KA-wavelets on semisimple Lie groups and quasi-orthogonality of matrix coefficients

Takeshi Kawazoe
Keio University at Fujisawa
5322 Endo, Fujisawa, 252-8520 Japan
kawazoe@sfc.keio.ac.jp

§1 Introduction.

First we brief the history of continuous wavelet transforms. Originally the (continuous) wavelet transform, introduced by Morlet around 1980, was the following one. We denote by $H^2(\mathbf{R})$ the closed subspace of $L^2(\mathbf{R})$ consisting of all L^2 functions f on $\mathbf{R}$ with $\mathrm{supp}(\hat{f}) \subset [0, \infty)$, and we fix $\psi \in H^2(\mathbf{R})$ satisfying the so-called admissible condition

$$c_\psi = \int_0^\infty \frac{|\hat{\psi}(\lambda)|^2}{\lambda} d\lambda < \infty.$$

Then the wavelet transform W_ψ associated to ψ is defined on $H^2(\mathbf{R})$ as

$$W_\psi f(u, v) = \int_{-\infty}^{\infty} f(x) e^{-u/2} \bar{\psi}(e^{-u} x + v) dx \quad (u, v \in \mathbf{R}).$$

Theorem 1.1. W_ψ is an isometric isomorphism from $H^2(\mathbf{R})$ onto $L^2(\mathbf{R}^2)$: For any $f \in H^2(\mathbf{R})$

$$\|f\|^2 = \frac{1}{c_\psi} \|W_\psi f\|^2.$$

Furthermore, for any $f \in H^2(\mathbf{R})$ and $x \in \mathbf{R}$ at which f is continuous,

$$f(x) = \frac{1}{c_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (W_\psi f)(u, v) e^{-u/2} \bar{\psi}(e^{-u} x + v) du dv.$$

In [GMP] Grossmann-Morlet-Paul pointed out the group-theoretical interpretation of the wavelet transform W_ψ. Let G be the affine group $\mathbf{R}^2$ with multiplication law:

$$(u, v)(u', v') = (u + u', e^{-u'} v + v'),$$

and let $(T, H^2(\mathbf{R}))$ be an irreducible unitary representation of G defined by

$$(T(u,v)f)(x) = e^{-u/2}f(e^{-u}x+v) \quad (f \in H^2(\mathbf{R})).$$

In this scheme W_ψ can be rewritten as

$$W_\psi f(u,v) = \langle f, T(u,v)\psi\rangle,$$

where $\langle \cdot, \cdot \rangle$ is the inner product of $H^2(\mathbf{R})$. Furthermore, since *dudv* is a left invariant Haar measure on G, Theorem 1.1 yields the square-integrability and the orthogonality of the matrix coefficients $\langle f, T(u,v)\psi\rangle$ of T on G. In this sense the theory of the continuous wavelet transform W_ψ on $H^2(\mathbf{R})$ is nothing but the one of the square-integrable representation $(T, H^2(\mathbf{R}))$ of G.

General theory of square-integrable representations of locally compact groups has been investigated by various mathematicians; Weyl [W] for compact groups, Godement [G] for unimodular locally compact groups, and Duflo-Moore [DM] for general locally compact groups. Explicit theory based on the construction of the square-integrable representations was obtained by Harish-Chandra [HC] for semisimple Lie groups and by Moore-Wolf [MW] for nilpotent groups.

