Initial Value Problem For White Noise Operators And Quantum Stochastic Processes

UN CIG JI
DEPARTMENT OF MATHEMATICS
SOGANG UNIVERSITY
SEOUL, 121-742 KOREA
AND
NOBUAKI OBATA
GRADUATE SCHOOL OF MATHEMATICS
NAGOYA UNIVERSITY
NAGOYA, 464-8602 JAPAN

Introduction

Over the past few years the interest has been increasing in white noise approach to both classical and quantum stochastic differential equations. It is the fundamental idea of white noise theory, or also called Hida calculus [7], [8], that randomness is reduced to its elemental components represented by deterministic vectors in an infinite dimensional space and that stochastic analysis is translated into an infinite dimensional calculus. This approach has been discussed along with classical stochastic calculus, see e.g., [9], [10], [14], and references cited therein, and has created a completely new idea of nonlinear extension of stochastic calculus via quantum domain [1], [2]. Namely, by means of white noise theory a traditional quantum stochastic differential equation introduced by Hudson and Parthasarathy [11] is brought into a normal-ordered white noise differential equation:

$$\frac{d\Xi}{dt} = L_t \diamond \Xi, \qquad \Xi|_{t=0} = I,$$

where $\{L_t\}$ is a quantum stochastic process involving lower powers (at most one) of quantum white noises. This observation led us naturally to construct a general scheme of normal-ordered white noise differential equations. In fact, in the series of papers [3], [4], [19], [20], we have established unique existence of a solution in the space of white noise operators and a method of examining its regularity properties in terms of weighted Fock spaces. However, the results were obtained only for linear equations as above though such equations are already far beyond the traditional Itô theory in the sense that the coefficients $\{L_t\}$ may involve very singular noises such as higher powers or higher order derivatives of quantum white noises.

This paper aims at a small step towards a systematic study of nonlinear white noise differential equations. We shall focus on an initial value problem of the form:

$$\frac{d\Xi}{dt} = F(t, \Xi), \qquad \Xi|_{t=0} = \Xi_0, \qquad 0 \le t \le T.$$

For technical reason it seems reasonable to start with the case that $F : [0, T] \times \mathcal{L}(\mathcal{W}, \mathcal{W}^*) \to \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ is a continuous function, where $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ stands for the space of white noise operators. A difficulty is caused by the fact that $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ is not a Banach space but a nuclear Fréchet space. For example, it seems very hard to obtain efficient norm estimates for a formal solution constructed by successive approximations. We shall surmount this obstacle by exploiting symbol calculus, which is a peculiar tool in white noise theory with a useful theorem of characterization [16], see also [2]. The main result is stated in Theorem 10 in Section 5.

1 White Noise Distributions

As usual, let us start with the Gaussian space (E^*, μ), that is, $E^* = \mathcal{S}'(\mathbf{R})$ is the space of tempered distributions and μ is the Gaussian measure on E^* defined by

$$e^{-|\xi|_0^2/2} = \int_{E^*} e^{i\langle x, \xi \rangle} \mu(dx), \qquad \xi \in E,$$

where $|\xi|_0$ stands for the norm of $\xi \in H = L^2(\mathbf{R})$ and $\langle \cdot, \cdot \rangle$ for the canonical bilinear form on $E^* \times E = \mathcal{S}'(\mathbf{R}) \times \mathcal{S}(\mathbf{R})$. The probability space (E^*, μ) is called the *Gaussian space* and plays a key role in white noise theory. For example, a Gaussian random variable

$$B_t(x) = \langle x, 1_{[0,t]} \rangle, \qquad x \in E^*, \quad t \geq 0, \tag{1}$$

is defined in the sense of $L^2(E^*, \mu)$ and $\{B_t\}$ becomes a realization of a Brownian motion. However, the time derivative of the Brownian motion, called the *white noise*, is not well-defined in $L^2(E^*, \mu)$. In fact, we obtain from (1) a rather formal representation:

$$W_t(x) = \langle x, \delta_t \rangle, \qquad x \in E^*, \quad t \geq 0.$$

The above ill-definedness will be easily conquered by introducing a particular Gelfand triple:

$$\mathcal{W} \subset L^2(E^*, \mu) \subset \mathcal{W}^*, \tag{2}$$

where the white noise process becomes a smooth map $t \mapsto W_t \in \mathcal{W}^*$, and moreover, nonlinear functions of $\{W_t\}$ are managed in $\mathcal{W}^*$.

As for the construction of (2), we adopt a general framework due to Cochran, Kuo and Sengupta [5]. We first take a sequence of positive numbers $\alpha = \{\alpha(n)\}_{n=0}^{\infty}$ satisfying the following five conditions:

(A1) $\alpha(0) = 1 \leq \alpha(1) \leq \alpha(2) \leq \cdots$;

(A2) the generating function $G_\alpha(t) = \displaystyle\sum_{n=0}^{\infty} \frac{\alpha(n)}{n!} t^n$ has an infinite radius of convergence;

(A3) the power series $\widetilde{G}_\alpha(t) = \displaystyle\sum_{n=0}^{\infty} \frac{n^{2n}}{n!\alpha(n)} \left\{ \inf_{s>0} \frac{G_\alpha(s)}{s^n} \right\} t^n$ has a positive radius of convergence;

(A4) there exists a constant $C_{1\alpha} > 0$ such that $\alpha(n)\alpha(m) \leq C_{1\alpha}^{n+m} \alpha(n+m)$ for any n, m;

(A5) there exists a constant $C_{2\alpha} > 0$ such that $\alpha(n+m) \leq C_{2\alpha}^{n+m} \alpha(n)\alpha(m)$ for any n, m.

Given such a positive sequence, we define a weighted Fock space:

$$\Gamma_\alpha(E_p) = \left\{ \phi \sim (f_n)_{n=0}^\infty \,;\, f_n \in E_p^{\widehat{\otimes} n},\ \| \phi \|_{p,+}^2 \equiv \sum_{n=0}^\infty n!\, \alpha(n)\, | f_n |_p^2 < \infty \right\}, \tag{3}$$

where

$$E_p = \left\{ \xi \in H \,;\, | \xi |_p \equiv | A^p \xi |_0 < \infty \right\}, \qquad A = 1 + t^2 - \frac{d^2}{dt^2}.$$

We then define

$$\Gamma_\alpha(E) = \operatorname*{proj\,lim}_{p \to \infty} \Gamma_\alpha(E_p), \tag{4}$$

which bears a resemblance to $\mathcal{S}(\mathbf{R}) \equiv E = \operatorname{proj\,lim}_{p\to\infty} E_p$. The constant numbers

$$\| A^{-1} \|_{\mathrm{OP}} = \frac{1}{2}, \qquad \| A^{-q} \|_{\mathrm{HS}}^2 = \sum_{j=0}^\infty \frac{1}{(2j+2)^{2q}}, \quad q > \frac{1}{2},$$

with the simple inequality:

$$| \xi |_p \le \| A^{-1} \|_{\mathrm{OP}}^q \, | \xi |_{p+q}, \qquad \xi \in E, \quad p \in \mathbf{R}, \quad q \ge 0, \tag{5}$$

will be used in various norm estimates below.

