On Inductive Limits of Topological Algebraic Structures in relation to the Product Topologies

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Abstract. In infinite-dimensional harmonic analysis, we encounter naturally inductive limits of certain topological algebraic objects, such as Lie groups, Banach algebras, topological semigroups and so on. In such cases, the inductive limit algebraic structures are not necessarily consistent with the inductive limit topologies, contrary to the affirmative statement in [Enc, Article 210]. This phenomenon is studied in [TSH] in the case of topological groups.

We study in this paper similar situations for other categories of topological algebraic structures. Further, in relation to this, we study certain properties of general topological spaces for the 'commutativity' of (1) taking direct products and (2) taking inductive limits.

This paper is a summarized version of [HSTH].

§1. Inductive limits and direct products

1.1. Preliminaries. Let us consider an inductive system in a certain category $\mathcal{C}$, of topological spaces, of topological groups, of topological vector spaces, or of topological algebras, etc., as

$$\{ (X_\alpha, \tau_{X_\alpha}),\ \alpha \in A\,;\ \phi_{\beta,\alpha},\ \alpha \preceq \beta,\ \alpha, \beta \in A \},$$

where the index set A is a directed set, each X_α is an object in $\mathcal{C}$ with topology τ_{X_α}, and $\phi_{\beta,\alpha}$ is a (continuous) homomorphism $X_\alpha \to X_\beta$ in $\mathcal{C}$ satisfying the consistency condition: $\phi_{\gamma,\beta} \circ \phi_{\beta,\alpha} = \phi_{\gamma,\alpha}$ for any $\alpha \preceq \beta \preceq \gamma$.

Then, on an inductive limit space $X := \varinjlim X_\alpha$, we define the corresponding algebraic structure. On the other hand, we have also an inductive limit topology, denoted as $\varinjlim \tau_{X_\alpha}$ or simply as τ^X_{ind}, in which a subset D of X is open, by definition, if and only if $\phi_\alpha^{-1}(D) \subset X_\alpha$ is open in τ_{X_α} for each $\alpha \in A$. Here, ϕ_α denotes the canonical homomorphism from X_α to X.

In this paper, we study about the harmonicity of the limit topology τ^X_{ind} with the algebraic structure on X. Furthermore, we consider an appropriate variant of τ^X_{ind} in each category $\mathcal{C}$ (denote it by $\tau^X_{\mathcal{C}}$ provisionally here) and study various kinds of harmonicity, and propose several problems.

Meantime, we find that one of the important points of discussions is the problem of commutativity of (1) taking the inductive limit $\tau^X_{\mathcal{C}}$ and (2) taking direct products. This commutativity is expressed symbolically as $\tau_C^X \times \tau_C^Y \cong \tau_C^{X\times Y}$, for two inductive systems $\{(X_\alpha, \tau_{X_\alpha}), \alpha \in A\}$ and $\{(Y_\alpha, \tau_{Y_\alpha}), \alpha \in A\}$ with $Y = \varinjlim Y_\alpha$. In the case where this commutativity holds, we say that the condition (DPA) (= *Direct Product is Admitted*) holds for $\tau_C^{\{*\}}$.

More in detail, let us explain our problems in the following.

1.2. Inductive limits of topological groups.

Let $\{(G_\alpha, \tau_{G_\alpha}); \alpha \in A\}$ be an inductive system of topological groups with a directed set A as index set. Here τ_{G_α} denotes the group topology on G_α and we are given an inductive system of continuous group homomorphisms $\phi_{\alpha_2,\alpha_1}; G_{\alpha_1} \to G_{\alpha_2}$ $(\alpha_1, \alpha_2 \in A, \alpha_1 \preceq \alpha_2)$ satisfying $\phi_{\alpha_3,\alpha_2} \circ \phi_{\alpha_2,\alpha_1} = \phi_{\alpha_3,\alpha_1}$ for $\alpha_1 \preceq \alpha_2 \preceq \alpha_3$. Put $G := \varinjlim G_\alpha$ and $\tau_{ind}^G := \varinjlim \tau_{G_\alpha}$ the inductive limit of groups and that of topologies respectively. Then, as seen in [TSH], the multiplication $G \times G \ni (g,h) \mapsto gh \in G$ is not necessarily continuous with respect to the inductive limit topology τ_{ind}^G, or more exactly, with respect to $(\tau_{ind}^G \times \tau_{ind}^G, \tau_{ind}^G)$.

Inspired by this rather critical phenomenon, we start to study the inductive limit topologies in detail in more general setting.

1.3. A continuity criterion.

Let $\{(X_\alpha, \tau_{X_\alpha}); \alpha \in A\}$ be an inductive system of topological spaces. Take another inductive system $\{(Z_\alpha, \tau_{Z_\alpha}); \alpha \in A\}$ of topological spaces with the same index set A and with an inductive system of continuous maps $\phi'_{\alpha_2,\alpha_1} : Z_{\alpha_1} \to Z_{\alpha_2}$. Then, assume that we are given a system of maps F_α of X_α to Z_α for $\alpha \in A$ which is *consistent* in the sense that $F_{\alpha_2} \circ \phi_{\alpha_2,\alpha_1} = \phi'_{\alpha_2,\alpha_1} \circ F_{\alpha_1}$ for $\alpha_1, \alpha_2 \in A, \alpha_1 \preceq \alpha_2$. Then this system induces a map $F : X \to Z := \varinjlim Z_\alpha$ such that $F \circ \phi_\alpha = \phi'_\alpha \circ F_\alpha \, (\alpha \in A)$, where ϕ_α (resp. ϕ'_α) denotes the natural map from X_α to X (resp. Z_α to Z), continuous with respect to $(\tau_{X_\alpha}, \tau_{ind}^X)$ (resp. to $(\tau_{Z_\alpha}, \tau_{ind}^Z)$). Furthermore the following fact is easy to prove.