How to extend the theory of square-integrable representations of locally compact groups G ? One of the ways is to replace the square-integrability on G by the one on a quotient space G/H for a closed subgroup H of G. More generally, find a representation $(T, \mathcal{H})$ of G, a measurable subset (S, ds) of G, and $\psi \in \mathcal{H}$ for which, for any $f \in \mathcal{H}$

$$(\star) \quad \|f\|^2 = \frac{1}{c_{S,\psi}} \int_S |\langle f, T(s)\psi\rangle|^2 ds.$$

Then, it is easy to see that the transform defined by $\langle f, T(s)\psi\rangle$ is an isometric isomorphism from $\mathcal{H}$ onto $L^2(S, ds)$, and each $f \in \mathcal{H}$ has an L^2 decomposition in the weak sense:

$$f = \frac{1}{c_{S,\psi}} \int_S \langle f, T(s)\psi\rangle T(s)\psi ds.$$

For the last decade researches has been done in this scheme and many wavelet transforms has been constructed on locally compact groups, for example, on $\mathbf{R}_+^* \times SO(n)$ by Murenzi [M], on $\mathbf{R}_+^* \times SO(1,n)$ by A-J. Unterberger [U], on $\mathbf{R}_+^* \times SO(1,n) \times \mathbf{R}^{n+1}$ by Bhonke [B], on $S \times V$, V is a vector space and S is a subgroup of $GL(V)$, by De Bièvre [DB], on $SO(2,1) \times \mathbf{R}^3$ by Ali, Antoine, Gazeau [AAG], on $\mathbf{R}_+^* \times SO(n) \times H_n$ by Kalisa-Toréssani [KT], Toréssani [T1,2], on $GL(n, \mathbf{R})$ by Bernier-Taylor [BT], on $SO(2,1)$ by Wu-Zhong [WZ], and on Iwasawa AN groups by Kawazoe [K3] and Liu [L].

In this paper we shall consider the case that G is a semisimple Lie group and $S = KA$, where K and A are respectively the maximal compact and abelian subgroups of G. More precisely, let G be a semisimple Lie group with finite center and $G = KAK$ the Cartan decomposition of G. dg denotes a Haar measure on G and $dg = D(a)dkdadk$ the corresponding decomposition of dg. Then we take $S = KA$ and $ds = D(a)dkda$ in the above scheme, and we try to find a representation $(T, \mathcal{H})$ of G and $\psi \in \mathcal{H}$ satisfying $(\star)$. Unfortunately, the condition $(\star)$ is very strong, so I feel that we have no answer for T and ψ. Therefore, we shall consider a weak condition; there exist constants $0 < C_1, C_2 < \infty$ such that

$$(\star\star) \quad C_1\|f\|^2 \leq \int_S |\langle f, T(s)\psi\rangle|^2 ds \leq C_2\|f\|^2$$

and we shall obtain a sufficient condition on ψ for which $\langle f, T(s)\psi\rangle$ satisfies $(\star\star)$ (see Theorem 3.1). In §4 we shall treat the case of $G = SU(1,1)$ and $(T_{1/2}, \mathcal{H}_{1/2})$ the limit of the holomorphic discrete series of G. We note that $T_{1/2}$ is not square-integrable on G. Then we shall find a $\psi \in \mathcal{H}_{1/2}$ satisfying $(\star\star)$. Moreover, we shall deduce that, if we ignore a finite dimensional subspace of $\mathcal{H}_{1/2}$, then we can find a $\psi \in \mathcal{H}_{1/2}$ satisfying $(\star)$ (see Theorem 4.4). In this process we use the facts that some differences of the matrix coefficients of $T_{1/2}$ are square-integrable on $\mathbf{R}$ with respect to $D(a)da$ and moreover, they satisfy a quasi-orthogonality. These facts are summarized in Lemmas 4.1, 4.2, and 4.3.

After the lecture, the author noticed that J.-P. Antoine and P. Vandergheynst [AV1,2] had the same idea and they obtained an example in the case of $SO(3,1)$.

§2. Notation.

