Hypergroup Actions and Wavelets

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Abstract

In analogy to wavelet transforms, we use group–like structures in order to introduce a class of integral transformations. We consider them in the context of Hilbert spaces and study their inversion.

0 Introduction

Wavelet analysis was introduced as a mathematical tool by A. Grossmann, J. Morlet, and T. Paul in [4] and was motivated by applications in signal processing. Many examples of important transformations can be recognized as wavelet transforms or are closely related to them (see [5], [6]). The mathematics of wavelet transform, as given in [5], is based on the theory of square integrable representations of locally compact groups and has a considerable range of generality.
In this paper we consider some integral transformations of wavelet type acting on the space of square integrable functions on a *commutative hypergroup*. They generalize the *classical wavelet transform* and the *windowed Fourier transform*. This work was motivated by a preprint of M. Rösler [10] and a series of papers by K. Trimèche (see [12], [13], [14], [15], [11]).

The first section recalls some results about commutative hypergroups. In the second section we define the left–transform. In the third section we discuss some special cases of the left–transform corresponding to transitive group actions.

1 Commutative hypergroups

Throughout this paper the following notation will be used: Let K be a locally compact space and denote by $C_b(K)$, $C_0(K)$, and $C_c(K)$ the spaces of continuous functions on K which are bounded, vanishing at infinity, and with compact support respectively. The symbol $M(K)$ denotes the space of Borel measures on K, $M_+(K)$, $M^b(K)$, and $M_+^b(K)$ are its subsets consisting of positive, bounded, and bounded positive measures, respectively. The σ–algebra of Borel measurable sets of K is denoted by $\mathcal{B}(K)$.

The notion of a hypergroup generalizes that of a locally compact group. (For additional reading on hypergroups we recommend [1] and [7].) A hypergroup K is a locally compact topological space with an axiomatically defined convolution $*$ on the Banach space $M^b(K)$ of bounded measures. With this operation, $M^b(K)$ forms a Banach algebra. The convolution $*$ satisfies several requirements which are natural for locally compact groups: For example, $*$ is weakly continuous, the convolution of probability measures is again a probability measure, there exists $e \in K$ such that the Dirac measure ε_e is the unit of the algebra $(M^b(K), *)$. Furthermore, there also exists a homeomorphism $^- : K \to K$ with $\int_K f(z^-)\varepsilon_x * \varepsilon_y(dz) = \int_K f(z)\varepsilon_{y^-} * \varepsilon_{x^-}(dz)$ for all $x, y \in K$, $f \in C_b(K)$. (In the case that K is a group, $^-$ is given by inversion.)

The hypergroup K is commutative if the algebra $(M^b(K), *)$ is commutative. If K is commutative then there exists (up to a constant) a uniquely determined measure $m \in M_+(K)$ satisfying $\varepsilon_x * m = m$ for all $x \in K$; m is called the *Haar measure.* As in the case of groups, family $(T_x)_{x\in K}$ of translation operators can be defined: For each $x \in K$ the corresponding T_x acts on suitable classes of functions by $f \mapsto T_x f$, $(T_x f)(y) = \int_K f d\varepsilon_x * \varepsilon_y$. Translation operators are contractions on $L^2(K, m)$ and $T_x^* = T_{x^-}$ holds for all $x \in K$. For commutative hypergroups, a Fourier transform and a Plancherel identity are available. A bounded measurable function $\chi : K \to \mathbb{C}$ is called *character*, if $\chi(e) = 1$, $\overline{\chi(x)} = \chi(x^-)$, and $T_x\chi = \chi(x)\chi$ are satisfied for all $x \in K$. The set $\widehat{K}$ of characters is endowed with the compact open topology. The Fourier transform $L^1(K, m) \to C_0(\widehat{K})$, $f \mapsto \hat{f}$ is defined by $\hat{f}(\chi) := \int_K \overline{\chi(x)} f(x) m(dx)$. There exists a unique measure $\pi \in M_+(\widehat{K})$ (*the Plancherel measure*), such that the Fourier transform maps $L^1(K, m) \cap L^2(K, m)$ into $L^2(\widehat{K}, \pi)$ L^2–isometrically; it can be extended to a unitary operator $\mathcal{F} : L^2(K, m) \mapsto L^2(\widehat{K}, \pi)$. Similarly, the inverse Fourier transform $L^1(\widehat{K}, \pi) \to C_0(K)$, $g \mapsto \check{g}$, $\check{g}(\chi) := \int \chi(x)g(\chi)\pi(d\chi)$ maps $L^1(\widehat{K}, \pi) \cap L^2(\widehat{K}, \pi)$ into $L^2(K, m)$ also L^2–isometrically. Its extension to $L^2(\widehat{K}, \pi)$ is the unitary operator $\mathcal{F}^{-1}$. We point out that in general the support S of the Plancherel measure is a *proper* subset of $\widehat{K}$. Translation operators are diagonalized by $\mathcal{F}$ in the following sense: For all $x \in K$ the operator $\mathcal{F}T_x\mathcal{F}^{-1}$ acts on $L^2(\widehat{K}, \pi)$ as the multiplication by the function $\widehat{K} \to \mathbb{C}$, $\chi \mapsto \chi(x)$.

