $\nu_\bullet$ is a convolution semigroup. We have to show that $t \mapsto \nu_t$ is continuous.
$\mathbb{H}$ is completely metrizable. Hence ν_t is tight for any $t \geq 0$, therefore the support is σ-compact. Hence w.l.o.g. we assume $\mathbb{H} = \bigcup K^{(m)}$ with an increasing sequence of compact sets $K^{(m)} \subseteq \mathbb{H}$. Hence in order to prove continuity it suffices to show that for any $t \geq 0$, for any sequence $t_n \to t$ and for any $K^{(m)}$ the restrictions $\nu_{t_n}|_{K^{(m)}} =: \kappa_n^{(m)}$ converge weakly to $\nu_t|_{K^{(m)}} =: \kappa^{(m)}$: [Indeed, we have the representations $\nu_{t_n} = \lim_{m\geq 1} \kappa_n^{(m)}$ and $\nu_t = \lim_{m \geq 1} \kappa^{(m)}$ with non-negative measures (convergence in norm). And therefore, if we can prove $\langle \kappa_n^{(m)}, f\rangle \overset{n\to\infty}{\longrightarrow} \langle \kappa^{(m)}, f\rangle$ for $f \in C^b(\mathbb{H})$, for all $m \in \mathbb{N}$, we easily conclude $\langle \nu_{t_n}, f\rangle \to \langle \nu_t, f\rangle$]
For any compact set $K \subseteq \mathbb{H}$ the restriction $\varphi|_K$ defines a topological isomorphism $K \to \varphi(K) =: K^\# \subseteq \mathbb{G}$. Hence for compact sets $K \subseteq \mathbb{H}$ we observe according to the portemanteau theorem applied to the continuous function $s \mapsto \varphi(\nu_s) = \mu_s$ that

$$\limsup \nu_{t_n}(K) = \limsup (\mu_{t_n})(K^\#) \leq \mu_t(K^\#) = \varphi(\nu_t)(\varphi(K)) = \nu_t(K).$$

Therefore, again by the portemanteau theorem applied to the restrictions $\nu_s|_K$ we conclude continuity of $s \mapsto \nu_s|_K$ for all compact $K \subseteq \mathbb{H}$.
In particular, $\kappa_n^{(m)} \overset{n\to\infty}{\longrightarrow} \kappa^{(m)}, m \in \mathbb{N}$, as asserted. ∎

Now we are ready to prove the following

2.3. Theorem. Suppose $\mathbb{G}$ to be a locally compact group and let $\mu_\bullet$ be a continuous convolution semigroup with trivial idempotent.

a) Let $\mathbb{G}$ be second countable and let $\mu_\bullet$ be (a, α)-semistable. Then there exist a completely metrizable topological contractible group $\mathbb{H}$ with contractive automorphism $\tilde{a}$, and a continuous injection $\varphi : \mathbb{H} \hookrightarrow \mathbb{G}$ such that $\varphi(\mathbb{H}) = C(a)$ and $\varphi \circ \tilde{a} = a \circ \varphi$. Furthermore there exists an $(\tilde{a}, \alpha)$-semistable continuous convolution semigroup $\nu_\bullet \subseteq \mathcal{M}^1(\mathbb{H})$ with $\varphi(\nu_t) = \mu_t, t \geq 0$.

b) In particular, if $\mathbb{G}$ is a Lie group then $\mathbb{H}$ is a homogeneous (Lie) group.

c) Analogously, if $\mathbb{G}$ is totally disconnected then $\mathbb{H}$ is totally disconnected too.

d) Let $T = (a_t)_{t>0}$ be a continuous group in $\mathrm{Aut}(\mathbb{G})$ and let $\mu_\bullet$ be T- stable. Then $\mathbb{H} = C(T)$ is a closed subgroup, (isomorphic to) a homogeneous group, φ is the canonical injection and $\nu_\bullet$ is the restriction $\mu_\bullet|_{\mathbb{H}}$.

e) Let $\mathbb{G}$ be a p-adic Lie group. Then $\mathbb{H} = C(a)$ is a closed subgroup hence again φ is the canonical injection and $\nu_\bullet$ is the restriction $\mu_\bullet|_{\mathbb{H}}$.

Note again that in case b) (and d)) the investigations of (semi-)stable laws are completely reduced to simply connected nilpotent Lie groups.

[a) is an immediate consequence of Proposition 2.2 above. According to 1.5 $\mu_\bullet$ is concentrated on $C(a)$. Now b) and c) are immediate consequences of a), for d) see [6]. e) follows by [18], cf. [2].]

Note that within the category of complete and metrizable groups our knowledge of the structure of contractible groups is considerably poor. However, for special cases — if the group $\mathbb{H} = \tilde{C}(a)$ with the natural topology is locally compact — we obtain a reduction of the problems and a complete overview of possible semistable laws. We describe the situation for totally disconnected groups:

Semistable convolution semigroups on contractible totally disconnected groups

A locally compact totally disconnected group $\mathbb{G}$ is contractible with contractive $a \in \mathrm{Aut}(\mathbb{G})$ iff $\mathbb{G}$ admits a *filtration* $(G_n)_{n\in\mathbb{Z}}$ adapted to a, i.e. if there exist compact open subgroups $G_n \subseteq G_{n-1}$ with $\bigcap G_n = \{e\}$ aund $\bigcup G_n = \mathbb{G}$, such that $a(G_n) = G_{n+1}, n \in \mathbb{Z}$. The filtration $(G_n)_{n\in\mathbb{Z}}$ is said to be *normal* if G_n are compact open normal subgroups in $\mathbb{G}$. (See [15].)

3.1. Remark. Let $\mathbb{G}$ be a contractible totally disconnected locally compact group with contractive $a \in \mathrm{Aut}(\mathbb{G})$ and filtration $(G_n)_{n\in\mathbb{Z}}$. Assume the filtration to be *normal.* Then $\mathbb{G} = \lim_{n\in\mathbb{N}} \mathbb{G}/G_n$ is a projective limit of discrete groups. Therefore any continuous convolution semigroup $(\mu_t)_{t\geq 0}$ on $\mathbb{G}$ is a limit of Poisson semigroups $\mu_\bullet^{(n)}$ on $\mathbb{G}/G_n$. (Convolution semigroups on discrete groups are Poisson.) Let $\mu_\bullet$ be (a, α)-semistable. Then the automorphism a is not representable as limit of automorphisms of the factor groups $\mathbb{G}/G_n$ and $\mu_\bullet^{(n)}$ can not be semistable. [Semistable Lévy measures are infinite or trivial, hence semistable laws on discrete groups are trivial.]

For totally disconnected locally compact groups admitting a contractive automorphism a we obtain a complete description of all possible semistable laws. Let $(G_n)_{n\in\mathbb{Z}}$ be a filtration of $\mathbb{G}$ adapted to a. Then $Z := G_0 \backslash G_1$ is a cross-section for the orbits $\{a^n(x) : n \in \mathbb{Z}\}$, $x \in \mathbb{G}\backslash\{e\}$. Let us remark that Z is locally compact.

