Triple Systems of Hecke Type and Hypergroups

by

Aloys Krieg

1 Introduction

One of the most important classes of hypergroups is given by double coset spaces (cf. [1]). In this note we will consider double coset spaces with different subgroups on the left and right hand side (cf. [4]) as they already appeared in the description of all normal subhypergroups arising from Hecke algebras (cf. [6], Theorem 4 c). This construction does not any longer yield an algebra in general. But we obtain an associative triple system as its algebraic structure in a natural way (cf. [7], [8]). This triple system can be embedded into a usual double coset hypergroup (cf. Theorem 2). For the sake of simplicity we only deal with discrete hypergroups arising from Hecke algebras as in [6].

2 Associative triple systems of Hecke type

We start with a multiplicative group G with unit element e. The set

$$\begin{aligned}\mathbb{C}[G] &:= \{\varphi : G \to \mathbb{C};\ \text{support}(\varphi) \text{ finite }\}\\ &= \left\{\sum_{g\in G}\varphi(g)\delta_g;\quad \varphi(g) \in \mathbb{C} \text{ non-zero for finitely many } g \in G\right\},\end{aligned}$$

where δ_g stands for the *Kronecker delta*, is a $\mathbb{C}$-vector space. Extending the product

$$\delta_g \cdot \delta_h := \delta_{gh}$$

to $\mathbb{C}[G]$ by linearity, we obtain an associative $\mathbb{C}$-algebra with unit element δ_e, the so-called *group algebra* or *group ring* of G (cf. [9]).

Now let us consider two subgroups U and V of G and *double cosets*

$$UgV := \{ugv;\ u \in U, v \in V\}, \quad g \in G.$$

Two double cosets are either disjoint or equal. Let

$$K := U\backslash G/V := \{UgV;\ g \in G\}$$

stand for the space of (U, V)-double cosets in G equipped with the discrete topology.

$$\mathcal{H}(U\backslash G/V) := \{\varphi : U\backslash G/V \to \mathbb{C};\ \text{support}(\varphi)\ \text{finite}\}$$
$$= \left\{ \sum_{UgV\subset G} \varphi(UgV)\delta_{UgV};\ \varphi(UgV) \in \mathbb{C}\ \text{non-zero for finitely many}\ UgV \subset G \right\}$$

is a $\mathbb{C}$-vector space. If $V = U$ we use the abbreviation $\mathcal{H}(G/\!\!/U) = \mathcal{H}(U\backslash G/U)$ just as in [6].

For the introduction of a product we need the so-called *Hecke condition*: (G, U) is a *Hecke pair* if $[U : U \cap g^{-1}Ug] < \infty$ for every $g \in G$. Now assume additionally that V and W are subgroups of G, which are *commensurable* with U, i.e. the intersection of any two of the subgroups has finite index in both. Then (G, V) and (G, W) as well as $(G, U \cap V \cap W)$ are Hecke pairs, too. Given $a, b \in G$ we obtain finite disjoint decompositions of the double cosets

$$UaV = \bigcup_{j=1}^{m} Ua_j,\quad m = \text{ind}_U UaV,\quad VbW = \bigcup_{k=1}^{n} Vb_k,\quad n = \text{ind}_V VbW.$$

Then define

$$\delta_{UaV} \cdot \delta_{VbW} := \sum_{UcW\subset G} \mu(c)\ \delta_{UcW}, \tag{1}$$
$$\mu(c) := \sharp\{(j, k);\ Ua_jb_k = Uc\} \in \mathbb{N}_0.$$

It can be shown that the definition of $\mu(c)$ does not depend on the choice of the representatives c, a_j, b_k. This product is extended linearly. Moreover we observe

$$\text{ind}_U UaV \cdot \text{ind}_V VbW = \sum_{UcW\subset G} \mu(c)\text{ind}_U UcW. \tag{2}$$

If X is another subgroup of G, which is commensurable with U, we obtain

$$(\varphi_1 \cdot \varphi_2) \cdot \varphi_3 = \varphi_1 \cdot (\varphi_2 \cdot \varphi_3) \in \mathcal{H}(U\backslash G/X) \tag{3}$$

for all $\varphi_1 \in \mathcal{H}(U\backslash G/V)$, $\varphi_2 \in \mathcal{H}(V\backslash G/W)$, $\varphi_3 \in \mathcal{H}(W\backslash G/X)$ (cf. [4], [10]).