We denote by $\mathcal{W}_\alpha$ the complexification of $\Gamma_\alpha(E)$ defined in (4). It is easily proved that $\mathcal{W}_\alpha$ is a nuclear space whose topology is given by the family of norms $\{ \| \cdot \|_{p,+} \,;\, p \in \mathbf{R} \}$ defined in (3). Taking the celebrated Wiener–Itô–Segal isomorphism $L^2(E^*, \mu) \cong \Gamma(H_{\mathbf{C}})$ into account, where $\Gamma(H_{\mathbf{C}})$ is the usual Fock space, i.e., the weighted Fock space with weight one, we obtain a Gelfand triple:

$$\mathcal{W}_\alpha \subset \Gamma(H_{\mathbf{C}}) \cong L^2(E^*, \mu) \subset \mathcal{W}_\alpha^*. \tag{6}$$

This is called the *Cochran-Kuo-Sengupta space* (or *CKS-space* shortly) associated with α. If there is no danger of confusion, we simply set $\mathcal{W} = \mathcal{W}_\alpha$. The canonical bilinear form on $\mathcal{W}^* \times \mathcal{W}$ is denoted by $\langle\langle \cdot, \cdot \rangle\rangle$. Then

$$\langle\langle \Phi, \phi \rangle\rangle = \sum_{n=0}^\infty n!\, \langle F_n, f_n \rangle, \qquad \Phi \sim (F_n) \in \mathcal{W}^*, \quad \phi \sim (f_n) \in \mathcal{W},$$

and it holds that

$$|\langle\langle \Phi, \phi \rangle\rangle| \le \| \Phi \|_{-p,-} \, \| \phi \|_{p,+}, \qquad \| \Phi \|_{-p,-}^2 = \sum_{n=0}^\infty \frac{n!}{\alpha(n)}\, | F_n |_{-p}^2 .$$

As is easily verified, the Brownian motion $t \mapsto B_t$ is differentiable in $\mathcal{W}^*$ and the white noise process $t \mapsto W_t \in \mathcal{W}^*$ is defined.

Here we mention some special cases. The *Hida-Kubo-Takenaka space* [13] is the CKS-space with $\alpha(n) \equiv 1$ and is denoted by $\mathcal{W} = (E)$. The *Kondratiev-Streit space* [12] is also the CKS-space with $\alpha(n) = (n!)^{\beta}$, $0 \le \beta < 1$, and is denoted by $\mathcal{W} = (E)_{\beta}$. Another interesting example is given by the k-th order Bell numbers $\{B_k(n)\}$ defined by

$$G_{\mathrm{Bell}(k)}(t) = \frac{\overbrace{\exp(\exp(\cdots(\exp}^{k-\text{times}} t)\cdots))}{\exp(\exp(\cdots(\exp 0)\cdots))} = \sum_{n=0}^{\infty} \frac{B_k(n)}{n!} t^n. \tag{7}$$

We record some properties of the generating function $G_\alpha(t)$ defined in (A2), whose proofs are straightforward.

Lemma 1 *Let $\alpha = \{\alpha(n)\}$ be a positive sequence satisfying (A1) and (A2), and $G_\alpha(t)$ the generating function defined therein. Then,*

(1) $G_\alpha(0) = 1$ *and* $G_\alpha(s) \le G_\alpha(t)$ *for* $0 \le s \le t$;

(2) $e^s G_\alpha(t) \le G_\alpha(s+t)$ *and* $e^t \le G_\alpha(t)$ *for* $s, t \ge 0$;

(3) $c[G_\alpha(t) - 1] \le G_\alpha(ct) - 1$ *for any* $c \ge 1$ *and* $t \ge 0$.

Lemma 2 *Let $\alpha = \{\alpha(n)\}$ be a positive sequence and $G_\alpha(t)$ the generating function defined therein. If α satisfies conditions (A1), (A2) and (A4), then*

$$G_\alpha(s)G_\alpha(t) \le G_\alpha(C_{1\alpha}(s+t)), \qquad s, t \ge 0.$$

If conditions (A1), (A2) and (A5) are fulfilled, then

$$G_\alpha(s+t) \le G_\alpha(C_{2\alpha}s)G_\alpha(C_{2\alpha}t), \qquad s, t \ge 0.$$

2 White Noise Operators

A continuous linear operator from $\mathcal{W}$ into $\mathcal{W}^*$ is called a *white noise operator*. The space of such operators is denoted by $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ and is equipped with the topology of uniform convergence on every bounded subset. In other words, the topology of $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ is defined by the seminorms:

$$\| \Xi \|_{B,B'} = \sup \{ |\langle\langle \Xi\phi, \psi \rangle\rangle| \,;\, \phi \in B, \, \psi \in B' \},$$

where B, B' run over all bounded subsets of $\mathcal{W}$. Similarly, the topology of $\mathcal{L}(\mathcal{W}, \mathcal{W})$ is defined by

$$\| \Xi \|_{B,p} = \sup \{ \| \Xi\phi \|_p \,;\, \phi \in B \},$$

where B runs over all bounded subsets of $\mathcal{W}$ and $p \ge 0$. Note that the canonical inclusion $\mathcal{L}(\mathcal{W}, \mathcal{W}) \to \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ is continuous.

