ON THE REGULARITY OF THE BERGMAN KERNEL ON THE BOUNDARY

JOE KAMIMOTO

1. Introduction

In this article, we study the regularity of the Bergman kernel and the Szegö kernel on the boundary of weakly pseudoconvex tube domains off the diagonal.

Let Ω be a domain in $\mathbb{C}^n$. The Bergman space $B(\Omega)$ is the closed subspace of $L^2(\Omega)$ consisting of holomorphic L^2-functions on Ω. The Bergman projection is the orthogonal projection $\mathbb{B} : L^2(\Omega) \to B(\Omega)$. It is known that the projection $\mathbb{B}$ can be represented by using some integral kernel:

$$\mathbb{B}f(z) = \int_\Omega B(z,w)f(w)dV(w) \quad \text{for } f \in L^2(\Omega),$$

where $B : \Omega \times \Omega \to \mathbb{C}$ is called the *Bergman kernel* of the domain Ω and dV is the Lebesgue measure on Ω.

The regularity of the Bergman kernel on the boundary off the diagonal is deeply connected with many other subjects in the $\bar{\partial}$-Neumann problem. In 1972 Kerzman [15] proved the Bergman kernel of a C^∞-smoothly bounded strictly pseudoconvex domain Ω in $\mathbb{C}^n$ is C^∞-smooth up to the boundary off the diagonal: i.e.

$$B \in C^\infty(\bar{\Omega} \times \bar{\Omega} \setminus \Delta), \tag{1.1}$$

where $\Delta = \{(z, z); z \in \partial\Omega\}$. His proof is based on a certain pseudolocal estimate of the $\bar{\partial}$-Neumann problem. Later Bell [1] and Boas [3] independently showed (1.1) in the case of domains of finite type (in the sense of Kohn or D'Angelo) by generalizing the argument of Kerzman.

Let us consider this kind of question in the real analytic category. For a set K in $\mathbb{C}^n$, $C^\omega(K)$ means the set of real analytic functions in some open neighborhood of K. In the case of C^ω-smoothly bounded strictly pseudoconvex domains in $\mathbb{C}^n$, the Bergman kernel is known in [20],[21],[22],[2] to satisfy

$$B \in C^\omega(\bar{\Omega} \times \bar{\Omega} \setminus \Delta). \tag{1.2}$$

In weakly pseudoconvex and of finite type case, it had also been expected that (1.2) always holds. Surprisingly Christ and Geller [7], in 1992, showed that the Bergman kernel does not satisfy (1.2) for the domain $\Omega_m = \{(z_1, z_2); \Im(z_2) > [\Re(z_1)]^{2m}\}$ $(m = 2, 3, \dots)$, which is a very simple weakly pseudoconvex domain of finite type. In general, necessary and sufficient conditions for (1.2) are yet to be known until now.

The following question is the first step for this problem: Find many perturbations of Ω_m whose Bergman kernels do not have the real analytic property (1.2). The following theorem partially answers this question.

Theorem 1.1. *For any weakly pseudoconvex tube domain Ω in $\mathbb{C}^2$ with real analytic boundary, there exist points on $\partial\Omega \times \partial\Omega \setminus \Delta$ where the Bergman kernel is not real analytic.*

In more detail, we can determine the set of the failure of the real analyticity and the best order of the Gevrey class (see Section 4). We remark that our theorem is established for both cases of bounded and unbounded bases of Ω.

Next let us consider an analogous problem about the Szegö kernel. Suppose that Ω has C^∞-smooth boundary equipped with a surface element $d\sigma$. The Hardy space $H^2(\Omega)$ is the subspace of $L^2(\partial\Omega)$ consisting L^2-boundary values of holomorphic functions. The Szegö projection is the orthogonal projection $\mathbb{S} : L^2(\partial\Omega) \to H^2(\Omega)$. The projection $\mathbb{S}$ can be represented by using some integral kernel:

$$\mathbb{S}f(z) = \int_{\partial\Omega} S(z, w) f(w) d\sigma(w) \quad \text{for } f \in L^2(\partial\Omega),$$

where $S : \Omega \times \Omega \to \mathbb{C}$ is called the *Szegö kernel* of the domain Ω (with respect to $d\sigma$).

There are many analogous studies about the Szegö kernel (refer to the Introduction in [7]). Christ and Geller [7] also showed the failure of the real analyticity of the Szegö kernel of Ω_m ($m = 2, 3, \dots$). We also give a similar result about the Szegö kernel.