First we note (Cf. [15]):

3.2. Proposition. Let η be a positive measure on $\mathcal{B}(\mathbb{G})$ with $\eta(\{e\}) = 0$, let $\alpha \in]0, 1[$ and $a \in \mathrm{Aut}(\mathbb{G})$. Then the following assertions are equivalent:

(i) $\eta(\mathbf{C}\mathbf{p}U) < \infty$ for all $U \in \mathfrak{U}(e)$; and $a(\eta) = \alpha \cdot \eta$;

(ii) there exists some finite positive measure κ on $\mathcal{B}(\mathbb{G})$ such that $\kappa(\mathbb{G}\backslash Z) = 0$ and such that $\eta = \sum_{k\in\mathbb{Z}} \alpha^{-k} \cdot a^k(\kappa)$. In fact, we have $\kappa = \eta|_Z$.

Since in the case of totally disconnected groups convolution semigroups are uniquely determined by their Lévy measures, proposition 3.2 provides a complete description of the possible semistable laws:

3.3. Corollary. Fix $\alpha \in (0, 1)$. By $\kappa \mapsto \eta := \sum_{k\in\mathbb{Z}} \alpha^{-k} \cdot a^k(\kappa)$ there is given a bijection between the finite measures κ on $\mathcal{B}(\mathbb{G})$ concentrated on Z and the Lévy measures η on $\mathbb{G}$ such that $a(\eta) = \alpha \cdot \eta$ and $\kappa = \eta|_Z$, hence between $\kappa \in \mathcal{M}^1(\mathbb{G})$ with $\kappa(\mathbf{C}\mathbf{p}Z) = 0$ and (a, α)-semistable continuous convolution semigroups $\mu_\bullet$.

We consider two examples of contractible totally disconnected groups:

3.4. Example. (*Semistable laws on the p-adics*) For some prime power p let $\mathbb{Q}_p$ denote the locally compact field of p-adic numbers. For any $t \in \mathbb{Q}_p$ we define the "homothetic" transformation $H_t : x \mapsto t \cdot x$. Via the mapping $t \mapsto H_t$ we obtain $\mathrm{Aut}(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times$, cf. [8], (26.18 d).) Let $|\cdot|_p$ denote the p-adic valuation of $\mathbb{Q}_p$. In view of $|H_t(x)|_p = |t|_p \cdot |x|_p$, the automorphism H_t is contractive iff $|t|_p < 1$. $\mathbb{Q}_p$ is totally disconnected. Moreover the subset $\Delta = \mathbb{Z}_p = \{x : |x|_p \leq 1\}$ of p-adic integers is a compact open subgroup of $\mathbb{Q}_p$; (see [8], § 10).

$\mathbb{Q}_p$ may be considered as the subset of the direct product $\otimes_{k\in\mathbb{Z}}\{0, \ldots, p-1\}$ consisting of sequences $\widehat{x} = (x(k))_{k\in\mathbb{Z}}$ such that $x(k) = 0, k \leq K$ for some $K = K(x) \in \mathbb{Z}$. (It is sometimes convenient to represent x equivalently as formal power series $\sum_{k\in\mathbb{Z}} x(k) \cdot p^k$ with $x(k) = 0$ for $k \leq K$.) Let $n := n(x) := \min\{k \in \mathbb{Z} : x(k) \neq 0\}$ if $x \neq 0$. The p-adic valuation is given as $|x|_p := p^{-n(x)}$, if $x \neq 0$, and $|0|_p := 0$.

The field of rational numbers $\mathbb{Q}$ is canonically densely embedded in $\mathbb{Q}_p$ and hence — endowed with the continuously extended algebraic operations of $\mathbb{Q}$ — the $|\cdot|_p$-closure $\mathbb{Q}_p$ is a locally compact totally disconnected topological field, and $\mathbb{Z}$ is dense in $\mathbb{Z}_p$.

Put $\Delta_n := \{x : |x|_p \leq p^{-n}\}, n \in \mathbb{Z}$, then $(\Delta_n)_{n\in\mathbb{Z}}$ is a nested sequence of compact open subgroups with $\bigcap \Delta_n = \{0\}, \bigcup \Delta_n = \mathbb{Q}_p$. And any compact subgroup is of the form Δ_n for some $n \in \mathbb{Z}$ ([8], 10.6).

Obviously, $H_{p^n}(\Delta_0) = \Delta_n, n \in \mathbb{Z}$, more generally, $H_t\Delta_0 = \Delta_n$ if $|t|_p = p^{-n}$. Hence in particular $(\Delta_n)_{n\in\mathbb{Z}}$ is a (normal) filtration adapted to $a := H_p$. For any $t \in \mathbb{Q}_p^\times$ with $|t|_p < 1$ the automorphism H_t is contractive. In particular, H_p is contractive.

If $|t|_p = p^{-d}, d \in \mathbb{N}$, then $(G_{(n)} := \Delta_{nd})_{n\in\mathbb{Z}}$ is a filtration adapted to H_t. We observe $G_{(n)}/G_{(n+1)} = \mathbb{Z}/(p^d \cdot \mathbb{Z})$.

The Haar measure ω_{Δ_n} is absolutely continuous to the Haar measure $\omega_{\mathbb{Q}_p}$:

Normalize $\omega_{\mathbb{Q}_p}$ such that $\omega_{\mathbb{Q}_p}(\Delta_0) = 1$. Then ω_{Δ_0} is the restriction $\omega_{\mathbb{Q}_p}|_{\Delta_0}$. And $\omega_{\Delta_n} = H_{p^n}(\omega_{\Delta_0}) = H_{p^n}(\omega_{\mathbb{Q}_p}|_{\Delta_0}) = \Delta(H_{p^n}) \cdot \omega_{\mathbb{Q}_p}|_{\Delta_n} = p^n \cdot \omega_{\mathbb{Q}_p}|_{\Delta_n}$.

In other words, $\quad \int_{\mathbb{Q}_p} f d\omega_{\Delta_n} = p^n \cdot \int_{|x|_p \leq p^{-n}} f d\omega_{\mathbb{Q}_p} \quad$ for $f \in L^1(\mathbb{Q}_p, \omega_{\mathbb{Q}_p})$.

Next we investigate in some details the following example of a semistable continuous convolution semigroup on the additive group $\mathbb{G} = (\mathbb{Q}_p, +)$.

Let $d \in \mathbb{N}$ and $t \in \mathbb{Q}_p$ with $|t|_p = p^{-d}$, put $a := H_t \in \mathrm{Aut}(\mathbb{Q}_p)$, and let $(G_{(n)} := \Delta_{nd})_{n\in\mathbb{Z}}$ be the corresponding filtration.