Lemma 1.1. *If every map $F_\alpha : X_\alpha \to Z_\alpha$ is continuous in $(\tau_{X_\alpha}, \tau_{Z_\alpha})$ for $\alpha \in A$, then the induced map $F : X \to Z$ is continuous in $(\tau_{ind}^X, \tau_{ind}^Z)$.*

Let us apply this lemma to the above case of inductive limits of topological groups, by setting

$$(X_\alpha, \tau_{X_\alpha}) = (G_\alpha \times G_\alpha, \tau_{G_\alpha} \times \tau_{G_\alpha}), \quad (Z_\alpha, \tau_{Z_\alpha}) = (G_\alpha, \tau_{G_\alpha}),$$

and $F_\alpha : X_\alpha \to Z_\alpha$ as $F_\alpha(g_\alpha, h_\alpha) = g_\alpha h_\alpha$. Then, since τ_{G_α} is a group topology on G_α, the map F_α is continuous for each $\alpha \in A$, and so, as their natural limit, the multiplication map $F(g,h) = gh$ of $X = G \times G$ to $Z = G$ is continuous, by Lemma 1.1, with respect to the topologies $\tau_{ind}^{G\times G} := \varinjlim (\tau_{G_\alpha} \times \tau_{G_\alpha})$ on $G \times G = X$

and $\tau^G_{ind} := \varinjlim \tau_{G_\alpha}$ on $G = Z$.

1.4. Direct products of inductive limits of topologies.

On the other hand, it is easy to see the following fact for the direct product of inductive limits of topologies. Take two inductive limits of topological spaces $(X, \tau^X_{ind}) = \left(\varinjlim X_\alpha, \varinjlim \tau_{X_\alpha}\right)$ and $(Y, \tau^Y_{ind}) = \left(\varinjlim Y_\alpha, \varinjlim \tau_{Y_\alpha}\right)$, and consider their direct products.

Proposition 1.2. *The product space $X \times Y$ is naturally identified with the inductive limit space $\varinjlim (X_\alpha \times Y_\alpha)$. On this space the direct product of inductive limit topologies $\tau^X_{ind} \times \tau^Y_{ind} = \left(\varinjlim \tau_{X_\alpha}\right) \times \left(\varinjlim \tau_{Y_\alpha}\right)$ is weaker than or equal to the inductive limit of product topologies $\tau^{X\times Y}_{ind} := \varinjlim (\tau_{X_\alpha} \times \tau_{Y_\alpha})$, or in a symbolic notation, $\tau^X_{ind} \times \tau^Y_{ind} \preceq \tau^{X\times Y}_{ind}$. In particular, for a subset of product type $D \times E \subset X \times Y$, it is open in the former topology if and only if so is in the latter.*

For an inductive limit of topological groups $G := \varinjlim G_\alpha$, taking into account the above result, we see from Lemma 1.1 that, in the case where the multiplication $G \times G \ni (g, h) \mapsto gh \in G$ is not continuous with respect to τ^G_{ind}, the product topology $\tau^G_{ind} \times \tau^G_{ind}$ should be strictly weaker than the inductive limit topology $\tau^{G\times G}_{ind} := \varinjlim (\tau_{G_\alpha} \times \tau_{G_\alpha})$. Thus we come naturally to the following problem.

Problem A. *Let the notations be as above. Then, give a necessary and sufficient condition for the equivalence of two topologies $\tau^X_{ind} \times \tau^Y_{ind}$ and $\tau^{X\times Y}_{ind} := \varinjlim (\tau_{X_\alpha} \times \tau_{Y_\alpha})$ on $X \times Y$, where $(X, \tau^X_{ind}) = \left(\varinjlim X_\alpha, \varinjlim \tau_{X_\alpha}\right)$ and $(Y, \tau^Y_{ind}) = \left(\varinjlim Y_\alpha, \varinjlim \tau_{Y_\alpha}\right)$.*

1.5. Examples and further problems.

Let us examine the simple example, Example 1.2 in [TSH], from the stand point of general topology.

Example 1.1. Let $G_n = F^n \times \mathbf{Q}, F = \mathbf{R}$, $\mathbf{Q}$ or $\mathbf{T}$ with the usual non-discrete topology τ_n for $n \in \mathrm{N}$. Then, $G = \varinjlim G_n = (\prod' F) \times \mathbf{Q}$, where $\prod' F$ denotes the restricted direct product of countable number of F's. The multiplication on G is not continuous with respect to $\tau^G_{ind} = \varinjlim \tau_{G_n}$. Hence, $\tau^G_{ind} \times \tau^G_{ind} \prec \tau^{G\times G}_{ind}$.

Furthermore, considering G_n as a topological space and express it as a direct product of two spaces as $X_n \times Y$, with $X_n = F^n, Y = \mathbf{Q}$. Then, $X := \varinjlim X_n = \varinjlim F^n = \prod' F$, and we see that the direct product topology $\tau^X_{ind} \times \tau_Y$ is strictly weaker than $\tau^{X\times Y}_{ind} = \varinjlim (\tau_{X_n} \times \tau_Y)$ at every point of $X \times Y$, by reexamining the proof in Example 1.2 in [TSH] for non-continuity of the multiplication on G.

In the above case, the topological space Y is fixed, and so the following problem is also important to study.

Problem B. *Let* $(X, \tau_{ind}^X) = \left(\varinjlim X_\alpha, \varinjlim \tau_{X_\alpha}\right)$ *be an inductive limit of topological spaces and* (Y, τ_Y) *a fixed topological space. Then, give a necessary and sufficient condition for the equivalence of two topologies* $\tau_{ind}^X \times \tau_Y$ *and* $\tau_{ind}^{X\times Y} := \varinjlim (\tau_{X_\alpha} \times \tau_Y)$ *on* $X \times Y$.

The former Problem A contains this Problem B, but it is worth to study Problem B by itself. We may expect that a solution to Problem B helps to solve Problem A. However the situation is not so simple that Problem A is reduced to Problem B, because, for instance, the topology τ_Y cannot be in general recovered from the system $\tau_{Y_n} = \tau_Y|_{Y_n}$. So we propose the following problem.

Problem C. *Let* (Y, τ_Y) *be a topological space and* $\{(Y_\alpha, \tau_{Y_\alpha}); \alpha \in A\}$ *be an inductive system of topological spaces such that* $Y_\alpha \subset Y$ *and* $Y = \varinjlim Y_\alpha$ *as sets. Assume that the restriction* $\tau_Y|_{Y_\alpha}$ *of the topology* τ_Y *onto* Y_α *is equal to* τ_{Y_α}. *Then,* $\tau_Y \preceq \tau_{ind}^Y := \varinjlim \tau_{Y_\alpha}$. *Look for a necessary and sufficient condition for the equivalence of these two topologies on* Y.

