Harmonic Analysis on Complex Random Systems

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Abstract White noise analysis has an aspect of harmonic analysis arising from the infinite dimensional rotation group $O(E)$ which is formed by all the linear isomorphisms of a basic nuclear space $E \subset L^2(R^d)$. In fact, the white noise measure μ is kept invariant under the action of the group $O^*(E^*)$ consisting of the adjoint transformations g^* of the members g in $O(E)$.

In this report, particular attentions will be paid to a subgroup generated by the so-called whiskers. A whisker, we mean, is a continuous one-parameter subgroup $\{g_t\}$ of $O(E)$, where each member g_t comes from a diffeomorphism of the time (or space-time) parameter space of the white noise. The most important whisker is the time shift. With this choice of a whisker, one can define a one-parameter unitary group $\{U_t\}$ acting on the Hilbert space $L^2(E^*, \mu)$ and speak of the spectral multiplicity. This notion enables us to consider a sort of degree of complexity of random evolutional phenomena that propagate as the time or space-time parameter moves.

Another interesting subgroup of $O(E)$ is the conformal group $C(d)$ generated by certain various whiskers involving the shift. The group structure of $C(d)$ is well known, since it is locally isomorphic to the Lie group $SO(d+1,1)$, so that it is ready to be applied to white noise theory. Indeed, this group $C(d)$ plays important roles, in particular, in the investigations of reversibility and of variations of a random field $X(C)$ when C is deformed by the action of the group $C(d)$.

Together with some other significant examples of whiskers, we can carry on an **essentially infinite dimensional harmonic analysis** in line with the white noise analysis.

§1. Introduction and background

The subject of *harmonic analysis on white noise space* has undergone a vast development: Laplacians, Fourier transform and operator theory in general. While, complexity or complex system is proposing interesting future directions in various fields in science. We shall, in this note, focus our attention to random phenomena, namely *random complex systems* and in fact, they can be discussed in line with white noise analysis. Note that the white noise analysis has an aspect of an infinite dimensional harmonic analysis that arises from the infinite dimensional rotation group. Thus, our present aim is to investigate complex random systems expressed in terms of white noise by appealing to the theory of infinite diemsional rotation group.

We shall briefly review the white noise space and the rotation group as background.

White noise is a measure space (E^*, μ), where E^* is a space of genralized functions on R^d and it is taken to be the dual space of some nuclear space E, and where μ is a measure on E^* determined by a characteristic functional

$$C(\xi) = \exp\left[-\frac{1}{2}\|\xi\|^2\right], \quad \xi \in E.$$

Set $(L^2) = L^2(E^*, \mu)$. Then, we have a Fock space:

$$(L^2) = \oplus_n H_n.$$

A Gel'fand triple

$$(S) \subset (L^2) \subset (S)^*$$

defines the space $(S)^*$ of *generalized white noise functionals.*

To have a visualized expression of $(S)^*$-functional φ is an S-transform (Kubo–Takenaka) defined by

$$(S\varphi)(\xi) = C(\xi) \int \exp[\langle x, \xi \rangle]\varphi(x) d\mu(x).$$

The S-transform is usuful to define operators, like annihilation operator ∂_t and creation operator ∂_t^*, that act on the space $(S)^*$. Indeed, S is a bijective mapping from $(S)^*$ to its range.

We then come to the rotation group $O(E)$ of E. Let g be a linear homeomorphism of E such that

$$\|g\xi\| = \|\xi\|, \quad \xi \in E.$$

Then, g is called a *rotation* of E. The collection $O(E)$ of all rotations of E forms a group under the usual product. Also, the compact-open topology is introduce to $O(E)$, so that it is a topological group.

Definition. The topological group $O(E)$ is called the *rotation group* of E. If E is not specified, it is called an *infinite dimensional rortation group* and is denoted by O_∞.

Let g^* be the adjoint operator of g. Necessarily g^* is a continuous linear operator acting on the space E^*.