An (a, α)-semistable continuous convolution semigroup $\mu_\bullet$ is defined by the Lèvy measure $\eta = c \cdot \sum_{k \in \mathbb{Z}} \alpha^{-k} \cdot a^k(\nu)$, $0 < \alpha < 1$, $c \geq 0$, $\nu \in \mathcal{M}^1(G_{(0)} \backslash G_{(1)})$. (Cf. 3.3.)
We call $\lambda \in \mathcal{M}^1(\mathbb{Q}_p)$ *rotation invariant* if $H_x(\lambda) = \lambda$ for all $x \in \mathbb{U}$, where $\mathbb{U} = \{t : |t|_p = 1\}$ is the group of units in $\mathbb{Q}_p^\times \cong \mathrm{Aut}(\mathbb{Q}_p, +)$. (Cf. also [1], [19].)
The orbits $\mathbb{U} \cdot x$ are given by $\{y \in \mathbb{Q}_p : |y|_p = |x|_p\}$, hence a function $f : \mathbb{Q}_p \to \mathbb{C}$ is $\mathbb{U}$-invariant iff $f(\cdot) = \varphi(|\cdot|_p)$ for some function $\varphi : \mathbb{R}_+ \to \mathbb{C}$. Since for any $n \in \mathbb{Z}$ we have $u \cdot \Delta_n = \Delta_n$, $u \in \mathbb{U}$, we easily conclude that ω_{Δ_n} is rotation invariant.
Obviously, μ_t is rotation invariant if ν has this property, ν as above. We consider the special rotation invariant measure $\nu := \frac{p^d}{p^d-1} \cdot (\omega_{G_{(0)}} - \frac{1}{p^d}\omega_{G_{(1)}}) \in \mathcal{M}^1(G_{(0)})$. As easily seen, since $G_{(0)}/G_{(1)} \simeq \mathbb{Z}/p^d \cdot \mathbb{Z} \simeq \{0, \ldots, p^d - 1\}$, we have
$\omega_{G_{(0)}} = \sum_{k=0}^{p^d-1} \frac{1}{p^d} \cdot \varepsilon_{x_j} * \omega_{G_{(1)}}$. Hence $\nu = \frac{1}{p^d-1} \cdot \sum_{k=1}^{p^d-1} \varepsilon_{x_j} * \omega_{G_{(1)}}$.
$(\mathbb{Q}_p, +)$ is a locally compact *Abelian* group, hence $\mu_\bullet$ may be represented in terms of the Fourier transform: Following the representation in [8], § 25, we obtain the following description of $\widehat{\mathbb{Q}}_p$:
Fix a nontrivial continuous character $\varphi_1 : \mathbb{Q}_p \to \mathbb{T}$ with kernel $\ker \varphi_1 = \Delta_0$. ($\mathbb{T}$ denoting the torus $\{z \in \mathbb{C} : |z| = 1\}$.)
For $y \in \mathbb{Q}_p$ define $\varphi_y : x \mapsto \varphi_1(H_y(x)) = \varphi_1(y \cdot x)$. Any continuous character is obtained in that way and by $y \mapsto \varphi_y$ we obtain an isomorphism, hence $\widehat{\mathbb{Q}}_p \cong \mathbb{Q}_p$.
Let $a = H_t \in \mathrm{Aut}(\mathbb{Q}_p)$ then we observe $\varphi_y(ax) = \varphi_{a(y)}(x) =: a^*(\varphi_y)(x)$. Hence $(\mathbb{Q}_p^\times, \cdot) \cong \mathrm{Aut}(\mathbb{Q}_p)$ acts in a natural way on $\widehat{\mathbb{Q}}_p$.
Now we have the means to compute explicitely the Fourier transform $\widehat{\nu}$ since
$\widehat{\omega}_{G_{(n)}}(\varphi_y) = 1$ iff $y \in G_{(-n)}$, and $= 0$ else. Hence
$\widehat{\nu}(\varphi_y) = \frac{p^d}{p^d-1}\widehat{\omega}_{G_{(0)}}(\varphi_y) - \frac{1}{p^d-1}\widehat{\omega}_{G_{(1)}}(\varphi_y)$. Therefore
$\widehat{\mu}_t(\varphi_y) = \exp t \cdot \int (\varphi_y - 1) d\eta$
$\quad = \exp(tc \cdot \sum_{k \in \mathbb{Z}} \alpha^{-k} (\frac{p^d}{p^d-1}(\widehat{\omega}_{G_{(k)}}(\varphi_y) - 1) - \frac{1}{p^d-1}(\widehat{\omega}_{G_{(k+1)}}(\varphi_y) - 1))$
For simplification we assume now $d = 1, |t|_p = p^{-1}$, hence $G_{(n)} = \Delta_n, n \in \mathbb{Z}$. In this case, $\widehat{\omega}_{\Delta_k}(\varphi_y) - 1 = 0$ if $y \in \Delta_{-k}$ and = -1 else. And the representation yields: There exists some constant $C = C(\alpha, p) > 0$ such that $\widehat{\mu}_t(\varphi_y) = \exp(-t \cdot C \cdot \alpha^{-M})$ for $y \in \Delta_{-M} \backslash \Delta_{-M+1}$, i.e. for $|y|_p = p^M$. Define $\gamma := -\ln \alpha / \ln p > 0$, hence $\alpha = p^{-\gamma}$, then we obtain

$$\widehat{\mu}_t(\varphi_y) = \exp(-t \cdot C \cdot |y|_p^\gamma),\ y \in \mathbb{Q}_p$$

And conversely, $\widehat{\mu}_t(\varphi_y) = \exp(-t \cdot C|y|_p^\gamma)$ defines a rotation invariant (H_p, α)-semistable continuous convolution semigroup on $\mathbb{Q}_p$ for any $0 < \alpha < 1$ (and $\gamma = \gamma(\alpha)$ as above) and any $C > 0$.
At the first glance this representation is similar to the Fourier transform of (elliplically) symmetric stable laws on $\mathbb{R}$ or on real vector spaces $\mathbb{V}$. But note that there is an essential difference: In the real or vector space case we have $0 < \gamma \leq 2$, in the p-adic situation there is no restriction on $\gamma > 0$. Hence the similarity is only formal.

Some further remarks: The Lévy measure $\eta = c \cdot \sum_{k \in \mathbb{Z}} \alpha^{-k} a^k(\nu)$, with $a = H_p, \nu = \frac{p}{p-1} \cdot (\omega_{\Delta_0} - \frac{1}{p} \cdot \omega_{\Delta_1})$ as above, is absolutely continuous with respect to the Haar measure on $\mathbb{Q}_p$ and the density is given by $c \cdot \frac{p}{p-1} \cdot \sum_{k \in \mathbb{Z}} (\alpha/p)^{-k} \cdot 1_{\Delta_k \backslash \Delta_{k+1}}$, as easily seen inserting $d\omega_{\Delta_n}/d\omega_{\mathbb{Q}_p} = p^n \cdot 1_{\Delta_n}$ in the definition of η.

In fact, the Lévy measure η is absolutely continuous and unbounded, whence $\mu_t \ll \omega_{\mathbb{Q}_p}$ follows, cf. A. Janssen [10] resp. E. Siebert [14]. For more details see the investigations Albeverio et al. [1] and K. Yasuda [19] where Lévy processes with rotation invariant semistable continuous convolution semigroups are considered; these laws are called "*stable*" in [19] .

