Scaling Limit of the Spectral Distributions of the Laplacians on Large Graphs

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Abstract

We examine several scaling limits of the spectral distributions of Laplacians (or equivalently adjacency operators) on regular graphs and their second quantization on Fock spaces as the graphs grow infinitely in certain manners.

1 Introduction

The present note reports our recent development in asymptotic spectral theory for Laplacians on certain graphs. Main references are [10], [11] and [12], while the material in §5 first appears in published form in this note.

Let us begin with an abstract setting. A regular graph $\Gamma = (V, E)$, V and E being its vertex set and edge set respectively, has by definition the same degree at every vertex x : $\kappa := |\{y \in V | x \sim y\}|$. Here $x \sim y$ denotes that x and y are adjacent vertices. The Laplacian operator Δ on Γ acts on $f : V \longrightarrow \mathbf{C}$ as

$$(\Delta f)(x) := \sum_{y \sim x} f(y) - \kappa f(x) ,$$

which is a formal expression when Γ is an infinite graph.

Taking a state ϕ on the algebra generated by Δ and I (the identity), one considers the spectral distribution of Δ for which the distribution function is determined by values of ϕ at the projectors in the spectral decomposition of Δ. In this note, we will deal with vacuum states and analogs of Gibbs states. We are interested in asymptotic behaviour of the spectral distribution along a growing family of graphs, especially in the case where $\kappa \to \infty$. We try to read a statistical property of the spectral distribution through a scaling limit. The scaling agrees with that of the central limit theorem (CLT, for short). Actually, our problem is closely related to the CLT in algebraic probability theory which was initiated by von Waldenfels et al. (e.g. [7], [15]).

It is convenient to refer to Cayley graphs to see the way CLT comes out. Let G be a group generated by $\Omega = \{\omega_1, \cdots, \omega_\kappa\} \not\ni e$, assuming that $\Omega^{-1} = \Omega$ as a set. Two vertices $x, y \in G$ are defined to be adjacent if $yx^{-1} \in \Omega$. The Laplacian on this Cayley graph is expressed as

$$\Delta = \sum_{j=1}^{\kappa} \pi_L(\omega_j) - \kappa I \tag{1}$$

where π_L denotes the left regular representation of G. Let us take vacuum state $\phi := \langle \delta_e, \cdot\, \delta_e \rangle_{\ell_2(G)}$. According to the formulation of CLT, our problem is to discuss weak convergence of the spectral distribution of

$$\frac{\Delta - \phi(\Delta)}{\sqrt{\phi((\Delta - \phi(\Delta))^2)}} = \frac{1}{\sqrt{\kappa}} \sum_{j=1}^{\kappa} \pi_L(\omega_j) \tag{2}$$

with respect to ϕ as G grows in a certain manner with $\kappa \to \infty$. Noncommuting summands $\pi_L(\omega_j)$ have a sort of (in)dependence reflecting the structure of G. It may reveal a new convolution structure of the limit distribution, yielding Gauss and Wigner as the extremal ones (see [8], [5]). Furthermore, replacing π_L and ϕ by other representations and states will be also interesting.

2 Preliminaries

2.1 Symmetric group and Young diagram

Let $\mathcal{S}_n$ denote the symmetric group of degree n and $\mathcal{S}_\infty := \bigcup_{n=1}^{\infty} \mathcal{S}_n$ their inductive limit. We follow the convention that a Young diagram is expressed as a finite array of left-aligned nonincreasing rows. Let $\mathcal{Y}$ denote the set of Young diagrams and $\mathcal{D}$ the subset of $\mathcal{Y}$ whose element has no rows consisting of a single box. If $\lambda \in \mathcal{Y}$ contains $k^{(j)}$ rows of length j, we use the notation $\lambda = (1^{k^{(1)}} 2^{k^{(2)}} \cdots j^{k^{(j)}} \cdots)$. The number of boxes contained in λ is $|\lambda| := \sum_j j k^{(j)}$. The conjugacy classes in $\mathcal{S}_\infty$ except the trivial one $\{e\}$ are parametrized by the diagrams in $\mathcal{D}$. Let C_λ be the conjugacy class in $\mathcal{S}_\infty$ corresponding to $\lambda \in \mathcal{D}$ and set $C_\lambda^{(n)} := \mathcal{S}_n \cap C_\lambda$ for $n \geq |\lambda|$. $C_\lambda^{(n)}$ is also a conjugacy class in $\mathcal{S}_n$. One sees