Let G be a semisimple Lie group with finite center and $G = KAN$ the Iwasawa decomposition of G. Let Σ be the set of roots for (G, A) and Σ^+ the one of positive roots corresponding to N. Let A^+ denote the closed positive Weyl chamber in A and $G = KA^+K$ the Cartan decomposition of G. Let

dg denote a Haar measure on G, and dk, da, and dn ones for K, A, and, N respectively. We normalize dk as $\int_K dk = 1$. According to the Iwasawa and Cartan decompositions of G, there are decompositions of dg such that

$$dg = e^{\rho(\log a)} dk da dn = D(a) dk da dk',$$

where $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha$ and

$$D(a) = \prod_{\alpha \in \Sigma^+} (\sinh \alpha(\log a))^{m_\alpha},$$

m_α stands for the multiplicity of α.

§3. KA-wavelets.

Let $(T, \mathcal{H})$ be a unitary representation of G and

$$\mathcal{H} = \oplus_{\tau \in \hat{K}} \mathcal{H}_\tau,$$

the K-type decomposition of $\mathcal{H}$. In the following argument we assume that

$$[T, \tau] \leq 1,$$

and we denote by $\hat{K}_T$ the set of all $\tau \in \hat{K}$ such that $[T, \tau] = 1$. Then, as a representation of K, $(T|_K, \mathcal{H}_\tau)$ is equivalent with τ for each $\tau \in \hat{K}_T$. We choose a complete orthonormal basis of $\mathcal{H}$ such that

$$\{e_n^\tau; e_n^\tau \in \mathcal{H}_\tau, 1 \leq n \leq \dim \tau, \ \tau \in \hat{K}_T\}$$

and we denote by I the set of the indexes $\{(\tau, n); 1 \leq n \leq \dim \tau, \ \tau \in \hat{K}_T\}$. For each $f \in \mathcal{H}$ the Fourier expansion of f is given by

$$f = \sum_{(\tau,n) \in I(f)} f_n^\tau e_n^\tau,$$

where $f_n^\tau = \langle f, e_n^\tau \rangle_{\mathcal{H}}$ and $I(f)$ the subset of I consisting of all (τ, n) such that $f_n^\tau \neq 0$. Here we put

$$I_A(f) = \{(\tau, n); (T(\cdot)f)_n^\tau = \langle T(\cdot)f, e_n^\tau \rangle \text{ is not identically } 0 \text{ on } A\}.$$

We say that $\psi \in \mathcal{H}$ is admissible if there exist constants $0 < C_1, C_2 < \infty$ such that, if $(\tau, n) \in I_A(\psi)$,

$$C_1 \leq c_{\psi,\tau,n} = \int_A |\langle T(a)\psi, e_n^\tau\rangle|^2 D(a)da \leq C_2.$$

We put

$$\mathcal{H}_\psi = \{f \in \mathcal{H}; I(f) \subset I_A(\psi)\}.$$

Then, by using the bounded constants $c_{\psi,\tau,n}$ we shall define a Fourier multiplier M_ψ on $\mathcal{H}_\psi$ as follows. For each $f = \sum_{(\tau,n)\in I(f)} f_n^\tau e_n^\tau$ in $\mathcal{H}_\psi$

$$M_\psi f = \sum_{(\tau,n)\in I(f)} c_{\psi,\tau,n}^{-1/2} f_n^\tau e_n^\tau.$$

Theorem 3.1. Let ψ be admissible in $\mathcal{H}$. Then for any $f \in \mathcal{H}_\psi$

(1)

$$C_1\|f\|^2 \leq \int\int_{KA} |\langle f, T(ka)\psi\rangle|^2 D(a)dkda \leq C_2\|f\|^2,$$

(2)

$$\|f\|^2 = \int\int_{KA} |\langle f, M_\psi T(ka)\psi\rangle|^2 D(a)dkda,$$

(3)

$$f = \int\int_{KA} \langle f, M_\psi T(ka)\psi\rangle M_\psi T(ka)\psi \; D(a)dkda.$$

Proof. We note that

$$T(k^{-1})f = \sum_{(\tau,n)\in I(f)} f_n^\tau T(k^{-1})e_n^\tau = \sum_{(\tau,n)\in I(f),(\tau',n')\in I} f_n^\tau \langle T(k^{-1})e_n^\tau, e_{n'}^{\tau'}\rangle e_{n'}^{\tau'}.$$