The following example points out once more the typical structure of totally disconnected contractible locally compact groups :

3.5. Example. (Cf. [15]). Let F be a finite group of order $r > 1$. By Λ we denote the set of all sequences $\widehat{x} = (x(k))_{k\in\mathbb{Z}} \in F^{\mathbb{Z}}$ such that $x(k) = e$ for all $k < k_0$ and for some $k_0 \in \mathbb{Z} \cup \{+\infty\}$. Defining the product of two such sequences componentwise, Λ becomes a group. Every subset $\Lambda_{(n)} := \{\widehat{x} = x(k) = e$ for all $k < n\}, n \in \mathbb{Z}$, is a normal subgroup of Λ. If n tends to $+\infty$ then the groups $\Lambda_{(n)}$ decrease to the identity e of Λ; if n tends to $-\infty$ then the groups $\Lambda_{(n)}$ increase to Λ.
We furnish Λ with the (unique) topology that turns Λ into a topological T_0 - group and has $(\Lambda_{(n)})_{n\in\mathbb{Z}}$ as a basis of the identity $\widehat{e}$ (cf. [8], (4.5) and (4.21)). Then Λ is a totally disconnected topological group.
Every factor group $\Lambda_{(n)}/\Lambda_{(n+1)}$ is finite (it is isomorphic with F); hence $\Lambda_{(0)}$ is totally bounded. Moreover Λ is complete with respect to its left uniform structure.
Thus $\Lambda_{(0)}$ is compact, and therefore Λ is locally compact.
Now let $\rho((x(k))_{k\in\mathbb{Z}}) := (x(k-1))_{k\in\mathbb{Z}}$ for all $\widehat{x} = (x(k))_{k\in\mathbb{Z}}$ in Λ (the shift restricted to Λ). It is easy to see that ρ is an automorphism of Λ such that $\rho(\Lambda_{(n)}) = \Lambda_{(n+1)}$ for all $n \in \mathbb{Z}$. Consequently, ρ is bicontinuous and contractive; and $(\Lambda_{(n)})_{n\in\mathbb{Z}}$ is a normal filtration of Λ adapted to ρ. In fact, it is easily verified that $\Lambda = \tilde{C}(a)$, and $\rho = \tilde{a}$ (cf. 2.1) where a denotes the shift on the direct product $F^{\mathbb{Z}}$. (See 4.1 below).

For later use we mention the following simple lemma generalizing 3.2, which enables us to construct semistable laws on general locally compact groups in concrete situations. Let $\mathcal{S}(a,\alpha) = \mathcal{S}(a,\alpha)(\mathbb{G}) := \{A \in \mathcal{GF}(\mathbb{G}) : a(A) = \alpha \cdot A\}$ denote the set of (a,α)-semistable generating functionals.

3.6. Lemma. Assume that $\mathbb{G}$ is a locally compact group, $a \in \operatorname{Aut}(\mathbb{G}), B \in \mathcal{GF}(\mathbb{G}), \alpha \in (0,1)$. Assume that for $f \in \mathcal{D}(\mathbb{G})$ the series $\sum_{k=-\infty}^{\infty} \alpha^{-k} \cdot \langle\, a^k(B), f\rangle$ is absolutely convergent.
Then $A : f \mapsto \langle A, f\rangle := \sum_{-\infty}^{\infty} \alpha^{-k} \cdot \langle B, f \circ a^k\rangle$ belongs to $\mathcal{S}(a,\alpha)$.
[As easily seen, A is almost positive and normalized (cf. [12], [9]). Hence $A \in \mathcal{GF}(\mathbb{G})$. And $a(A) = \alpha \cdot A$ obviously follows.]

(Semi-)stability on solenoidal groups

There exist compact connected finite-dimensional groups and stable semigroups of probabilities $\mu_\bullet$ with $\operatorname{supp}(\mu_t) = \mathbb{G}, t > 0$. ($\mathbb{G}$ cannot be a Lie group.) The corresponding group of automorphims $T = (a_t)_{t>0}$ is contractive on a dense subgroup (the range of the exponential map), but not contractive on $\mathbb{G}$. $t \mapsto a_t$ is not continuous in this example, and $\mathbb{G}$ is not second countable.

3.7. Example. Choose $\mathbb{R}_d$, the real line with the discrete topology, and let $\mathbb{G}$ be the solenoidal group $\mathbb{G} = (\mathbb{R}_d)^\wedge$ ($= \beta(\mathbb{R})$, the Bohr compactification of $\mathbb{R}$). Then $\psi : \mathbb{R}_d \to \mathbb{R}, \psi(x) := x$, is a continuous injective homomorphism, therefore the dual

homomorphism $\varphi : \widehat{\mathbb{R}}\ (\cong \mathbb{R}) \to (\mathbb{R}_d)^\wedge = \mathbb{G}$ is continuous, injective and has dense range. (Indeed $\mathbb{G}$ is one-dimensional and $\varphi : \mathbb{R} \to \mathbb{G}$ is just the exponential map.).
Now let $(\nu_t)_{t\geq 0}$ be strictly stable on $\mathbb{R}$, i.e. let $b_t = H_{t^\alpha} : x \mapsto t^\alpha \cdot x, t > 0, x \in \mathbb{R}$ and assume $b_t(\nu_s) = \nu_{st}$. b_t can be regarded as automorphism of $\mathbb{R}_d$, therefore the dual map $\widehat{b_t} =: a_t : \mathbb{G} \to \mathbb{G}$ is an automorphism of $\mathbb{G}$.
$y \in \mathbb{R}_d$ is identified with a character γ_y of $\mathbb{G}$, defined on the dense range $\varphi(\mathbb{R}_d)$ by $\langle \varphi(x), \gamma_y \rangle = e^{ixy}, x \in \mathbb{R}_d$. Therefore $\langle a_t(g), \gamma_y \rangle = \langle g, \gamma_{H_{t^\alpha}(y)} \rangle$ for all $t > 0$, $y \in \mathbb{R}_d$, $g \in \mathbb{G}$.
Define $\mu_t := \varphi(\nu_t)_{t\geq 0}$. Obviously $a_t(\mu_s) = a_t(\varphi(\nu_s)) = \varphi(b_t(\nu_s)) = \varphi(\nu_{ts}) = \mu_{ts}$ for $t, s > 0$. So $(\mu_s)_{s\geq 0}$ is stable w.r.t. $T = (a_t)_{t>0}$.
The group T is not contractive on (the compact group) $\mathbb{G}$, but T acts contractively on the range $\varphi(\mathbb{R})$: for $x \in \mathbb{R}$ we observe $a_t(\varphi(x)) = \varphi(t^\alpha \cdot x) \overset{t\to 0}{\longrightarrow} \varphi(0) = e$.
On the other hand $\mu_t = \varphi(\nu_t)$ is concentrated on $\varphi(\mathbb{R})$. ($\varphi(\mathbb{R})$ is σ-compact and hence measurable.) According to 2.3 any (semi-)stable law on $\mathbb{G}$ arises in that way.
We note that $t \mapsto a_t$ is not continuous: There exist elements $g \in \mathbb{G}$ which are non-continuous characters on $\mathbb{R}$. But the set of continuity points $S(T)$ is dense.

Semistability on infinite products of compact groups

If $\mathbb{G}$ is a (real or p-adic) Lie group (not necessarily contractible) we have a more or less complete survey over semistable laws supported by $\mathbb{G}$. (See e.g. [3], [6], [7], [14], [2]). Beyond this class of groups there exist semistable laws, but the properties may differ in a characteristic manner. To point out those differences we investigate as a particular example infinite products $\mathbb{G} = K^{\mathbb{Z}}$ where $K \neq \{e\}$ is a compact group. Let a denote the shift, $a(\widehat{x})(k) := \widehat{x}(k+1)$ for $\widehat{x} \in \mathbb{G}$, $\widehat{x} : \mathbb{Z} \to K$.