Theorem 1.2. *For any weakly pseudoconvex tube domain Ω in $\mathbb{C}^2$ with real analytic boundary, there exist points on $\partial\Omega \times \partial\Omega \setminus \Delta$ where the Szegö kernel (with respect to some surface element) is not real analytic.*

Note that the above Szegö kernel is C^∞-smooth on $\bar{\Omega} \times \bar{\Omega} \setminus \Delta$ by [17]. The real analyticity of the Szegö kernel is deeply connected with the analytic hypoellipticity of the tangential Cauchy-Riemann operator $\bar{\partial}_b$ on the boundary. It was shown in [7] that the *CR* manifold $\partial\Omega_m$ ($m = 2, 3, \dots$) is a counterexample to the analytic hypoellipticity of $\bar{\partial}_b$ by regarding the Szegö kernel as a singular solution of $\bar{\partial}_b u = 0$. More generally Christ [4] directly constructed singular solutions for $\bar{\partial}_b \bar{\partial}_b^* u = 0$ and $\bar{\partial}_b^* u \notin C^\omega$ in the case of weakly pseudoconvex domain $\Omega_P = \{z \in \mathbb{C}^2; \Im(z_2) > P(\Re(z_1))\}$ where P is real analytic. (In [4] he mainly treated the case of bounded Reinhardt domains.) The singularity of his solutions closely resembles that of the Bergman kernel in our analysis.

Let us explain our analysis. In this ariticle we only consider the case of the Bergman kernel. Our analysis is based on integral representations of the Bergman kernel which were obtained in the case of general tube domains in [8],[19], etc. (Section 2). Christ and Geller [7] also used these representations, but their proof needed some kind of homogeneity of the domain Ω_m. In the case of general tube domains, this homogeneity cannot always be expected, so it seems difficult to apply their method directly. On the other hand the author [12] (see also [5]) computed some asymptotic expansion of the the Bergman kernel to see the situation of these singularities directly. This analysis is valid for our case. In order to apply the analysis of [5],[12], some appropriate localization of the singularity is necessary (Section 3). This property of localization implies that the failure of the real analyticity is determined by the local geometry of the boundary. After localizing integral representation, we compute some asymptotic expansion by the residue formula. In this expansion it can be directly understood that each term fails to be real analytic and the first term has the strongest singularity. Thus we can obtain Theorem 1.1 (Section 4). In the case of the Szegö kernel, similar integral representations were obtained in [16],[10],[19], etc., so Theorem 1.2 can be shown in a similar fashion.

Last we remark that Francsics and Hanges [9] obtained a very similar result to Theorem 1.1. They explain the regularity problem for the Bergman kernel by using symplectic geometry.

2. Integral representations

First let us recall an integral representation of the Bergman kernel for general tube domains. We set $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, $z_j = x_j + iy_j$ $(x_j, y_j \in \mathbb{R})$, $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$, $\bar{w} = (\bar{w}_1, \dots, \bar{w}_n) \in \mathbb{C}^n$, $t = (t_1, \dots, t_n) \in \mathbb{R}^n$ and $\langle z, t\rangle = \sum_{j=1}^n z_j t_j$.

Let $\Omega \subset \mathbb{C}^n$ be a tube domain whose base is $\omega \subset \mathbb{R}^n$; that is

$$\Omega = \mathbb{R}^n + i\omega.$$

From [8],[19], the Bergman kernel $B(z, w)$ of Ω can be expressed as follows:

$$B(z, w) = \frac{1}{(2\pi)^n} \int_{\Lambda^*} e^{i\langle z-\bar{w}, t\rangle} \frac{dt}{D(t)}, \tag{2.1}$$

with

$$D(t) = \int_\omega e^{-2\langle t, y\rangle} dy,$$

where $\Lambda^* = \{t \in \mathbb{R}^n; D(t) < \infty\}$.

Next in order to prove the theorem, we will rewrite the above representation by using appropriate transformations. From now on we assume that Ω is a pseudoconvex tube domain in $\mathbb{C}^2$ with real analytic boundary. Then it is well known that the base ω is convex in $\mathbb{R}^2$. Let $z^0 = (z_1^0, z_2^0)$ be a boundary point of Ω. By a translation of coordinate axes, we may assume that $\Im(z_1^0) = \Im(z_2^0) = 0$. Then the *maximum cone* Λ of $\omega \subset \mathbb{R}^2$ is defined by

$$\Lambda = \{y \in \mathbb{R}^2 \colon (sy_1, sy_2) \in \omega \text{ for any } s > 0\}.$$

and the set Λ^* becomes the *dual cone* of Λ, i.e.

$$\Lambda^* = \{t \in \mathbb{R}^2; \langle t, y\rangle \geq 0 \text{ for any } y \in \Lambda\}.$$

First we consider the case where the base ω is unbounded. By a linear transformation in $\mathbb{R}^2$, ω can be transformed into ω_f, which has the following properties: ω_f is expressed as

$$\omega_f = \{y \in \mathbb{R}^2; y_2 > f(y_1)\},$$

where $f \in C^\omega((a_-, a_+))$, with $-\infty \leq a_- < 0 < a_+ \leq \infty$, satisfying that $f(0) = f'(0) = 0$ and $f(x) \to \infty$ as $x \to a_\pm$; moreover the maximum cone of ω_f is $\Lambda_R = \{y \in \mathbb{R}^2; y_2 \geq R|y_1| > 0\}$ for $R \geq 0$ or $\Lambda_\infty := \{(0, y_2); y_2 > 0\}$. The dual cone of Λ_R is $\Lambda_R^* = \{t \in \mathbb{R}^2; t_2 \geq R^{-1}|t_1| > 0\}$ and $\Lambda_\infty^* = \{t \in \mathbb{R}^2; t_2 \geq 0\}$. From (2.1), the Bergman kernel of $\Omega_f := \mathbb{R}^2 + i\omega_f$ can be expressed as