We explain the basic idea of this paper by means of the examples of the classical wavelet transform and of the windowed Fourier transform on $\mathbb{R}$:

1. Given a function $0 \neq v \in L^2(\mathbb{R})$, we define $L_v : L^2(\mathbb{R}) \to C(\mathbb{R}\times\mathbb{R}\setminus\{0\})$, $h \mapsto L_v h$ as
$$(L_v h)(b, a) = \int \frac{1}{|a|}\overline{v(\frac{r}{a})}h(r + b)dr, \qquad \forall \qquad h \in L^2(\mathbb{R}),$$
$b \in \mathbb{R}$, and $a \in \mathbb{R} \setminus \{0\}$. The function $(b, a) \mapsto (L_v h)(b, a)$ is up to the factor $(b, a) \mapsto |a|^{\frac{1}{2}}$, the usual wavelet transform of h. Let us introduce on $L^2(\mathbb{R})$ the families $(T_b)_{b\in\mathbb{R}}$ and $(D_a)_{a\in\mathbb{R}\setminus\{0\}}$ of translation and dilation operators respectively as $(T_b f)(r) := f(b + r)$, $(D_a f)(r) := \frac{1}{|a|}f(\frac{r}{a})$ for all $f \in L^2(\mathbb{R})$, $r \in \mathbb{R}$. With these operators we may write $(L_v h)(b, a) = \langle D_a v, T_b h\rangle$ for all $h \in L^2(\mathbb{R})$, $b \in \mathbb{R}$, $a \in \mathbb{R} \setminus \{0\}$.

2. Given a function $0 \neq v \in L^2(\mathbb{R})$ we define the transform $W_v : L^2(\mathbb{R}) \to C(\mathbb{R}\times\mathbb{R})$,

as $(W_v h)(b,a) = \int_{\mathbb{R}} e^{iar}\overline{v(r)}h(r+b)dr$, which is up to a factor the windowed Fourier transform. Again, using dilation (in this case modulation) operators $(D'_a)_{a\in\mathbb{R}}$ given by $(D'_a f)(r) := e^{-iar}f(r)$ for all $f \in L^2(\mathbb{R})$, $r \in \mathbb{R}$, and $a \in \mathbb{R}$, the transform W_v may be written as $(W_v h)(b,a) = \langle D'_a v, T_b h\rangle$ for all $h \in L^2(\mathbb{R})$, $a, b \in \mathbb{R}$.

The following remarkable observation should be pointed out: If we define the actions β and β' of the groups $(\mathbb{R}\setminus\{0\},\cdot)$ and $(\mathbb{R},+)$ on the dual $\widehat{\mathbb{R}}$ of $\mathbb{R}$ as

$$\begin{array}{ll} \beta : \widehat{\mathbb{R}} \times \mathbb{R}\setminus\{0\} \mapsto \widehat{\mathbb{R}} & \beta(\chi, a) := \chi \cdot a, \\ \beta' : \widehat{\mathbb{R}} \times \mathbb{R} \mapsto \widehat{\mathbb{R}} & \beta'(\chi, a) := \chi + a, \end{array}$$

then for each $g \in L^2(\mathbb{R})$ we obtain all dilations $(D_a)_{a\in\mathbb{R}\setminus\{0\}}$ as $\mathcal{F}D_a\mathcal{F}^{-1}g = g(\beta(.,a))$ and dilations $(D'_a)_{a\in\mathbb{R}}$ as $\mathcal{F}D'_a\mathcal{F}^{-1}g = g(\beta'(.,a))$. In both cases the dilations are unitarily equivalent via $\mathcal{F}$ to operators on $L^2(\widehat{\mathbb{R}})$, induced by an action of a group on $\widehat{\mathbb{R}}$.

Motivated by this observation we start with a commutative hypergroup K, a function $v \in L^2(K,m)$, and an action β of a locally compact group G on $\widehat{K}$. We study the linear operator $L_v : L^2(K,m) \to \mathbb{C}^{K\times G}$, given by $(L_v h)(b,a) := \langle D_a v, T_b h\rangle$ for all $h \in L^2(K,m)$, $(b,a) \in K \times G$. Here $(D_a)_{a\in G} \subset B(L^2(K))$ are dilations defined by $\mathcal{F}D_a\mathcal{F}^{-1}g := g(\beta(.,a))$ for all $g \in L^2(\widehat{K},\pi)$, and $(T_b)_{b\in K}$ are the usual translations of the hypergroup K.