4.1. Proposition. a) There exist non-trivial (a, α)-semistable laws on any group representable as infinite product $\mathbb{G} = K^{\mathbb{Z}}$, in particular on the infinite-dimensional torus $\mathbb{T}^{\mathbb{Z}}$, where a denotes the shift and $\alpha \in (0, 1)$.
b) Analogously, there exist non-trivial stable laws on any group $\mathbb{G} = K^{\mathbb{R}}$, for a nontrivial compact group K; in particular on the infinite-dimensional torus $\mathbb{T}^{\mathbb{R}}$. In this case the automorphism group T is the (non-continuous) group of shifts.
[a) Let $K \neq \{e\}$ be a compact group (e.g. $K = \mathbb{T}$). Define $\mathbb{G} := K^{\mathbb{Z}}$ and let $a : \mathbb{G} \to \mathbb{G}$ be the shift $a(\widehat{x})(k) := \widehat{x}(k+1)$, $k \in \mathbb{Z}$, for $\widehat{x} \in \mathbb{G}$, $\widehat{x} : \mathbb{Z} \to K$. For any $n_1 < n_2 \in \mathbb{Z}$, let $J := \{n_1, \dots n_2\}$ and $\mathcal{K}_J := K^J$. Obviously, $\mathcal{D}(\mathbb{G}) = \mathcal{E}(\mathbb{G}) = \{f = f' \circ \pi_J$ for some $J \subseteq \mathbb{Z}$ and $f' \in \mathcal{D}(\mathcal{K}_J)\}$ ($\pi_J : K^{\mathbb{Z}} \to K^J$ denotes the canonical projection). Consequently, for any generating functional $B^\# \in \mathcal{GF}(\mathcal{K}_J)$ we define $B \in \mathcal{GF}(\mathbb{G})$ via $\langle B, f \rangle := \langle B^\#, f' \rangle$ where $f = f' \circ \pi_J$.
For any $f \in \mathcal{D}(\mathbb{G})$ obviously $\sum_{k\in\mathbb{Z}} \alpha^{-k} \cdot \langle B, f \circ a^k \rangle$ converges (indeed the entries are zero, except a finite number.) Therefore $A = \Sigma \alpha^{-k} \cdot a^k(B)$ is a semistable generating functional on $\mathbb{G}$ (cf. 3.6).
b) Let $\mathbb{G} = K^{\mathbb{R}}$ be represented as $\mathbb{G} = \{\widehat{x} : \mathbb{R}_+^\times \to K\}$ and define $T = (a_t)_{t>0}$ to be the group of shifts $a_t(\widehat{x})(s) := \widehat{x}(ts), \widehat{x} \in \mathbb{G}, t, s > 0$ (with multiplicative parametrization). T is a non-continuous group in $\mathrm{Aut}(\mathbb{G})$ fulfilling $a_t a_s = a_{ts}$. (If K is finite or $K = \mathbb{T}^m$ then there exist only trivial *continuous* groups in $\mathrm{Aut}(\mathbb{G})$).

Let $\lambda_\bullet^{(r)}$ be a continuous convolution semigroup on K $(\cong K^{(r)})$ and define $\mu_s := \otimes_{r>0}\lambda_s^{(r)}$ for any coordinate $r > 0$. Then, as immediately seen, $a_t(\mu_s) = \otimes_{r>0}\lambda_s^{(r/t)}$, $t > 0$. Hence for fixed $\gamma > 0$, we have

$$a_{t^\gamma}(\mu_s) = \mu_{s\cdot t}, s \geq 0, t > 0 \quad \text{iff} \quad \lambda_{s\cdot t}^{(r)} = \lambda_s^{(r/t^\gamma)} \ (r > 0).$$

In analogy to a), let $\nu_\bullet$ be an arbitrary continuous convolution semigroup on K with generating functional B. Then $\mu_s^{(\gamma)} := \otimes_{r>0}a_r(\nu_{s/r^{1/\gamma}})$ fulfil the relations $a_{t^\gamma}(\mu_s^{(\gamma)}) = \mu_{s\cdot t}^{(\gamma)}$.
If we identify K with $K^1 \subseteq \mathbb{G}$ and consider $B \in \mathcal{GF}(K)$ as generating functional $B \in \mathcal{GF}(\mathbb{G})$ then (e.g. for $\gamma = 1$) the generating functional of $\mu_\bullet^{(1)}$ is given by $A = \sum_{r>0} r^{-1} \cdot a_r(B)$. In fact, $f \in \mathcal{D}(\mathbb{G})$ depends only on finitely many coordinates, $\{r_1, \ldots, r_r\}$ say. Hence $\langle A, f\rangle$ is well defined and we have $a_t(A) = t \cdot A$ for all $t > 0$.]

4.2. Remark. In a) assume in particular $J = \{0\}$, consider $K = K^{(0)}$ as subgroup of $\mathbb{G}$. Let $B = B^{(0)} \in \mathcal{GF}(K)$ denote the generating functional of a continuous convolution semigroup $\mu_\bullet = \mu_\bullet^{(0)} \subseteq \mathcal{M}^1(K)$. Then the continuous convolution semigroup generated by A has product form $\mu_t = \otimes_{k\in\mathbb{Z}}\mu_t^{(k)}$, with $\mu_t^{(k)} = \mu_{\alpha^{-k}t}$.

$C(a) \cap C(a^{-1})$ *on infinite products* $K^{\mathbb{Z}}$

We consider the subgroups $\mathcal{F}_l := \mathcal{F}_l(a) := \{\widehat{x} \in \mathbb{G} : \lim_{k\to\infty}\widehat{x}(k) = e\}$, $\mathcal{F}_r := \{\widehat{x} \in \mathbb{G} : \lim_{k\to-\infty}\widehat{x}(k) = e\}$, $\mathcal{F}_0 := \{\widehat{x} \in \mathbb{G} : \lim_{|k|\to\infty}\widehat{x}(k) = e\} = \mathcal{F}_l \cap \mathcal{F}_r$ and $\mathcal{F} := \{\widehat{x} \in \mathbb{G} : \widehat{x}(k) \neq e \text{ finitely often}\}$.
Obviously, $C(a) = \mathcal{F}_l$, $C(a^{-1}) = \mathcal{F}_r$, and we observe $\mathcal{F} = \mathcal{F}_0$ iff K is finite.