$$B(z, w) = \frac{1}{2\pi^2}\int_0^\infty \int_{-Rt_2}^{Rt_2} e^{i(z-\bar{w}, t)} \frac{t_2}{\mathcal{D}_f(t_1, t_2)} dt_1 dt_2,$$

where

$$\mathcal{D}_f(t_1, t_2) = \int_{a_-}^{a_+} e^{-2t_2 f(\xi) - 2t_1\xi} d\xi.$$

Next we consider the case where ω is bounded. In a similar fashion, ω can be transformed into $\omega_{f,\tilde{f}}$, which has the following properties: $\omega_{f,\tilde{f}}$ is expressed as

$$\omega_{f,\tilde{f}} = \{y \in \mathbb{R}^2; f(y_1) < y_2 < \tilde{f}(y_1)\},$$

where $f, \tilde{f} \in C^\omega((a_-, a_+))$, with $-\infty < a_- < 0 < a_+ < \infty$, satisfy that $f(0) = f'(0) = 0$ and $f(a_\pm) = \tilde{f}(a_\pm)$, respectively. Here the maximum cone is $\Lambda_0 = \emptyset$ and the dual cone of Λ_0 is $\Lambda_0^* = \mathbb{R}^2$. From (2.1), the Bergman kernel of $\Omega_{f,\tilde{f}} := \mathbb{R}^2 + i\omega_{f,\tilde{f}}$ can be expressed as

$$B(z, w) = \frac{1}{2\pi^2}\iint_{\mathbb{R}^2} e^{i(z-\bar{w}, t)} \frac{t_2}{\mathcal{D}_f(t_1, t_2) - \mathcal{D}_{\tilde{f}}(t_1, t_2)} dt_1 dt_2,$$

where $\mathcal{D}_f, \mathcal{D}_{\tilde{f}}$ are as above.

Since linear transformations have no essential influence on the argument of regularity of the Bergman kernel, it suffices to investigate the real analyticity in the above two cases.

3. LOCALIZATION

In this section we show that the singularity of the Bergman kernel at the boundary can be locally determined.

We set $\zeta = (\zeta_1, \zeta_2)$ where $\zeta_j = (z_j - \bar{w}_j)/2i$ and $\mathcal{B}(\zeta) = B(z, w)$. Let $\rho, \delta_\pm$ be constants such that $0 < \rho \leq R$ and $a_- \leq \delta_- < 0 < \delta_+ \leq a_+$. We set $\delta =$

$\min\{-\delta_-, \delta_+\}$ and $\tilde{\delta} = \max\{-\delta_-, \delta_+\}$. For $\rho, \delta_\pm$, define the function $\mathcal{B}(\zeta; \rho, \delta_\pm)$ by

$$\mathcal{B}(\zeta; \rho, \delta_\pm) = \frac{1}{2\pi^2}\int_0^\infty \int_{-\rho t_2}^{\rho t_2} e^{-2\langle\zeta, t\rangle}\frac{t_2}{\mathcal{D}_f(t_1, t_2; \delta_\pm)}dt_1 dt_2. \tag{3.1}$$

where

$$\mathcal{D}_f(t_1, t_2; \delta_\pm)(:= \mathcal{D}_f(\delta_\pm)) = \int_{\delta_-}^{\delta_+} e^{-2t_2 f(\xi) - 2t_1\xi} d\xi.$$

Note that $\mathcal{B}(\zeta; R, a_\pm) = \mathcal{B}(\zeta)$ in the case of ω_f.

Now the singularity of the Bergman kernel $\mathcal{B}(\zeta)$ at $K_0 := \{(0, 0)\} + i\mathbb{R}^2$ is locally described as follows.

Proposition 3.1. *For any $\delta_\pm$, there exists a positive constant ρ_0 such that if $0 < \rho \leq \rho_0$, then $\mathcal{B}(\zeta) - \mathcal{B}(\zeta; \rho, \delta_\pm)$ is real analytic in ζ in some neighborhood of K_0.*

The proof of this proposition is seen in [11].

4. Proof of Theorem 1.1.

4.1. **Preliminaries.** Since ω is convex and $f(x)$ is real analytic in (a_-, a_+) with $f(0) = f'(0) = 0$, there exist a natural number m and a real analytic function $g(x)$ such that $g(0) > 0$ and $f(x) = x^{2m}g(x)$ in (a_-, a_+). Note that z^0 is of type $2m$ (in the sense of D'Angelo). (If $f^{(k)}(0) = 0$ for any $k \in \mathbb{N}$, then the real analyticity of $f(x)$ implies that $\Omega_f = \{z \in \mathbb{C}^2; \Im(z_2) > 0\}$ whose Bergman space is $\{0\}$.)