2 The left–transform

Let (K,m) be a commutative hypergroup K equipped with a fixed Haar measure m. We assume that a locally compact group G acts continuously on the support of the Plancherel measure $S = \operatorname{supp}\pi \subset \widehat{K}$: That means that there exists a continuous mapping $\beta : S \times G \to S$, $(\chi, a) \mapsto \chi^a$ satisfying $(\chi^{a_1})^{a_2} = \chi^{a_1 a_2}$ for all $\chi \in S$ and $a_1, a_2 \in G$.

Let μ be a fixed left Haar measure of G. We introduce the set $\{\mu^\chi : \chi \in S\}$ of image measures of μ induced by the mappings $G \to S$, $a \mapsto \chi^a$: For each $\chi \in S$ we obtain $\mu^\chi(B) = \mu(\{a \in G : \chi^a \in B\})$ for all $B \in \mathcal{B}(S)$. Let us also define the set $\{\pi^a : a \in G\}$ of image measures of $\pi|_S$ induced by the mappings $S \to S$, $\chi \mapsto \chi^a$. For each $a \in G$ we obtain $\pi^a(B) = \pi(\{\chi \in S : \chi^a \in B\})$ for all $B \in \mathcal{B}(S)$. We suppose the following assumption to be satisfied:

Assumption 1. *For all $a \in G$ the measure π^a is absolutely continuous with respect to $\pi|_S$ and the corresponding Radon–Nikodym derivative satisfies $\frac{d\pi^a}{d\pi|_S} \in L^\infty(S,\pi|_S)$.*

For each $a \in G$ and $f \in \mathbb{C}^S$ we define the function $f^a \in \mathbb{C}^S$ as $f^a(\chi) := f(\chi^a)$ for all $\chi \in S$. Due to the above assumption, the mapping $f \mapsto f^a$ defines a continuous linear operator $L^2(S,\pi|_S) \to L^2(S,\pi|_S)$ for each $a \in G$. Since the Hilbert spaces $L^2(S,\pi|_S)$ and $L^2(\widehat{K},\pi)$ are naturally isomorphic we may consider the mapping $f \mapsto f^a$ as a continuous linear operator on $L^2(\widehat{K},\pi)$.

Definition.
(i) The operators $(D_a)_{a\in G} \subset B(L^2(K,m))$, defined by $D_a : L^2(K,m) \to L^2(K,m)$, $h \mapsto \mathcal{F}^{-1}(\mathcal{F}h)^a$ for all $a \in G$, are called dilation operators.

(ii) For each $v \in L^2(K,m)$, the linear mapping $L_v : L^2(K,m) \to \mathbb{C}^{K\times G}$, $h \mapsto L_v h$, given by $(L_v h)(b,a) := \langle D_a v, T_b h\rangle$ for all $(b,a) \in K \times G$, is called the left-transform corresponding to v.
(iii) The elements of $\mathcal{A} := \{v \in L^2(K,m) : (\chi \mapsto \int_G |\mathcal{F}v(\chi^a)|^2 \mu(da)) \in L^\infty(S, \pi|_S)\}$ are called admissible vectors. The elements of $\mathcal{A}\setminus\{0\}$ are called wavelets.

Given $v_1, v_2 \in \mathcal{A}$, we define $C_{v_2,v_1} : S \to \mathbb{C}$ as $C_{v_2,v_1}(\chi) := \int_G \overline{(\mathcal{F}v_2)(\chi^a)}(\mathcal{F}v_1)(\chi^a)\mu(da)$ for all $\chi \in S$. It follows from Cauchy–Schwarz inequality that $C_{v_2,v_1} \in L^\infty(S, \pi|_S)$. We remark that the function C_{v_2,v_1} is constant on each orbit:

Lemma 1. *For all $v_1, v_2 \in \mathcal{A}$, $\tilde{\chi} \in S$ and $\chi_0 \in \beta(\tilde{\chi}, G)$ we have $C_{v_2,v_1}(\chi_0) = C_{v_2,v_1}(\tilde{\chi})$.*

Proof. For $\chi_0 \in \beta(\tilde{\chi}, G)$ there exists $a_0 \in G$ with $\chi_0 = \tilde{\chi}^{a_0}$, and it follows that

$$C_{v_2,v_1}(\chi_0) = \int_G \overline{(\mathcal{F}v_2)(\chi_0^a)}(\mathcal{F}v_1)(\chi_0^a)\mu(da) = \int_G \overline{(\mathcal{F}v_2)(\tilde{\chi}^{a_0 a})}(\mathcal{F}v_1)(\tilde{\chi}^{a_0 a})\mu(da).$$