If $\mathbb{G}$ is a Lie group then $C(\tau) \cap C(\tau^{-1}) = \{e\}$ for all $\tau \in \mathrm{Aut}(\mathbb{G})$. [This is easily proved e.g. repeating the arguments in [16], example 1.] Hence (a, α)- and (a^{-1}, β)-semistable laws are concentrated on subgroups with trivial intersection.
In contrast, for $\mathbb{G} = K^{\mathbb{Z}}$ and if a denotes the shift as above then $\mathcal{F}$ and hence $\mathcal{F}_0 = C(a) \cap C(a^{-1})$ are dense in $\mathbb{G}$. However, for semistable laws in productform we obtain:
4.3. Proposition. Let $\rho_\bullet$ and $\sigma_\bullet$ be non-degenerate (a, α)- and (a^{-1}, β)-semistable continuous convolution semigroups of product form considered in 4.2. Then, for $s, t > 0$, ρ_t and σ_s are concentrated on the disjoint measurable subsets $C(a)\backslash\mathcal{F}_0$ and $C(a^{-1})\backslash\mathcal{F}_0$ respectively.
Proof: In fact, if K is finite, the assertion follows since by construction semistable laws have infinite Lévy measures and are thus diffuse measures ([10], [14]). On the other hand, in this case $\mathcal{F}_0 = \mathcal{F}$ is countable. Whence $\rho_t(\mathcal{F}) = \sigma_s(\mathcal{F}) = 0, t, s > 0$.
If K is infinite, assume according to 4.2 $\rho_t = \otimes_{k\in\mathbb{Z}}\mu_t^{(k)}$ with $\mu_t^{(k)} = \mu_{\alpha^{-k}t}$ (where $\mu_\bullet$ is a continuous convolution semigroup in $\mathcal{M}^1(K) \cong \mathcal{M}^1(K^{(k)})$). And assume an analogous representation for $\sigma_\bullet$. We have to show $\rho_t(\mathcal{F}_0) = \sigma_s(\mathcal{F}_0) = 0$ for $s, t > 0$.
Since μ_t is non-degenerate the limit set $\mathrm{LIM}\{\mu_t : t \to \infty\}$ is contained in $\{\varepsilon_x * \omega_H\}$ for some non-trivial subgroup $H \subset K$. Therefore, as easily seen, for a neighbourhood $U \in \mathfrak{U}(e)$ in K we have $\limsup \mu_t\{U\} < 1$. I.e. $\mu_t\{U\} \leq \kappa < 1$ for sufficiently large t, hence $\mu_t^{(k)}\{U\} = \mu_{\alpha^{-k}t}\{U\} \leq \kappa$ for sufficiently large k. For any $L \in \mathbb{N}$ we conclude

$$\rho_t\{\prod_{|j|\leq L} K \times \prod_{|j|>L} U\} = (\otimes_{j\in\mathbb{Z}}\mu_t^{(j)})\{\prod_{|j|\leq L} K \times \prod_{|j|>L} U\} = 0$$

since $\prod_{|j|>L} \mu_{t\cdot\alpha^{-j}}(U) = 0$.
Whence $\mu_t\{\mathcal{F}_0\} = 0$ for $t > 0$ as asserted since $\mathcal{F}_0 \subseteq \bigcup_{L\in\mathbb{N}} \prod_{|j|\leq L} K \times \prod_{|j|>L} V$ for any $V \in \mathfrak{U}\{e\}$. ∎

Marginals of semistable laws on ifinite products

4.4. Remarks. **a**) If K is a finite group, then K^n is finite for $n \in \mathbb{N}$, hence $\mathcal{S}(a,\alpha)(K^n)$ is trivial but $K^{\mathbb{Z}} = \mathbb{G}$ possesses non-trivial semistable laws. But according to 3.1 no finite-dimensional marginal distribution is semistable.
b) Finite-dimensional tori $\mathbb{T}^d, d \geq 2$, admit automorphisms with dense contractible subgroups and semistable laws on $\mathbb{T}^d$ are homomorphic images of operator semistable laws on subspaces of $\mathbf{V} = \mathbb{R}^d$.
Let a denote the shift on the *infinite-dimensional torus* $\mathbb{G} = \mathbb{T}^{\mathbb{Z}}$ acting contractively on the dense subgroup $\mathcal{F}_l$. Again, also in this case finite-dimensional marginals of (a,α)-semistable laws need not be semistable: Let $B \in \mathcal{GF}(\mathbb{G})$ be a generating functional such that the generated continuous convolution semigroup is concentrated on a finite-dimensional torus $\mathbb{H} := \mathbb{T}^I$, I finite $\subseteq \mathbb{Z}$, e.g. on $\mathbb{T}^{\{0\}}$. Assume $\alpha \in (0,1)$ and put $A := \sum_{k\in\mathbb{Z}} \alpha^{-k} \cdot a^k(B)$. According to 3.6 resp. 4.2 the continuous convolution semigroup generated by A is (a,α)-semistable. If B is a Poissongenerator then for any finite $I \subseteq \mathbb{Z}$ the projection onto $\mathbb{T}^I$ is Poisson and hence not semistable.

Limit laws on infinite-dimensional tori $\mathbb{T}^{\mathbb{Z}}$ and on $\mathbb{R}^{\mathbb{Z}}$

5.1. Example. $\mathbb{G} = \mathbb{T}^{\mathbb{Z}}$ is arcwise connected, with (infinite-dimensional Abelian) Lie algebra $\mathbb{R}^{\mathbb{Z}}$. In this case, the exponential map $\pi = \exp : \mathbf{V} := \mathbb{R}^{\mathbb{Z}} \to \mathbb{T}^{\mathbb{Z}}, \widehat{\phi} := (\phi(k) : k \in \mathbb{Z}) \mapsto (e^{i\cdot\phi(k)} : k \in \mathbb{Z})$, is surjective. There exists a linear subspace $\overset{o}{\mathcal{F}_l} := \{\widehat{\phi} \in \mathbb{R}^{\mathbb{Z}} : \lim_{k\to-\infty} \phi(k) = 0\}$ of $\mathbb{R}^{\mathbb{Z}}$ and an automorphism $\overset{o}{a}$, the shift on $\mathbb{R}^{\mathbb{Z}} = \mathbf{V}$, which acts contractively on $\overset{o}{\mathcal{F}_l}$, such that $a \circ \exp = \exp \circ \overset{o}{a}$ and such that $\exp(\overset{o}{\mathcal{F}_l}) = \mathcal{F}_l$. The restriction of the exponential map to $\overset{o}{\mathcal{F}_l}, \exp : \overset{o}{\mathcal{F}_l} \to \mathcal{F}_l$ is surjective but not injective. Moreover, it is not possible to describe $\overset{o}{a}$ by its action on finite dimensional subspaces. Also on $\mathbf{V} = \mathbb{R}^{\mathbb{Z}}$, finite-dimensional marginal distributions of $(\overset{o}{a}, \alpha)$-semistable laws need not be semistable as shown analogously to the situation $\mathbb{T}^{\mathbb{Z}}$ in 4.4.b)

We avoided to develop a theory of generating functionals for the (non locally compact) group $\mathbb{R}^{\mathbb{Z}}$. Indeed, $\mathbf{V} = \mathbb{R}^{\mathbb{Z}}$ is a nuclear vector space and $\mathbb{G} = \mathbb{T}^{\mathbb{Z}}$ is a compact Abelian group. Hence Fourier transforms are available, and Fourier transforms in both cases are determined by finite-dimensional projections.
Let $\xi \in \widehat{\mathbf{V}}$, i.e. let $\phi \in \mathbf{V}'$ be a continuous linear functional, and $\langle \xi, X \rangle := e^{i\cdot\langle \phi, X\rangle}$, and let $\pi = \pi_I$ be a finite-dimensional projection. If ϕ is constant on cosets of ker π then $\mathbb{G} \ni x = \pi(X) \mapsto e^{i\cdot\langle \phi, X\rangle} =: \langle x, \widehat{\pi}(\xi)\rangle$ defines a character $\bar{\xi} = \widehat{\pi}(\xi)$ of $\mathbb{G}$; and any continuous character arises in this way.
Hence, with the notations introduced above the Fourier transforms fulfil
$\widehat{\overset{o}{\lambda}}(\xi) = \widehat{\lambda}(\pi(\xi))$ for $\overset{o}{\lambda} \in \mathcal{M}^1(\mathbf{V})$ resp. $\lambda = \exp(\overset{o}{\lambda}) \in \mathcal{M}^1(\mathbb{G})$.
Analogously, let $\mu_\bullet$ denote the Poisson semigroup on $\mathbb{G}$ with Fourier transform