Then the orthogonality of the matrix coefficients of $T|_K$ yields that

$$\begin{aligned}
&\int\int_{KA} |\langle f, T(ka)\psi\rangle|^2 D(a)dkda \\
= &\int_A \sum_{(\tau,n)\in I(f)} |f_n^\tau|^2 |\langle T(a)\psi, e_n^\tau\rangle|^2 D(a)da \\
= &\sum_{(\tau,n)\in I(f)} |f_n^\tau|^2 \left(\int_A |\langle T(a)\psi, e_n^\tau\rangle|^2 D(a)da\right)
\end{aligned}$$

Since

$$\|f\|^2 = \sum_{(\tau,n)\in I(f)} |f_n^\tau|^2 \text{ and } I(f) \subset I_A(\psi),$$

(1) easily follows from the definition of the admissible vector ψ. We replace f by $M_\psi f$ in the above calculation. Then $|f_n^\tau|^2$ in the last equation turns to $|f_n^\tau|^2 c_{\psi,\tau,n}^{-1}$ and then, $c_{\psi,\tau,n}^{-1}$ cancels the integral over A. Thereby (2) follows. As for (3) we put $\mathcal{H}(f) = \mathrm{Span}\{e_n^\tau; (\tau, n) \in I(f)\}$ and define an opertor Q on $\mathcal{H}(f)$ by

$$h \mapsto \int\int_{KA} \langle f, M_\psi T(ka)\psi\rangle \langle h, M_\psi T(ka)\rangle\psi \ D(a)dkda.$$

Then (2) and the Schwarz inequality yield that Q is bounded and $\|Q\| \leq \|f\|^2$, and thereby, there exists $f_0 \in \mathcal{H}(f)$ such that $Q(h) = \langle h, f_0\rangle$ and $\|f_0\| = \|Q\|$. Since $Q(f) = \langle f, f_0\rangle = \|f\|^2$ by (2), it easily follows that $f = f_0$ (cf. [K]). Clearly, $Q(h) = \langle h, f\rangle$ means (3).

Remark 3.2. When $(T, \mathcal{H})$ is an irreducible square-integrable representation of G, it is well-known that each $\psi \in \mathcal{H}$ is admissible and satisfies

$$c_{\psi,\tau,n} = d_T^{-1}\|\psi\|^2,$$

where c_T is the formal degree of T (cf.[V]). Furthermore, applying the orthogonality of the matrix coefficients on G, we can replace the integrals over KA in Theorem 3.1 by the ones over G.

§4. Example in $SU(1, 1)$.

Let G be $SU(1, 1)$. Then

$$K = \{k_\theta = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}; 0 \leq \theta < 4\pi\},$$

$$A = \{a_t = \begin{pmatrix} \cosh t/2 & \sinh t/2 \\ \sinh t/2 & \cosh t/2 \end{pmatrix}; t \in \mathbf{R}\},$$

and $A^+ = \{a_t; t > 0\}$. In what follows we put

$$x = \tanh t.$$

Let $(T_h, \mathcal{H}_h)$ $(h \in \mathbf{Z}/2, h \geq 1)$ be the holomorphic discrete series of G realized on the weighted Bergman space $\mathcal{H}_h$ on the unit disk $D = G/K$:

$$\mathcal{H}_h = \{f : D \to \mathbf{C}; f \text{ is holomorphic on } D \text{ and}$$
$$\|f\|_h^2 = \Gamma(2h-1)^{-1} \int_D |f(z)|^2 (1-|z|^2)^{2(h-1)} dz < \infty\},$$

and $(T_{1/2}, \mathcal{H}_{1/2})$ the limit of holomorphic discrete series of G realized on the Hardy space $\mathcal{H}_{1/2}$ on D:

$$\mathcal{H}_{1/2} = \{f : D \to \mathbf{C}; f \text{ is holomorphic on } D \text{ and}$$
$$\|f\|_{1/2}^2 = \lim_{h \to 1/2} \|f\|_h^2 < \infty\}.$$