We conclude that

$$\int_G \overline{(\mathcal{F}v_2)(\tilde{\chi}^{a_0 a})}(\mathcal{F}v_1)(\tilde{\chi}^{a_0 a})\mu(da) = \int_G \overline{(\mathcal{F}v_2)(\tilde{\chi}^a)}(\mathcal{F}v_1)(\tilde{\chi}^a)\mu(da) = C_{v_2,v_1}(\tilde{\chi})$$

since μ is a left Haar measure on G. (The same argument implies that $\mu^\chi = \mu^{\tilde{\chi}}$ for $\chi \in \beta(\tilde{\chi}, G)$). ■

For an admissible vector v the left–transform can actually be discussed in the framework of Hilbert spaces:

Proposition 1:
(i) Given $v \in \mathcal{A}$ the mapping $h \mapsto L_v h$ defines a bounded linear operator from $L^2(K,m)$ into $L^2(K \times G, m \otimes \mu)$.
(ii) $\langle L_{v_1} h_1, L_{v_2} h_2\rangle = \int_S \overline{(\mathcal{F}h_1)(\chi)}(\mathcal{F}h_2)(\chi) C_{v_2,v_1}(\chi)\pi(d\chi)$ holds for all $v_1, v_2 \in \mathcal{A}$ and $h_1, h_2 \in L^2(K,m)$.

Proof. (i) Let $v \in \mathcal{A}$ and $h \in L^2(K,m)$. The function $L_v h$ is measurable since

$$\begin{aligned} L_v h(b,a) &= \langle D_a v, T_b h\rangle = \langle \mathcal{F}D_a v, \mathcal{F}T_b h\rangle = \int_{\widehat{K}} \overline{(\mathcal{F}v)^a(\chi)}\chi(b)(\mathcal{F}h)(\chi)\pi(d\chi) \\ &= \int_S \overline{(\mathcal{F}v)(\chi^a)}\chi(b)(\mathcal{F}h)(\chi)\pi(d\chi), \end{aligned}$$

and the integrand $K \times G \times S \to \mathbb{C}$, $(b,a,\chi) \mapsto \overline{(\mathcal{F}v)(\chi^a)}\chi(b)(\mathcal{F}h)(\chi)$ is measurable in view of continuity of $\beta : (\chi, a) \mapsto \chi^a$.

Now $L_v h \in L^2(K \times G, m \otimes \mu)$ is seen as follows:

$$\begin{aligned} \infty > \int_S |(\mathcal{F}h)(\chi)|^2 C_{v,v}(\chi)\pi(d\chi) &= \int_S |(\mathcal{F}h)(\chi)|^2 \int_G |((\mathcal{F}v)(\chi^a)|^2 \mu(da)\pi(d\chi) \\ &= \int_G \int_S |\overline{(\mathcal{F}v)(\chi^a)} \cdot (\mathcal{F}h)(\chi)|^2 \pi(d\chi)\mu(da) \\ &= \int_G \int_{\widehat{K}} |\overline{(\mathcal{F}v)^a} \cdot (\mathcal{F}h)|^2 d\pi\mu(da). \end{aligned}$$

showing that $\overline{(\mathcal{F}v)^a} \cdot (\mathcal{F}h) \in L^2(\widehat{K})$ for μ–almost all $a \in G$. The isometry of $\mathcal{F}$ ensures that

$$\begin{aligned}
\infty > \int_G \int_{\widehat{K}} |\overline{(\mathcal{F}v)^a} \cdot (\mathcal{F}h)|^2 d\pi \mu(da) &= \int_G \int_K |(\overline{(\mathcal{F}v)^a}\mathcal{F}h)^\vee(b)|^2 m(db)\mu(da) \\
&= \int_G \int_K |\langle(\mathcal{F}v)^a, \mathcal{F}T_b h\rangle|^2 m(db)\mu(da) \\
&= \int_{K\times G} |(L_v h)(b,a)|^2 m \otimes \mu(d(b,a)).
\end{aligned}$$

The first equality holds since

$$\langle(\mathcal{F}v)^a, \mathcal{F}T_b h\rangle = \int_{\widehat{K}} \overline{(\mathcal{F}v)^a(\chi)}\chi(b)(\mathcal{F}h)(\chi)\pi(d\chi) = (\overline{(\mathcal{F}v)^a}\mathcal{F}h)^\vee(b).$$

(ii) Polarizing

$$\langle L_v h, L_v h\rangle = \int_S |(\mathcal{F}h)(\chi)|^2 C_{v,v}(\chi)\pi(d\chi) \quad \forall v \in \mathcal{A}, h \in L^2(K,m), \tag{1}$$

we obtain

$$\langle L_{v_1}h_1, L_{v_2}h_2\rangle = \int_S \overline{(\mathcal{F}h_1)}(\chi)(\mathcal{F}h_2)(\chi)C_{v_2,v_1}(\chi)\pi(d\chi) \qquad \forall v_1, v_2 \in \mathcal{A}, h_1, h_2 \in L^2(K,m).$$

■

Remark. (The inversion of the left–transform.) Let us suppose that for a given $v_2 \in \mathcal{A}$ there exists $v_1 \in \mathcal{A}$ satisfying $C_{v_2,v_1} = 1$. In this situation we obviously obtain $\langle L_{v_1}h_1, L_{v_2}h_2\rangle = \langle h_1, h_2\rangle$ for all $h_1, h_2 \in L^2(K,m)$, which means $L_{v_1}^* L_{v_2} = \mathbb{1}$.

