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SPECTRAL SYNTHESIS FOR L^1-ALGEBRAS AND FOURIER ALGEBRAS OF LOCALLY COMPACT GROUPS

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1. Introduction

The purpose of these notes is to report on progress that has been achieved during the past twenty years in spectral synthesis for L^1- and Fourier algebras of (non-abelian) locally compact groups. However, some of these results, in particular for Fourier algebras, are very recent.

To start with, let G be a locally compact abelian group and $L^1(G)$ the convolution algebra of integrable functions on G. Then the spectrum (or Gelfand space) of $L^1(G)$ can be identified with the dual group $\widehat{G}$ of G by means of the mapping $\alpha \to \varphi_\alpha$, where $\varphi_\alpha(f) = \widehat{f}(\alpha) = \int_G f(x)\alpha(x)dx$ for $f \in L^1(G)$ and $x \in G$. Spectral synthesis problems concern the extent to which a closed ideal I of $L^1(G)$ is determined by its hull $h(I) = \{\alpha \in \widehat{G} : \widehat{f}(\alpha) = 0 \text{ for all } f \in I\}$ in $\widehat{G}$. We refer the reader to [3] or to Section 2 for the notion of spectral set and Ditkin set for $L^1(G)$.

Since Malliavin's [20] famous discovery that, given any non-compact locally compact abelian group G (equivalently, $\widehat{G}$ is non-discrete), there exists a closed subset of $\widehat{G}$ which fails to be a spectral set for $L^1(G)$, there has been much effort in producing spectral sets and Ditkin sets. Specifically, so-called injection and projection theorems for spectral sets and Ditkin sets (see [3], [23] and [24]) as well as results about unions of such sets have been established (see [3]). As general references to spectral synthesis we mention [3], [10] and [24]. One of the major unsettled problems (even for $G = \mathbb{Z}$) is whether every spectral is actually a Ditkin set. In Sections 2 and 3 we discuss analogous problems for Fourier algebras and for L^1-algebras of (non-abelian) locally compact groups.

2. Fourier Algebras

For a locally compact group G, let $A(G)$ and $B(G)$ denote the Fourier algebra and the Fourier-Stieltjes algebra of G as introduced and first systematically studied by Eymard [5]. Recall that $B(G)$ is the linear span of all continuous positive definite functions on G and therefore is the Banach space dual of $C^*(G)$, the group C^*-algebra of G. Then $A(G)$ is the closed ideal of $B(G)$ generated by the functions in $B(G)$ with compact support. It turns out that

$A(G)$ consists precisely of all coefficient functions of the left regular representation λ of G on $L^2(G)$, and $A(G)$ can be identified with the predual of the von Neumann algebra $VN(G)$ generated by λ. When G is abelian and $\widehat{G}$ denotes the dual group of G, then $A(G)$ and $B(G)$ are isomorphic (by means of the Fourier transform) to $L^1(\widehat{G})$ and $M(\widehat{G})$.

$A(G)$ is a regular semisimple commutative Banach algebra with spectrum $\Delta(A(G)) = G$ [5, Théorème 3.34 and Lemme 3.2]. In fact, the mapping $x \to \varphi_x$, where $\varphi_x(u) = u(x)$ for $u \in A(G)$, provides a homeomorphism between G and $\Delta(A(G))$. Thus, associated to every closed subset E of G, is a largest and a smallest ideal, $I(E)$ and $J(E)$, of $A(G)$ with zero set equal to E. More precisely,

$$I(E) = \{u \in A(G) : u(x) = 0 \text{ for all } x \in E\}$$

and

$$J(E) = \{u \in A(G) \cap C_c(G) : u \text{ vanishes on a neighbourhood of } E\}.$$

E is called a *spectral set* or *set of synthesis* if $I(E) = \overline{J(E)}$, and E is said to be a *Ditkin set* if $u \in \overline{uJ(E)}$ for every $u \in I(E)$. Obviously, each Ditkin set is a spectral set. In addition, there are local variants of these notions (see [3, 4, 9, 16]). They are obtained by replacing $I(E)$ with $I(E) \cap C_c(G)$. When G is abelian, the local notions agree with the former ones. For any regular semisimple commutative Banach algebra A it is customary to say that spectral synthesis (respectively, local spectral synthesis) holds for A whenever every closed subset of $\Delta(A)$ is a spectral set (respectively, local spectral set).