If $V = U$ we have the Hecke algebra $\mathcal{H}(G/\!\!/U)$ of the Hecke pair (G, U) just as in [5], [10].

In the general case again, there is a linear isomorphism

$$J = J_{U,V} : \mathcal{H}(U\backslash G/V) \to \mathcal{H}(V\backslash G/U),\quad \delta_{UaV} \mapsto \delta_{Va^{-1}U},$$

satisfying

$$J(\varphi_1 \cdot \varphi_2) = J(\varphi_2) \cdot J(\varphi_1), \quad J \circ J = \mathrm{id} \tag{4}$$

(cf. [4]).

This becomes the foundation of our algebraic structure. A $\mathbb{C}$-vector space $\mathcal{A}$ equipped with a trilinear triple product

$$\mathcal{A} \times \mathcal{A} \times \mathcal{A} \to \mathcal{A}, \quad (x, y, z) \mapsto < x, y, z >,$$

is called an *associative triple system* (of the second kind) if

$$<< u, v, w >, x, y > = < u, < x, w, v >, y > = < u, v, < w, x, y >>$$

holds for all $u, v, w, x, y \in \mathcal{A}$ (cf. [7], [8]). The notions of homomorphisms and sub-triple systems are then defined in the obvious way. Now (3) and (4) imply

Theorem 1 ([4]). *Let U and V be commensurable subgroups of a group G such that (G, U) is a Hecke pair. Then $\mathcal{H}(U\backslash G/V)$ is an associative triple system by*

$$< \varphi_1, \varphi_2, \varphi_3 > := \varphi_1 \cdot J(\varphi_2) \cdot \varphi_3.$$

The notion of associative triple systems comes from the following idea: Start with an associative $\mathbb{C}$-algebra $\mathcal{A}$ with an involution j on $\mathcal{A}$, i.e. $j : \mathcal{A} \to \mathcal{A}$ is linear and satisfies $j(xy) = j(y)j(x)$ as well as $j(j(x)) = x$ for all $x, y \in \mathcal{A}$. Then $(\mathcal{A}, j)$ becomes an associative triple system by

$$< x, y, z > := xj(y)z.$$

On the other hand Loos [7] showed that each associative triple system can be obtained as a sub-triple system of $(\mathcal{A}, j)$ for suitable $\mathcal{A}$ and j. In the case of Hecke triple systems we can simplify his construction considerably.

Theorem 2. *Let U and V be commensurable subgroups of a group G and $r := \sqrt{[U : U \cap V] \cdot [V : U \cap V]}$. Assume that (G, U) is a Hecke pair. Then*

$$\phi : (\mathcal{H}(U\backslash G/V), J) \to (\mathcal{H}(G/\!\!/(U \cap V)), J)$$
$$\varphi = \sum_{UgV \subset G} \varphi(UgV)\delta_{UgV} \mapsto \frac{1}{r} \sum_{(U\cap V)g(U\cap V) \subset G} \varphi(UgV)\delta_{(U\cap V)g(U\cap V)},$$

is an injective homomorphism of the associative triple systems.

Proof. Obviously ϕ is well-defined, linear and injective. It suffices to show that

$$\phi(\delta_{UaV}) \cdot J(\phi(\delta_{UbV})) \cdot \phi(\delta_{UcV}) = \phi(\delta_{UaV} \cdot J(\delta_{UbV}) \cdot \delta_{UcV}) \tag{5}$$

holds for all $a, b, c \in G$. Assume that

$$UaV = \bigcup_{j=1}^{\alpha} Ua_j, \quad UbV = \bigcup_{k=1}^{\beta} b_k V, \quad UcV = \bigcup_{l=1}^{\gamma} Uc_l$$
$$U = \bigcup_{\nu=1}^{s} (U \cap V)u_\nu, \quad V = \bigcup_{\mu=1}^{t} v_\mu (U \cap V)$$

are disjoint coset decompositions. Then

$$UaV = \bigcup_{j=1}^{\alpha} \bigcup_{\nu=1}^{s} (U \cap V) u_\nu a_j,$$
$$Vb^{-1}U = \bigcup_{k=1}^{\beta} \bigcup_{\mu=1}^{t} (U \cap V) v_\mu^{-1} b_k^{-1}$$
$$UcW = \bigcup_{l=1}^{\gamma} \bigcup_{\rho=1}^{s} (U \cap V) u_\rho c_l$$