For $h \in \mathbf{Z}/2, h \geq 1/2$ we denote by $\langle \cdot, \cdot \rangle_h$ the inner product of $\mathcal{H}_h$ and we put

$$e_n^h(z) = \left(\frac{\Gamma(2h+n)}{\Gamma(2h)\Gamma(n+1)} \right)^{1/2} z^n \quad (n \in \mathbf{N}).$$

Then $\{e_n^h; n \in \mathbf{N}\}$ is an orthonormal basis of $\mathcal{H}_h$. For simplicity we denote

$$\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{1/2} \text{ and } e_n(z) = e_n^{1/2}(z) = z^n.$$

According to this basis the matrix coefficients of T_h are given as follows (see [Sa]):

$$\begin{aligned}\langle T_h(g) e_n^h, e_m^h \rangle_h &= e^{i(n\theta + m\theta')} \langle T_h(a_t) e_n^h, e_m^h \rangle_h \quad (g = k_\theta a_t k_{\theta'}) \\ &= e^{i(n\theta + m\theta')} M(h; n, m; x),\end{aligned}$$

where for $n \geq m$,

$$M(h; n, m; x) = C_{n,m}^h (1-x^2)^h (-x)^{n-m} F(-m, n+2h, n-m+1; x^2),$$

$$C_{n,m}^h = \left(\frac{\Gamma(n+1)\Gamma(n+2h)}{\Gamma(m+1)\Gamma(m+2h)} \right)^{1/2} \frac{1}{\Gamma(n-m+1)}$$

and $F(a. b, c; x)$ is the hypergeometric function, and for $m > n$ we change n and m by m and n respectively. Since

$$D(a_t)dt = \sinh(2t)dt = \frac{2x}{(1-x^2)^2} dx,$$

$M(h;n,m;x)$ $(n,m \in \mathbf{N})$ are square-integrable on G if and only if $h > 1/2$. Here we note that for $n \geq m$,

$$\begin{aligned}
&\lim_{x\to 1}(1-x^2)^{-h}M(h;n,m;x) \\
=\ & C^h_{n,m}(-1)^n\frac{\Gamma(1-m+n)\Gamma(m+2h)}{\Gamma(2h)\Gamma(n+1)} \\
=\ & (-1)^n\frac{1}{\Gamma(2h)}\left(\frac{\Gamma(n+2h)\Gamma(m+2h)}{\Gamma(n+1)\Gamma(m+1)}\right)^{1/2} \\
=\ & (-1)^n D^h_{n,m}
\end{aligned}$$

and for $m > n$, $\lim_{x\to 1}(1-x^2)^{-h}M(h;n,m;x) = (-1)^m D^h_{m,n} = (-1)^m D^h_{n,m}$ Then we shall define the normalized matrix coefficients $NM(h;n,m,x)$ as

$$NM(h;n,m;x) = (D^h_{n,m})^{-1}M(h;n,m;x)$$

and the differences of the normalized matrix coefficients $DM(h;n,m;x)$ as

$$DM(h;n,m;x) = NM(h;n,m;x) - NM(h;n+2,m;x).$$

The key lemmas are the following.