A useful tool for analyzing white noise operators is the operator symbol. With each $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ we associate a $\mathbf{C}$-valued function on $E_{\mathbf{C}} \times E_{\mathbf{C}}$ defined by

$$\widehat{\Xi}(\xi, \eta) = \langle\langle \Xi\phi_\xi, \phi_\eta \rangle\rangle, \qquad \xi, \eta \in E_{\mathbf{C}},$$

where ϕ_ξ is the exponential vector defined by

$$\phi_\xi(x) = e^{\langle x, \xi \rangle - \langle \xi, \xi \rangle /2} \quad \sim \quad \left(1, \xi, \frac{\xi^{\otimes 2}}{2!}, \cdots, \frac{\xi^{\otimes n}}{n!}, \cdots \right).$$

The above $\widehat{\Xi}$ is called the *symbol* of Ξ. Every operator in $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ is uniquely specified by its symbols since the exponential vectors $\{\phi_\xi\,;\, \xi \in E_{\mathbf{C}}\}$ span a dense subspace of $\mathcal{W} = \mathcal{W}_\alpha$ for any α. The following analytic characterization theorem for operator symbol is a peculiar consequence of white noise theory.

Theorem 3 [2] *A function $\Theta : E_{\mathbf{C}} \times E_{\mathbf{C}} \to \mathbf{C}$ is the symbol of a white noise operator $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$, i.e., $\Theta = \widehat{\Xi}$, if and only if the following two conditions are satisfied:*

(O1) *for any $\xi, \xi_1, \eta, \eta_1 \in E_{\mathbf{C}}$ the function $(z, w) \mapsto \Theta(z\xi + \xi_1, w\eta + \eta_1)$ is entire holomorphic on $\mathbf{C} \times \mathbf{C}$;*

(O2) *there exist constant numbers $C \geq 0$ and $p \geq 0$ such that*

$$|\Theta(\xi, \eta)|^2 \leq C G_\alpha(|\xi|_p^2) G_\alpha(|\eta|_p^2), \qquad \xi, \eta \in E_{\mathbf{C}}.$$

In that case

$$\| \Xi\phi \|^2_{-(p+q),-} \leq C \widetilde{G}^2_\alpha(\| A^{-q} \|^2_{\mathrm{HS}}) \| \phi \|^2_{p+q,+}, \qquad \phi \in \mathcal{W},$$

where $q > 1/2$ is taken in such a way that $\widetilde{G}_\alpha(\| A^{-q} \|^2_{\mathrm{HS}}) < \infty$.

Among white noise operators the most fundamental are the annihilation and creation operators at a point $t \in \mathbf{R}$. Let us now recall the definitions. For any $\phi \in \mathcal{W}$ the limit

$$a_t\phi(x) = \lim_{\theta \to 0} \frac{\phi(x + \theta\delta_t) - \phi(x)}{\theta}, \qquad x \in E^*, \quad t \in \mathbf{R},$$

always exists and a_t becomes a continuous operator from $\mathcal{W}$ into itself, i.e., $a_t \in \mathcal{L}(\mathcal{W}, \mathcal{W})$. Hence by duality $a_t^* \in \mathcal{L}(\mathcal{W}^*, \mathcal{W}^*)$. These operators a_t and a_t^* are called the *annihilation operator* and the *creation operator* at a time point t, respectively.

3 Stochastic Processes as Continuous Flows

Following [17] we introduce some notions. A continuous map $t \mapsto \Phi_t \in \mathcal{W}^*$ defined on an interval is reasonably called a *classical stochastic process* (in the sense of white noise theory). Basic examples are the Brownian motion $\{B_t\}$ and the white noise process $\{W_t\}$. Similarly, a continuous map $t \mapsto \Xi_t \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ defined on an interval is called a *quantum stochastic process* (in the sense of white noise theory). The annihilation operators $\{a_t\}$ and the creation operators $\{a_t^*\}$ form quantum stochastic processes. In some literature the pair $\{a_t, a_t^*\}$ is called the *quantum white noise process.* Moreover, we have

Proposition 4 *Both maps $t \mapsto a_t \in \mathcal{L}(\mathcal{W}, \mathcal{W})$ and $t \mapsto a_t^* \in \mathcal{L}(\mathcal{W}^*, \mathcal{W}^*)$ are infinitely many times differentiable.*

The proof is easy with the help of the norm estimates of derivatives of the delta function, see [18, Appendix]. We next mention a criterion of the continuity of $t \mapsto \Xi_t$ in terms of operator symbols. The proof is a straightforward modification of the argument for the Kondratiev-Streit space [20, Theorem 1.8].

Lemma 5 *Let T be a locally compact space. Then a function $t \mapsto \Xi_t \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$, $t \in T$, is continuous if and only if for any $t_0 \in T$ there exist $K \geq 0$, $p \geq 0$ and an open neighborhood U_0 of t_0 such that*

$$|\widehat{\Xi}_t(\xi, \eta)|^2 \leq K G_\alpha(|\xi|_p^2) G_\alpha(|\eta|_p^2), \qquad \xi, \eta \in E_{\mathbf{C}}, \quad t \in U_0,$$

and

$$\lim_{t \to t_0} \widehat{\Xi}_t(\xi, \eta) = \widehat{\Xi}_{t_0}(\xi, \eta), \qquad \xi, \eta \in E_{\mathbf{C}}.$$

Although an immediate consequence from the above, the next result is also useful.

Lemma 6 *Let $\Xi_n, \Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$, $n = 1, 2, \cdots$. Then Ξ_n converges to Ξ in $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ if and only if there exist $K \geq 0$, $p \geq 0$ such that*

$$|\widehat{\Xi}_n(\xi, \eta)|^2 \leq K G_\alpha(|\xi|_p^2) G_\alpha(|\eta|_p^2), \qquad \xi, \eta \in E_{\mathbf{C}}, \quad n = 1, 2, \cdots,$$

and

$$\lim_{n \to \infty} \widehat{\Xi}_n(\xi, \eta) = \widehat{\Xi}(\xi, \eta), \qquad \xi, \eta \in E_{\mathbf{C}}.$$

We are now in a position to clarify the classical-quantum correspondence in white noise theory. It can be verified that the pointwise multiplication in $\mathcal{W}$ gives rise to a continuous bilinear map: $\mathcal{W} \times \mathcal{W} \to \mathcal{W}$. Hence by duality, for $\Phi \in \mathcal{W}^*$ and $\phi \in \mathcal{W}$ there exists a unique element denoted by $\Phi\phi \in \mathcal{W}^*$ such that

$$\langle\langle \Phi, \phi\psi \rangle\rangle = \langle\langle \Phi\phi, \psi \rangle\rangle, \qquad \psi \in \mathcal{W}.$$

Moreover, the map $\widetilde{\Phi} : \phi \mapsto \Phi\phi$ becomes a white noise operator, i.e., $\widetilde{\Phi} \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$. Thus, every $\Phi \in \mathcal{W}^*$ gives rise to a white noise operator by multiplication and we obtain a continuous inclusion $\mathcal{W}^* \hookrightarrow \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$. In this sense, every classical stochastic process $\{\Phi_t\}$ is identified with a quantum stochastic process. Conversely, given a quantum stochastic process $\{\Xi_t\}$ and a white noise function $\phi \in \mathcal{W}$, $\{\Phi_t = \Xi_t \phi\}$ becomes a classical stochastic process. In particular, a classical stochastic process $\{\Phi_t\}$ is recovered from the corresponding quantum stochastic process $\{\widetilde{\Phi}_t\}$ as $\Phi_t = \widetilde{\Phi}_t \phi_0$, where ϕ_0 is the vacuum vector. We often identify $\widetilde{\Phi}_t$ with Φ_t and denote them by the common symbol for simplicity.