$\widehat{\mu}_t = \exp(t(\widehat{\lambda}-1))$, then $\widehat{\mu}_t^{\circ}(\xi) = \exp(t(\widehat{\lambda}^{\circ}(\pi(\xi))-1))$ defines the Poisson semigroup $\overset{\circ}{\mu}_{\bullet} = \exp \bullet(\overset{\circ}{\lambda} - \varepsilon_0)$ on $\mathbb{V}$.
Assume λ to be concentrated on $\mathbb{T}^{\{0\}}$, put $B := \lambda - \varepsilon_e$ and define $A \in \mathcal{GF}(\mathbb{G})$ as in 4.1.a). Assume further $\operatorname{supp}(\overset{\circ}{\lambda}) \subseteq [0, 2\pi]$. Then for $\bar{\xi} := \widehat{\pi}(\xi) \in \widehat{\mathbb{G}}, \bar{\xi} = (\bar{\xi}(k))_{k\in\mathbb{Z}} \in (\mathbb{T}^{\mathbb{Z}})^{\wedge} \cong \mathbb{Z}^{*\mathbb{Z}}$ (weak product), we obtain
$\widehat{A}(\bar{\xi}) = \sum \alpha^{-k} \cdot (a^k(\lambda - \varepsilon_e))^{\wedge}(\bar{\xi}) = \sum \alpha^{-k} \cdot (\widehat{\lambda}(\bar{\xi}(k)) - 1)$, and analogously,
$\widehat{A}^{\circ}(\xi) = \sum \alpha^{-k} \cdot (\overset{\circ}{a}{}^k(\overset{\circ}{\lambda} - \varepsilon_0))^{\wedge}(\xi) = \sum \alpha^{-k} \cdot (\widehat{\lambda}^{\circ}(\xi(k)) - 1)$.

Let $\nu_{\bullet}$ denote the semigroup on $\mathbb{G}$ defined by $\widehat{\nu}_t = e^{t\widehat{A}}$. Then by $\overset{\circ}{\nu}_t{}^{\wedge} := \exp(t\widehat{A}^{\circ})$ there is defined a continuous convolution semigroup $\overset{\circ}{\nu}_{\bullet} \subseteq \mathcal{M}^1(\mathbb{V})$ with $\pi(\overset{\circ}{\nu}_t) = \nu_t, t \geq 0$. (Fourier transforms $\widehat{A}$ resp. $\widehat{A}^{\circ}$ of generating functionals are logarithms of Fourier transforms of the generated probability measures, defined by $\widehat{\nu}_t = \exp(t \cdot \widehat{A})$ resp. $\widehat{\nu}_t^{\circ} = \exp(t \cdot \widehat{A}^{\circ})$. Hence $\widehat{A}^{\circ}$ is well defined, even if we avoided here to define generating functionals $\overset{\circ}{A}$ on $\mathbb{V}$.)
As immediately seen, $\nu_{\bullet}$ and $\overset{\circ}{\nu}_{\bullet}$ are (a, α)- resp. $(\overset{\circ}{a}, \alpha)$-semistable. But for any finite-dimensional projection $p : \mathbb{V} \to \mathbb{R}^I$ the Lévy measure of $p(\overset{\circ}{A})$ is concentrated on the compact subset $[0, 2\pi]^I \subseteq \mathbb{R}^I$, hence $p(\nu_{\bullet})$ can not be semistable.

Central limit laws and rescaled canonical random walks on Lie groups

6.1. Let $\mathbb{H}$ be a Lie group with Lie algebra $\mathbb{V}$. Let U and V be neighbourhoods of e and 0 in $\mathbb{H}$ and $\mathbb{V}$ respectively such that $\exp : V \to U$ is bijective. Let $\overset{\circ}{\gamma}_t$ be a Gaussian convolution semigroup on $\mathbb{V}$ with covariance I w.r.t. a basis $\{X_1, \ldots X_d\}$. Consider $\Delta = \frac{1}{2}\Sigma X_i^2$ as Laplacian on $\mathbb{H}$ and on $\mathbb{V}$ simultaneously. Hence Δ generates symmetric Gaussian semigroups $(\mu_{\bullet})$ in $\mathcal{M}^1(\mathbb{H})$ and $(\overset{\circ}{\gamma}_{\bullet})$ in $\mathcal{M}^1(\mathbb{V})$.
According to the usual central limit theorem (on vector spaces) $\overset{\circ}{\gamma}_t$ is representable as limit distribution of a canonical sequence of rescaled random walks:
Consider $\{\pm X_i : i = 1, \ldots d\}$, the nearest neighbours of 0 in $\mathbb{V}$ $(= \mathbb{R}^d)$. Let $(Y_j)_{j\geq 1}$ be a sequence of i.i.d. r.v. with distribution $\overset{\circ}{\nu}_0 = \frac{1}{2d}\sum \varepsilon_{\pm X_i}$. Then for all n $\{Y_j^{(n)} := n^{-1/2}Y_j\}_{n\geq 1}$ is an i.i.d. sequence on the (rescaled) lattice $n^{-1/2}\mathbb{Z}^d$ (w.r.t. the fixed basis $X_i, 1 \leq i \leq d$) with distribution $\overset{\circ}{\nu}_n = \frac{1}{2d}\sum \varepsilon_{\pm n^{-1/2}\cdot X_i}$.
Define $\xi_i(\cdot)$ to be the curves $\xi_i(t) := \exp(tX_i)_{t\in\mathbb{R}}$ in $\mathbb{H}$, put $\Psi_i^{(n)} := \exp(Y_i^{(n)})$, then $(\prod_{1\leq i\leq m} \Psi_i^{(n)})_{m\geq 1}$ is a sequence of random walks on $\mathbb{H}$ with distribution $\nu_n = \frac{1}{2d}\sum \varepsilon_{\xi_i(\pm n^{-1/2})}$. (In some sense rescaled nearest neighbour random walks, but not necessarily concentrated on sublattices of $\mathbb{H}$).
In $\mathcal{M}^1(\mathbb{V})$ the CLT yields convergence of distributions of the rescaled random walks $n^{-1/2}\sum_0^{[nt]} Y_j = \sum_0^{[nt]} Y_j^{(n)}$, $\overset{\circ}{\nu}_n{}^{[nt]} \to \overset{\circ}{\gamma}_t, t \geq 0$.
According to E. Siebert's characterization of limit laws (cf. e.g. [13], [5]) this is equivalent to $n \cdot (\overset{\circ}{\nu}_n - \varepsilon_0) \to \Delta$ (for C_b^{∞}-functions on $\mathbb{V}$ with support in V).
Since exp is (locally) bijective, again by Siebert's theorem this is equivalent to $n \cdot (\nu_n - \varepsilon_e) \to \Delta$ (for C_b^{∞}-functions on $\mathbb{H}$ with support in U).
And again we obtain equivalence to $\nu_n^{[nt]} \to \mu_t, t \geq 0$.

Hence *Gaussian distributions* μ_t on a Lie group $\mathbb{H}$ are representable as limits of distributions of the rescaled random walks $\prod_{1\le i\le[nt]} \Psi_i^{(n)}$, and vice versa.