Proposition 2.1. *Let G be an arbitrary locally compact group. Then*

(i) *Local spectral synthesis holds for $A(G)$ if and only if G is discrete.*

(ii) *Spectral synthesis holds for $A(G)$ if and only if G is discrete and $u \in \overline{uA(G)}$ for every $u \in A(G)$.*

The additional condition in (ii) is of course satisfied if $A(G)$ has an approximate identity in the weakest possible sense. It is not unlikely that this condition is fulfilled for most groups. In contrast, by a result of Leptin [15], $A(G)$ has a norm bounded approximate identity precisely when G is amenable.

The above proposition can be found in [13]. We indicate the proof of (i). Thus, suppose that local spectral synthesis holds for $A(G)$. Using the fact that this property is inherited by quotient groups and by closed subgroups, it was shown earlier (see [16] and [7]) that G must be totally disconnected (indeed, a connected Lie group is generated by its one-parameter subgroups). Fix a compact open subgroup K of G and suppose that K is infinite. Then, by a deep theorem of Zelmanov [27, Theorem 2], K contains an infinite abelian (closed) subgroup H. Now, local spectral synthesis, and hence spectral synthesis, holds for $A(H)$, contradicting Malliavin's theorem. Thus K is finite, whence G is discrete.

Proposition 2.1 and the results that have been established for $L^1(H)$, H abelian, suggest a study of (local) spectral sets and (local) Ditkin sets for

Fourier algebras. In this context, the desire to not having to treat the local variants separately, lead to the following generalization of the notions of spectral set and Ditkin set [13].

Recall that $A(G)^* = VN(G)$ and that there is natural action of $B(G)$ on $VN(G)$ given by

$$\langle u \cdot T, v\rangle = \langle T, uv\rangle,$$

$T \in VN(G), u \in B(G), v \in A(G)$. Let X be an $A(G)$-invariant linear subspace of $VN(G)$. A closed subset E of G is called an *X-spectral set* or *set of X-synthesis* for $A(G)$ if each $T \in X$ with support (in the sense of [5]) in E belongs to $I(E)^{\perp}$, the annihiltator of $I(E)$ in $VN(G)$. E is called an *X-Ditkin set* if for every $T \in X$ and $u \in I(E)$ there exists a net $(u_\alpha)_\alpha$ in $J(E)$ such that $\langle T, uu_\alpha\rangle \to \langle T, u\rangle$. These notions reduce to the previous ones when taking for X all of $VN(G)$ and the subspace of operators with compact support in $VN(G)$, respectively.

Returning to locally compact abelian groups, it is worthwhile to mention that while the union of two Ditkin sets is Ditkin, it is an open question whether the union of two spectral sets is again spectral. In a more general context, however, Atzmon [1] has given an example of a regular semisimple commutative Banach algebra with unit and of two sets of synthesis in $\Delta(A)$ the union of which fails to be of synthesis.

Regarding unions of spectral sets and Ditkin sets for Fourier algebras, we now have the following results [13, Theorems 2.9 amd 2.10].

Theorem 2.2. *Let G be a locally compact group and X an $A(G)$-invariant linear subspace of $VN(G)$. Suppose that E_1 and E_2 are closed subsets of G such that $E_1 \cap E_2$ is X-Ditkin. Then $E_1 \cup E_2$ is an X-spectral set if and only if both E_1 and E_2 are X-spectral sets.*

Theorem 2.3. *Let G and X be as in Theorem 2.2, and let E and F be closed subsets of G such that $E \cap F$ is an X-Ditkin set. Then $E \cup F$ is X-Ditkin if and only if both E and F are X-Ditkin sets.*

The preceding two theorems have been known before in the special case where $X = VN(G)$ [26, Theorems 1 and 4]. Such results can be used in both directions. In particular, it follows that, if $A(G)$ has an approximate identity, then each open and closed subset of G is a Ditkin set. Moreover, under the same hypothesis, it follows that finite subsets of G are spectral sets, since singletons are known to be sets of synthesis [5, Corollaire 4.10].