4 Integration of Quantum Stochastic Processes

Let $L^1_{\rm loc}(\mathbf{R})$ be the space of all $\mathbf{C}$-valued locally integrable functions on $\mathbf{R}$. We begin with the following

Lemma 7 *Let $\{L_t\}$ be a quantum stochastic process defined on an interval $I \subset \mathbf{R}$. Then for any $a, t \in I$ and $f \in L^1_{\rm loc}(\mathbf{R})$ there exists a unique operator $\Xi_{a,t}(f) \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ such that*

$$\langle\langle \Xi_{a,t}(f)\phi, \psi \rangle\rangle = \int_a^t f(s) \langle\langle L_s \phi, \psi \rangle\rangle ds, \qquad \phi, \psi \in \mathcal{W}. \tag{8}$$

Moreover, $t \mapsto \Xi_{a,t}(f) \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ is continuous.

PROOF. Let $[a,b] \subset I$ be a closed finite interval. Since $s \mapsto L_s$ is continuous, the interval $[a,b]$ is mapped to a compact subset $K \subset \mathcal{L}(\mathcal{W},\mathcal{W}^*) \cong (\mathcal{W}\otimes\mathcal{W})^*$. Hence there exists some $p \geq 0$ such that

$$C \equiv \sup_{a\leq s\leq b} \| L_s \|_{-p} < \infty.$$

Then for any $s \in [a,b]$ we have

$$|\langle\langle L_s\phi,\ \psi\rangle\rangle| = |\langle\langle L_s,\ \phi\otimes\psi\rangle\rangle| \leq \| L_s \|_{-p} \| \phi\otimes\psi \|_p \leq C \| \phi \|_p \| \psi \|_p,$$

and

$$\left| \int_a^t f(s)\langle\langle L_s\phi,\ \psi\rangle\rangle ds \right| \leq C \| \phi \|_p \| \psi \|_p \int_a^t |f(s)| ds, \qquad \phi,\psi \in \mathcal{W}, \quad a \leq t \leq b.$$

Namely, the right hand side of (8) is a continuous bilinear form on $\mathcal{W}$ and, therefore, a white noise operator $\Xi_{a,t}(f) \in \mathcal{L}(\mathcal{W},\mathcal{W}^*)$ is specified as in (8). Moreover, we obtain

$$|\langle\langle(\Xi_{a,t}(f) - \Xi_{a,s}(f))\phi,\ \psi\rangle\rangle| \leq C \| \phi \|_p \| \psi \|_p \int_s^t |f(u)|\, du, \quad \phi,\psi \in \mathcal{W}, \quad a \leq s < t \leq b.$$

Then for bounded subsets $B_1, B_2 \subset \mathcal{W}$ we have

$$\| \Xi_{a,t}(f) - \Xi_{a,s}(f) \|_{B_1,B_2} \leq C \| B_1 \|_p \| B_2 \|_p \int_s^t |f(s)|\, ds, \qquad a \leq s < t \leq b, \tag{9}$$

where $\| B \|_p = \sup\{\| \phi \|_p\ ;\ \phi \in B\} < \infty$ for any bounded subset $B \subset \mathcal{W}$. The continuity of $t \mapsto \Xi_{a,t}$ then follows from (9) immediately. ∎

The white noise operator $\Xi_{a,t}(f)$ defined in (8) is denoted by

$$\Xi_{a,t}(f) = \int_a^t f(s) L_s\, ds.$$

We can now mention an analogue of the fundamental theorem of calculus.

Theorem 8 *Assume that two quantum stochastic processes $\{L_t\}$ and $\{\Xi_t\}$ are related as*

$$\Xi_t = \int_a^t L_s\, ds.$$

Then, the map $t \mapsto \Xi_t \in \mathcal{L}(\mathcal{W},\mathcal{W}^*)$ is differentiable and

$$\frac{d}{dt}\Xi_t = L_t$$

holds in $\mathcal{L}(\mathcal{W},\mathcal{W}^*)$.

PROOF. For the differentiability at t it is sufficient to show that given bounded subsets $B_1, B_2 \in \mathcal{W}$

$$\lim_{h\to 0} \left\| \frac{\Xi_{t+h} - \Xi_t}{h} - L_t \right\|_{B_1,B_2} = 0. \tag{10}$$

It follows from definition that

$$\left\langle\!\!\left\langle \left(\frac{\Xi_{t+h}-\Xi_t}{h} - L_t\right)\phi,\, \psi \right\rangle\!\!\right\rangle = \frac{1}{h}\int_t^{t+h} \langle\!\langle (L_s - L_t)\phi,\, \psi\rangle\!\rangle\, ds, \qquad \phi, \psi \in \mathcal{W}.$$

Since $s \mapsto L_s$ is continuous, given $\epsilon > 0$ there exists some $\delta > 0$ such that $\| L_s - L_t \|_{B_1,B_2} < \epsilon$ for $|s-t| < \delta$. Then, for $0 < |h| < \delta$ we have

$$\left\| \frac{\Xi_{t+h}-\Xi_t}{h} - L_t \right\|_{B_1,B_2} \le \frac{1}{h}\int_t^{t+h} \| L_s - L_t \|_{B_1,B_2}\, ds < \epsilon,$$

which proves (10). ∎

Lemma 9 *If $\{L_t\}$ is a quantum stochastic process, so are both $\{L_t a_t\}$ and $\{a_t^* L_t\}$.*