Proposition. *The group $O^*(E^*)$ is isomorphic to $O(E)$ under the correspondence $g^* \leftrightarrow g^{-1}$.*

With the help of the characteristic functional we can prove

Theorem 1. *The white noise measure μ is invariant under the action of the group $O^*(E^*)$:*

$$g^* \cdot \mu = \mu.$$

Hence, the operator U_g given by

$$U_g\varphi(x) = \varphi(g^*x)$$

is unitary. We can therefore introduce the unitary representation of the group $O(E)$ on the Hilbert space (L^2).

§2. Subgroups of $O(E)$ and their roles

The group $O(E)$ is, in a sense, quite big; in fact, it is not even locally compact, and its structure is very complex. It would be a good idea to take subgroups separately and investigate their roles in white noise analysis.

B. Finite dimesional subgroups

Take a finite dimensional subspace, say E_n isomorphic to R^n. The collection of rotations g such that their restrictions to E_n are its rotations and identity on $E_n^{\perp}$ forms a subgroup, denoted by G_n. Obviously, G_n is isomorphic to the linear group $SO(n)$.

I. Hyperfinite dimensional subgrpoup

Set

$$G_\infty = \vee_n G_n.$$

Then, the infinite dimensional Laplace-Beltrami operator Δ_∞ is determined by the subgroup G_∞ and is expressed in the form

$$\Delta_\infty = \int \partial_t^* \partial_t dt.$$

Also, we can prove (see [2]) the unitary representation $\{U_g, g \in G_\infty\}$ on H_n, $n \geq 1$, is irreducible. As a result, Δ_∞ takes a constant value, in fact $-n$, on the subspace H_n.

II. Infinite dimensional subgroup: The Lévy group

As is well known the Lévy group $\mathcal{G}$ is essentially infinite dimensional. Its action can generally not be approximated by finite dimensional rotations. Contrary to the case I above, the Lévy Laplacian Δ_L acts effectively on the space $(S)^*$ and annihilates the basic space (L^2). There is a formal expression (due to H.-H. Kuo) of the Lévy Laplacian that helps to understand its actions.

$$\Delta_L = \int (\partial_t)^2 (dt)^2.$$

It is noted that the subgroups that have appeared so far depend on the choice of a complete orthonormal system for $L^2(R^d)$.

III. Ultra infinite dimensional subgroups: Whiskers

There are significant one-parameter subgroups that come from the diffeomorphisms of the parameter space R^d. They are called whiskers. The most important whisker is the shift. Define S_t^j by

$$S_t^j \xi(u) = \xi(u - te_j), \quad \xi \in E; \quad t \in R; \quad j = 1, 2, ..., d,$$

where e_j is the j-th coordinate vector of R^d. There are many other whiskers that have good relations (commutation relations) with shift. A significant class of whiskers is isomorphic to the conformal group $C(d)$.

As we shall discuss in what follows, the shift expresses the change of time or space-time and illustrates the propagation of random phenomena.

§3. Complex systems

What we shall be concerned with are random complex systems which are time-oriented or space-time-oriented. Assume further that the systems in question are functionals of white noise. This means that we tacitly assume that white noise input is provided behind the system. The observed data shall be expressed as a stochastic process $X(t)$ depending on the time t or a random field $X(C)$ indexed by a manifold C, say a contour, that runs through a Euclidean space. Mathematically they are functionals, maybe generalized functionals, of white noise.

There are various approaches to those random complex systems; among others we propose the *innovation approach*. The original idea came from P. Lévy's paper [4], where he has proposed a *stochastic infinitesimal equation* for a stochastic process $X(t)$. This can also be extended to the case of a random firld $X(C)$, although the existence of the proposed equation can not always be expected. With the help of the innovation we can measure thr complexity of random complex systems. In some cases we can form the innovation for our purpose, and they are now in order.

Starting from a Brownian motion or a white noise , which is a basic elementary stochastic process or generalized stochastic process, resp., we discuss functions of Brownian motion (or white noise) taking the time development (shift) into account.