Lemma 4.1. Let notations be as above. Then

$$DM(h;n,m;x) = \frac{(1-x^2)^{1/2}}{x}$$
$$\times\left(\frac{m}{2h}NM(h+1/2;n,m-1;x) - \frac{m+2h}{2h}NM(h+1/2;n+1,m;x)\right).$$

Proof. We realize T_h on the circle and let $z = e^{i\theta}$ $(0 \leq \theta < 2\pi)$ (see [Sa]). We first note that

$$(D^h_{n,m})^{-1}e^h_n = \left(\frac{\Gamma(2h)\Gamma(m+1)}{\Gamma(m+2h)}\right)^{1/2} z^n,$$

$$(D^h_{n+2,m})^{-1}e^h_{n+2} = \left(\frac{\Gamma(2h)\Gamma(m+1)}{\Gamma(m+2h)}\right)^{1/2} z^{n+2};$$

and moreover,

$$
\begin{aligned}
& T_h(a_t)(z^n - z^{n+2}) \\
= & \frac{1}{(-z\sinh t/2 + \cosh t/2)^{2h}} \left(\frac{z\cosh t/2 - \sinh t/2}{-z\sinh t/2 + \cosh t/2} \right)^n \\
& \qquad \times \left(1 - \left(\frac{z\cosh t/2 - \sinh t/2}{-z\sinh t/2 + \cosh t/2} \right)^2 \right) \\
= & \frac{1}{(-z\sinh t/2 + \cosh t/2)^{2h}} \left(\frac{z\cosh t/2 - \sinh t/2}{-z\sinh t/2 + \cosh t/2} \right)^n \\
& \qquad \times \frac{1 - z^2}{(-z\sinh t/2 + \cosh t/2)^2} \\
= & \frac{1}{\sinh t/2} \frac{1}{(-z\sinh t/2 + \cosh t/2)^{2h+1}} \left(\frac{z\cosh t/2 - \sinh t/2}{-z\sinh t/2 + \cosh t/2} \right)^n \\
& \qquad \times \left(- \left(\frac{z\cosh t/2 - \sinh t/2}{-z\sinh t/2 + \cosh t/2} \right) + z \right).
\end{aligned}
$$

On the other hand, we easily see that

$$
\begin{aligned}
& \langle \left(\frac{\Gamma(2h)\Gamma(m+1)}{\Gamma(m+2h)} \right)^{1/2} z^{n+1}, e_m^h \rangle_h \\
= & \langle (D_{n+1,m}^{h+1/2})^{-1} e_{n+1}^{h+1/2}, e_m^{h+1/2} \rangle_h \\
= & \frac{m+2h}{2h} \langle (D_{n+1,m}^{h+1/2})^{-1} e_{n+1}^{h+1/2}, e_m^{h+1/2} \rangle_{h+1/2}
\end{aligned}
$$

and

$$
\begin{aligned}
& \langle \left(\frac{\Gamma(2h)\Gamma(m+1)}{\Gamma(m+2h)} \right)^{1/2} z^{n}, e_{m-1}^h \rangle_h \\
= & \langle (D_{n,m-1}^{h+1/2})^{-1} e_{n}^{h+1/2}, e_{m-1}^{h+1/2} \rangle_h \\
= & \frac{m}{2h} \langle (D_{n,m-1}^{h+1/2})^{-1} e_{n}^{h+1/2}, e_{m-1}^{h+1/2} \rangle_{h+1/2}.
\end{aligned}
$$

Then the desired result follows.

Lemma 4.2. Let notations be as above. Then for each $n, m \in \mathbf{N}$,

$$
0 < \int_0^1 DM(h; n, m; x)^2 \frac{2x}{(1-x^2)^2} dx < \infty,
$$

and especially, for $m > n$

$$\int_0^1 DM(h; n, m; x)^2 \frac{2x}{(1-x^2)^2} dx$$

$$= \Gamma(2h)^2 2(n+h+1) \frac{\Gamma(m+1)}{\Gamma(m+2h)} \frac{\Gamma(n+1)}{\Gamma(n+2h+2)}.$$