As pointed out in the introduction, when A is a locally compact abelian group, a second possibility to produce new sets of synthesis or Ditkin sets for $L^1(A)$ is to apply injection and projection theorems for such sets. To establish similar results for Fourier algebras turns out to be considerably more difficult and so far, as we shall outline in the sequel, there are only partial analogues due to Lohoué [16], Derighetti [4] and Kaniuth and Lau [13, 14].

We start with projection theorems. Thus, let G be a locally compact group, N a closed normal subgroup and $q : G \to G/N$ the quotient homomorphism.

The problem is whether, for a closed subset E of G/N, E is a (local) spectral set or (local) Ditkin set for $A(G/N)$ if and only if $q^{-1}(E)$ is a (local) spectral set or (local) Ditkin set for $A(G)$. The main difficulty in relating $A(G)$ and $A(G/N)$ is that, except when N is compact, there is no homomorphism from $A(G)$ onto $A(G/N)$. However, there is a natural homomorphism from $A(G) \cap C_c(G)$ onto $A(G/N) \cap C_c(G/N)$ given by $u \to T_N u$, where $T_N u(xN) = \int_N u(xn)dn, x \in G$. This homomorphism has been exploited by Lohoué to prove the following projection theorem for local spectral sets [16, Théorème].

Theorem 2.4. *Let G be a locally compact group, N a closed normal subgroup of G and $q : G \to G/N$ the quotient homomorphism. Then, for any closed subset E of G/N, E is a local spectral set for $A(G/N)$ if and only if $q^{-1}(E)$ is a local spectral set for $A(G)$.*

To prepare for the setting of injection theorems, let H be a closed subgroup of the locally compact group G, and let

$$r : A(G) \to A(H), \ u \to u|H$$

be the restriction map. r is norm decreasing and surjective. More precisely, given $v \in A(H)$, there exists $u \in A(G)$ such that $r(u) = v$ and $\|u\|_{A(G)} = \|v\|_{A(H)}$ [9, Theorem 1b; 21, Theorem 4.21]. Thus the adjoint map

$$r^* : VN(H) \to VN(G), \langle r^*(S), u\rangle = \langle S, r(u)\rangle,$$

$u \in A(G), S \in VN(H)$, is injective. The range of r^* equals $VN_H(G)$, the weak-$*$-closure of the linear span of all operators $\lambda(h), h \in H$, in $VN(G)$. Moreover, r^* maps the subspace of operators with compact support in $VN(H)$ onto the subspace of operators with compact support in $VN_H(G)$.

For any $A(G)$-invariant subspace X of $VN(G)$, let

$$X_H = r^{*^{-1}}(X),$$

an $A(H)$-invariant subspace of $VN(H)$. Now we are ready to formulate the injection theorem for X-spectral sets [13, Theorem 3.4].

Theorem 2.5. *Let X be an $A(G)$-invariant linear subspace of $VN(G)$. Let H be a closed subgroup of G and E a closed subset of H. Then E is an X-spectral set for $A(G)$ if and only if E is an X_H-spectral set for $A(H)$.*

The proof exploits properties of the map r^* as well as the fact that the subgroup H is a set of synthesis for $A(G)$ [25, Theorem 3]. Thus, as special cases, we obtain injection theorems for spectral sets and for local spectral sets. The latter has previously been shown by Derighetti [4, Proposition 8].

An injection theorem for local Ditkin sets has been proved by Derighetti [4, Théorème 12] whenever the subgroup H is normal in G. Recently, this theorem was generalized to the effect that the hypothesis that H be normal is weakened and that X-Ditkin sets, for arbitrary X, are considered.

To elaborate the condition on H, we have to introduce some more notation. Let $P(G)$ denote the set of all continuous positive definite functions on G, and, for a closed subgroup H of G, let

$$P_H(G) = \{u \in P(G) : u(h) = 1 \text{ for all } h \in H\}.$$

We say that G has the *H-separation property* if for every $x \in G, x \notin H$, there exists $u \in P_H(G)$ such that $u(x) \neq 1$. When G has the H-separation property for every closed subgroup H of G, we refer to G as a group with the *separation property.* If H is either normal, or compact, or open in G, then G has the H-separation property. Such subgroups H subsume in the class of neutral subgroups which are defined as follows. A closed subgroup H of G is called *neutral* in G if there exists a neighbourhood basis $\mathcal{V}$ of the identity of G such that $VH = HV$ for all $V \in \mathcal{V}$. Now, if G is any locally compact group and H a neutral subgroup of G, then G has the H-separation property [14, Proposition 2.2]. On the other hand, for connected groups the separation property to hold is a very restrictive condition. Indeed, by Theorem 1.1 of [14], an almost connected locally compact group G has the separation property if and only if G contains an open normal subgroup N of finite index such that N is a direct product of a compact group and a vector group.