3 Transitive group action

In this section a special group action is considered: We suppose that there is essentially only one orbit in S, which implies that the function C_{v_2,v_1} is constant $\pi|_S$–almost everywhere on S. This assumption is analogous to that of irreducibility for square integrable group representations.

Assumption 2. *The action β of G on S is assumed to be transitive, which means that there exists $\tilde{\chi} \in S$ with $\pi(\widehat{K} \setminus \beta(\tilde{\chi}, G)) = 0$. Furthermore, we assume the measures $\mu^{\tilde{\chi}} \in M_+(S)$ and $\pi|_S$ to be equivalent.*

Remark A similar condition is discussed in the case of groups in [2] Proposition 2.
We denote by R the function given as $R : \widehat{K} \to \mathbb{R}_+$, $R(\chi) := \frac{d\mu^{\tilde{\chi}}}{d\pi|_S}(\chi)$ for all $\chi \in S$, and $R(\chi) := 0$ for all $\chi \in \widehat{K} \setminus S$. Obviously $R > 0$ π–almost everywhere on $\widehat{K}$.

Lemma 2. *Assumption 2 implies:*
(i) $\mathcal{A} = \{v \in L^2(K,m) : R^{\frac{1}{2}}\mathcal{F}v \in L^2(\widehat{K},\pi)\}$. In particular $\mathcal{A}$ is a dense linear subspace of $L^2(K,m)$.
(ii) If v is a wavelet then L_v is, up to a positive factor, an isometric operator.

Proof. (i): Let us choose an arbitrary $\tilde{\chi} \in S$ satisfying $\pi(\widehat{K} \setminus \beta(\tilde{\chi}, G)) = 0$ and $v \in L^2(K, m)$. From

$$\int_G |(\mathcal{F}v)(\tilde{\chi}^a)|^2 \mu(da) = \int_S |\mathcal{F}v|^2 d\mu^{\tilde{\chi}} = \int_S |\mathcal{F}v|^2 \frac{\mu^{\tilde{\chi}}}{d\pi_S} d\pi|_S = \int_{\widehat{K}} |\mathcal{F}v|^2 R d\pi$$

it follows that $v \in \mathcal{A}$ if and only if the above integrals are finite.
(ii): Since $0 \neq v \in \mathcal{A}$ we obtain from the above arguments that $L^\infty(S, \pi|_S) \ni C_{v,v} = \int_{\widehat{K}} |v(\chi')|^2 \underbrace{R(\chi')}_{>0} \pi(d\chi') > 0$. It follows for all $h \in L^2(K, m)$ that

$$\langle L_v h, L_v h \rangle = \int_S |(\mathcal{F}h)(\chi)|^2 C_{v,v}(\chi) \pi(d\chi) = \|h\|^2 \underbrace{\int_{\widehat{K}} |(\mathcal{F}v)(\chi')|^2 R(\chi') \pi(d\chi')}_{>0}. \qquad \blacksquare$$

Polarizing the last equality, we are led to the following orthogonality relation:

$$\langle L_{v_1} h_1, L_{v_2} h_2 \rangle = \langle R^{\frac{1}{2}} \mathcal{F}v_2, R^{\frac{1}{2}} \mathcal{F}v_1 \rangle \langle h_1, h_2 \rangle \qquad \forall v_1, v_2 \in \mathcal{A}, \quad h_1, h_2 \in L^2(K \times G, m \otimes \mu).$$

For admissible vectors we may normalize the left–transform and obtain an isometric operator:

Definition. *Let Assumption 2 be satisfied and $v \in L^2(K, m)$ be a wavelet. The isometric operator $\mathcal{L}_v := \frac{1}{\|L_v v\|} L_v$ is called the wavelet transform corresponding to the wavelet v.*

Remark. As in the case of groups the wavelet transform $\mathcal{L}_v$ is inverted on its range by its adjoint $\mathcal{L}_v^*$, what means $\mathcal{L}_v^* \mathcal{L}_v = \mathbb{1}$; here

$$\mathcal{L}_v^* \xi = \frac{1}{\|L_v v\|} \int_{K \times G} \xi(b, a) T_{b^-} D_a v \ \ m \otimes \mu(d(b, a))$$

holds in the weak sense for all $\xi \in L^2(K \times G, m \otimes \mu)$. The range of $\mathcal{L}_v$ consists precisely of those $\xi \in L^2(K \times G, m \otimes \mu)$ satisfying $\mathcal{L}_v \mathcal{L}_v^* \xi = \xi$, where the last assertion is equivalent to

$$\xi(b, a) = \int_{K \times G} \frac{\langle T_{b^-} D_a v, T_{b'^-} D_{a'} v \rangle}{\|L_v v\|^2} \xi(b', a') m \otimes \mu(d(b', a')) \qquad \forall (b, a) \in K \times G.$$