$$|C_\lambda^{(n)}| = n^{\underline{|\lambda|}} \Big/ \prod_{j \geq 2} j^{k^{(j)}} k^{(j)}!$$

for $\lambda = (2^{k^{(2)}} 3^{k^{(3)}} \cdots)$ with $n^{\underline{r}} := n(n-1)\cdots(n-r+1)$. π_L denoting the left regular representation of $\mathcal{S}_\infty$, we set

$$A_\lambda^{(n)} := \sum_{x \in C_\lambda^{(n)}} \pi_L(x) \quad \text{and formally} \quad A_\lambda := \sum_{x \in C_\lambda} \pi_L(x) \tag{3}$$

for $\lambda \in \mathcal{D}$. The representation matrix of $A_\lambda^{(n)}|_{\ell^2(\mathcal{S}_n)}$ with respect to the basis $\{\delta_x | x \in \mathcal{S}_n\}$ is an adjacency matrix of the group association scheme of $\mathcal{S}_n$. The complex linear hull of these adjacency matrices is closed under multiplication and hence becomes an algebra. (See [1].) We call A_λ also an adjacency operator on $\mathcal{S}_\infty$.

Regarding $\mathcal{Y}$ as a vertex set and joining two Young diagrams if one diagram is made by adding a box to the other, one obtains the Young graph (or Young lattice). Later in §5, we will mention the Young graph equipped with multiplicity (or colour) on each edge.

2.2 Distance-regular graph

Let S be a v-set (i.e. $|S| = v$) and set $V := \{x \subset S | |x| = d\}$ as a vertex set. (Assume $2d \leq v$ without loss of generality.) $x, y \in V$ are defined to be adjacent if $|x \cap y| = d - 1$. Obviously, $|V| = \binom{v}{d}$ and $\kappa = d(v - d)$ (degree). This graph $J(v, d)$ is called a Johnson graph. The Laplacian on $J(v, d)$ describes the classical Bernoulli-Laplace model imitating a kind of diffusion of sparse gases.

We give a quick review on distance-regular graphs (DRG, for short), among which $J(v, d)$ plays a central role in this note. See [1] for details. Let $\Gamma = (V, E)$ be a finite connected graph. $\partial(x, y)$ denotes the distance (i.e. minimal length) between $x, y \in V$ and $\mathrm{diam}\Gamma := \max_{x,y \in V} \partial(x, y)$ the diameter of Γ. Γ is called a DRG with diameter d if, for $\forall h, i, j \in \{0, 1, \cdots, d\}$, $|\{z \in V | \partial(x, z) = i, \partial(z, y) = j\}| =: p_{ij}^h$ does not depend on the choice of x, y whenever $\partial(x, y) = h$. In particular, $p_{11}^0 = \kappa$ (degree of Γ). Set $\kappa_i := p_{ii}^0$. The ith adjacency operator A_i $(i = 0, 1, \cdots, d)$ is defined as

$$(A_i f)(x) := \sum_{\partial(x,y)=i} f(y) \qquad \text{for} \quad f : V \longrightarrow \mathbf{C} \, .$$