PROOF. We only prove that $t \mapsto L_t a_t \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ is continuous, for the rest is obtained by duality. To this end we fix $t \in \mathbf{R}$ and a finite interval $[a, b]$ containing t inside, and choose $p \ge 0$ and $C \ge 0$ as in the proof of Lemma 7. Let $B_1, B_2 \subset \mathcal{W}$ be bounded subsets. Then we have

$$\begin{aligned} \| L_s a_s - L_t a_t \|_{B_1,B_2} &\le \| L_s(a_s - a_t) \|_{B_1,B_2} + \| (L_s - L_t)a_t \|_{B_1,B_2} \\ &\le \| L_s \|_{-p} \| a_s - a_t \|_{B_1,p} \| B_2 \|_p + \| L_s - L_t \|_{a_t B_1, B_2} \\ &\le C \| a_s - a_t \|_{B_1,p} \| B_2 \|_p + \| L_s - L_t \|_{a_t B_1, B_2}, \end{aligned} \tag{11}$$

where $\| B_2 \|_p < \infty$ and $a_t B_1 \subset \mathcal{W}$ is bounded. Then the continuity of $t \mapsto L_t a_t \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ follows immediately from (11). ∎

In Hudson–Parthasarathy calculus (see also [15], [21]) a fundamental role is played by the following three quantum stochastic processes:

$$A_t = \int_0^t a_s\, ds, \qquad A_t^* = \int_0^t a_s^*\, ds, \qquad \Lambda_t = \int_0^t a_s^* a_s\, ds,$$

which are called the *annihilation process*, the *creation process*, and the *number (gauge) process*, respectively. It follows from Proposition 4, Theorem 8 and Lemma 9 that

$$\frac{d}{dt} A_t = a_t, \qquad \frac{d}{dt} A_t^* = a_t^*, \qquad \frac{d}{dt}\Lambda_t = a_t^* a_t,$$

hold in $\mathcal{L}(\mathcal{W}, \mathcal{W})$, $\mathcal{L}(\mathcal{W}^*, \mathcal{W}^*)$ and $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$, respectively. These relations play a key role to go beyond the traditional Itô theory by means of white noise theory.

5 Initial Value Problem

We now study the initial value problem:

$$\frac{d\Xi}{dt} = F(t, \Xi), \qquad \Xi|_{t=0} = \Xi_0, \qquad 0 \le t \le T, \tag{12}$$

where $F : [0, T] \times \mathcal{L}(\mathcal{W}, \mathcal{W}^*) \to \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ is a continuous function and Ξ_0 is a white noise operator. A solution of (12) must be a C^1-map defined on $[0, T]$ with values in $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$, hence by Theorem 8, the initial value problem (12) is equivalent to

$$\Xi_t = \Xi_0 + \int_0^t F(s, \Xi_s)\, ds. \tag{13}$$

Since the solution depends on the "regularity property" of the initial value Ξ_0, we need to consider two weight sequences $\alpha = \{\alpha(n)\}$ and $\omega = \{\omega(n)\}$ with conditions (A1)–(A5), the generating functions of which are related in such a way that

$$G_\alpha(t) = \exp \gamma\{G_\omega(t) - 1\}, \tag{14}$$

where $\gamma > 0$ is a certain constant. In that case, we have continuous inclusions:

$$\mathcal{W}_\alpha \subset \mathcal{W}_\omega \subset L^2(E^*, \mu) \cong \Gamma(H_{\mathbf{C}}) \subset \mathcal{W}_\omega^* \subset \mathcal{W}_\alpha^*$$

and

$$\mathcal{L}(\mathcal{W}_\omega, \mathcal{W}_\omega^*) \subset \mathcal{L}(\mathcal{W}_\alpha, \mathcal{W}_\alpha^*).$$

Such a situation is abstracted from the case of Bell numbers, see (7) for definition. In fact, we have a simple recurrence formula:

$$G_{\mathrm{Bell}(k+1)}(t) = \exp \gamma_k \left\{ G_{\mathrm{Bell}(k)}(t) - 1 \right\}, \quad k \ge 1; \quad G_{\mathrm{Bell}(1)}(t) = e^t,$$

where $\gamma_{k+1} = \exp \gamma_k$ for $k \ge 1$ and $\gamma_1 = 1$.

Theorem 10 *Let $\alpha = \{\alpha(n)\}$ and $\omega = \{\omega(n)\}$ be two weight sequences with conditions (A1)–(A5) such that their generating functions are related as in (14). Let $F : [0, T] \times \mathcal{L}(\mathcal{W}_\alpha, \mathcal{W}_\alpha^*) \to \mathcal{L}(\mathcal{W}_\alpha, \mathcal{W}_\alpha^*)$ be a continuous function and assume that there exist $p \ge 0$ and a nonnegative function $K \in L^1[0, T]$ such that*

$$|\widehat{F}(s, \Xi_1)(\xi, \eta) - \widehat{F}(s, \Xi_2)(\xi, \eta)|^2 \le K(s) G_\omega(|\xi|_p^2) G_\omega(|\eta|_p^2) |\widehat{\Xi}_1(\xi, \eta) - \widehat{\Xi}_2(\xi, \eta)|^2, \tag{15}$$

and

$$|\widehat{F}(s, \Xi)(\xi, \eta)|^2 \le K(s) G_\omega(|\xi|_p^2) G_\omega(|\eta|_p^2)(1 + |\widehat{\Xi}(\xi, \eta)|^2), \tag{16}$$

for all $\xi, \eta \in E_{\mathbf{C}}$, $\Xi \in \mathcal{L}(\mathcal{W}_\omega, \mathcal{W}_\omega^*)$, and $s \in [0, T]$. Then, for any $\Xi_0 \in \mathcal{L}(\mathcal{W}_\omega, \mathcal{W}_\omega^*)$ the initial value problem (12) has a unique solution in $\mathcal{L}(\mathcal{W}_\alpha, \mathcal{W}_\alpha^*)$.