Proof. The case of $m > n$: We note that

$$\frac{(1-x^2)^{1/2}}{x} \frac{m}{2h} NM(h+1/2; n, m-1; x)$$

$$= Ax^{m-n-2}(1-x^2)^{h+1} G_n(m-n+2h, m-n; x^2)$$

and

$$\frac{(1-x^2)^{1/2}}{x} \frac{m+2h}{2h} NM(h+1/2; n+1, m; x)$$

$$= \frac{m+2h}{n+2h+1} Ax^{m-n-2}(1-x^2)^{h+1} G_{n+1}(m-n+2h, m-n; x^2),$$

where

$$A = \frac{\Gamma(2h)\Gamma(m+1)}{\Gamma(n+2h+1)\Gamma(m-n)}$$

and $G_n(x) = G_n(\alpha, \gamma, x)$ ($\alpha = m-n+2h, \gamma = m-n$) is the Jacobi polynomial. Hence,

$$\begin{aligned} I &= \int_0^1 DM(h; n, m; x)^2 \frac{2x}{(1-x^2)^2} dx \\ &= A^2 \int_0^1 x^{2(m-n-2)}(1-x^2)^{2h} \left(G_n(x^2) - \frac{m+2h}{n+2h+1} G_{n+1}(x^2) \right)^2 2x dx \\ &= A^2 \int_0^1 x^{\gamma-1}(1-x)^{\alpha-\gamma} \left(G_n(x) - \frac{m+2h}{n+2h+1} G_{n+1}(x) \right)^2 \frac{dx}{x}. \end{aligned}$$

We here consider the case of $m > n+1$. Then, $\gamma - 2 = m-n-2 \geq 0$. We note that $G_n^2 = (G_n - 1)G_n + G_n$ and $(G_n - 1)/x$ is the polynomial of degree $n-1$. So the orthogonality relations for the Jacobi polynomials and the definition of $G_n(x)$;

$$G_n(x) = \frac{\Gamma(\gamma)}{\Gamma(\gamma+n)} x^{1-\gamma}(1-x)^{\gamma-\alpha}\left(\frac{d}{dx}\right)^n \left(x^{\gamma+n-1}(1-x)^{\alpha+n-\gamma}\right)$$

yield that

$$\begin{aligned}
& \int_0^1 x^{\gamma-1}(1-x)^{\alpha-\gamma} G_n(x)^2 \frac{dx}{x} \\
= & \int_0^1 x^{\gamma-1}(1-x)^{\alpha-\gamma} G_n(x) \frac{dx}{x} \\
= & \Gamma(m-n)^2 \frac{\Gamma(n+1)\Gamma(n+2h+1)}{\Gamma(m+1)\Gamma(m+2h)} \frac{m}{m-n-1} \\
= & B,
\end{aligned}$$

and similarly,

$$\begin{aligned}
& \int_0^1 x^{\gamma-1}(1-x)^{\alpha-\gamma} G_{n+1}(x)^2 \frac{dx}{x} \\
= & \int_0^1 x^{\gamma-1}(1-x)^{\alpha-\gamma} G_n(x) G_{n+1}(x) \frac{dx}{x} \\
= & \int_0^1 x^{\gamma-1}(1-x)^{\alpha-\gamma} G_{n+1}(x) \frac{dx}{x} \\
= & \frac{n+2h+1}{m+2h} \frac{n+1}{m} B.
\end{aligned}$$

Therefore,

$$\begin{aligned}
I &= A^2 B \left(1 - 2\frac{n+1}{m} + \frac{n+1}{m}\frac{m+2h}{n+2h+1}\right) \\
&= A^2 B \frac{2(m-n-1)(n+h+1)}{m(n+2h+1)}
\end{aligned}$$

and hence, the desired result follows.

In the case of $m = n+1$ we note that $(G_n(x) - G_{n+1}(x))/x$ is a polynomial of degree n and thus, the integral I is well-defined. Then the analytic continuation on γ, letting $\gamma \to 1$ in the previous case, yields the desired formula for $m = n+1$.

The case of $m \leq n$: Since $M(h+1/2; n, m-1; x)$ and $M(h+1/2; n+1, m; x)$ have the term x^{n-m+1} and $n-m+1 \geq 1$, it easily follows from Lemma 4.1 that the desired integral is positive and finite.