We set

$$K(z, t) = \mathcal{B}((z, t + if(y)); (0, 0)).$$

Suppose that z^0 is a weakly pseudoconvex point of type $2m$ ($m \geq 2$). Now fix $\delta_\pm = \pm\delta_0$ with $0 < \delta_0 \leq \min\{-a_-, a_+\}$ and set $\hat{\tau} = \tau^{1/(2m)}$. For $\tau_0, \rho_0 > 0$, define the function $K(z, t; \tau_0, \rho_0)$ by

$$K(z, t; \tau_0, \rho_0) = \frac{1}{2\pi^2}\int_{\tau_0}^\infty e^{it\tau}e^{-f(y)\tau}F(z; \hat{\tau}, \rho_0)\tau^{1+\frac{1}{m}}d\tau, \tag{4.1}$$

$$F(z; \hat{\tau}, \rho_0) = \int_{-\rho_0\hat{\tau}^{2m-1}}^{\rho_0\hat{\tau}^{2m-1}} \frac{e^{iz\hat{\tau}v}}{\varphi(v; \hat{\tau})}dv \tag{4.2}$$

$$\varphi(v, \hat{\tau}) = \int_{-\delta_0\hat{\tau}}^{\delta_0\hat{\tau}} e^{-2g(w/\hat{\tau})w^{2m} - 2vw}dw.$$

Recalling the definitions of $\mathcal{B}(\zeta)$ and $\mathcal{B}(\zeta; \rho, \delta_\pm)$ in Section 3, we have

$$K(z, t) = \mathcal{B}(z/2i, (t + if(y))/2i),$$
$$K(z, t; 0, \rho) = \mathcal{B}(z/2i, (t + if(y))/2i; \rho, \pm\delta_0).$$

By Proposition 3.1, there exists $\rho_0 > 0$ such that if $\rho \leq \rho_0$, then $K(\cdot, \cdot; 0, \rho) - K(\cdot, \cdot)$ is real analytic around $(0, 0)$. Moreover it is easy to check the real analyticity of $K(\cdot, \cdot; \tau_0, \rho_0) - K(\cdot, \cdot; 0, \rho_0)$ for any $\tau_0 \geq 0$.

In a small neighborhood of $(0,0)$, if $K(\cdot,\cdot)$ is real analytic away from $(0,0)$, then so is $K(\cdot,\cdot;\tau_0,\rho_0)$. Our goal is to show the following theorem.

Theorem 4.1. *There exist positive numbers x_0,ρ_0,τ_0 such that $K(z,t;\rho_0,\tau_0)$ is not real analytic in (z,t) on the set $\Xi(x_0)=\{(x+i0,0);0<|x|\leq x_0\}$, moreover it belongs to s-th order Gevrey class for $s\geq 2m$, but no better, on $\Xi(x_0)$.*

Remark. If the boundary $\partial\Omega$ is locally regarded as $\mathbb{C}\times\mathbb{R}$ as above, the Bergman kernel $B((x+iy,t+if(y));(u+iv,s+if(v)))$ fails to be real analytic on the set

$$\{(x+iy,t;u+iv,s);y=v=0,t=s\}\cup\{\text{diagonal}\}$$

in some small neighborhood of $(0,0)$.

4.2. **Analysis of $\varphi(v,\hat{\tau})$.** In order to prove the theorems, it is necessary to analyze the function $\varphi(v,\hat{\tau})$. Note that a similar analysis is done in [4]. We express some positive constants depending on X by $C(X)$ or $C_j(X)$. The proofs of the lemmas blow are seen in [11].

When $\hat{\tau}$ is sufficiently large, the function $\varphi(v,\hat{\tau})$ can be well approximated by the entire function:

$$\varphi(v)=\int_{-\infty}^{\infty}e^{-2gw^{2m}-2vw}dw\quad(m=2,3,\dots),$$

where $g:=g(0)$. Indeed the Lemmas 4.2,4.3, below, show this nature. There are many studies of the properties of $\varphi(v)$ (refer to the Introduction in [13]).

First let us consider the zeros of $\varphi(\cdot,\hat{\tau})$. It is known that all zeros of φ exist on the imaginary axis ([18]) and are simple ([14]). The set of the zeros of φ is denoted by $\{\pm ia_j^*;0<a_j^*<a_{j+1}^*(j\in\mathbb{N})\}$ (Note that φ is an even function). For $\eta,\sigma>0$, set $R(\eta,\sigma)=\{v\in\mathbb{C};|\Re(v)|<\eta,|\Im(v)|<\sigma\}$. Let $\{\pm ia_{\pm j};0\leq\Re(a_{\pm j})\leq\Re(a_{\pm(j+1)})\}$ be the set of zeros of $\varphi(\cdot,\hat{\tau})$ in $R(\eta,\sigma)$. Note that the values of $a_{\pm j}$ depend on $\hat{\tau}$.