In particular, $A_0 = I$, $A_1 = A$ (adjacency operator) and $\Delta = A - \kappa I$. The condition of distance-regularity is translated into a linearizing formula for adjacency operators :

$$A_i A_j = \sum_{h=0}^{d} p_{ij}^h A_h \, .$$

The commutative algebra $\mathcal{A}(\Gamma)$ generated by A and I is called the adjacency algebra of Γ. Clearly, $\{A_0, A_1, \cdots, A_d\}$ is a linear basis of $\mathcal{A}(\Gamma)$. Then one sees that $\mathrm{diam}\Gamma + 1 = \dim \mathcal{A}(\Gamma) =$ the number of distinct eigenvalues of A. (For a general graph, the former '$=$' should be replaced by '$\leq$'. A DRG has high symmetry and its eigenvalues are thus degenerated.) Letting θ_0 $(= \kappa) > \theta_1 > \cdots > \theta_d$ be distinct eigenvalues of A and E_j the orthogonal projector on $\ell^2(V)$ corresponding to θ_j, one has

$$A = \sum_{j=0}^{d} \theta_j E_j \, , \qquad A_i = \sum_{j=0}^{d} v_i(\theta_j) E_j \quad (i = 0, 1, \cdots, d) \, .$$

Here v_i is shown to be a polynomial of degree i such that $v_i(A) = A_i$. $\{E_0, E_1, \cdots, E_d\}$ also forms a linear basis of $\mathcal{A}(\Gamma)$.

3 Central Limit Theorem for Adjacency Operators on $\mathcal{S}_\infty$

It is quite interesting to seek out statistical properties of large symmetric groups as is seen in [13], [2], [3] etc. In this section, we report the main result in [10] which extends the result in [13]. We follow the notations in §§2.1.

Let $\phi := \langle \delta_e, \cdot \delta_e \rangle_{\ell^2(\mathcal{S}_\infty)}$ be the vacuum state. For each $\lambda \in \mathcal{D}$, one sees

$$\phi(A_\lambda^{(n)}) = 0 \ , \quad \phi(A_\lambda^{(n)2}) = |C_\lambda^{(n)}|$$

as the mean and the variance of $A_\lambda^{(n)}$ with respect to ϕ respectively. Hence we consider an asymptotic spectral behaviour of $A_\lambda^{(n)}/\sqrt{|C_\lambda^{(n)}|}$ as $n \to \infty$ from the viewpoint of CLT. Let $H_r(x)$ denote the Hermite polynomial of degree r obeying the recurrence formula :

$$H_{r+1}(x) = xH_r(x) - rH_{r-1}(x) \ , \quad H_0(x) = 1 \ , \quad H_1(x) = x \ .$$

Theorem 1 ([10]) *For all $\lambda_1, \cdots, \lambda_m \in \mathcal{D}$ and for all $p_1, \cdots, p_m \in \mathbf{N}$, we have*

$$\begin{aligned} &\lim_{n\to\infty} \phi\Big(\Big(\frac{A_{\lambda_1}^{(n)}}{\sqrt{|C_{\lambda_1}^{(n)}|}}\Big)^{p_1} \cdots \Big(\frac{A_{\lambda_m}^{(n)}}{\sqrt{|C_{\lambda_m}^{(n)}|}}\Big)^{p_m}\Big) \\ &= \prod_{j\geq 2} \int_{-\infty}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}} \Big(\frac{H_{k_1^{(j)}}(x)}{\sqrt{k_1^{(j)}!}}\Big)^{p_1} \cdots \Big(\frac{H_{k_m^{(j)}}(x)}{\sqrt{k_m^{(j)}!}}\Big)^{p_m} dx \ , \end{aligned} \tag{4}$$

where $\lambda_i = (2^{k_i^{(2)}} 3^{k_i^{(3)}} \cdots j^{k_i^{(j)}} \cdots)$ $(i = 1, \cdots, m)$.

From (4) we can read how adjacency operators $A_{\lambda_1}, \cdots, A_{\lambda_m}$ are correlated with respect to ϕ. The strucure of the right hand side of (4) tells that rows of different length among $\lambda_1, \cdots, \lambda_m$ essentially play independent roles while there remain some interfering effects among rows of the same length and in different diagrams. In our computation, this asymptotic independence along length j is attributed to disjoint union structure of a certain graph. The left hand side of (4) can be expressed in terms of the irreducible characters of $\mathcal{S}_n$ and the Plancherel measure on $\hat{\mathcal{S}}_n$. Under this formulation, Kerov showed in [13] the corresponding result to (4) for one-row Young diagrams.