This completes the proof of the lemma.

Lemma 4.3. Let notations be as above and suppose that

$$n, m \in 2\mathrm{N} \quad \text{or} \quad n, m \in 2\mathrm{N}+1.$$

Then, for $p > n, m$

$$\int_0^1 DM(h; n, p, x) DM(h; m, p, x) \frac{2x}{(1-x^2)^2} dx$$

$$= \delta_{nm} \Gamma(2h)^2 2(n+h+1) \frac{\Gamma(p+1)}{\Gamma(p+2h)} \frac{\Gamma(n+1)}{\Gamma(n+2h+2)}.$$

Proof. When $n = m$, it follows from Lemma 4.2. We may suppose that $n > m$ and hence, $n - m \geq 2$ and even. Then, applying the same argument used in the proof of Lemma 4.2, we see that the desired integral equals to

$$\int_0^1 x^{p-n-1+(n-m)/2} (1-x)^{2h}$$

$$\times \left(G_n(x) - \frac{p+2h}{n+2h+1} G_{n+1}(x) \right) \left(G_m(x) - \frac{p+2h}{m+2h+1} G_{m+1}(x) \right) \frac{dx}{x}.$$

Since $(n-m)/2$ is integer, $0 \leq (n-m)/2 - 1 \leq n-1$, and

$$\left(G_m(x) - \frac{p+2h}{m+2h+1} G_{m+1}(x) \right)$$

is a polynomial of degree $m+1 < n$, the orthogonality relations for the Jacobi polynomials yield that the integral equals to 0.

We here note that, if $h = 1/2$, then $D^h_{n,m} = 1$ and hence,

$$\begin{aligned} DM(1/2; n, m; x) &= M(1/2; n, m; x) - M(1/2; n+2, m; x) \\ &= \langle T_{1/2}(a_t)(e_n - e_{n+2}), e_m \rangle. \end{aligned}$$

Therefore, Lemma 4.2 implies that

$$0 < \int_A |\langle T_{1/2}(a_t)(e_n - e_{n+2}), e_m\rangle|^2 D(a_t)dt < \infty$$

and for $m > n$ this integral equals to

$$\frac{(2n+3)}{(n+1)(n+2)}.$$

Furthermore, these differences $\langle T_{1/2}(a_t)(e_n - e_{n+2}), e_m\rangle$ satisfy the quasi-orthogonality relations stated in Lemma 4.3 with $h = 1/2$. Thereby, as an application of Theorem 3.1, we see the following.

Theorem 4.4. Let $G = SU(1,1)$ and $(T_{1/2}, \mathcal{H}_{1/2})$ the limit of the discrete series of G.
(1) Let ψ be a finite linear combination of $e_{n+2} - e_n$. Then there exist constants $0 < C_1, C_2 < \infty$ such that for any f in $\mathcal{H}_{1/2}$

$$C_1\|f\|^2 \leq \int\int_{KA} |\langle f, T_{1/2}(ka_t)\psi\rangle|^2 \sinh 2t \; dkdt \leq C_2\|f\|^2.$$

(2) Let

$$\psi = \sum c_n \left(\frac{(2n+3)}{(n+1)(n+2)}\right)^{-1/2} (e_{n+2} - e_n),$$

where the sum is taken over $0 \leq n \leq N, n \in 2\mathrm{N}$ or $0 \leq n \leq N, n \in 2\mathrm{N}+1$, and let $\|\psi\|_0^2 = \sum |c_n|^2$. Then for any f in the L^2-span of $\{e_p, p \geq N+1\}$,

$$f(x) = \frac{1}{\|\psi\|_0} \int\int_{KA} \langle f, T_{1/2}(ka_t)\psi\rangle T_{1/2}(ka_t)\psi \; \sinh 2t \; dkdt.$$

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