1.6. A characterization of the product topology $\tau_{ind}^X \times \tau_{ind}^Y$.

For the product $X \times Y$ of two inductive limits of topological spaces $(X, \tau_{ind}^X) = \left(\varinjlim X_\alpha, \varinjlim \tau_{X_\alpha}\right)$ and $(Y, \tau_{ind}^Y) = \left(\varinjlim Y_\alpha, \varinjlim \tau_{Y_\alpha}\right)$, we have by Proposition 1.2, the relation $\tau_{ind}^X \times \tau_{ind}^Y \preceq \tau_{ind}^{X\times Y} := \varinjlim (\tau_{X_\alpha} \times \tau_{Y_\alpha})$.

Further we can characterize the product topology as the strongest topology on $X \times Y$ among direct product topologies weaker than $\tau_{ind}^{X\times Y}$. More exactly, we have the following.

Theorem 1.3. *Let* τ'_X *and* τ'_Y *be topologies on* X *and* Y *respectively such that* $\tau'_X \times \tau'_Y \preceq \tau_{ind}^{X\times Y}$. *Then,* $\tau'_X \preceq \tau_{ind}^X$, $\tau'_Y \preceq \tau_{ind}^Y$, *and so* $\tau'_X \times \tau'_Y \preceq \tau_{ind}^X \times \tau_{ind}^Y$.

The above facts evoke studies on inductive limit topologies in various kinds of categories, such as the Bamboo-Shoot topology τ_{BS}^G in the category of topological groups in [TSH] and its generalization, the locally convex vector topology τ_{lcv}^X in the category of locally convex topological vector spaces, and so on.

§2. Inductive limit topologies in various categories

As mentioned in **1.2**, for an inductive limit $G = \varinjlim G_n$ of topological groups

$G_n, n \geq 1$, the multiplication map is not necessarily continuous with respect to the inductive limit topology $\tau_{ind}^G = \varinjlim \tau_{G_n}$. So we have introduced in [TSH] a so-called Bamboo-Shoot topology τ_{BS}^G on G as the strongest group topology $\preceq \tau_{ind}^G$, under the condition (PTA) on the inductive system $\{ G_n \}$.

In these respects, it is also natural to ask the similar question for other topological algebraic objects, such as topological vector spaces (= TVSs), topological semigroups, topological rings, and topological algebras etc.

2.1. Case of locally convex topological vector spaces.

A good category of TVSs is the category of locally convex topological vector spaces (= LCTVSs) over a field $F = \mathbf{R}$ or $\mathbf{C}$. In that category, we know well how to define an inductive limit of topologies.

Let $\{(X_\alpha, \tau_{X_\alpha}); \alpha \in A\}$ be an inductive system of LCTVSs with $\phi_{\alpha_2,\alpha_1} : X_{\alpha_1} \to X_{\alpha_2}$, $\alpha_1, \alpha_2 \in A, \alpha_1 \preceq \alpha_2$, a homomorphism in the category of LCTVSs, that is, a continuous linear map. On the vector space $X = \varinjlim X_\alpha$, we usually consider a locally convex vector topology as follows.

On the limit space $X = \varinjlim X_\alpha$ of an inductive system $\{ X_\alpha \}$ of LCTVSs, a locally convex vector topology, denoted by lcv-$\varinjlim \tau_{X_\alpha}$ or τ_{lcv}^X, is defined as the one for which a fundamental system of neighbourhood of the null element 0 is given as $\{U \subset X$; τ_{ind}^X-open, convex, balanced (i.e., $\lambda x \in U$ for $x \in U, \lambda \in F, |\lambda| \leq 1$), and absorbing $\}$ (cf. [Yo, I.1, Definition 6, p.27]). Further we have also a simple characterization of neighboufoods of $0 \in X$, as is given in [Tr, §13, p.126].

Now we propose the following problem.

Problem D. *Assume that every space X_α in an inductive system of LCTVSs has an additional structure or operation of the same kind, which induces as its inductive limit such a structure or an operation on the limit space $X := \varinjlim X_\alpha$. Is this structure or operation consistent with the lcv-limit topology τ_{lcv}^X ?*

2.2. Multiplication or product in an inductive system.

Let us first consider two concrete cases to show what kind of things we want to study.

Let M be a non-compact differentiable manifold, and $M_n \nearrow M, n \geq 1$, be an increasing sequence of relatively compact, open submanifolds such that the closure $\overline{M_n}$ is contained in M_{n+1}. The space of complex-valued test functions (C^∞-functions with compact supports) on M, denoted by $\mathcal{D}(M)$, is a LCTVS obtaind as an inductive limit of the inductive system $X_n = \mathcal{D}(\overline{M_n}) := \{\varphi \in C^\infty(M); \mathrm{supp}(\varphi) \subset \overline{M_n}\}$, $n \in \mathrm{N}$. Here $\mathcal{D}(\overline{M_n})$ is topologized in a usual manner by means of a countable number of seminorms.

Let us consider two kinds of operations in $X = \mathcal{D}(M)$. First one is the pointwise multiplication $T : X \times X \to X$, given as $T(\varphi_1, \varphi_2)(p) = \varphi_1(p)\,\varphi_2(p)$ $(p \in M)$, and the second one is the convolution $T(\varphi_1, \varphi_2) = \varphi_1 * \varphi_2$ in the case of $M = \mathbf{R}^k$. We ask if they are continuous or not in $(\tau_{lcv}^X \times \tau_{lcv}^X, \tau_{lcv}^X)$.

Note that, for the first T, $\mathrm{supp}(\varphi_1\varphi_2) \subset \mathrm{supp}(\varphi_1) \cap \mathrm{supp}(\varphi_2)$, and so it maps $X_n \times X_n$ into X_n. On the other hand, for the second T, $\mathrm{supp}(\varphi_1 * \varphi_2)$ becomes bigger and is in general comparable to $\mathrm{supp}(\varphi_1) + \mathrm{supp}(\varphi_2)$, and so T maps $X_n \times X_n$ into $X_{\beta(n)}$ with a $\beta(n) > n$.