Lemma 4.2. *For any $\eta>0$ and $N\in\mathbb{N}$, there exists $\hat{\tau}_0>0$ such that if $\hat{\tau}>\hat{\tau}_0$, then in $R(\eta,\sigma_N)$ with $\sigma_N=(a_N^*+a_{N+1}^*)/2$*

(i) *the number of zeros of $\varphi(\cdot,\hat{\tau})$ is $2N$,*
(ii) *$|a_{\pm j}-a_j^*|<C_1(\eta,N)/\hat{\tau}$ for $j=1,\dots,N$,*
(iii) *all zeros of $\varphi(\cdot,\hat{\tau})$ are simple,*
(iv) *$|\varphi_v(ia_{\pm j},\hat{\tau})-\varphi'(ia_j^*)|<C_2(\eta,N)/\hat{\tau}$ for $j=1,\dots,N$, where $\varphi_v(v,\hat{\tau})$ is the partial derivative of $\varphi(v,\hat{\tau})$ in v.*

Next let us consider the behavior of $\varphi(\cdot,\hat{\tau})$ at infinity in the directions $\arg v=0,\pi$. The following lemma shows that this behavior is similar to that of $\varphi(v)$ in these directions (see Theorem 3.1 in [12]).

Lemma 4.3. *There are positive constants α_0,ρ_0,R such that if $|v|/\hat{\tau}\leq\rho_0$, $|v|>R$ and $|\arg v|<\alpha_0$ or $|\arg v-\pi|<\alpha_0$, then*

$$C_1<|v|^{(m-1)/(2m-1)}e^{-a|v|^{2m/(2m-1)}}\cdot|\varphi(v,\hat{\tau})|<C_2,$$

where a, C_1, C_2 are positive constants independent of $\hat{\tau}, v$.

4.3. **Analysis of** $F(v;\hat{\tau},\rho_0)$. Fix any positive integer N and set $\sigma_N=(a_N^*+a_{N+1}^*)/2$ ($\pm ia_j^*$'s are zeros of $\varphi(v)$). For the computation below, we prepare integral curves $\Gamma_{\pm}^{(N)}$ as follows. $\Gamma_{\pm}^{(N)}$ consist three parts $\Gamma_{\pm 1}^{(N)}, \Gamma_{\pm 2}^{(N)}, \Gamma_{\pm 3}^{(N)}$. First $\Gamma_{\pm 1}^{(N)}$ follow the line $\{v;\Re(v)=-\rho_0\hat{\tau}^{2m-1}\}$ from $-\rho_0\hat{\tau}^{2m-1}+i0$ to $-\rho_0\hat{\tau}^{2m-1}\pm i\sigma_N$. Second $\Gamma_{\pm 2}^{(N)}$ follow the lines $\{v;\Im(v)=\pm\sigma_N\}$ from $-\rho_0\hat{\tau}^{2m-1}\pm i\sigma_N$ to $\rho_0\hat{\tau}^{2m-1}\pm i\sigma_N$. Third $\Gamma_{\pm 3}^{(N)}$ follow the line $\{v;\Re(v)=\rho_0\hat{\tau}^{2m-1}\}$ from $\rho_0\hat{\tau}^{2m-1}\pm i\sigma_N$ to $\rho_0\hat{\tau}^{2m-1}+i0$. (See Figure 1.)

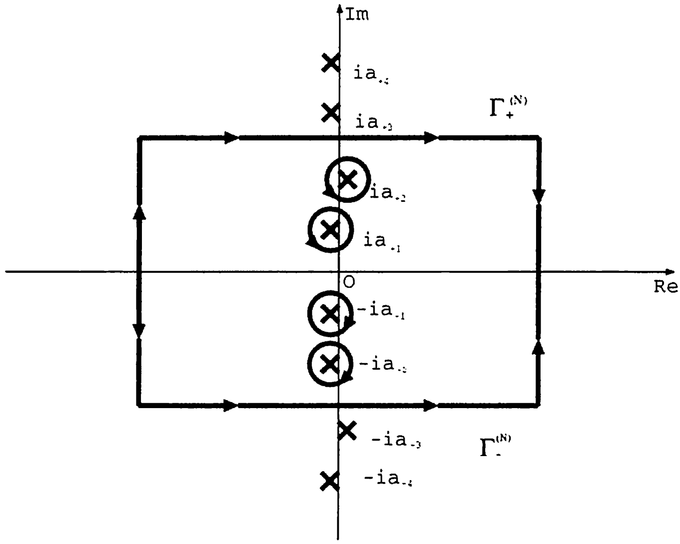

FIGURE 1. Integral contours $\Gamma_{\pm}^{(N)}$.

