Irreducible Bounded Representations of Exponential Solvable Lie Groups

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Introduction

In this survey we present the theory of irreducible bounded representations of exponential solvable Lie groups. For these groups the exponential mapping from the Lie algebra $\mathfrak{g}$ of G into G is a diffeomorphism and the unitary dual is explicitly known thanks to the work of Mackey, Dixmier, Kirillov, Bernat, Pukanszky and Vergne in the years 1950 to 1970.
In the first part of the paper we recall the structure of exponential solvable Lie groups G and in the second part we explain Kirillov's theory, i.e. we give the description of the irreducible unitary representations of G using the orbit method. In the last part the algebraically irreducible (or simple) modules of the group algebra $L^1(G)$ are presented together with what is known about topologically irreducible bounded representations of G. The theory of the simple $L^1(G)$ modules, (G exponential), has been developed by Leptin and Poguntke from 1975 to 1981 and Poguntke published a classification of these modules in 1983. It turns out that irreducible unitary and simple modules can be realized in the framework of induced representations. This is no longer true for general bounded irreducible representations on Banach spaces.
In recent years, the method of Poguntke has been used to study these representations. For so called non-*-regular exponential groups, more complicated representations appear, which are not subrepresentations of induced representations and which are constructed by using irreducible non bounded representations of vector groups on Banach spaces.
Many interesting problems remain to be solved. For instance: Is it possible to characterize the separable Banach spaces, on which exponential solvable groups act irreducibly? This problem is closely related to the invariant subspace problem. Is it possible to give explicit descriptions of some of these strange representations for lower dimensional groups?
No proofs will be given in this survey article, they can be found in the literature or they will be published elsewhere.

1. The Structure of Exponential Solvable Lie Groups.

1.1 Let $\mathfrak{g}$ be a real finite dimensional Lie algebra. We let $\mathfrak{g}^1 = \mathfrak{g}$ and we define the central descending series $\mathfrak{g}^j, j = 1, 2, \cdots$, of $\mathfrak{g}$ by $\mathfrak{g}^{j+1} = [\mathfrak{g}, \mathfrak{g}^j]$. We say that $\mathfrak{g}$ is nilpotent of step k if there exists $k \in \mathbb{N}$ such that $\mathfrak{g}^{k+1} = (0)$ and $\mathfrak{g}^k \neq (0)$.

1.2. We say that $\mathfrak{g}$ is solvable if the descending series $\mathfrak{s}^1 = \mathfrak{g}, \mathfrak{s}^{j+1} = [\mathfrak{s}^j, \mathfrak{s}^j], j = 1, 2, \cdots$, stops with $\mathfrak{s}^{l+1} = (0)$ for some $l \in \mathbb{N}$.

1.3. A sequence of ideals of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{a}_1 \supset \cdots \supset \mathfrak{a}_j \supset \cdots \supset \mathfrak{a}_{m+1} = (0)$$

is called a Jordan-Hölder series or J.H. series, if for every $j = 1, \cdots, m$, the $\mathfrak{g}$-module $\mathfrak{a}_j/\mathfrak{a}_{j+1}$ is irreducible. A theorem of Lie says that for solvable Lie algebras every irreducible complex finite dimensional Lie algebra module is of dimension 1 (see [Di.3]). Hence for every J.H.-series $(\mathfrak{a}_j)_j$ of a real solvable Lie algebra the dimension of $\mathfrak{a}_j/\mathfrak{a}_{j+1}$ is equal to 1 or 2 for every j. We call these irreducible modules the roots of $\mathfrak{g}$. Let us denote by Λ the set of all the roots of $\mathfrak{g}$. If $\mathfrak{a}_j/\mathfrak{a}_{j+1}$ is one dimensional, then the corresponding root λ_j is just a real character of $\mathfrak{g}$. If $\mathfrak{a}_j/\mathfrak{a}_{j+1}$ is two dimensional then we can describe the root $\lambda_j = \lambda$ in the following way. There exist two real linear functionals l_λ and p_λ of $\mathfrak{g}$ and two vectors $X = X_j$ and $Y = Y_j$ in $\mathfrak{a}_j$, such that $\{X, Y\}$ is a basis of $\mathfrak{a}_j$ mod $\mathfrak{a}_{j+1}$ and such that

$$[U, X + iV] = (l_\lambda(U) + ip_\lambda(U))(X + iY) \bmod (a_{j+1})_{\mathbb{C}}, U \in \mathfrak{g},$$

(where $V_{\mathbb{C}}$ indicates the complexification of a real vector space V). In this way we may consider the roots λ of $\mathfrak{g}$ as linear functionals (a real one in the one dimensional case and as complex valued one $\lambda \simeq l_\lambda + ip_\lambda$ in the two dimensional case).