PROOF. In principle, the proof is based on the standard Picard–Lindelöf method of successive approximations (see e.g., [6]) applied to the operator symbols. We define

$$\Xi_t^{(0)} = \Xi_0,$$
$$\Xi_t^{(n)} = \Xi_0 + \int_0^t F(s, \Xi_s^{(n-1)})\, ds, \qquad n \ge 1.$$

We first prove that $\Xi_t^{(n)} \in \mathcal{L}(\mathcal{W}_\omega, \mathcal{W}_\omega^*)$ for $n = 1, 2, \cdots$. Since $\Xi_0 \in \mathcal{L}(\mathcal{W}_\omega, \mathcal{W}_\omega^*)$ by assumption, we may choose $K_0 \ge 0$ and $p_0 \ge 0$ such that

$$|\widehat{\Xi}_0(\xi, \eta)|^2 \le K_0 G_\omega(|\xi|_{p_0}^2) G_\omega(|\eta|_{p_0}^2). \tag{17}$$

Hence by (16) we have

$$|\widehat{F}(s,\Xi_0)(\xi,\eta)|^2 \le K(s)G_\omega(|\,\xi\,|_p^2)G_\omega(|\,\eta\,|_p^2)\left(1+K_0G_\omega(|\,\xi\,|_{p_0}^2)G_\omega(|\,\eta\,|_{p_0}^2)\right). \tag{18}$$

By Lemma 2 we see that

$$G_\omega(|\,\xi\,|_p^2)G_\omega(|\,\xi\,|_{p_0}^2) \le G_\omega(C_{1\omega}(|\,\xi\,|_p^2+|\,\xi\,|_{p_0}^2)) \le G_\omega(|\,\xi\,|_{p_1}^2),$$

where $p_1 \ge \max\{p,p_0\}$ is chosen in such a way that $2C_{1\omega}\|\,A^{-1}\,\|_{\mathrm{OP}}^{p_1-\max\{p,p_0\}} \le 1$, see also (5). Then (18) becomes

$$\begin{aligned}|\widehat{F}(s,\Xi_0)(\xi,\eta)|^2 &\le K(s)\left\{G_\omega(|\,\xi\,|_p^2)G_\omega(|\,\eta\,|_p^2)+K_0G_\omega(|\,\xi\,|_{p_1}^2)G_\omega(|\,\eta\,|_{p_1}^2)\right\}\\ &\le (1+K_0)K(s)G_\omega(|\,\xi\,|_{p_1}^2)G_\omega(|\,\eta\,|_{p_1}^2),\end{aligned} \tag{19}$$

and by integration,

$$\begin{aligned}|\widehat{\Xi_t^{(1)}}(\xi,\eta)|^2 &\le 2|\widehat{\Xi}_0(\xi,\eta)|^2+2\left|\int_0^t \widehat{F}(s,\Xi_0)(\xi,\eta)\,ds\right|^2\\ &\le 2|\widehat{\Xi}_0(\xi,\eta)|^2+2T\bar{K}(1+K_0)G_\omega(|\,\xi\,|_{p_1}^2)G_\omega(|\,\eta\,|_{p_1}^2),\end{aligned} \tag{20}$$

where

$$\bar{K}=\int_0^T K(s)\,ds.$$

Combining (17) and (20), we come to

$$|\widehat{\Xi_t^{(1)}}(\xi,\eta)|^2 \le K_1G_\omega(|\,\xi\,|_{p_1}^2)G_\omega(|\,\eta\,|_{p_1}^2), \qquad 0\le t\le T, \quad \xi,\eta\in E_{\mathbf{C}}, \tag{21}$$

where $K_1 = 2K_0 + 2T\bar{K}(1+K_0)$ is a constant. It then follows from the characterization theorem for operator symbols (Theorem 3) that $\Xi_t^{(1)} \in \mathcal{L}(\mathcal{W}_\omega,\mathcal{W}_\omega^*)$. Comparing (21) with (17), we see that the above argument can be repeated to conclude that $\Xi_t^{(n)} \in \mathcal{L}(\mathcal{W}_\omega,\mathcal{W}_\omega^*)$ for all n.

For simplicity we put

$$\Theta_n(t;\xi,\eta)=\widehat{\Xi_t^{(n)}}(\xi,\eta)=\langle\langle \Xi_t^{(n)}\phi_\xi,\,\phi_\eta\rangle\rangle, \qquad \xi,\eta\in E_{\mathbf{C}}, \quad 0\le t\le T.$$

We shall prove that the limit

$$\Theta_t(\xi,\eta)=\lim_{n\to\infty}\Theta_n(t;\xi,\eta)$$

exists. Since

$$\Theta_n(t;\xi,\eta)=\widehat{\Xi}_0(\xi,\eta)+\int_0^t \widehat{F}(s,\Xi_s^{(n-1)})(\xi,\eta)\,ds \tag{22}$$

by definition, in view of assumption (15) we have

$$\begin{aligned}|\Theta_n(t;\xi,\eta)-\Theta_{n-1}(t;\xi,\eta)|^2 &= \left|\int_0^t\left\{\widehat{F}(s,\Xi_s^{(n-1)})(\xi,\eta)-\widehat{F}(s,\Xi_s^{(n-2)})(\xi,\eta)\right\}ds\right|^2\\ &\le TG_\omega(|\,\xi\,|_p^2)G_\omega(|\,\eta\,|_p^2)\int_0^t K(s)\,|\Theta_{n-1}(s;\xi,\eta)-\Theta_{n-2}(s;\xi,\eta)|^2\,ds,\end{aligned} \tag{23}$$

and moreover, repeating this argument yields

$$\begin{aligned}
&|\Theta_n(t;\xi,\eta) - \Theta_{n-1}(t;\xi,\eta)|^2 \\
&\quad \le \left\{TG_\omega(|\xi|_p^2)G_\omega(|\eta|_p^2)\right\}^{n-1} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-2}} dt_{n-1} \\
&\qquad\qquad \times K(t_1)K(t_2)\cdots K(t_{n-1})|\Theta_1(t_{n-1};\xi,\eta) - \Theta_0(t_{n-1};\xi,\eta)|^2. \qquad (24)
\end{aligned}$$

As for the last quantity, we see from (19) that

$$\begin{aligned}
|\Theta_1(t;\xi,\eta) - \Theta_0(t;\xi,\eta)|^2 &= \left|\int_0^t \widehat{F}(s,\Xi_0)(\xi,\eta)\,ds\right|^2 \\
&\le T\int_0^T |\widehat{F}(s,\Xi_0)(\xi,\eta)|^2 ds \\
&\le T\bar{K}(1+K_0)G_\omega(|\xi|_{p_1}^2)G_\omega(|\eta|_{p_1}^2) \equiv H(\xi,\eta).
\end{aligned}$$