Define the functions $I_{\pm}^{(N)}(z;\hat{\tau},\rho_0)$ by

$$I_{\pm}^{(N)}(z;\hat{\tau},\rho_0)=\sum_{k=1}^{3} I_{\pm k}^{(N)}(z;\hat{\tau},\rho_0),$$

where

$$I_{\pm k}^{(N)}(z;\hat{\tau},\rho_0)=\int_{\Gamma_{\pm k}^{(N)}}\frac{e^{iz\hat{\tau}v}}{\varphi(v,\hat{\tau})}dv \quad \text{for } k=1,2,3.$$

First we consider the case where $x = \Re(z) > 0$. For $N \in \mathbb{N}$, we set $\eta_N = \max\{2R, 2\sigma_N/\tan\alpha_0\}$, where R is as in Lemma 4.3. By Lemma 4.2, for any $N \in \mathbb{N}$, there exists $\hat{\tau}_N > 0$ such that if $\hat{\tau} > \hat{\tau}_N$, then in the region $R(\eta, \sigma_N)$, the number of zeros of $\varphi(\cdot, \hat{\tau})$ is $2N$, $|a_{+j} - a_j^*| < 10^{-1}\min\{a_j^* - a_{j-1}^*, a_{j+1}^* - a_j^*\}$, all zeros of $\varphi(\cdot, \hat{\tau})$ are simple and $|\varphi_v(ia_{+j}, \hat{\tau}) - \varphi'(ia_j^*)| < 10^{-1}|\varphi'(ia_j^*)|$ for $j = 1, \ldots, N$.

Suppose that $\hat{\tau} > \hat{\tau}_N$. By deforming the original integral curve in (4.2) into $\Gamma_+^{(N)}$, the residue formula implies

$$F(z; \hat{\tau}, \rho_0) = 2\pi i \sum_{j=1}^{N} \frac{e^{-a_{+j} z \hat{\tau}}}{\varphi_v(ia_{+j}, \hat{\tau})} + I_+^{(N)}(z; \hat{\tau}, \rho_0). \tag{4.3}$$

In fact the function $e^{iz\hat{\tau}v}/\varphi(v, \hat{\tau})$ in v has simple poles with residue $2\pi i e^{-a_{+j} z\hat{\tau}}/\varphi_v(ia_{+j}, \hat{\tau})$ at $v = ia_{+j}$. Hereafter we use C_N for various constants depending on N.

First $I_{+j}^{(N)}$ $(j = 1, 3, x > 0)$ can be estimated as follows.

$$\begin{aligned} |I_{+j}^{(N)}(x + i0; \hat{\tau}, \rho_0)| &\le \int_0^{\sigma_N} \frac{e^{-x\hat{\tau}q}}{|\varphi(-\rho_0\hat{\tau}^{2m-1} + iq, \hat{\tau})|} dq \\ &\le C\hat{\tau}^{m-1} e^{-a\hat{\rho}_0\tau} \int_0^{\sigma_N} e^{-x\hat{\tau}q} dq \\ &\le C_N \hat{\tau}^{m-1} e^{-a\hat{\rho}_0\tau}, \end{aligned}$$

by using Lemma 4.3. Second $I_{+2}^{(N)}$ $(x > 0)$ can be estimated as follows.

$$\begin{aligned} |I_{+2}^{(N)}(x + i0; \hat{\tau}, \rho_0)| &\le e^{-x\sigma_N\hat{\tau}} \int_{-\rho_0\hat{\tau}^{2m-1}}^{\rho_0\hat{\tau}^{2m-1}} \frac{dp}{|\varphi(p + i\sigma_N, \hat{\tau})|} \\ &\le C_N e^{-x\sigma_N\hat{\tau}}. \end{aligned}$$

In the case where $x < 0$, we can obtain the same inequality by deforming the integral curve into $\Gamma_-^{(N)}$.

Therefore if $x \neq 0$, then we have

$$\left| I_{\sigma(x)}^{(N)}(x + i0; \hat{\tau}, \rho_0) \right| \le C_N e^{-|x|\sigma_N\hat{\tau}}, \tag{4.4}$$

where $\sigma(x)$ is the sign of x.

4.4. **Proof of Theorem 1.1.** Fix any $N \in \mathbb{N}$ and suppose that $x = \Re(z) > 0$. Substituting (4.3) into (4.1), we have

$$K(z, t; \tau_N, \rho_0) = \sum_{j=1}^{N} K_j(z, t; \tau_N) + R_N(z, t; \tau_N, \rho_0), \tag{4.5}$$

where

$$K_j(z, t; \tau_N) = \frac{i}{\pi} \int_{\tau_N}^{\infty} e^{it\tau} e^{-f(y)\tau} e^{-a_{+j} z\hat{\tau}} \frac{\tau^{1+1/m}}{\varphi_v(ia_{+j}, \hat{\tau})} d\tau$$

for $j = 1, \dots, N$ and

$$R_N(z,t;\tau_N,\rho_0) = \frac{i}{\pi}\int_{\tau_N}^{\infty} e^{it\tau}e^{-f(y)\tau}I_+^{(N)}(z;\hat{\tau},\rho_0)\tau^{1+1/m}d\tau.$$

In the case where $x < 0$, if we replace a_{+j}, $I_+^{(N)}$ with $-a_{-j}$, $I_-^{(N)}$ respectively, then the equation (4.5) holds. Now we show the following proposition.