3.1 A remark on discretization

The most important feature of the classical wavelet transform is the discretization technique, since multiresolution analysis based on orthogonal wavelets provide tools for the design of fast algorithms. The discretization of the classical wavelet transform is possible due to Poisson's summation formula on $\mathbb{R}$. Unfortunately, no corresponding result is available for commutative hypergroups. For this reason, no straightforward discretization technique can be done in the context of hypergroups and we can present only a discretization of the diation parameter. An alternative approach to discretization is based on a direct construction of the so–called wavelet frames. This construction is known in some special cases, see [10].

Let the assumptions 1 and 2 be satisfied and v be a wavelet. A *discretization* of $\mathcal{L}_v$ is given by a set $\mathcal{D} \subset K \times G$ such that $\mathcal{L}_v h|_{\mathcal{D}}$ determines $\mathcal{L}_v h$ uniquely. The most desirable case is that where $\mathcal{D}$ is discrete and $h \mapsto \mathcal{L}_v h|_{\mathcal{D}}$ is a bounded injective operator from $L^2(K, m)$ into $l^2(\mathcal{D})$. In our setting, we consider only the case $\mathcal{D} = K \times G_d$, where $G_d \subset G$ is a discrete subgroup of G. The group action β is restricted to the action β_d of the discrete subgroup G_d. The first assumption still holds for β_d, but the transitivity of β_d (second assumption) fails in general. However, for $v \in \mathcal{A}_d$ (admissible vector for β_d) the operator $h \mapsto \mathcal{L}_v h|_{\mathcal{D}}$ mapping from $L^2(K)$ into $L^2(K \times G_d)$ is still bounded. It is also injective, if $\inf_{\chi \in S} C_{v,v}(\chi) > 0$. This follows from (1):

$$\langle L_v h, L_v h \rangle_{L^2(K \times G_d)} = \int_S |(\mathcal{F}v)(\chi)|^2 C_{v,v}(\chi) \pi(d\chi) \geq \|h\|^2 \inf_{\chi \in S} C_{v,v}(\chi) \ \forall\, h \in L^2(K).$$

Note that here $C_{v,v}$ also corresponds to β_d and is given by:

$$C_{v,v}(\chi) = \int_{G_d} |\mathcal{F}v(\chi^a)|^2 \mu_{G_d}(da) \qquad \forall \chi \in S.$$

4 Examples

Example 1. (The wavelet transform on $\mathbb{R}$). The hypergroup, endowed with the Haar measure m, is given as $(K, m(dr)) := (\mathbb{R}, dr)$; this choice implies $(\widehat{K}, \pi(d\chi)) := (\mathbb{R}, \frac{1}{2\pi} d\chi)$ and $S = \widehat{K}$. The translations $(T_b)_{b \in K}$ are given as $(T_b h)(r) = h(b + r)$ for all $h \in L^2(K, m)$, $b \in K$. Let us define $(G, \mu(da)) := (\mathbb{R} \setminus \{0\}, \frac{1}{|a|} da)$. The group G acts on $\widehat{K}$ by multiplication: $\beta : (\chi, a) \mapsto \chi \cdot a$. Assumptions 1 and 2 are automatically satisfied. We obtain for all $a \in G$ $\pi^a(d\chi) := \frac{1}{2\pi|a|} d\chi$, and, putting $\tilde{\chi} := 1$, the image measure $\mu^{\tilde{\chi}}$ is given by $\mu^{\tilde{\chi}}(d\chi) = \frac{1}{|\chi|} d\chi$. The dilation (here modulation) operators are easily seen as acting as $(D_a h)(r) = (\mathcal{F}^{-1}(\mathcal{F}h)(. \cdot a))(r) = \frac{1}{|a|} h(\frac{r}{a})$ for all $a \in G$, $r \in K$, $h \in L^2(K, m)$. Given $v \in L^2(K, m)$, we obtain the left–transform of $h \in L^2(K, m)$ as

$$(L_v h)(b, a) = \langle D_a v, T_b h \rangle = \int_{\mathbf{R}} \frac{1}{|a|} \overline{v}(ra^{-1}) h(r + b) dr = \int_{\mathbf{R}} \frac{1}{|a|} \overline{v}(\frac{u - b}{a}) h(u) du$$

for all $(b, a) \in K \times G$. The function R is calculated by $R(\chi) := \frac{d\mu^{\tilde{\chi}}}{d\pi}(\chi) = \frac{2\pi}{|\chi|}$ for all $\chi \in \widehat{K}$. By definition, $0 \neq v \in L^2(K, m)$ is a wavelet if

$$\int_{\widehat{K}} R(\chi) |\mathcal{F}v(\chi)|^2 \pi(d\chi) = \int_{\mathbf{R}} \frac{2\pi}{|\chi|} |\mathcal{F}v(\chi)|^2 \frac{1}{2\pi} d\chi = \int_{\mathbf{R}} |\mathcal{F}v(\chi)|^2 \frac{1}{|\chi|} d\chi < \infty.$$