4 Central Limit Theorems on Distance-Regular Graphs

Since the Laplacian Δ on a DRG does not yield such a canonical decomposition as (1) or (3), the original feature of CLT which describes a macroscopic effect of sums of small 'independent' fluctuations through appropriate scaling may seem to go somewhat backward. However it has a good meaning to consider

$$(\Delta - \Phi(\Delta))/\sqrt{\Phi((\Delta - \Phi(\Delta))^2)} \tag{5}$$

with respect to some state Φ on adjacency algebra $\mathcal{A}(\Gamma)$ in the situation that DRG Γ grows in some manner. Then the (in)dependence of summands should be transformed into topological structure of the graph. In this section, we survey our results concerning the Johnson graph as examples of such CLT on a DRG as (5). We follow the notations in §§2.2.

4.1 Vacuum state

For DRG Γ, we define vacuum state Φ_0 on $\mathcal{A}(\Gamma)$ as

$$\begin{aligned}\Phi_0(X) &:= \frac{1}{|V|}\mathrm{tr}X \qquad (X \in \mathcal{A}(\Gamma)) \\ &= \langle \delta_x, X\delta_x \rangle_{\ell^2(V)} \qquad (X \in \mathcal{A}(\Gamma)) \qquad \text{for all } x \in V\ .\end{aligned}$$

Theorem 2 ([11]) *Let $\Gamma = J(2d,d)$ (Johnson graph) and $\Phi = \Phi_0$ (vacuum state) in (5). Then the spectral distribution of (5) with respect to Φ_0 converges weakly to*

$$e^{-(\xi+1)}I_{[-1,\infty)}(\xi)d\xi$$

as $d \to \infty$. Here $I_{\cdot}$ denotes an indicator function.

4.2 Gibbs state

We announce the main result in [12]. For DRG Γ with diameter d, we define linear functional Φ_q on $\mathcal{A}(\Gamma)$ by

$$\Phi_q(A_h) := \kappa_h q^h \qquad (h = 0, 1, \cdots, d)$$

where q is a parameter. It is shown that, for $\Gamma = J(v,d)$ and $0 \le q \le 1$, Φ_q is actually a state (namely, enjoys positivity) on $\mathcal{A}(J(v,d))$. Φ_q is regarded as analogue of the Gibbs state with inverse temperature parameter $\beta = -\log q$ ($q = 0 \iff$ vacuum state Φ_0).

Theorem 3 ([12]) *Let $\Gamma = J(2d,d)$ and $\Phi = \Phi_q$ in (5) where $0 \le q \le 1$. Then the spectral distribution of (5) with respect to Φ_q converges weakly to the following as $d \to \infty$:*

(Case 1) if $q = r/d^\alpha$ where $r \geq 0$ and $\alpha > 1$ are constants,

$$e^{-(\xi+1)} I_{[-1,\infty)}(\xi) d\xi \ ; \tag{6}$$

(Case 2) if $q = r/d$ where $r \geq 0$ is a constant,

$$\sqrt{2r+1} e^{-(\xi\sqrt{2r+1}+2r+1)} J_0(i2\sqrt{r(\xi\sqrt{2r+1}+r+1)}) I_{[-(r+1)/\sqrt{2r+1},\infty)}(\xi) d\xi \ . \tag{7}$$

Here

$$J_0(z) := \sum_{k=0}^{\infty} \frac{(-z^2/4)^k}{(k!)^2} \qquad (z \in \mathbf{C})$$

is the 0th Bessel function.

In both cases, $d \to \infty$ and $q \to 0$ hence "temperature of the graph" tends to 0.