Proposition 2.1. *In the space of test functions $X = \mathcal{D}(M)$, the multiplication map $T(\varphi_1, \varphi_2) = \varphi_1\,\varphi_2$ is continuous in $(\tau_{lcv}^X \times \tau_{lcv}^X, \tau_{lcv}^X)$.*

Proposition 2.2. *In the space of test functions $X = \mathcal{D}(\mathbf{R}^k)$, the convolution map $T(\varphi, \psi) = \varphi * \psi$ is continuous in $(\tau_{lcv}^X \times \tau_{lcv}^X, \tau_{lcv}^X)$.*

In the above two cases, the proofs are not routine as may be expected. Here multiplications T are both commutative, but in our proofs the commutativity is not important but the special structure of the space $\mathcal{D}(M)$ is fully used. So, the proofs can not be generalized directly in the following general situation.

Problem E. *Assume that an inductive system $\{X_\alpha ; \alpha \in A\}$ of LCTVSs has multiplications, consistent in the sense that, for any α, there exists a $\beta(\alpha)$ such that $T_\alpha : X_\alpha \times X_\alpha \to X_{\beta(\alpha)}$ is a continuous bilinear map, and that, for any $\alpha_1, \alpha_2 \in A$, there exists a $\gamma \in A$ such that $\gamma \succeq \alpha_j$, $\beta(\gamma) \succeq \beta(\alpha_j)$, $j = 1, 2$, and T_{α_j}'s are naturally induced from T_γ. Then the system $\{T_\alpha\}$ induces as its inductive limit a multiplication T on $X = \varinjlim X_\alpha$.*

Is the limit map T continuous with respect to $\tau_{lcv}^X = \mathrm{lcv}\text{-}\varinjlim \tau_{X_\alpha}$?

2.3. Multiplication map between two spaces of test functions.

Let M and M' be two differentiable manifolds. We assume that at least one of them, say M', is non-compact.

The space of testing functions $X = \mathcal{D}(M)$ is equipped with a locally convex vector topology τ'_X, where $\tau'_X = \tau_X$ the usual C^∞-topology in the case M is compact, and $\tau'_X = \tau_{lcv}^X := \mathrm{lcv}\text{-}\varinjlim \tau_{X_n}$ with $X_n = \mathcal{D}(M_n)$ as above in the case M is non-compact. The space $Y = \mathcal{D}(M')$ is equipped with the lcv-limit topology $\tau_{lcv}^Y := \mathrm{lcv}\text{-}\varinjlim \tau_{Y_n}$ with $Y_n = \mathcal{D}(M'_n)$, where $\{M'_n ; n = 1, 2, \dots\}$ is a sequence of relatively compact open submanifolds such that $\overline{M'_n} \subset M'_{n+1}$ and $M' = \cup_{n \geq 1} M'_n$. We can give to the product space $X \times Y = \mathcal{D}(M) \times \mathcal{D}(M')$ the lcv-limit topology $\tau_{lcv}^{X \times Y}$ which is equal to $\mathrm{lcv}\text{-}\varinjlim (\tau_X \times \tau_{Y_n})$ if M is compact, and to $\mathrm{lcv}\text{-}\varinjlim (\tau_{X_n} \times \tau_{Y_n})$ if M is non-compact.

Now put $Z := \mathcal{D}(M \times M')$. Then, we ask if the multiplication (or product) map $T : X \times Y \to Z$, given as $T(\varphi, \psi)(p, p') = \varphi(p) \cdot \psi(p')$, $p \in M, p' \in M'$, for

$\varphi \in X, \psi \in Y$, is continuous with respect to $(\tau'_X \times \tau^Y_{lcv}, \tau^Z_{lcv})$.

Theorem 2.3. *Let M and M' be two differentiable manifolds. Assume that one of them, say M', is non-compact. Then, the multiplication map $T : \mathcal{D}(M) \times \mathcal{D}(M') \ni (\varphi, \psi) \mapsto \varphi \cdot \psi \in \mathcal{D}(M \times M')$ is not continuous in $(\tau'_X \times \tau^Y_{lcv}, \tau^Z_{lcv})$, where $X = \mathcal{D}(M), Y = \mathcal{D}(M')$, $Z = \mathcal{D}(M \times M')$, and $\tau'_X = \tau_X$ or $\tau'_X = \tau^X_{lcv}$ according as M is compact or not.*

The proof is interesting but we have no space to write it down here.

Taking into account Propositions 2.1, 2.2 and Theorem 2.3, we propose the following problem.

Problem F. *Take three inductive systems of LCTVSs $\{ (X_\alpha, \tau_{X_\alpha}); \alpha \in A \}$, $\{ (Y_\alpha, \tau_{Y_\alpha}); \alpha \in A \}$, and $\{ (Z_\alpha, \tau_{Z_\alpha}); \alpha \in A \}$, and let their inductive limits be (X, τ^X_{lcv}), (Y, τ^Y_{lcv}) and (Z, τ^Z_{lcv}). Assume that, for every $\alpha \in A$, there exists a continuous multiplication (bilinear map) $T_\alpha : X_\alpha \times Y_\alpha \to Z_{\beta(\alpha)}$ with a $\beta(\alpha) \succeq \alpha$, which are consistent with these inductive systems so that there exists a multiplication $T : X \times Y \to Z$ as their inductive limit. Then, under what conditions, T is continuous in $(\tau^X_{lcv} \times \tau^Y_{lcv}, \tau^Z_{lcv})$?*

Remark 2.1. In comparison to the so-called kernel theorem for distributions (cf. [Tr, Th.51.7]), we give a remark. In the situation in Theorem 2.3 with M' non-compact, take a distribution S on $M \times M'$ or $S \in \mathcal{D}'(M \times M')$. Then the bilinear functional $\mathcal{D}(M) \times \mathcal{D}(M') \ni (\varphi, \psi) \mapsto S(T(\varphi, \psi))$ is not necessarily continuous in the product topology, because so is not the bilinear map $T : \mathcal{D}(M) \times \mathcal{D}(M') \to \mathcal{D}(M \times M')$.

2.4. Spaces of finitely many times differentiable functions.

Let r be a non-negative integer and M' is a non-compact $C^{(r)}$-class differentiable manifold. Let us consider the space $Y = C^{(r)}_c(M')$ of $C^{(r)}$-class functions with compact supports. For $r = 0$, Y is nothing but the space of continuous functions with compact supports. Further let $Z = C^{(\infty,r)}_c(M \times M')$ be the space of functions $f(x, y)$ in $(x, y) \in M \times M'$, which is simultaneously of class $C^{(\infty)}$ in $x \in M$ and of class $C^{(r)}$ in $y \in M'$, and compactly supported. We topologize Y and Z respectively as inductive limits of sequences of Banach spaces $Y_n = C^{(r)}(\overline{M'_n})$, and $Z_n = C^{(\infty,r)}(\overline{M_n} \times \overline{M'_n})$.