Example 2. (The windowed Fourier transform on $\mathbb{R}$). We choose $(K, m(dr))$, $(\widehat{K}, \pi(d\chi))$ and $(T_b)_{b \in K}$ as in the previous example. Let us define the group as $(G, \mu(da)) := (\mathbb{R}, da)$. The group G acts on $\widehat{K}$ by addition: $\beta : (\chi, a) \mapsto \chi + a$. Assumptions 1 and 2 are then satisfied. We obtain $\pi^a(d\chi) := \frac{1}{2\pi} d\chi$ for all $a \in G$, and, putting $\tilde{\chi} := 0$, the image measure $\mu^{\tilde{\chi}}$ is found as $\mu^{\tilde{\chi}}(d\chi) = d\chi$. The dilation (here modulation) operators are easily seen as acting as $(D_a h)(r) = (\mathcal{F}^{-1}(\mathcal{F}h)(. + a))(r) = e^{-iar} h(r)$ for all $a \in G$, $r \in K$. For a given $v \in L^2(K, m)$, we obtain the left–transform of h as

$$(L_v h)(b, a) = \langle D_a v, T_b h \rangle = \int_{\mathbf{R}} \overline{e^{-iar} v(r)} h(r + b) dr$$

for all $(b,a) \in K \times G$. Since $R(\chi) := \frac{d\mu^{\tilde{\chi}}}{d\pi}(\chi) = 2\pi$ each $0 \neq v \in L^2(K,m)$ is a wavelet.

Example 3. (Radial wavelet transform, a special case of [10]). The Bessel–Kingman hypergroup K with parameter $\alpha > -\frac{1}{2}$ is given as $K := \mathbb{R}_+$, the Haar measure is just $m(dr) := r^{2\alpha+1}dr$, and the convolution $*$ of point measures satisfies

$$(\varepsilon_x * \varepsilon_y)(dr) = \frac{\Gamma(\alpha+1)}{\Gamma(\frac{1}{2})\Gamma(\alpha+\frac{1}{2})2^{\alpha-1}} \frac{[(r^2-(x-y)^2)((x+y)^2-r^2)]^{\alpha-\frac{1}{2}}}{(xyr)^{2\alpha}} 1_{[|x-y|,x+y]}dr.$$

The set of characters of K is just

$$\{r \mapsto j_\alpha(\chi \cdot r) \mid \chi \in \mathbb{R}_+, j_\alpha \text{ is the modified Bessel function of order } \alpha\},$$

$$j_\alpha(z) := \sum_{k\geq 0} \frac{(-1)^k\Gamma(\alpha+1)}{2^{2k}k!\Gamma(\alpha+k+1)} z^{2k} \qquad \forall z \in \mathbb{C}$$

and via this parameterization the dual $\widehat{K}$ can be identified topologically with $\mathbb{R}_+$. The Plancherel measure π, associated with (K,m), is given by $\pi(d\chi) = \frac{\chi^{2\alpha+1}}{(2^\alpha\Gamma(\alpha+1))^2}d\chi$, and its support S is equal to $\widehat{K}$. Let the group $G := \mathbb{R}_+ \setminus \{0\}$ act on $\widehat{K}$ by multiplication: $\beta : (\chi, a) \mapsto \chi \cdot a$. We fix the Haar measure μ on G as $\mu(da) := \frac{1}{a}da$. Assumption 1 is satisfied since $\pi^a(d\chi) := \frac{\chi^{2\alpha+1}}{(2^\alpha\Gamma(\alpha+1))^2a^{2\alpha+2}}d\chi$ for all $a \in G$. The dilation operators can be obtained explicitly: It follows from

$$\begin{aligned}
(D_ah)(r) &= (\mathcal{F}^{-1}(\mathcal{F}h)(\cdot\, a))(r)\\
&= \int_0^\infty j_\alpha(\chi r)(\mathcal{F}h)(\chi a)\frac{\chi^{2\alpha+1}}{(2^\alpha\Gamma(\alpha+1))^2}d\chi\\
&= \int_0^\infty j_\alpha(a\chi\frac{r}{a})(\mathcal{F}h)(\chi a)\frac{(\chi a)^{2\alpha+1}}{(2^\alpha\Gamma(\alpha+1))^2a^{2\alpha+1}}d\chi\\
&= \frac{a^{-1}}{a^{2\alpha+1}}\underbrace{\int_0^\infty j_\alpha(\chi\frac{r}{a})(\mathcal{F}h)(\chi)\frac{\chi^{2\alpha+1}}{(2^\alpha\Gamma(\alpha+1))^2}d\chi}_{h(\frac{r}{a})} \qquad \forall h \in C_c(K)
\end{aligned}$$