are disjoint decompositions, too. In view of (1) the coefficient of $(U \cap V)g(U \cap V)$ on the left hand side of (5) is

$$\frac{1}{r^3} \sharp \{(\nu, j, \mu, k, \rho, l); \quad (U \cap V) u_\nu a_j v_\mu^{-1} b_k^{-1} u_\rho c_l = (U \cap V) g\}$$
$$= \frac{1}{r^3} \cdot \sharp \{(j, \mu, k, \rho, l); \quad U a_j v_\mu^{-1} b_k^{-1} u_\rho c_l = Ug\}$$
$$= \frac{st}{r^3} \cdot \sharp \{(j', k', l); \quad U a_{j'} b_{k'}^{-1} c_l = Ug\}.$$

By virtue of $st = r^2$ and (1) this is also the coefficient of $(U \cap V)g(U \cap V)$ on the right hand side of (5). Thus the claim follows. □

3 Associative Banach triple systems of Hecke type

Consider the data of section 2. Given an arbitrary mapping $\varphi : U\backslash G/V \to \mathbb{C}$ define its *norm* by

(6) $$\|\varphi\| := \sum_{(U \cap V)a \subset G} \varphi(UaV) \in [0; \infty].$$

Then

$$\hat{\mathcal{H}}(U\backslash G/V) := \{\varphi : U\backslash G/V \to \mathbb{C}; \quad \|\varphi\| < \infty\}$$

equipped with $\|\cdot\|$ is obviously a Banach space containing $\mathcal{H}(U\backslash G/V)$ as a dense subset. Extending the product form $\mathcal{H}(U\backslash G/V)$ we conclude

$$\| < \varphi_1, \varphi_2, \varphi_3 > \| \leq \|\varphi_1\| \cdot \|\varphi_2\| \cdot \|\varphi_3\|$$

for all $\varphi_1, \varphi_2, \varphi_3 \in \hat{\mathcal{H}}(U\backslash G/V)$ from Theorem 1, Theorem 2 and [6], Theorem 2.

A Banach space $\mathcal{A}$, which is an associative triple system and satisfies

$$\| < x, y, z > \| \ \leq \ \|x\| \cdot \|y\| \cdot \|z\| \quad \text{for all } x, y, z \in \mathcal{A}$$

is called an *associative Banach triple system* (cf. [2]). Thus we have

Corollary 1. *Let U and V be commensurable subgroups of a group G such that (G, U) is a Hecke pair. Then $\hat{\mathcal{H}}(U\backslash G/V)$ is an associative Banach triple system containing $\mathcal{H}(U\backslash G/V)$ as a dense subset.*

4 Hypergroups

Consider again the data of section 2. Let ε stand for the point measure. Given $a, b \in G$ use (1) in order to define

$$\varepsilon_{UaV} * \varepsilon_{VbW} := \sum_{UcW \subset G} \frac{\mu(c) \cdot \text{ind}_U(UcW)}{\text{ind}_U(UaV) \cdot \text{ind}_V(VbW)}\, \varepsilon_{UcW}. \tag{7}$$

It follows from (2) that the right hand side of (7) is a probability measure again.

Recall the definition of a *hypergroup* and in particular of the discrete double coset hypergroup $(G/\!/(U \cap V), *)$ from [1], Chapter 1.1. Thus Theorem 2, Corollary 1 and [6], Theorem 3, lead to

Theorem 3. *Let U and V be commensurable subgroups of a group G and $r := \sqrt{[U : U \cap V] \cdot [V : U \cap V]}$. Assume that (G, U) is a Hecke pair. Then*

$$\Phi : \hat{\mathcal{H}}(U\backslash G/V) \to (G/\!/(U \cap V), *), \quad \varphi \mapsto \frac{1}{r} \sum_{(U\cap V)a\subset G} \varphi(UaV)\, \varepsilon_{(U\cap V)a(U\cap V)},$$

is an injective homomorphism of the associative triple systems.

Note that a hypergroup with the attached involution naturally defines an associative triple system. Thus we can view $(U\backslash G/V, *)$ as an *associative hypergroup triple system.*

5 Examples

The notion of Hecke algebras originates from the theory of modular forms. It should be noted that the consideration of (U, V)-double cosets there also plays an essential role when dealing with congruence subgroups (cf. [3], III.7.3, [10], section 3.4).