Returning to $A(G)$, the following injection theorem for X-Ditkin sets has been proved in [14, Theorem 3.5].

Theorem 2.6. *Let G be a locally compact group and let X be an $A(G)$-invariant linear subspace of $VN(G)$. Let H be a closed subgroup of G and E a closed subset of H.*

(i) *If E is X-Ditkin for $A(G)$, then E is X_H-Ditkin for $A(H)$.*

(ii) *Suppose that G has the H-separation property and that $u \in \overline{uA(G)}$ for every $u \in I(H)$. Then, if E is X_H-Ditkin for $A(H)$, then it is also X-Ditkin for $A(G)$.*

Since, due to the regularity of $A(G)$, for each compactly supported function $u \in A(G)$ there exists $v \in A(G)$ such that $u = uv$, Theorem 2.6 includes Derighetti's injection theorem for local Ditkin sets alluded to above.

In establishing Theorem 2.6, rather than the separation property itself the following equivalent property is used. There exists a projection P from $VN(G)$ onto $VN_H(G)$ such that, in the weak-$*$-operator topology on $\mathcal{B}(VN(G))$, P is the limit of operators $T \to u \cdot T$, where $u \in P_H(G)$.

We finish this section by pointing out that the H-separation property of a locally compact group G deserves further investigation since it appears to play an important role in the ideal theory of Fourier algebras. For instance, it has been shown in [14, Theorem 3.4] that if G has the H-separation property, then the ideal $I(H)$ has an approximate identity with norm bound 2, the best possible bound whenever G/H is infinite.

3. L^1-Algebras

In this section we turn to L^1-algebras of (non-abelian) locally compact groups and discuss analogous issues as in the previous section for Fourier algebras. To start with, however, let A be an arbitrary semisimple Banach

$*$-algebra, and let $\widehat{A}$ denote the set of equivalence classes of irreducible $*$-representations of A. The primitive ideal space of A, $\mathrm{Prim}_* A$, consists of all kernels, $\ker\pi, \pi \in \widehat{A}$, and carries the hull-kernel topology. For each closed subset E of $\mathrm{Prim}_* A$, let

$$k(E) = \cap\{P : P \in E\},$$

the largest ideal of A with hull equal to E. Whenever $k(E)$ is the only closed ideal of A with hull E, then E is called a *spectral set* (or *set of synthesis*) for A. Also, we say that sepctral synthesis holds for A if every closed subset of $\mathrm{Prim}_* A$ is a spectral set.

Now, let G be a locally compact group and recall that there is a one-to-one correspondence between $\widehat{G}$, the set of equivalence classes of irreducible unitary representations of G, and $\widehat{L^1(G)}$. When G is type I and $L^1(G)$ is $*$-regular, the map $\pi \to \ker\pi$ from $\widehat{G}$ onto $\mathrm{Prim}_* L^1(G)$ is a homeomorphism and $\widehat{G}$ and $\mathrm{Prim}_* L^1(G)$ are usually identified.

It is easy to see that if G is compact, and hence $\mathrm{Prim}_* L^1(G)$ is discrete, then spectral synthesis synthesis holds for $L^1(G)$. However, it is worth mentioning that spectral synthesis may fail for a semisimple Banach $*$-algebra with discrete primitive ideal space. An example has been presented in [22]. The obvious question is whether spectral synthesis for $L^1(G)$ forces the locally compact group G to be compact. Somewhat surprising, the answer is negative. In [6] the following example was given of a non-compact locally compact group for which spectral synthesis holds.

Example 3.1. Let p be a prime and let N be the field of p-adic numbers. Let K denote the subset of elements of N of valuation 1. Then K is a compact group under multiplication. Form the semi-direct product $G = K \ltimes N$, where K acts on the additive group N by multiplication. The group G is often referred to as Fell's example of a non-compact group with countable dual. In fact,

$$\widehat{G} = \widehat{K} \cup \{\pi_j : j \in \mathbb{Z}\},$$

where each π_j is induced from some character of N. Both $\widehat{K}$ and $\{\pi_j : j \in \mathbb{Z}\}$ are discrete, $\widehat{K}$ is closed and a sequence $(\pi_{j_k})_k$ converges to some (and hence all) $\sigma \in \widehat{K}$ if and only if $j_k \to -\infty$.