1) Gaussian system

Let $X(t)$ be a Gaussian process with mean $E(X(t)) = 0$. Assume that $X(t)$ is separable and has *unit multipliocity* in the time domain. Then, there exists a white noise $\dot{B}(t)$ such that

$$X(t) = \int^t F(t,u)\dot{B}(u)du,$$

where $F(t,u)$ is a non random kernel function. In addition, $\{X(u), u \leq t\}$ has the same information as $\{\dot{B}(u), u \leq t\}$ for every t. A representation satisfying these conditions is called *canonical.*

The notion of multiplicity can be understood in such a way that associated with each t is a projection $E(t)$ corresponding to the space spanned by the variables $X(s)$, $s \leq t$, (if necessary $E(t)$ is modified so as to be right continuous) so that the spectrum as well as the (spectral) multiplicity can be defined by the Helliger–Hahn theorem.

The unit multiplicity means that the given Gaussian process represented by a single Brownian motion (white noise) which we could call an elemental stochastic process. There are many Gaussian processes with higher multiplicity and number of the multiplicity expresses the "degree of complexity."

2) Nonlinear functionals of white noise

There are a lot of significant stochastic processes that are expressed by nonlinear functionals of a white noise (Brownian motion). There is requested a calculus, called white noise analysis, where a white noise $\{\dot{B}(t)\}$ is taken to be the system of variables.

In order to establish the causal calculus of complex systems of the above form of a stochastic process, it is necessary to generalize the notion the multiplicity. Namely, a one-parameter unitary group $\{U(t), t \in R\}$, acting on the space of white noise functionals and representing the time propagation, is introduced. Actually, $U(t)$ is defined so as to hold the relation $U(t)\dot{B}(s) = \dot{B}(t+s)$.

Once the unitary group is introduced, one can see a cyclic subspace of the form

$$H(f) = \mathrm{span}\{U(t)f, t \in R\}.$$

Again the Hellinger-Hahn theorem claims that there is a system $\{H(f_n); n = 1, 2....\}$ such that it is an orthogonal system and that the entire complex system in question is expressed as the direct sum of those cyclic subspaces. Those subspaces are arranged in the order of the spectral measures. The number of the cyclic subspaces is the *multiplicity* in the general sense. This multiplicity is different from the Gaussian case, but it also serves to the measurement of complexity.

Remark. A stochastic process formed by some nonlinear functional for which its innovation is actually obtained (see [3]) can be discussed directly for degree of complexity.

Example. The Wiener expansion. There is a famous application called the Wiener expansion. We want to identify an unknown system that permits white noise input as is illustrated below.

input $\longrightarrow$ nonlinear system $\longrightarrow$ output

Let the known nonlinear systems be provided in advance. If the same input as that to the nonlinear system is given, then their outputs can be compared to those of the unknown system. Thus, the Wiener expansion provides a tool to identify a random complex system that admits white noise input. Nonlinear system has usually infinite multiplicity which means we need, theoretically speaking, infinitely many known systems.

§4. Reversibility and irreversibility: Roles of whiskers

Reversibility and irreversibility of random evolutional phenomena may be expressed in terms of the $\dot{B}(t)$ instead of the time parameter t itself and both properties are defined with respect to the conformal transformations mapping a time interval onto another in a time reverse order.

We start our discussion with a simple example in Gaussian case where the time interval is taken to be $[0, 1]$ to fix the idea.

1) A Brownian motion $\{B(t), t \in [0, 1]\}$ is certainly irreversible, since it is an accumulated sum of independent variables $\dot{B}(t)$'s at every instant t, and both variance and entropy increase as t proceeds.