that $(D_ah)(r) = \frac{1}{a^{2\alpha+2}}h(\frac{r}{a})$ for all $h \in L^2(K,m)$, $a \in G$, and $r \in K$. Finally to see Assumption 2 is satisfied, we set $\tilde{\chi} := 1$ and obtain $\mu^{\tilde{\chi}}(d\chi) = \frac{1}{\chi}d\chi$. This implies that $R(\chi) := \frac{d\mu^{\tilde{\chi}}}{d\pi}(\chi) = \frac{1}{\chi}\frac{(2^\alpha\Gamma(\alpha+1))^2}{\chi^{2\alpha+1}} > 0$ for all $\chi \in \widehat{K}$. The function $0 \neq v \in L^2(K,m)$ is a wavelet if

$$\infty > \int_{\widehat{K}} R|\mathcal{F}v|^2d\pi = \int_0^\infty \frac{1}{\chi}\frac{(2^\alpha\Gamma(\alpha+1))^2}{\chi^{2\alpha+1}}|\mathcal{F}v(\chi)|^2\frac{\chi^{2\alpha+1}}{(2^\alpha\Gamma(\alpha+1))^2}d\chi = \int_0^\infty |\mathcal{F}v(\chi)|^2\frac{1}{\chi}d\chi.$$

Example 4. Here we consider the wavelet transform on Chébli–Trimèche hypergroups. This is a generalization of the previous example. A Chébli–Trimèche hypergroup K with Haar measure m is given by $(K, m(dr)) := (\mathbb{R}_+, A(r)dr)$. The mapping $A : \mathbb{R}_+ \to \mathbb{R}_+$, called the Chébli–Trimèche function, is assumed to satisfy several conditions. (For the exact definition of Chébli–Trimèche hypergroups we refer the reader to [1], p. 209). The set of characters $\widehat{K}$ is identified with $\mathbb{R}_+ \cup i[0,\rho]$, (the constant $\rho \in \mathbb{R}_+$ is called the index of the hypergroup). By this identification the support of the Plancherel measure is given by $S := \mathbb{R}_+$. Furthermore there exists a function $C : \mathbb{R}_+ \to \mathbb{C}$ with $\pi|_S(d\chi) = |C(\chi)|^{-2} d\chi$. By a result of Trimèche (see [12]):

$$\sup_{\chi>0} \frac{|C(\frac{\chi}{a})|^{-2}}{|C(\chi)|^{-2}} < \infty \qquad \forall a \in \mathbb{R}_+ \setminus \{0\}. \tag{2}$$

Let us define the action of the group $G := \mathbb{R}_+ \setminus \{0\}$ on S by multiplication: $\beta : (\chi, a) \mapsto \chi \cdot a$. It follows from

$$\begin{aligned}
\int_S f(\chi)\pi^a(d\chi) &= \int_S f(\chi^a)\pi(d\chi) = \int_0^\infty f(\chi \cdot a)|C(\chi)|^{-2} d\chi \\
&= \int_0^\infty f(\chi \cdot a)|C(\frac{\chi \cdot a}{a})|^{-2} d\chi \\
&= \int_0^\infty f(\chi)|C(\frac{\chi}{a})|^{-2} \frac{1}{a} d\chi \qquad \forall f \in C_c(\widehat{K})
\end{aligned}$$

that $\pi^a(d\chi) = \frac{|C(\frac{\chi}{a})|^{-2}}{a} d\chi$ for all $a \in G$. We conclude that Assumption 1 is satisfied since in view of (2) $\frac{d\pi^a}{d\pi} \in L^\infty(S, \pi|_S)$ holds for all $a \in G$. As in the previous example, we endow the group G with the Haar measure $\mu(da) = \frac{1}{a} da$. Choosing $S \ni \tilde{\chi} := 1$ the action β is easily seen to be transitive. It follows from $\frac{d\mu^{\tilde{\chi}}}{d\pi|_S}(\chi) = \frac{1}{\chi} \frac{1}{|C(\chi)|^{-2}} > 0$ that Assumption 2 is satisfied. The function $0 \neq v \in L^2(K, m)$ is a wavelet if

$$\int_{\widehat{K}} R|\mathcal{F}v|^2 d\pi = \int_0^\infty \frac{1}{\chi} \frac{1}{|C(\chi)|^{-2}} |\mathcal{F}v(\chi)|^2 |C(\chi)|^{-2} d\chi = \int_0^\infty \frac{1}{\chi} |\mathcal{F}v(\chi)|^2 d\chi < \infty.$$

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