Using this description of the topology of $\widehat{G}$, the projection theorem for spectral sets (see Theorem 3.5 below) and the fact that $L^1(G)$ has the so-called Wiener property (compare [17]), it is not difficult to show that every closed subset of $\widehat{G} = \mathrm{Prim}_* L^1(G)$ is a spectral set.

When looking carefully at the preceding example, an interesting problem arises. Suppose that $L^1(G)$ contains a closed ideal I such that $\mathrm{Prim}_* I$ and $\mathrm{Prim}_* L^1(G)/I$ are both discrete. Does then spectral synthesis hold for $L^1(G)$? An affirmative answer would cover Example 3.1.

Notice that the group G of Example 3.1 has an abelian normal subgroup with compact abelian quotient group. In contrast, for nilpotent locally compact groups it can be deduced from Malliavin's theorem that spectral synthesis fails for $L^1(G)$ whenever G is non-compact [12]. In the course of investigations to relate spectral synthesis to properties of certain topologies on the space of all closed ideals of the enveloping C^*-algebra $C^*(G)$, this latter result was recently generalized as follows [6, Theorem 3.7].

Theorem 3.2. *Let G be a locally compact group and suppose that G contains a compact normal subgroup K such that N/K is a finite extension of a nilpotent group. If spectral synthesis holds for $L^1(G)$, G must be compact.*

Apart from nilpotent groups this comprises, for instance, the class of Moore groups (that is, groups with finite dimensional irreducible representations).

An apparently very difficult problem for L^1-algebras of locally compact groups G is the existence of a smallest (closed) ideal $j(E)$ for a given hull $E \subseteq \mathrm{Prim}_* L^1(G)$. The next theorem is due to Ludwig [18].

Theorem 3.3. *Let G be a locally compact group of polynomial growth, and suppose that $L^1(G)$ is symmetric. Then, given a closed subset E of $\mathrm{Prim}_* L^1(G)$, there exists a smallest closed ideal whose hull is equal to E.*

We remind the reader that a locally compact group G is polynomially growing if for every compact subset K of G, the Haar measure of powers $K^n, n \in \mathbb{N}$, grows at most polynomially in n. Moreover, a Banach $*$-algebra A is called symmetric if every selfadjoint element of A has a real spectrum. Several classes of locally compact groups, among them nilpotent groups and motion groups, satisfy both of these hypotheses (see [17]). A main tool in proving Theorem 3.3 is Dixmier's functional calculus for groups of polynomial growth. Unfortunately, the ideal $j(E)$ is only described in terms of a generating set. This fact seems to be responsable for that, so far, there are no results on unions of spectral sets.

On the other hand, the existence of such smallest closed ideals turned out to be very useful in establishing injection and projection theorems for spectral sets. Naturally, for L^1-algebras of non-abelian locally compact groups, the setting is much more complicated than for Fourier algebras, and this is what we are now going to describe.

Let N be a closed normal subgroup of G, and let $q : G \to G/N$ denote the quotient homomorphism and $T : L^1(G) \to L^1(G/N)$ the corresponding homomorphism of L^1-algebras. Then there is a canonical embedding

$$i : \mathrm{Prim}_* L^1(G/N) \to \mathrm{Prim}_* L^1(G)$$

given by $i(\ker \pi) = \ker(\pi \circ q) = T^{-1}(\ker \pi)$. Then $i(\mathrm{Prim}_* L^1(G/N))$ is closed in $\mathrm{Prim}_* L^1(G)$ and i is a homeomorphism onto its range. In this situation, Hauenschild and Ludwig have proved the following injection theorem for spectral sets [8, Theorem 3.2].