6.2. Remark. If Δ is a sub-Laplacian then the corresponding Gaussian semigroup $\overset{o}{\gamma}_t$ and the random walks $\overset{o}{\nu}_n{}^m$ are concentrated on a subspace of $\mathbb{V}$. But μ_t may have full support on $\mathbb{H}$.

6.3. Let $\mathbb{G}$ be a connected compact group with Lie algebra $\mathbb{V}$. $\mathbb{G}$ and $\mathbb{V}$ are projective limits $\mathbb{G} = \lim_{\leftarrow} \mathbb{G}^\alpha, \mathbb{G}^\alpha = \mathbb{G}/K_\alpha$, resp. $\mathbb{V} = \lim_{\leftarrow} \mathbb{V}^\alpha$. For fixed α let $\{X_1^\alpha, \dots X_{d_\alpha}^\alpha\}$ be a basis of $\mathbb{V}^\alpha$, let $\exp_\alpha : \mathbb{V}^\alpha \to \mathbb{G}^\alpha$ be the exponential mapping.
Let $(\mu_t)_{t\ge0}$ be a Gaussian convolution semigroup on $\mathbb{G}$ and let for fixed α μ_t^α be the projected measures on $\mathbb{G}^\alpha$ with Laplacian $\Delta^\alpha = \sum(X_i^\alpha)^2$. And let $\overset{o}{\gamma}_t{}^\alpha$ be defined analogously. (W.l.o.g. we assume the basis of $\mathbb{V}^\alpha$ to be suitably chosen.) According to step 6.1 there exist random walks $(\nu_n^\alpha)^{[nt]}$ on $\mathbb{G}^\alpha$ and $(\overset{o}{\nu}_n{}^\alpha)^{[nt]}$ on $\mathbb{V}^\alpha$ converging to μ_t^α resp. to $\overset{o}{\gamma}_t{}^\alpha, t \ge 0$.

In particular we are interested in the following
6.4. Example. a) If $\mathbb{G} = \prod G_n$ is a product of compact connected Lie groups $G_n, n \in \mathbb{N}$, then we obtain a projective basis $\{X_i : i \ge 1\}$ of $\mathbb{V}$, such that $\{X_i : d_n + 1 \le i \le d_{n+1}\}$ is a basis of G_n, hence $\{X_i : 1 \le i \le d_{i+1}\}$ is a basis of $\prod_{1\le j\le n} G_j =: \mathbb{G}^n$.
If $\mu_t = \otimes_{k\in\mathbb{Z}} \mu_t^{(n)}$ is a product of Gaussian semigroups $\mu_t^{(n)} \in \mathcal{M}^1(G_n)$ with (Laplacian) generating functional Δ then the basis $\{X_i\}$ can be chosen in such a way that the Laplacians Δ_n corresponding to the projection $\mu_t^{(n)}$ to $\mathbb{G}^n$ have the form $\sum_1^{d_n} X_i^2$.
In this case the approximating random walks admit a construction without making explicit use of the particular Lie groups: Elements of $\mathbb{V}$ may be represented as sequences $(c_j) \in \mathbb{R}^{\mathbb{Z}}$, formally as $\sum c_j X_j$. The random walks defined on $\mathbb{G}^n$ according to 6.3 form a projective family $(\overset{o}{\pi}_m(\overset{o}{\nu}_n)^{[nt]})_{m=1,2,\dots}$, where $\overset{o}{\pi}_m : \mathbb{V} \to \mathbb{V}^m$ denote the canonical projections and $\overset{o}{\nu}_n \in \mathcal{M}^1(\mathbb{V})$ are of product form $\otimes_{k\in\mathbb{N}} \overset{o}{\nu}_n{}^{(k)}$.
Note that $\mathbb{V} = \lim_{\leftarrow} \mathbb{V}^n$ is a nuclear vector space, hence the projective families define probabilities $\overset{o}{\nu}_n$ on $\mathbb{V}$. And analogously, $(\pi_m(\nu_n)^{[nt]})_{m=1,2,\dots}$ form a projective spectrum on $\mathbb{G}$ with $\nu_n = \otimes_{k\in\mathbb{N}} \nu_n^{(k)}$.
Furthermore, $\overset{o}{\pi}_m(\overset{o}{\nu}_n)^{[nt]} \to \overset{o}{\pi}_m(\gamma_t)$, $t \ge 0$, iff $\pi_m(\nu_n)^{[nt]} \to \pi_m(\mu_t)$, $t \ge 0$, for all $m \in \mathbb{N}$. But this is equivalent to the convergence $\nu_n^{[nt]} \to \mu_t, t \ge 0$.

Putting things together, for *Gaussian laws* we obtain equivalence of convergence of the random walks on $\mathbb{G}$ and $\mathbb{V}$ respectively, in other words,

$$\overset{o}{\nu}_n{}^{[nt]} \to \overset{o}{\gamma}_t, t \ge 0 \quad \text{iff} \quad \nu_n^{[nt]} \to \mu_t, t \ge 0 \qquad (*)$$

b) If we are in a situation analogous to 6.1, i.e. if $\mathbb{G} = K^{\mathbb{Z}}$, $\Delta = \Delta_0$ is a Laplacian on $K = K^{(0)}$, and $\Delta_n := \alpha^{-n} \cdot a^n(\Delta), n \in \mathbb{Z}$, ($a$ denoting again the shift), then the limits $\mu_\bullet$ and $\overset{o}{\gamma}_\bullet$ are Gaussian and (a, α)- resp. $(\overset{o}{a}, \alpha)$-semistable on $\mathbb{G}$ and $\mathbb{V}$ respectively.
In this situation, as easily seen, $\overset{o}{\nu}_n{}^{(k)}$ and $\nu_n^{(k)}$ in a) are representable as $(2d)^{-1} \cdot \sum_1^d \varepsilon_{\pm\alpha^{-k/2} \cdot n^{-1/2} \cdot X_i}$ and $(2d)^{-1} \cdot \sum_1^d \varepsilon_{\xi_i(\pm\alpha^{-k/2} \cdot n^{-1/2})}$, shifted by $\overset{o}{a}{}^k$ and a^k respectively. And we obtain $(*)$ with $\nu_k = \otimes_{k\in\mathbb{Z}} \nu_n^{(k)}$ and $\overset{o}{\nu}_k = \otimes_{k\in\mathbb{Z}} \overset{o}{\nu}_n{}^{(k)}$.

6.5. Remark. Note that the equivalence (∗) can only be proved for Gaussian limits: The construction makes heavy use of the fact that for finite-dimensional projections (at least for large n) $\mathrm{supp}(\nu_n)$ and $\mathrm{supp}(\overset{o}{\nu}_n)$ are contained in neighbourhoods U and V on which exp is bijective. Hence considering finite-dimensional projections we conclude that the limits have to be Gaussian:
If $\mathbb{H}$ is a compact connected Lie group with Lie algebra $\mathbb{V}$ and $\exp : V \to U$ bijective then (U and) V must be bounded. Hence in particular, $\overset{o}{\nu}_n$ being concentrated on V has finite second moments. And therefore, if $\overset{o}{\nu}_n^{[nt]} \to \overset{o}{\gamma}_t$, $t \geq 0$, for some convolution semigroup $\overset{o}{\gamma}_\bullet$, then $\overset{o}{\nu}_n$ belongs to the domain of attraction of $\overset{o}{\gamma}_t$ and has finite second moments, hence the limit must be Gaussian.

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