Theorem 2.4. *Let M be a differentiable manifold and M' be a non-compact $C^{(r)}$-class manifold for some r, $0 \le r < \infty$. Put $X = \mathcal{D}(M), Y = C^{(r)}_c(M')$ and $Z = C^{(\infty,r)}_c(M \times M')$. Then, the multiplication map $T : X \times Y \ni (\varphi, \psi) \mapsto \varphi \cdot \phi \in Z$ is not continuous in $(\tau'_X \times \tau^Y_{lcv}, \tau^Z_{lcv})$, where $\tau'_X = \tau_X$ if M is compact,* *and* $\tau'_X = \tau^X_{lcv}$ *if* M *is non-compact.*

§3. Bamboo-Shoot topology τ^G_{BS} and locally convex topology τ^X_{lcv}

3.1. Bamboo-Shoot topology for PTA-groups.

For an inductive system of topological groups $\{(G_\alpha, \tau_{G_\alpha}); \alpha \in A\}$, assume that the index set A is cofinal to a sub-directed-set isomorphic to $\mathbf{N}$. Then we introduced in [TSH, §2] a condition called (PTA), and under this condition, we defined the so-called Bamboo-Shoot topology τ^G_{BS} on $G = \varinjlim G_\alpha$, and proved that it is the strongest one among group topologies weaker than or equal to the inductive limit topology τ^G_{ind} on G.

3.2. Bamboo-Shoot topology and locally convex topology.

The group topology τ^G_{BS} has an intimate relation to the locally convex vector topology τ^X_{lcv} as in the following problem.

Problem G. *Let* $\{(X_n, \|\cdot\|_n); n \in \mathbf{N}\}$ *be an inductive system of Banach algebras. Then* $X = \varinjlim X_n$ *has naturally a structure of algebra. Take an inductive system of topological subgroups* G_n *of* $(X_n^\times, \tau_{X_n^\times})$ *the group of all invertible elements in* X_n*, with the restriction* $\tau_{X_n^\times}$ *of* $\|\cdot\|_n$*-topology on* $X_n^\times$*. In the case where the condition* (PTA) *holds, what is the relation between the Bamboo-Shoot topology* τ^G_{BS} *on* $G = \varinjlim G_n$ *and the restriction* $\tau^X_{lcv}|_G$ *onto* G *of the locally convex vector topology* τ^X_{lcv} *?*

A. Yamasaki[Ya] and T. Edamatsu[Ed] studied certain special cases of this problem.

Slitely generalizing the situation, we also propose the following proplem.

Problem H. *Assume that every* (X_n, τ_{X_n}) *is locally convex as a TVS. Then, with the locally convex limit topology* τ^X_{lcv}*, does the algebra* X *become a topological algebra ?*

Furthermore, let $G_n := X_n^\times$ *be the set of all invertible elements in* X_n*. Then,* G_n *is a topological group with the relative topology* $\tau_{G_n} := \tau_{X_n}|_{G_n}$*, and they form an inductive system of topological groups. Then, under the condition* (PTA)*, what is the relation between the Bamboo-Shoot topology* τ^G_{BS} *on* G *and the restriction* $\tau^X_{lcv}|_G$ *onto* G *of the locally convex limit topology* τ^X_{lcv} *on* X *?*

We also remark here that studies in different directions on inifinite dimensional Lie groups, containing the theory of their representations, are continued for example in [Boy] and in [NRW].

3.3. Extension of Bamboo-Shoot topologies and their products.

In the category of topological groups, we can extend in an abstract way the notion of Bamboo-Shoot topology on an inductive limit group $G = \varinjlim G_\alpha$ for any (not necessarily countable) inductive system $\{ (G_\alpha, \tau_{G_\alpha}), \alpha \in A; \phi_{\beta,\alpha}, \alpha \preceq \beta \}$.

In fact, we see easily from axioms of neighbouhood system of the unit element for a topological group (e.g., (GT1) $\sim$ (GT5) in [TSH, §1.3]) that there exists, on an inductive limit group $G = \varinjlim G_\alpha$, the strongest group topology under the condition that every canonical homomorphism $\phi_\alpha : G_\alpha \to G$ is continuous. We call it the *extended Bamboo-Shoot topology* and denote it again by τ_{BS}^G.

In the case where the inductive system is countable and the condition (PTA) holds for it, this topology coincides with the Bamboo-Shoot topology τ_{BS}^G constructed explicitly in [TSH].

In the category of topological groups, the problem similar to Problem A is affirmatively solved as follows. Let $\{ (G_\alpha, \tau_{G_\alpha}) ; \alpha \in A \}$ and $\{ (H_\alpha, \tau_{H_\alpha}) ; \alpha \in A \}$ be inductive systems of topological groups. Let $G = \varinjlim G_\alpha$ and $H = \varinjlim H_\alpha$ be their inductive limit groups, and the canonical homomorphisms be $\phi_\alpha : G_\alpha \to G$ and $\psi_\alpha : H_\alpha \to H$.

Then, we have the direct product of inductive systems as $\{ (G_\alpha \times H_\alpha, \tau_{G_\alpha \times H_\alpha}) ; \alpha \in A \}$ with $\tau_{G_\alpha \times H_\alpha} = \tau_{G_\alpha} \times \tau_{H_\alpha}$. Its inductive limit is canonically identified with the direct product $G \times H$.