Remark (communicated to the author by P.Biane) Checking the characteristic function of (7), one sees that (7) is expressed as

$$\delta_{-(r+1)/\sqrt{2r+1}} * \mu_r * \nu_r \quad \text{where} \quad \mu_r(d\xi) := \sqrt{2r+1} e^{-\xi\sqrt{2r+1}} I_{[0,\infty)} d\xi$$

and ν_r is the infinitely divisible distribution whose characteristic function is given by

$$\exp \int_0^\infty (e^{is\xi} - 1) r\sqrt{2r+1} e^{-\xi\sqrt{2r+1}} d\xi \ .$$

Note that

$$\delta_{-(r+1)/\sqrt{2r+1}} * \mu_r \longrightarrow (6) \quad \text{and} \quad \nu_r \longrightarrow \delta_0 \quad \text{as} \quad r \to 0 \ .$$

5 Second Quantization and Central Limit Theorem

In this section, we give some observations on CLT for the second quantizations of discrete Laplacians.

5.1 Second quantization

Let $\mathcal{F}(\mathcal{H})$ be the Boson Fock space over Hilbert space $\mathcal{H}$:

$$\mathcal{F}(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \mathcal{H}^{\circ n} \ , \qquad \mathcal{H}^{\circ 0} := \mathbf{C}\mathbf{1}$$

where $\circ$ denotes the symmetric tensor product and $\mathbf{1}$ the vacuum vector. The creator $a^*(\xi)$ and annihilator $a(\xi)$ on $\mathcal{F}(\mathcal{H})$ are defined by

$$a^*(\xi)\xi_1 \circ \cdots \circ \xi_n := \sqrt{n+1}\xi \circ \xi_1 \circ \cdots \circ \xi_n \ , \qquad a^*(\xi)\mathbf{1} := \xi$$
$$a(\xi)\xi_1 \circ \cdots \circ \xi_n := \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \langle \xi, \xi_j \rangle_{\mathcal{H}} \xi_1 \circ \cdots \circ \check{\xi_j} \circ \cdots \circ \xi_n \ , \quad a(\xi)\mathbf{1} := 0$$

$(\xi, \xi_1, \cdots, \xi_n \in \mathcal{H})$. Here $\check{}$ indicates the conventional notation for removal of a component. The exponential vector defined as

$$e(\xi) := \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \xi^{\circ n} \quad \text{satisfies} \quad \langle e(\xi), e(\eta) \rangle_{\mathcal{F}(\mathcal{H})} = e^{\langle \xi, \eta \rangle_{\mathcal{H}}}$$

$(\xi, \eta \in \mathcal{H})$. The (differential) second quantization of operator A on $\mathcal{H}$ is

$$d\Gamma(A) := \sum_{n=1}^{\infty} \sum_{j=1}^{n} I \otimes \cdots \otimes I \otimes A \otimes I \otimes \cdots \otimes I ,$$

where A sits on the jth component in the product of the right hand side.

Let us work on Cayley graph (G, Ω), i.e. Ω is a generator set of group G such that $\Omega^{-1} = \Omega \not\ni e$. Assume that Ω is an infinite set. For each $n \in \mathrm{N}$, take finite subset Ω_n of Ω such that $\Omega_n^{-1} = \Omega_n$ and $\Omega_n \nearrow \Omega$ (as a set) as $n \to \infty$. (Recall the discussion in §3 on the conjugacy classes in $\mathcal{S}_\infty$.) We consider adjacency operators on $\ell^2(G)$:

$$A := \sum_{\omega \in \Omega} \pi_L(\omega) \quad \text{(formally)} \quad \text{and} \quad A_n := \sum_{\omega \in \Omega_n} \pi_L(\omega) . \tag{8}$$