Proposition 4.4. *For any $N \in \mathbb{N}$, there exist $x_0 > 0$, $k_0 \in \mathbb{N}$ such that if $0 < |x| \le x_0$ and $k \ge k_0$, then*

$$C_j^{(1)}\frac{\Gamma(2mk+4m+2)}{(|x|a_j^*)^{2mk+4m+2}} \le \left|\frac{\partial^k}{\partial t^k}K_j(x+i0,0;\tau_N)\right| \le C_j^{(2)}\frac{\Gamma(2mk+4m+2)}{(|x|a_j^*)^{2mk+4m+2}} \tag{4.6}$$

for $j = 1, \dots, N$, where $C_j^{(1)}, C_j^{(2)} > 0$ are constants depending on j, and

$$\left|\frac{\partial^k}{\partial t^k}R_N(x+i0,0;\tau_N,\rho_0)\right| \le C_N\frac{\Gamma(2mk+4m+2)}{(|x|\sigma_N)^{2mk+4m+2}}, \tag{4.7}$$

where $C_N > 0$ is a constant depending on N.

If we admit the above proposition, each K_j does not satisfy the Cauchy inequality on the set $\Xi(x_0) = \{(x+i0,0): 0 < |x| \le x_0\}$ and the singularity of K_j becomes *weaker* as j increases. Thus we can obtain Theorem 4.1, that is, K fails to be real analytic and moreover it belongs to s-th order Gevrey class for $s \ge 2m$, but no better, on $\Xi(x_0)$.

Proof of Proposition 4.4. We only consider the case where $x > 0$. There is a function $f_j(\hat{\tau})$ $(j = 1, \dots, N)$ and a constant $c_N > 0$ such that $a_{+j}\hat{\tau} = a_j^*\hat{\tau} + f_j(\hat{\tau})$ and $|f_j(\hat{\tau})| < c_N$ for $\hat{\tau} > \hat{\tau}_N$. We take $x_0 > 0$ such that $c_N x_0 < 1/100$. Then $|e^{-f_j(\hat{\tau})x} - 1| < 1/10$. If $0 < x < x_0$, then

$$\begin{aligned}
&\left|\frac{e^{-f_j(\hat{\tau})x}}{\varphi_v(ia_{+j},\hat{\tau})} - \frac{1}{\varphi'(ia_j^*)}\right| \\
&\le \frac{|e^{-f_j(\hat{\tau})x}\varphi'(ia_j^*) - \varphi_v(ia_{+j},\hat{\tau})|}{|\varphi_v(ia_{+j},\hat{\tau})||\varphi'(ia_j^*)|} \\
&\le \frac{10}{9}\frac{|e^{-f_j(\hat{\tau})x} - 1||\varphi'(ia_j^*)| + |\varphi'(ia_j^*) - \varphi_v(ia_{+j},\hat{\tau})|}{|\varphi'(ia_j^*)|^2} \\
&\le \frac{2}{9}\frac{1}{|\varphi'(ia_j^*)|}
\end{aligned}$$

Note that we took $\hat{\tau}_N$ as in Subsection 4.3. By using the above inequality, if $0 < x < x_0$, then

$$\left|\frac{\partial^k}{\partial t^k} K_j(x+i0,0;\tau_N)\right| = \frac{1}{\pi}\left|\int_{\tau_N}^{\infty} \tau^{k+1+1/m} e^{-a_j^* x\hat{\tau}} \frac{e^{-f_j(\hat{\tau})x}}{\varphi_e(ia_{+j},\hat{\tau})} d\tau\right|$$
$$\geq \frac{7}{9\pi}\frac{1}{|\varphi'(ia_j^*)|}\int_{\tau_N}^{\infty} \tau^{k+1+1/m} e^{-a_j^* x\hat{\tau}} d\tau$$
$$= \frac{7}{9\pi}\frac{1}{|\varphi'(ia_j^*)|}\left\{\frac{\Gamma(2mk+4m+2)}{(xa_j^*)^{2mk+4m+2}} - H_{j,N,k}\right\}.$$

Here it is easy to obtain

$$|H_{j,N,k}| = \left|\int_0^{\tau_N^{1/(2m)}} \tau^{2mk+4m+1} e^{-a_j^* x\tau} d\tau\right| \leq \frac{\tau_N^{k+2+1/m}}{2mk+4m+2}.$$

Therefore if k is sufficiently large, we can obtain the left inequality in (4.6) in the proposition. On the other hand, the right inequality in (4.6) can be shown as follows.

$$\left|\frac{\partial^k}{\partial t^k} K_j(x+i0,0;\tau_N)\right| \leq \frac{1}{\pi}\int_{\tau_N}^{\infty} \tau^{k+1+1/m} e^{-a_j^* x\hat{\tau}} \left|\frac{e^{-f_j(\hat{\tau})x}}{\varphi_e(ia_{+j},\hat{\tau})}\right| d\tau$$
$$\leq \frac{11}{9\pi}\frac{1}{|\varphi'(ia_j^*)|}\int_0^{\infty} \tau^{2mk+4m+1} e^{-a_j^* x\tau} d\tau$$
$$= \frac{11}{9\pi}\frac{1}{|\varphi'(ia_j^*)|}\frac{\Gamma(2mk+4m+2)}{(xa_j^*)^{2mk+4m+2}}.$$