Theorem 3.1. (i) *Let $G = \varinjlim G_\alpha$, $H = \varinjlim H_\alpha$, and $G \times H = \varinjlim (G_\alpha \times H_\alpha)$ be as above. Then the extended Bamboo-Shoot topologies τ_{BS}^G, τ_{BS}^H, and $\tau_{BS}^{G\times H}$ on G, H, and $G \times H$ respectively satisfy*

$$\tau_{BS}^G \times \tau_{BS}^H \cong \tau_{BS}^{G\times H} \qquad \text{on} \quad G \times H.$$

(ii) *In the case of countable inductive systems, if $\{ (G_n, \tau_{G_n}) ; n \in \mathbf{N} \}$ and $\{ (H_n, \tau_{H_n}) ; n \in \mathbf{N} \}$ satisfy the condition* (PTA), *then so does their direct product $\{ (G_n \times H_n, \tau_{G_n \times H_n}) ; n \in \mathbf{N} \}$.*

3.4. Direct product of locally convex vector topology.

Let $\{(X_\alpha, \tau_{X_\alpha}); \alpha \in A\}$ and $\{(Y_\alpha, \tau_{Y_\alpha}); \alpha \in A\}$ be inductive systems of LCTVSs, and put $X = \varinjlim X_\alpha$, $Y = \varinjlim Y_\alpha$. The direct product of these systems is defined as $\{(X_\alpha \times Y_\alpha, \tau_{X_\alpha \times Y_\alpha}); \alpha \in A\}$ with $\tau_{X_\alpha \times Y_\alpha} := \tau_{X_\alpha} \times \tau_{Y_\alpha}$. Then its inductive limit is isomorphic to the direct product $X \times Y$ as vector spaces. For topologies on this space, we already know that $\tau_{lcv}^X \times \tau_{lcv}^Y \preceq \tau_{lcv}^{X\times Y} := \text{lcv-}\varinjlim \tau_{X_\alpha \times Y_\alpha}$.

On the other hand, we can translate the proof of Theorem 3.1 appropriately in the category of LCTVSs, and see that the condition (DPA) holds in general for the 'lcv-limit functor' $\tau_{lcv}^{\{*\}}$ as follows.

Theorem 3.2. *Let $X = \varinjlim X_\alpha, Y = \varinjlim Y_\alpha$ be inductive limits in the category of LCTVSs. The direct product space $X \times Y$ is identified with the inductive limit of the direct product of inductive systems. Then, as locally convex vector topologies on $X \times Y$, there holds the equivalence*

$$\tau_{lcv}^X \times \tau_{lcv}^Y \cong \tau_{lcv}^{X \times Y} := \text{lcv-}\varinjlim \tau_{X_\alpha \times Y_\alpha}.$$

§4. Sufficient conditions for Problem A

For sufficient conditions for Problem A or B, the local compactness and the local sequential compactness play important roles. Here we study them for Problem A.

4.1. A sufficient condition for $\tau_{ind}^X \times \tau_{ind}^Y \cong \tau_{ind}^{X \times Y}$.

As in **1.4**, take two inductive systems of topological spaces and put $X = \varinjlim X_\alpha, Y = \varinjlim Y_\alpha$. First let us give a simple sufficient condition for the 'commutativity' of (1) taking inductive limits and (2) taking direct products, for inductive limits of topologies, that is, the condition (DPA) for $\tau_{ind}^{\{*\}}$.

Theorem 4.1. *Assume that A has a cofinal sub-directed-set isomrphic to* $\mathbf{N}$. *For two inductive systems of topological spaces, assume that every X_α and Y_α are locally compact Hausdorff spaces. Then, as topologies on $X \times Y$ with $X = \varinjlim X_\alpha, Y = \varinjlim Y_\alpha$, identified with $\varinjlim\,(X_\alpha \times Y_\alpha)$, the product topology $\tau_{ind}^X \times \tau_{ind}^Y$ and the inductive limit topology $\tau_{ind}^{X \times Y} := \varinjlim\,(\tau_{X_\alpha} \times \tau_{Y_\alpha})$ are mutually equivalent: $\tau_{ind}^X \times \tau_{ind}^Y \cong \tau_{ind}^{X \times Y}$, that is, the condition* (DPA) *holds.*

4.2. Other sufficient conditions.

We give other sufficient conditions assuming on X_n and Y_n a stronger condition (SC) than the local sequential compactness.

Definition 4.1. For a subset D of a topological space Z, its *sequential closure*, denoted by $\mathrm{scl}(D)$, is defined as

$\mathrm{scl}(D) := \{\, z \in Z\,;\, \exists z_n \in D \text{ such that } \lim_{n\to\infty} z_n = z \,\}$,

and D is called *sequentially compact* if every sequence in it has a subsequence converging to a point in D, and further Z is called *locally sequentially compact* if every point in it has an open neighbourhood U for which $\mathrm{scl}(U)$ is sequentially compact.

Our condition (SC) on Z is defined as follows.

(SC) *For every sequentially compact subset K and an open set O containing it, there exists an open set G such that $K \subset G \subset \mathrm{scl}(G) \subset O$ and that $\mathrm{scl}(G)$ is sequentially compact.*

Under this condition (SC), we can give two kinds of sufficient conditions for Problem A as follows. For an inductive system, assume that $A = \mathbf{N}$, and that $X_1 \subset \cdots \subset X_n \subset X_{n+1} \subset \cdots \subset X$ canonically by the identification through the canonical maps ϕ_n.

Theorem 4.2. *Let $A = \mathbf{N}$ for an inductive system of topological spaces, and assume that every (X_n, τ_{X_n}) and (Y_n, τ_{Y_n}) satisfies the condition* (SC). *Then, in the case where they all satisfy the first countability axiom, the condition* (DPA) *holds, i.e., for $X = \varinjlim X_n$ and $Y = \varinjlim Y_n$, there holds the equivalence $\tau_{ind}^X \times \tau_{ind}^Y \cong \tau_{ind}^{X \times Y} := \varinjlim (\tau_{X_n} \times \tau_{Y_n})$ on $X \times Y$.*

Theorem 4.3. *Let $A = \mathbf{N}$, and assume the condition* (SC) *for every (X_n, τ_{X_n}) and (Y_n, τ_{Y_n}). Then, in the case where the system satisfies $\tau_{X_{n+1}}|_{X_n} = \tau_{X_n}$, $\tau_{Y_{n+1}}|_{Y_n} = \tau_{Y_n}$ for $n \geq 1$, and the condition*

(Gδ) $\quad$ *X_n is a G_δ-set of X_{n+1}, and Y_n is a G_δ-set of Y_{n+1}, for $n \geq 1$,*

there holds for $X \times Y$ the equivalence $\tau_{ind}^X \times \tau_{ind}^Y \cong \tau_{ind}^{X \times Y} := \varinjlim (\tau_{X_n} \times \tau_{Y_n})$.

§5. The case of a fixed Y and Problem B

In the following, we study in detail Problems A and B, especially necessary conditions for converses of theorems in §4. In this section, we study the case where Y is fixed, or the case where $(Y_n, \tau_{Y_n}) = (Y, \tau_Y)$ for any $n \geq 1$. This is our Problem B.