1.4. In particular $\mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}]$ is contained in the kernel of every root. Since the algebra $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is non trivial if $\mathfrak{g} \neq (0)$ and abelian we have that at least one of the roots of $\mathfrak{g}$ is 0. The roots of $\mathfrak{g}$ give us also the spectrum $\sigma(ad(X))$ of $ad(X)(X \in \mathfrak{g})$ considered as linear operator on $\mathfrak{g}_{\mathbb{C}}$. In fact $\sigma(ad(X)) = \{\lambda(X), \lambda \in \Lambda\}$.

1.5. The nilradical $\mathfrak{n}$ of $\mathfrak{g}$ is the largest nilpotent ideal of $\mathfrak{g}$. In the solvable case, the nilradical is given by

$$\mathfrak{n} = \bigcap_{\lambda \in \Lambda} \ker(\lambda) \supset [\mathfrak{g}, \mathfrak{g}].$$

From now on we will only consider solvable Lie algebras.

1.6. Let us describe the Jordan decomposition of such an algebra. If $\mathfrak{g}$ is not nilpotent, we can choose an element T of $\mathfrak{g}$ which is in general position with respect to the roots of $\mathfrak{g}$, i.e. for every pair λ and μ of roots, considered as complex linear functionals, we always have that

$$\lambda(T) - \mu(T) \neq 0.$$

We take now the Jordan decomposition of $ad(T)$ on $\mathfrak{g}_{\mathbb{C}}$:

$$\mathfrak{g}_{\mathbb{C}} = \sum_{\lambda \in \Lambda} (\mathfrak{g}_{\mathbb{C}})_\lambda,$$

where

$$(\mathfrak{g}_{\mathbb{C}})_\lambda = \{U \in \mathfrak{g}_{\mathbb{C}}, (ad(T) - \lambda(T))^k(U) = 0 \text{ for some } k > 0\}.$$

We have the classical relations

$$[(\mathfrak{g}_{\mathbb{C}})_\lambda, (\mathfrak{g}_{\mathbb{C}})_\mu] \subset (\mathfrak{g}_{\mathbb{C}})_{\lambda+\mu}, \quad \lambda, \mu \in \Lambda.$$

Since T is in general position with respect to the roots of $\mathfrak{g}$, it follows that $(\mathfrak{g}_{\mathbb{C}})_0$ is a nilpotent subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Let now $\mathfrak{g}_0 = (\mathfrak{g}_{\mathbb{C}})_0 \cap \mathfrak{g}$ and for a root $\lambda \neq 0$, let

$$\mathfrak{g}_\lambda = \left((\mathfrak{g}_{\mathbb{C}})_\lambda + \overline{(\mathfrak{g}_{\mathbb{C}})_\lambda}\right) \cap \mathfrak{g} = \left((\mathfrak{g}_{\mathbb{C}})_\lambda + (\mathfrak{g}_{\mathbb{C}})_{\overline{\lambda}}\right) \cap \mathfrak{g}.$$

Let $\mathfrak{m} = \sum_{\lambda \neq 0} \mathfrak{g}_\lambda$. Then $[\mathfrak{g}_0, \mathfrak{m}] = \mathfrak{m}$ and so $\mathfrak{m}$ is contained in $[\mathfrak{g}, \mathfrak{g}]$ whence $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{m} = \mathfrak{g}_0 + [\mathfrak{g}, \mathfrak{g}]$.