Next by (4.4), we have

$$\begin{aligned}\left|\frac{\partial^k}{\partial t^k} R_N(x+i0,0;\tau_N,\rho_0)\right| &\leq C\int_{\tau_N}^{\infty} \tau^{k+1+1/m}\left|I_+^{(N)}(x+i0;\hat{\tau},\rho_0)\right| d\tau \\ &\leq C_N\int_0^{\infty} \tau^{k+1+1/m} e^{-x\sigma_N\hat{\tau}} d\tau \\ &\leq C_N\frac{\Gamma(2mk+4m+2)}{(x\sigma_N)^{2mk+4m+2}}.\end{aligned}$$

We have completed the proof of Proposition 4.4.

References

[1] S. R. Bell, *Differentiability of the Bergman kernel and pseudolocal estimates*, Math. Z. **192** (1986), 467–472.

[2] ———, *Extendibility of the Bergman kernel function*, Complex analysis, II (College Park, Md., 1985–86), 33–41, Lecture Notes in Math., **1276**, Springer, Berlin-New York, 1987.

[3] H. P. Boas, *Extension of Kerzman's theorem on differentiability of the Bergman kernel function*, Indiana Univ. Math. J. **36** (1987), 495–499.

[4] M. Christ, *Analytic hypoellipticity breaks down for weakly pseudoconvex Reinhardt domains*, International Math. Research Notices **1** (1991), 31-40.
[5] ———, *Remarks on the breakdown of analyticity for $\bar{\partial}_b$ and Szegö kernels*, Proceedings of 1990 Sendai conference on harmonic analysis (S. Igari, ed.), Lecture Notes in Math. Springer, 61-78.
[6] ———, *Remarks on analytic hypoellipticity of $\bar{\partial}_b$*, Modern method in complex analysis, Princeton Univ. Press, 41-62.
[7] M. Christ and D. Geller, *Counterexamples to analytic hypoellipticity for domains of finite type*, Ann. of Math. **235** (1992), 551-566.
[8] J. Faraut and A. Korányi, *Analysis on symmetric cones*, Oxford Univ. Press, New York (1994).
[9] G. Francsics and N. Hanges, *Analytic regularity for the Bergman kernel*, Proceedings of the Conference on Partial Differential Equations, (Saint-Jean-de-Monts, 1998), Exp. No. V, 11 pp., Univ. Nantes, Nantes, (1998).
[10] F. Haslinger, *Szegö kernels of certain unbounded domains in $\mathbb{C}^2$*, Rev. Roumaine Math. Pures Appl., **39** (1994), 939-950.
[11] J. Kamimoto, *Non-analytic Bergman and Szegö kernels for weakly pseudoconvex tube domains in $\mathbb{C}^2$*, to appear in Math. Z.
[12] ———, *On the singularities of non-analytic Szegö kernels*, J. Math. Sci. Univ. Tokyo, **6** (1999), 13-39.
[13] ———, *On an integral of Hardy and Littlewood*, Kyushu J. of Math. **52**, (1998) 249-263.
[14] J. Kamimoto, H. Ki and Y. O. Kim, *On the multiplicities of the zeros of Laguerre–Pólya functions*, Proc. of Amer. Math. Soc., **128** (2000), no. 1, 189–194.
[15] N. Kerzman, *The Bergman kernel function. Differentiability at the boundary*, Math. Ann. **195** (1972), 149–158.
[16] A. Nagel, *Vector fields and nonisotropic metrics*, Beijing Lectures in Harmonic Analysis, (E. M. Stein, ed.), Princeton University Press, Princeton, NXJ, (1986), 241-306.
[17] A. Nagel, J. P. Rosay, E. M. Stein and S. Wainger, *Estimates for the Bergman and Szegö kernels in $\mathbb{C}^2$*, Ann. of Math. **129** (1989), 113-150.
[18] G. Pólya, *Über trigonometrische Integrale mit nur reelen Nullstellen*, J. Reine Angew. Math. **58** (1927), 6-18.
[19] S. Saitoh, *Integral transforms, reproducing kernels and their applications*, Pitman Research Notes in Mathematics Series **369**, Addison Wesley Longman, UK (1997).
[20] D. Tartakoff, *Local analytic hypoellipticity for $\Box_b$ on non-degenerate Cauchy-Riemann manifolds*, Proc. Nat. Acad. Sci. U.S.A. **75** (1978), 3027-3028.
[21] ———, *The local real analyticity of solutions to $\Box_b$ and the $\bar{\partial}$-Neumann problem*, Acta Math., **145** (1980), 177-204.
[22] F. Treves, *Analytic hypoellipticity of a class of pseudodifferential operators*, Comm. in P.D.E., **3** (1978), 475-642.

DEPARTMENT OF MATHEMATICS, KUMAMOTO UNIVERSITY, KUMAMOTO 860-8555, JAPAN.
E-mail address: joe@sci.kumamoto-u.ac.jp