5.1. Comments to converses of Theorems 4.1, 4.2 and 4.3.

Statements for direct converses of these theorems contain necessarily a global characterization such as "*X_n is a locally compact space*". However, this kind of global characterization of spaces X_n and Y_n are not possible in its nature of inductive sequences of topological spaces, and so, possible converses should be at first stated in languages of local characterizations of these spaces. This can be seen from the following examples.

Example 5.1. Let $X = \mathbf{R}$ and $X_n = (-n, n) \cup \mathbf{Q}$ with an open interval $(-n, n)$, where X is equipped with a usual topology $\tau_{\mathbf{R}}$ of $\mathbf{R}$, and X_n with its relative topology $\tau_{X_n} = \tau_{\mathbf{R}}|_{X_n}$. Then, no X_n is locally compact, whereas so is the inductive limit space X (cf. Theorems 5.2 and 5.3). Note that the space

$(\mathbf{Q}, \tau_{\mathbf{Q}} = \tau_{\mathbf{R}}|_{\mathbf{Q}})$ is totally disconnected and normal.

Example 5.2. Let $Y = \prod'_{k\geq 1} \mathbf{R}_k$ with $\mathbf{R}_k = \mathbf{R}$ be the restricted direct product of $\mathbf{R}$. Put $Y_n = \prod_{k=1}^{n} \mathbf{R}_k = \mathbf{R}^n$, $Y'_n = \left(\prod_{k=1}^{n-1} \mathbf{R}_k\right) \times \mathbf{Q} \subset Y_n$, and imbed Y_n into Y_{n+1} as $Y_n \ni y \mapsto (y, 0) \in Y_{n+1}$. The space Y_n is equipped with the usual Euclidean metric, and the space Y'_n with its relative topology. Then, Y_n is locally compact, whereas no point of Y'_n has a compact neighbourhood. However the topological space Y considered as the inductive limit of (Y_n, τ_{Y_n}), $n \geq 1$, is also equal to the inductive limit of $(Y'_n, \tau_{Y'_n})$, $n \geq 1$, since there is a mixed inductive system given by $Y''_{2n+1} := Y_n$, $Y''_{2n} := Y'_n$, $(n \geq 1)$, which converges to (Y, τ_{ind}^{Y}).

Now let $\{X_n ; n \in \mathrm{N}\}$ be an inductive system of separable locally compact spaces and put $X = \varinjlim X_n$. Consider two inductive systems of direct product type as $\{X_n \times Y_m ; (n, m) \in \mathrm{N} \times \mathrm{N}\}$, and $\{X_n \times Y'_m ; (n, m) \in \mathrm{N} \times \mathrm{N}\}$, where $(n, m) \preceq (n', m')$ in $\mathrm{N} \times \mathrm{N}$ if and only if $n \leq n'$, $m \leq m'$. Then we get as their inductive limits the same space $X \times Y$. Denote by $\tau_{ind,1}^{X\times Y}$ and $\tau_{ind,2}^{X\times Y}$ the inductive limit topologies on $X \times Y$ corresponding to the first and the second system respectively. We assert that $\tau_{ind,1}^{X\times Y} \cong \tau_{ind,2}^{X\times Y} \cong \tau_{ind}^{X} \times \tau_{ind}^{Y}$.

In fact, the first equivalence is affirmed by considering a mixed inductive system (Z_n, τ_{Z_n}), $n > 1$, with $(Z_{2n+1}, \tau_{Z_{2n+1}}) := (X_n \times Y_n, \tau_{X_n} \times \tau_{Y_n})$, $(Z_{2n}, \tau_{Z_{2n}}) := (X_n \times Y'_n, \tau_{X_n} \times \tau_{Y'_n})$. Another equivalence $\tau_{ind,1}^{X\times Y} \cong \tau_{ind}^{X} \times \tau_{ind}^{Y}$ is guaranteed by Theorem 4.1 thanks to the local compactness of X_n's and Y_n's.

Furthermore, in the case the index m is fixed, as for the topologies on $\lim_{n\to\infty} (X_n \times Y_m) = X \times Y_m$ and on $\lim_{n\to\infty} (X_n \times Y'_m) = X \times Y'_m$, we get the equivalence $\tau_{ind}^{X} \times \tau_{Y_m} = \tau_{ind}^{X\times Y_m}$ by Theorem 4.1, but the inequivalence $\tau_{ind}^{X} \times \tau_{Y'_m} \prec \tau_{ind}^{X\times Y'_m}$ by Theorem 5.2 below.

5.2. A sufficient condition for $\tau_{ind}^{X} \times \tau_Y \cong \tau_{ind}^{X\times Y}$.

Let us now begin to treat Problem B. Fix a topological space (Y, τ_Y). Put $Z_n = X_n \times Y$, $\tau_{Z_n} = \tau_{X_n} \times \tau_Y$, and $Z = \varinjlim Z_n, \tau_{ind}^{Z} = \varinjlim \tau_{Z_n}$. We identify Z with $X \times Y$ and τ_{ind}^{Z} with $\tau_{ind}^{X\times Y}$. We know in general $\tau_{ind}^{X} \times \tau_Y \preceq \tau_{ind}^{X\times Y}$, and the problem is to guarantee the converse relation. A simple sufficient condition is given as follows.

Proposition 5.1. *Assume for the inductive system $\{(X_n, \tau_{X_n})\}$ that X_n is imbedded homeomprphically into X_{n+1} for $n \geq 1$, and for the counter part (Y, τ_Y) that Y is locally compact Hausdorff. Then there holds the equivalence $\tau_{ind}^{X} \times \tau_Y \cong \tau_{ind}^{X\times Y}$.*

5.3. Normalization of situations.

To simplify the situations we put some natural assumptions from the beginning.

First we assume for simplicity that the index set A contains a cofinal subset isomorphic to N as directed set, and so we take $A = \mathrm{N}$ later on except when the contrary is announced. It may be assumed without essential loss of generality that

(00-X) *each canonical map* $\phi_{n+1,n} : X_n \to X_{n+1}$ $(n \geq 1)$ *is injective,*

and so considering as $X_n \subset X_{n+1}$ and $X = \bigcup_{n\geq 1} X_n$, we can omit the notations $\phi_{m,n}$ and ϕ_n rather freely, and then,

(01-X) *each* $\phi_{n+1,n}$ *is a homeomorphism, or* $\tau_{X_{n+1}}|_{X_n} \cong \tau_{X_n}$.