2) Let a Brownian motion $B(t)$ be pinned at $t = 1$ to a position c, namely let $B(1) = c$. Then, we are given a Gaussian process, denoted by $X_c(t)$. The reversibility maybe understood to be an invariant property of a process under the simple time reflection. If so, we have

Proposition. *The probability distributipon of $X_0(t)$, $t \in [0, 1]$, is invariant under the time refelection: $t \mapsto 1 - t$.*

Proof easily comes from the computation of the covariace function:

$$\Gamma(t,s) = (t \wedge s)\{(1-t) \wedge (1-s)\}.$$

There are observations.

1. It is easily seen that a Brownian motion $B(t)$, which is an irreversible process, is viewed as a superposition of reversible processes $X_c(t), c \in R^1$, with the weight of the standard Gaussian measure $g(1,c)dc$ to which $B(1)$ is subject.

2. The (forward) canonical representation of $X(t)$ is expressed in the form

$$X_1(t) = (1-t)\int_0^t \frac{1}{1-u}\dot{B}_1(u)du, \quad t \in [0,1].$$

The above $B_1(t)$ is a new Brownian motion that has the same information as $X_1(t)$. While, the reversal canonical representation is given by

$$X_2(t) = t\int_t^1 \frac{1}{u}\dot{B}_2(u)du, \quad t \in [0,1].$$

Two representations given above express the same Brownian bridge as a Gaussian process and they are linked by the projective transformation of the parameter t (see [2:Chapter 5]). There, a role of whiskers can be seen.

The reversibility of a Gaussian process $X(t)$ in white noise analysis is to be considered in terms of the innovation. Since the time domain is limited to a finite interval, the innovation should be formed locally in time. This implies that there is a differential operator L_t such that

$$L_t X(t) = \dot{B}(t).$$

Now the reversibility of a Gaussian process may be dealt with as follows.

a) We understand that a Brownian bridge is an *elemental reversible Gaussian process.* Thus, starting from a Brownian bridge we may consider general reversible Gaussian processes.

b) We generalize the reversible property in such a way that the canonical kernels of forward and reversal representations are linked by conformal transformations.

Thus, in the present situation we may assume that

c) the system of the fundamental solutions of the differential equation

$$L_t f = 0$$

consists of polynomials in $(t-1)$.

Summing up we now have

Theorem 2. *Let a bridged Gaussian process $X(t)$ satisfy the conditions a), b) and assumption c). Assume that the order of the differential operator L_t is N uniformly in t. Then, the process $X(t)$ is reversible.*

PROOF. By assumption, we have the canonical representation of $X(t)$ (see [1]):

$$X(t) = \int_0^t R(t, u)\dot{B}(u)du,$$

where $R(t, u)$ is Riemann's function of the form

$$R(t, u) = \sum_{k=1}^{N} a_k \frac{(1-t)^k}{(1-u)^k},$$

where we may assume $a_1 = 1$ so that all the a_k's are uniquely determined. Then, as a generalization of the Proposition a conformal map of the interval $[0, 1]$ defines a new representation of $X(t)$ by using the forward and reversal canonical representations.

§5. Concluding remark

With a generalization explained at the end of the last section, we are suggested to think of reversibility of a random field $X(C)$. To fix the idea, C is taken to be a contour in the plane. To discuss reversibility, it is necessary to have an oriented family $\mathbf{C}$ of contours. Denote it by $\mathbf{C} = \{C_t, t_0 \le t \le t_1\}$ with the order $C_s < C_t$ for $s < t$ denoting C_s is inside of C_t. Most important requirement is that the C_t expands as t increases from C_0 to C_1 smoothly by the action of continuous family $\{g_t\}$ of conformal transformations. With this setup a reversibility of $X(C)$ can be discussed, where $X(C)$ is an integral of white noise over the domain (C) enclosed by a contour C (cf. causality).

It seems to be interesting to note that $X(C_t)$, $t_0 \le t \le t_1$, denotes a trajectory (path) of a Gaussian random field and on the set of the trajectories a Gaussian measure is naturally introduced. It is, therefore, our hope that we are ready to apply to the path integral. Actual computations have been given in the case where $\{C_T\}$ is a family of concentric circles.

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