Thus (24) becomes

$$\begin{aligned}
&|\Theta_n(t;\xi,\eta) - \Theta_{n-1}(t;\xi,\eta)|^2 \\
&\quad \le \left\{TG_\omega(|\xi|_p^2)G_\omega(|\eta|_p^2)\right\}^{n-1} \times \\
&\qquad\qquad \times H(\xi,\eta)\int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-2}} dt_{n-1}K(t_1)K(t_2)\cdots K(t_{n-1}) \\
&\quad \le \frac{1}{(n-1)!}\left\{T\bar{K}G_\omega(|\xi|_p^2)G_\omega(|\eta|_p^2)\right\}^{n-1} H(\xi,\eta). \qquad (25)
\end{aligned}$$

Let $0 < r < 1$. Then we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} |\Theta_n(t;\xi,\eta) - \Theta_{n-1}(t;\xi,\eta)| \\
&\qquad \le \left(\frac{r^2}{1-r^2}\right)^{1/2}\left(\sum_{n=1}^{\infty}\frac{1}{r^{2n}}|\Theta_n(t;\xi,\eta) - \Theta_{n-1}(t;\xi,\eta)|^2\right)^{1/2} \\
&\qquad \le \left(\frac{H(\xi,\eta)}{1-r^2}\right)^{1/2} \exp\left\{\frac{T\bar{K}}{2r^2}G_\omega(|\xi|_p^2)G_\omega(|\eta|_p^2)\right\}. \qquad (26)
\end{aligned}$$

This proves that

$$\Theta_t(\xi,\eta) = \lim_{n\to\infty}\Theta_n(t;\xi,\eta) = \widehat{\Xi}_0(\xi,\eta) + \sum_{n=1}^{\infty}\left\{\Theta_n(t;\xi,\eta) - \Theta_{n-1}(t;\xi,\eta)\right\} \qquad (27)$$

converges uniformly in t for any fixed $\xi, \eta \in E_{\mathbf{C}}$.

We next prove that that there exists a white noise operator $\Xi_t \in \mathcal{L}(\mathcal{W}_\alpha, \mathcal{W}_\alpha^*)$ such that $\Theta_t = \widehat{\Xi}_t$ for $0 \le t \le T$. Condition (O1) in Theorem 3 is easily checked from (26) since the convergence (27) is also uniform in (ξ,η) running over any compact subset of $\mathcal{W} \times \mathcal{W}$. As for condition (O2) we shall estimate $|\Theta_t(\xi,\eta)|^2$. First by (26) and (27) we have

$$\begin{aligned}|\Theta_t(\xi,\eta)|^2 &\le 2|\widehat{\Xi}_0(\xi,\eta)|^2 + 2\left|\sum_{n=1}^{\infty}\left\{\Theta_n(t;\xi,\eta) - \Theta_{n-1}(t;\xi,\eta)\right\}\right|^2 \\ &\le 2|\widehat{\Xi}_0(\xi,\eta)|^2 + \frac{2H(\xi,\eta)}{1-r^2}\exp\left\{\frac{T\bar{K}}{r^2}G_\omega(|\xi|_p^2)G_\omega(|\eta|_p^2)\right\}. \qquad (28)\end{aligned}$$

Using an elementary inequality: $t\,e^{\lambda t} \le e^{(\lambda+1)t}$ for $t \ge 0$, the second term of (28) becomes

$$\frac{2H(\xi,\eta)}{1-r^2}\exp\left\{\frac{T\bar{K}}{r^2}G_\omega(|\xi|_p^2)G_\omega(|\eta|_p^2)\right\} \le M_1 \exp\left\{M_2 G_\omega(|\xi|_{p_1}^2)G_\omega(|\eta|_{p_1}^2)\right\}, \qquad (29)$$

where $|\xi|_p \le |\xi|_{p_1}$ is used and

$$M_1 = \frac{2T\bar{K}(1+K_0)}{1-r^2}, \qquad M_2 = \frac{T\bar{K}}{r^2} + 1.$$

We choose $0 < r < 1$ in such a way that $M_2/\gamma \ge 1$, where γ is the constant defined in (14). Then by Lemmas 1 and 2 we have

$$\begin{aligned}M_2 G_\omega(|\xi|_{p_1}^2)G_\omega(|\eta|_{p_1}^2) &\le M_2 G_\omega\left(C_{1\omega}(|\xi|_{p_1}^2 + |\eta|_{p_1}^2)\right) \\ &= \gamma\left\{\frac{M_2}{\gamma}\left[G_\omega\left(C_{1\omega}(|\xi|_{p_1}^2 + |\eta|_{p_1}^2)\right) - 1\right]\right\} + M_2 \\ &\le \gamma\left\{G_\omega\left(\frac{M_2}{\gamma}C_{1\omega}(|\xi|_{p_1}^2 + |\eta|_{p_1}^2)\right) - 1\right\} + M_2. \qquad (30)\end{aligned}$$

We then take $q \ge 0$ in such a way that $(M_2/\gamma)\,C_{1\omega}\|A^{-1}\|_{\mathrm{OP}}^{2q} \le 1$. Then (30) becomes

$$M_2 G_\omega(|\xi|_{p_1}^2)G_\omega(|\eta|_{p_1}^2) \le \gamma\left\{G_\omega(|\xi|_{p_1+q}^2 + |\eta|_{p_1+q}^2) - 1\right\} + M_2,$$

and, in view of (14) we obtain

$$\exp\left\{M_2 G_\omega(|\xi|_{p_1}^2)G_\omega(|\eta|_{p_1}^2)\right\} \le e^{M_2}G_\alpha(|\xi|_{p_1+q}^2 + |\eta|_{p_1+q}^2). \qquad (31)$$

Consequently, combining (28), (29) and (31), we have

$$\begin{aligned}|\Theta_t(\xi,\eta)|^2 &\le 2|\widehat{\Xi}_0(\xi,\eta)|^2 + M_1 e^{M_2}G_\alpha(|\xi|_{p_1+q}^2 + |\eta|_{p_1+q}^2) \\ &\le 2K_0 G_\omega(|\xi|_{p_0}^2)G_\omega(|\eta|_{p_0}^2) + M_1 e^{M_2}G_\alpha(C_{2\alpha}|\xi|_{p_1+q}^2)G_\alpha(C_{2\alpha}|\eta|_{p_1+q}^2),\end{aligned}$$

where (17) and Lemma 2 are used. Taking $q_1 > p_1+q > p_0$ such that $C_{2\alpha}\|A^{-1}\|_{\mathrm{OP}}^{2(q_1-p_1-q)} \le 1$ and noting that $G_\omega(s) \le \gamma^{-1}e^{\gamma-1}G_\alpha(s)$ for $s \ge 0$, we come to

$$|\Theta_t(\xi,\eta)|^2 \le (2K_0\gamma^{-1}e^{\gamma-1} + M_1 e^{M_2})G_\alpha(|\xi|_{q_1}^2)G_\alpha(|\eta|_{q_1}^2). \qquad (32)$$