For (01-X), we remark that the topologies τ_{X_n} can be replaced by $\tau_{ind}^X|_{X_n}$ to get the same inductive limit topology τ_{ind}^X, and then (01-X) holds for new topologies on X_n's. From now on, we assume (00-X) and (01-X) for $\{ X_n \}$.

Taking an appropriate cofinal sequence if necessary, we may put the following assumption for $\{ X_n \}$ from the beginning:

(1-X) *for any* n, X_n *as a subset of* X_{n+1} *has no* $\tau_{X_{n+1}}$*-inner point of* X_{n+1}.

5.4. Necessary conditions for $\tau_{ind}^X \times \tau_Y \cong \tau_{ind}^{X\times Y}$.

We follow the discussion of A. Yamasaki in [Ya] to get the following necessary condition.

Theorem 5.2. *Let* $A = \mathrm{N}$ *and* Y *be fixed. Assume the condition* (1-X) *and the following:*

(2-x_0) *for* $n \gg 1$, $x_0 \in X_n$ *has a countable fundamental system of* τ_{X_n}*-neighbouhoods;*

(3-y_0) $y_0 \in Y$ *has a countable fundamental system of neighbourhoods consisting of closed ones;*

(4-y_0) $y_0 \in Y$ *does not have a sequentially compact neighbourhood.*

Then, $\tau_{ind}^X \times \tau_Y \prec \tau_{ind}^{X\times Y} := \varinjlim\, (\tau_{X_n} \times \tau_Y)$ *at* $(x_0, y_0) \in X \times Y$.

Reformulating the above result in a global form, we get a kind of converse, in the case of a fixed Y, of affirmative assertions in theorems in §4 as follows.

Theorem 5.3. *Assume* (1-X) *and the following:*

(2-X) *each* (X_n, τ_{X_n}) *satisfies the first countability axiom;*

(3-Y) Y *is regular and satisfies the first countability axiom.*

Then, $\tau_{ind}^X \times \tau_Y \prec \tau_{ind}^{X\times Y}$ *at any point* $(x, y) \in X \times Y$ *for which* $y \in Y$ *has no sequentially compact neighbourhood.*

§6. Necessary conditions for $\tau_{ind}^X \times \tau_{ind}^Y \cong \tau_{ind}^{X\times Y}$ and Problem A

Let $A = \mathbf{N}$. Let us consider two inductive systems $\{X_n\}$ and $\{Y_n\}$, and put $Z_n = X_n \times Y_n$ and identify $Z = \varinjlim Z_n$ with $X \times Y$, then $\tau_{ind}^Z = \tau_{ind}^{X\times Y}$. Assume (00-X) and (01-X) for $\{X_n\}$ and similarly (00-Y) and (01-Y) for $\{Y_n\}$, for simplicity.

6.1. Conditions for $\tau_{ind}^X \times \tau_{ind}^Y \prec \tau_{ind}^{X\times Y}$ at a point.

We study when the above two inductive limit topologies on $Z = X \times Y$ are different from each other at a point $z_0 = (x_0, y_0) \in Z$.

Theorem 6.1. *Assume the following:*

(1-X) *X_n has no $\tau_{X_{n+1}}$-inner point of X_{n+1} for $n \geq 1$;*

(2-X) *X_n satisfies the first countability axiom for $n \geq 1$;*

(3-Y_{n_0}) *Y_{n_0} is regular and satisfies the first countability axiom;*

(4-Y_{n_0}-y_0) *$y_0 \in Y_{n_0}$ has no sequentially compact neighbourhood;*

(5-Y_{n_0}) *Y_{n_0} is τ_{Y_n}-closed in Y_n for all $n > n_0$.*

Then, $\tau_{ind}^X \times \tau_{ind}^Y \prec \tau_{ind}^{X\times Y}$ at $(x_0, y_0) \in X \times Y$ for any $x_0 \in X_{n_0}$.

Reformulating the above result in a global form, we get a converse of Theorem 4.1 as follows.

Theorem 6.2. *Assume* (1-X) *and* (2-X) *and further assume the following:*

(3′-Y) *each (Y_n, τ_{Y_n}) is regular and satisfies the first contability axiom;*

(5′-Y) *Y_n is closed in $(Y_{n+1}, \tau_{Y_{n+1}})$, for $n \geq 1$.*

Then, if $y_0 \in Y$ has no sequentially compact neighbourhood in any (Y_n, τ_{Y_n}), there holds $\tau_{ind}^X \times \tau_{ind}^Y \prec \tau_{ind}^Z$ at $(x_0, y_0) \in Z$ for any $x_0 \in X$.

To get much faithful converses to Theorems 4.1, 4.2 and 4.3, we should get rid of the first countability axiom.

Theorem 6.3. *Let X_n and Y_n be all regular Hausdorff spaces satisfying the first countability axiom. Assume the conditions* (1-X) *and* (5′-X) *for $\{X_n\}$ and similarly* (1-Y) *and* (5′-Y) *for $\{Y_n\}$. Then $\tau_{ind}^X \times \tau_{ind}^Y \cong \tau_{ind}^{X\times Y}$ if and only if X_n and Y_n are all locally sequentially-compact.*

6.2. Case of metrizable spaces.

In the case of metrizable spaces, they are automatically regular and satisfy the first countability axiom, and furthermore sequential compactness is equivalent to compactness. Therefore, in that case, we get from Theorems 4.1 and 6.2 the following simple necessary and sufficient condition for the commutativity of "inductive limit" and "direct product": $\tau_{ind}^X \times \tau_{ind}^Y \cong \tau_{ind}^{X\times Y} := \varinjlim (\tau_{X_n} \times \tau_{Y_n})$.

Theorem 6.4. *Assume the conditions* (00-X), (01-X), (1-X) *and* (5′-X) *for* $\{ X_n \}$, *and similarly* (00-Y), (01-Y), (1-Y) *and* (5′-Y) *for* $\{ Y_n \}$. *Let* X_n *and* Y_n *be all metrizable spaces. Then,* $\tau_{ind}^X \times \tau_{ind}^Y \cong \tau_{ind}^{X\times Y}$ *if and only if* X_n *and* Y_n *are locally compact.*

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