Theorem 3.4. *Let N be a closed normal aubgroup of the locally compact group G, and let F be a closed subset of $\mathrm{Prim}_* L^1(G/N)$ and $E = i(F) \subseteq \mathrm{Prim}_* L^1(G)$.*

(i) *If E is a spectral set, then so is F.*

(ii) *Let F be a spectral set and suppose that G has polynomial growth and $L^1(G)$ is symmetric. Then E is a spectral set.*

In (ii), the condition that $L^1(G)$ is symmetric and G has polynomial growth can be replaced by the hypothesis that $i(\mathrm{Prim}_* L^1(G/N))$, the hull of the kernel of T, is a spectral set for $L^1(G)$ [8]. However, the only case where $i(\mathrm{Prim}_* L^1(G/N))$ is known to be a spectral set seems to be the indicated one.

Let us now turn to projection theorems. As before, let N be a closed normal subgroup of G. The action of G on N by inner automorphisms gives rise to actions of G on $L^1(N)$ and hence on the primitive ideal space $\mathrm{Prim}_* L^1(N)$. Now, if π is a representation of G, then the L^1-kernel of $\pi|N$ is a G-invariant ideal of $L^1(N)$. In particular, relating spectral sets for $L^1(G)$ to spectral sets for $L^1(N)$ leads to consider G-invariant subsets of $\mathrm{Prim}_* L^1(N)$.

Hauenschild and Ludwig have been the first to accomplish a projection theorem for spectral sets for non-abelian locally compact groups [8, Theorem 2.6]. Their result was subsequently improved by Bekka [2] as follows.

Theorem 3.5. *Let G be a locally compact group and N a closed normal subgroup of G. Let F be a closed G-invariant subset of* $\mathrm{Prim}_* L^1(N)$ *and*

$$E = \{\ker \pi : \pi \in \widehat{G} \text{ such that } \pi|N(k(F)) = 0\}.$$

(i) *Suppose that N has polynomial growth and $L^1(N)$ is symmetric. If E is a spectral set, then so is F.*

(ii) *Suppose that G has polynomial growth and $L^1(G)$ is symmetric. If F is a spectral set, then E is a spectral set.*

Part (i) is entirely due to Hauenschild and Ludwig. For the more sophisticated part (ii), they needed an additional hypothesis which Bekka was able to remove.

To indicate the difficulty, consider a G-invariant closed ideal J of $L^1(N)$. Regarding $L^1(N)$ as a subspace of $M(G)$, naturally associated to J is a closed ideal $e(J)$ of $L^1(G)$, the extension ideal. Indeed, $e(J)$ is defined to be the closed linear span of $C_c(G) * J$ in $L^1(G)$. Retaining the notation of Theorem 3.5, if $F = h(J)$ then $E = h(e(J))$. The main problem now is to show that $e(j(F)) = j(E)$. In [8] this equality was proved when G/N is solvable, and in some other less important cases. Taking into account that groups with polynomial growth are amenable, the essential missing step was to deal with compact quotients G/N. Bekka managed this by extending Dixmier's functional calculus to matrix valued functions.

Neither part (i) nor part (ii) of the theorem holds for arbitrary G or N (see [2] and [8]).

In Example 3.1, we have already given a sample of possible applications of the projection theorem. To conclude, we mention three further examples concerning singletons in $\mathrm{Prim}_* L^1(G)$. In treating two of them, (ii) and (iii), the projection theorem is substantial.

Example 3.6. (i) If G is a finitely generated nilpotent discrete group, then singletons in $\mathrm{Prim}_* L^1(G)$ are Ditkin sets. In fact, more generally, the so-called Helson-Reiter theorem holds for $L^1(G)$ [11].

(ii) In contrast, when G is a connected and simply connected nilpotent Lie group of nilpotence class ≥ 3, then singletons in $\mathrm{Prim}_* L^1(G)$ need not be spectral sets [19].

(iii) Let $G_n = SO(n) \ltimes \mathbb{R}^n$, $n \geq 2$, be the Euclidean motion group in dimension n. Using the two facts that the non-trivial orbits in $\widehat{\mathbb{R}^n} = \mathbb{R}^n$ are spheres and that $S^{n-1} \subseteq \mathbb{R}^n$ is a set of synthesis precisely when $n = 2$, it can be shown (see [2]) that all singletons in $\mathrm{Prim}_* L^1(G_2)$ are sets of synthesis, whereas, for $n \geq 3$, $\{\pi\} \subseteq \widehat{G_n}$ is spectral only if $\pi \in \widehat{SO(n)}$.

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