Next consider a Hecke pair (G, U) and a subgroup $U \subset H \subset G$ such that $H/\!/U$ is normal in $G/\!/U$. This means $HgH = HgU$ for all $g \in G$ due to [6], Theorem 4. In this case one can easily sharpen Theorem 2. The associative hypergroup triple systems $(H\backslash G/U, *)$ and $(G/\!/H, *)$ are then isomorphic. An explicit example of this type is

$$G = GL_n(\mathbb{F}_q), \quad H = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \in G \right\}, \quad U = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in G \right\}$$

(cf. [6], section 3).

Now we consider finite subgroups U and V of a group G. It follows from (1) and (7) that

$$\frac{1}{\mathrm{ind}_U UaV}\,\delta_{UaV}\cdot\frac{1}{\mathrm{ind}_V Vb^{-1}U}\,\delta_{Vb^{-1}U}\cdot\frac{1}{\mathrm{ind}_U UcV}\,\delta_{UcV}$$
$$=\frac{1}{\sharp U\cdot\sharp V}\sum_{u\in U,v\in V}\frac{1}{\mathrm{ind}_U Uavb^{-1}ucV}\,\delta_{Uavb^{-1}ucV},$$
$$\varepsilon_{UaV}*\varepsilon_{Vb^{-1}U}*\varepsilon_{UcV}=\frac{1}{\sharp U\cdot\sharp V}\sum_{u\in U,v\in V}\varepsilon_{Uavb^{-1}ucV}.$$

The elements

$$c_U:=\frac{1}{\sharp U}\sum_{u\in U}\delta_u,\quad c_V:=\frac{1}{\sharp V}\sum_{v\in V}\delta_v$$

are idempotents in $\mathbb{C}[G]$. We consider the associative triple system $(\mathbb{C}[G],J)$ with $J(\delta_g)=\delta_{g^{-1}}$. In view of $J(c_U)=c_U$ and $J(c_V)=c_V$ we observe that $c_U\cdot\mathbb{C}[G]\cdot c_V$ becomes a sub-triple system of $(\mathbb{C}[G],J)$. Thus a verification (cf. [5], I(6.6), [6], Theorem 5) yields

Theorem 4. *Let U and V be finite subgroups of a group G. Then*

$$\mathcal{H}(U\backslash G/V)\to c_U\cdot\mathbb{C}[G]\cdot c_V,\quad \varphi\mapsto\frac{1}{\sqrt{\sharp U\cdot\sharp V}}\sum_{g\in G}\varphi(UgV)\delta_g,$$

is an isomorphism of the associative triple systems.

References

[1] Bloom, W.R., Heyer, H.: Harmonic analysis of probability measures on hypergroups. de Gruyter, Berlin-New York 1995.

[2] Fernandez Lopez, A., Garcia, E.: Compact associative B^*-triple systems. Quarterly J. Math. Oxford **41** (1990), 61–69.

[3] Koecher, M., Krieg, A.: Elliptische Funktionen und Modulformen. Springer-Verlag, Berlin-Heidelberg-New York 1998.

[4] Krieg, A.: Associative triple systems of Hecke type. Algebras, Groups, Geom. **5** (1988), 341–357.

[5] Krieg, A.: Hecke algebras. Memoirs Amer. Math. Soc. **435** (1990).

[6] Krieg, A.: Hecke algebras and hypergroups. In H. Heyer, J. Marion (eds.): Analysis on infinite-dimensional Lie-groups and algebras. World Scientific, Singapore-New Jersey-London 1998, 197–206.

[7] Loos, O.: Assoziative Tripelsysteme. Manuscripta Math. **7** (1972), 103–112.

[8] Meyberg, K.: Lectures on algebras and triple systems. Lecture Notes, University of Virginia, Charlottesville 1972.

[9] Passman, D.S.: The algebraic structure of group rings. J. Wiley, New York 1977.

[10] Shimura, G.: Introduction to the arithmetic theory of automorphic functions. Iwanami Shoten and Princeton University Press, Tokyo-Princeton 1971.

Author's address:
Lehrstuhl A für Mathematik, RWTH Aachen, D-52056 Aachen, Germany
e-mail: krieg@mathA.rwth-aachen.de