The second quantizations of them on $\mathcal{F}(\ell^2(G))$ are expressed in terms of creators and annihilators as

$$d\Gamma(A) = \sum_{\omega \in \Omega} \sum_{x \in G} a^*_{\omega x} a_x \quad \text{(formally) and} \quad d\Gamma(A_n) = \sum_{\omega \in \Omega_n} \sum_{x \in G} a^*_{\omega x} a_x ,$$

where we set $a_x := a(\delta_x)$ and $a^*_x := a^*(\delta_x)$ $(x \in G)$ for simplicity. These operators describe the (nearest neighbour) random walk on G from the viewpoint of quantum fields. Setting

$$\Phi := \langle e^{-1/2} e(\delta_e), \cdot\, e^{-1/2} e(\delta_e) \rangle_{\mathcal{F}(\mathcal{H})} \qquad \text{coherent state}$$

(sorry for confusing usage of several 'e's), we have

$$\Phi(d\Gamma(A_n)) = 0 \quad \text{and} \quad \Phi(d\Gamma(A_n)^2) = |\Omega_n| .$$

Hence our problem of CLT is to discuss weak convergence of the spectral distribution of the operator:

$$d\Gamma(A_n/\sqrt{|\Omega_n|}) = \frac{1}{\sqrt{|\Omega_n|}} \sum_{\omega \in \Omega_n} \sum_{x \in G} a^*_{\omega x} a_x$$

with respect to Φ as $n \to \infty$. This can be solved by relating the moments of an operator on $\mathcal{H}$ to those of its second quantization.

5.2 Moments with respect to coherent state

In general, let $\mathcal{H}$ be a Hilbert space, $\xi \in \mathcal{H}$ a unit vector, and A a self-adjoint operator on $\mathcal{H}$. Set

$$\phi := \langle \xi, \cdot \xi \rangle_{\mathcal{H}} \quad \text{and} \quad \Phi := \langle e^{-1/2} e(\xi), \cdot e^{-1/2} e(\xi) \rangle_{\mathcal{F}(\mathcal{H})} . \tag{9}$$

The relation between the moments of A and $d\Gamma(A)$ are as follows.

Proposition 1 *Set $m_r := \phi(A^r)$ and $M_r := \Phi(d\Gamma(A)^r)$ for $r \in \mathbf{N}$. Then we have*

$$M_r = \sum_{|\lambda|=r, \lambda \in \mathcal{Y}} d(\lambda) m_1^{k(1)} m_2^{k(2)} \cdots m_r^{k(r)} \tag{10}$$

where $\lambda = (1^{k(1)} 2^{k(2)} \cdots r^{k(r)})$ in each term and

$$d(\lambda) := \frac{r!}{1!^{k(1)} 2!^{k(2)} \cdots r!^{k(r)} k(1)! k(2)! \cdots k(r)!} . \tag{11}$$

(10) is the same relation as that between moments of a probability measure and its cumulants. Note that one has

$$\Phi(e^{-itd\Gamma(A)}) = \exp\{\phi(e^{-itA}) - 1\} \qquad (\forall t \in \mathbf{R}) .$$

Combined with the following elementary formula, this yields Proposition 1.

Lemma 1

$$\frac{d^r}{dt^r} e^{f(t)} = e^{f(t)} \sum_{|\lambda|=r, \lambda \in \mathcal{Y}} d(\lambda) f'(t)^{k(1)} f''(t)^{k(2)} \cdots f^{(r)}(t)^{k(r)}$$

where $\lambda = (1^{k(1)} 2^{k(2)} \cdots r^{k(r)})$ and $d(\lambda)$ is given by (11).

Lemma 1 is easily shown by induction on r.