In other words, Θ_t satisfies condition (O2) in Theorem 3, and hence there exists a unique $\Xi_t \in \mathcal{L}(\mathcal{W}_\alpha, \mathcal{W}_\alpha^*)$ such that

$$\Theta_t(\xi,\eta) = \widehat{\Xi}_t(\xi,\eta), \qquad \xi,\eta \in E_{\mathbf{C}}, \quad t \in [0,T]. \qquad (33)$$

We now prove that $\{\Xi_t\}$ is a solution of (12). As is already obvious, $\Theta_n(t)$ also satisfies (32) commonly, and therefore by Lemma 6 we see that $\Xi_t^{(n)} \to \Xi_t$ in $\mathcal{L}(\mathcal{W}_\alpha, \mathcal{W}_\alpha^*)$ uniformly in t. Hence, letting $n \to \infty$ in (22), we conclude that

$$\Theta_t(\xi, \eta) = \widehat{\Xi}_0(\xi, \eta) + \int_0^t \widehat{F}(s, \Xi_s)(\xi, \eta)\, ds,$$

which means that $\{\Xi_t\}$ is a solution of (13), and hence of (12).

For the uniqueness we suppose that two quantum stochastic processes $\{\Xi_t\}$ and $\{X_t\}$ satisfy the same integral equation (13). A similar argument as in the derivation of (23) yields

$$|\widehat{\Xi}_t(\xi, \eta) - \widehat{X}_t(\xi, \eta)|^2 \le T G_\omega(|\xi|_p^2) G_\omega(|\eta|_p^2) \int_0^t K(s) |\widehat{\Xi}_s(\xi, \eta) - \widehat{X}_s(\xi, \eta)|^2\, ds,$$

from which $\widehat{\Xi}_t = \widehat{X}_t$ follows by a standard argument with the Gronwall inequality. ∎

We remind that Theorem 10 covers a simple example: Let $\{L_t\}, \{M_t\} \subset \mathcal{L}(\mathcal{W}_\omega, \mathcal{W}_\omega^*)$ be two quantum stochastic processes, where t runs over $[0, T]$. Then the initial value problem

$$\frac{d}{dt}\Xi_t = L_t \diamond \Xi_t + M_t, \qquad \Xi|_{t=0} = \Xi_0 \in \mathcal{L}(\mathcal{W}_\omega, \mathcal{W}_\omega^*), \tag{34}$$

has a unique solution in $\mathcal{L}(\mathcal{W}_\alpha, \mathcal{W}_\alpha^*)$. Note that equation (34) is a considerable generalization of a traditional quantum stochastic differential equation.

References

[1] L. Accardi, Y.-G. Lu and I. Volovich: *Non-linear extensions of classical and quantum stochastic calculus and essentially infinite dimensional analysis*, in "Probability Towards 2000 (L. Accardi and C. C. Heyde, Eds.)," Lect. Notes in Stat. Vol. 128, pp. 1–33, Springer–Verlag, 1998.

[2] D. M. Chung, U. C. Ji and N. Obata: *Higher powers of quantum white noises in terms of integral kernel operators*, Infinite Dimen. Anal. Quantum Prob. **1** (1998), 533–559.

[3] D. M. Chung, U. C. Ji and N. Obata: *Normal-ordered white noise differential equations II: Regularity properties of solutions*, in "Prob. Theory and Math. Stat. (B. Grigelionis et al. Eds.)," VSP/TEV, 1999. (in press)

[4] D. M. Chung, U. C. Ji and N. Obata: *Quantum stochastic analysis via white noise operators in weighted Fock space*, preprint, 2000.

[5] W. G. Cochran, H.-H. Kuo and A. Sengupta: *A new class of white noise generalized functions*, Infinite Dimen. Anal. Quantum Prob. **1** (1998), 43–67.

[6] P. Hartman: "Ordinary Differential Equations (second edition)," Birkhäuser, 1982.

[7] T. Hida: "Analysis of Brownian Functionals," Carleton Math. Lect. Notes, no. 13, Carleton University, Ottawa, 1975.

[8] T. Hida: *Harmonic analysis on complex random systems*, in this volume, 2000.

[9] T. Hida, H.-H. Kuo, J. Potthoff and L. Streit: "White Noise: An Infinite Dimensional Calculus," Kluwer Academic Publishers, 1993.

[10] H. Holden, B. Øksendal, J. Ubøe and T. Zhang: "Stochastic Partial Differential Equations," Birkhäuser, 1996.

[11] R. L. Hudson and K. R. Parthasarathy: *Quantum Itô's formula and stochastic evolutions*, Commun. Math. Phys. **93** (1984), 301–323.

[12] Yu. G. Kondratiev and L. Streit: *Spaces of white noise distributions: Constructions, descriptions, applications I*, Rep. Math. Phys. **33** (1993), 341–366.

[13] I. Kubo and S. Takenaka: *Calculus on Gaussian white noise I*, Proc. Japan Acad. **56A** (1980), 376–380.

[14] H.-H. Kuo: "White Noise Distribution Theory," CRC Press, 1996.

[15] P.-A. Meyer: "Quantum Probability for Probabilists," Lect. Notes in Math. Vol. 1538, Springer–Verlag, 1993.

[16] N. Obata: "White Noise Calculus and Fock Space," Lect. Notes in Math. Vol. 1577, Springer–Verlag, 1994.

[17] N. Obata: *Generalized quantum stochastic processes on Fock space*, Publ. RIMS **31** (1995), 667–702.

[18] N. Obata: *Integral kernel operators on Fock space – Generalizations and applications to quantum dynamics*, Acta Appl. Math. **47** (1997), 49–77.

[19] N. Obata: *Quantum stochastic differential equations in terms of quantum white noise*, Nonlinear Analysis, Theory, Methods and Applications **30** (1997), 279–290.

[20] N. Obata: *Wick product of white noise operators and quantum stochastic differential equations*, J. Math. Soc. Japan. **51** (1999), 613–641.

[21] K. R. Parthasarathy: "An Introduction to Quantum Stochastic Calculus," Birkhäuser, 1992.