Coming back to Cayley graph (G, Ω), we set $\xi = \delta_e$ in (9):

$$\phi = \langle \delta_e, \cdot \delta_e \rangle_{\ell^2(G)} , \quad \Phi = \langle e^{-1/2} e(\delta_e), \cdot e^{-1/2} e(\delta_e) \rangle_{\mathcal{F}(\ell^2(G))} ,$$

and consider A_n in (8). The limits of moments of $A_n / \sqrt{|\Omega_n|}$ with respect to ϕ are, if they exist, majorized by the Gaussian ones, i.e.

$$\lim_{n \to \infty} \phi((A_n / \sqrt{|\Omega_n|})^{2p}) \leq \frac{(2p)!}{2^p p!} \qquad (\forall p \in \mathbf{N})$$

(see [8]) where the right hand side is the $2p$th moment of the standard normal distribution. Applying Proposition 1 to the Gaussian case, in which $m_{2p} = (2p)!/(2^p p!)$ and the odd moments vanish, we have

$$M_{2p} = \frac{(2p)!}{2^p p!} B(p) \tag{12}$$

by using the pth Bell number $B(p)$ i.e. the number of classification of p objects. Taking into account the asymptotic of $B(p)$ as $p \to \infty$, we can majorize (12) and hence limiting moments of $d\Gamma(A_n)/\sqrt{|\Omega_n|}$ with respect to Φ.

Proposition 2 *If for* $\forall r \in \mathrm{N}$

$$\lim_{n\to\infty} \phi\Big(\Big(\frac{A_n}{\sqrt{|\Omega_n|}}\Big)^r\Big) =: m_r \quad \textit{exists, then} \quad \lim_{n\to\infty} \Phi\Big(\Big(\frac{d\Gamma(A_n)}{\sqrt{|\Omega_n|}}\Big)^r\Big) =: M_r$$

also exists for all $r \in \mathrm{N}$ *and satisfies*

$$\lim_{p\to\infty} M_{2p}^{1/2p} / \, 2p < \infty \,. \tag{13}$$

(13) is a modification of Carleman's condition. It ensures the unique existence of a probability whose rth moment is M_r (see e.g. [6]).

5.3 Branching, q-deformation

We end the section with two remarks.

Let the Young graph be equipped with multiplicity function $\kappa(\lambda, \mu)$ on each edge with $\lambda, \mu \in \mathcal{Y}$ such that $|\mu| = |\lambda| + 1$. Then the Young graph is simply called a branching. We refer to [14] for terminology and examples of branchings. $\lambda_0 \in \mathcal{Y}$ denotes the diagram consisting of a single box. To each path $u = (\lambda_0, \lambda_1, \cdots, \lambda_n)$, in which $|\lambda_{i+1}| = |\lambda_i| + 1$, going from λ_0 to $\lambda = \lambda_n$, one assigns the weight $w_u := \prod_{i=0}^{n-1} \kappa(\lambda_i, \lambda_{i+1})$. Then

$$d(\lambda) := \sum_{u=(\lambda_0,\cdots,\lambda_n),\lambda_n=\lambda} w_u \tag{14}$$

is called the combinatorial dimension function on the branching. If the multiplicity function is trivial i.e. $\kappa(\lambda, \mu) \equiv 1$, $d(\lambda)$ agrees with the number of standard tableaux in λ and hence with the dimension of the irreducible representation of $S_{|\lambda|}$ associated with λ. We see that $d(\lambda)$ in (11) is the combinatorial dimension function on the branching determined by the following multiplicity function. Let $\lambda, \mu \in \mathcal{Y}$ such that $|\mu| = |\lambda| + 1$.
(i) If μ is made by adding a box to a row (say, of length j) in λ and λ contains r rows of length j, then set $\kappa(\lambda, \mu) := r$.
(ii) If μ is made by adding a box to λ as the new bottom row, then set $\kappa(\lambda, \mu) := 1$.
This observation helps recurrent computation of $d(\lambda)$ in (11).

A parallel discussion to the preceding subsections can proceed if one considers the second quantization on a q-Fock space ($0 < q < 1$). See e.g. [4] for the structure of the inner product, the creators and the annihilators on a q-Fock space. An exponential vector and a coherent state in (9) are naturally q-deformed. Then it is shown that Proposition 1 and the branching in the last paragraph yield their 'q-analogue'. Namely, the combinatorial dimension function $d(\lambda)$ is given by (14), but the rule assigning the multiplicity function $\kappa(\lambda, \mu)$ should be slightly modified depending on q.

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