If $\mathfrak{g}$ is nilpotent, then of course every root is 0 and $\mathfrak{g} = \mathfrak{g}_0$. If not, let for $j = 1, \cdots, m$, $\mathfrak{v}_j$ be a one or two-dimensional subspace of $\mathfrak{a}_j$, such that $\mathfrak{a}_j = \mathfrak{v}_j \oplus \mathfrak{a}_{j+1}$. Then

$$\mathfrak{g} = \oplus_{j=1}^{m} \mathfrak{v}_j.$$

1.7. Let us now study simply connected solvable Lie groups. We say that a real finite dimensional connected Lie group G is nilpotent if its Lie algebra $\mathfrak{g} = Lie(G)$ is nilpotent. We can provide a nilpotent Lie algebra with a group structure using the Campbell-Baker-Hausdorff multiplication:

$$X \cdot Y = CBH(X,Y) = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] + \frac{1}{12}[Y,[Y,X]] + \cdots, \quad X, Y \in \mathfrak{g}.$$

This multiplication is a polynomial expression in X and Y, since $\mathfrak{g}$ is nilpotent. Hence $(\mathfrak{g}, CBH)$ becomes a Lie group, whose Lie algebra is $(\mathfrak{g}, [,])$. It is obvious that that for every $X \in \mathfrak{g}$, the mapping

$$E_X : \mathbb{R} \to \mathfrak{g}; t \mapsto tX,$$

is a group homomorphism from $(\mathbb{R}, +)$ to $(\mathfrak{g}, CBH)$. Hence the exponential mapping $\exp : \mathfrak{g} \to (\mathfrak{g}, CBH)$ is the identity mapping in this case and every simply connected Lie group whose Lie algebra is isomorphic to $(\mathfrak{g}, [,])$ is itself isomorphic to $(\mathfrak{g}, CBH)$.

1.8. If G is a simply connected solvable Lie group, we know (see [Di.3]), that the exponential mapping is a diffeomorphism if and only if all the roots of $\mathfrak{g} = Lie(G)$ are of the form $l_\lambda + i\omega_\lambda l_\lambda$, for some real constant ω_λ and a real valued character l_λ of $\mathfrak{g}$. More precisely, Dixmier has shown in ([Di. 3]) that for a simply connected solvable Lie group G the following conditions are equivalent:

i) The exponential mapping $\exp : \mathfrak{g} \to G$ is injective.
ii) The exponential mapping $\exp : \mathfrak{g} \to G$ is surjective.
iii) The exponential mapping $\exp : \mathfrak{g} \to G$ is a diffeomorphism.
iv) Every root λ of $\mathfrak{g}$ is of the form $\lambda = (1 + i\omega)l$ for some real linear form $l \in \mathfrak{g}^*$ and some $\omega \in \mathbb{R}$.
v) For every $X \in \mathfrak{g}$ the spectrum of the operator $ad(X)$ acting on $\mathfrak{g}_{\mathbb{C}}$ does not contain a number of the form $i\tau, \tau \in \mathbb{R} \setminus (0)$.

We call the solvable groups, which satisfy these conditions, (solvable) exponential.

Such an exponential group G can be realized on its Lie algebra $\mathfrak{g}$. The Cambell-Baker-Hausdorff multiplication, which converges on a neighbourhood of 0, extends to a unique analytic map on $\mathfrak{g} \times \mathfrak{g}$ and in this way G is isomorphic to the group $(\mathfrak{g}, CBH)$, the exponential mapping for the latter group being the identity.

1.9. A general solvable simply connected Lie group is as a variety always diffeomorphic to a vector space. Indeed, let us take a Jordan-decomposition $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{m} = \mathfrak{g}_0 + \mathfrak{n}$, for some nilpotent ideal $\mathfrak{n}$ of $\mathfrak{g}$ containing $[\mathfrak{g}, \mathfrak{g}]$. Choose a subspace $\mathfrak{t}$ of $\mathfrak{g}_0$, such that

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{n}.$$

For $S, T \in \mathfrak{t}$, we write

$$CBH(S,T) = CBH(S+T, Q(S,T)),$$

where $Q(S,T) = CBH(-S-T, CBH(S,T)) \in [\mathfrak{g}_0, \mathfrak{g}_0]$ is a polynomial expression of brackets in S and T. For a vector U in $\mathfrak{n}$ and $T \in \mathfrak{t}$, let

$$^{T}U = \exp\ (ad(-T))U = \sum_{j=0}^{\infty} \frac{ad(-T)^j}{j!}(U)$$

We obtain a group multiplication on $\mathfrak{s} = \mathfrak{t} \oplus \mathfrak{n}$ by the following rule:

$$(T,U) \cdot (T',U') = (T+T', CBH(Q(T,T'), CBH(^{T'}U, U')); T, T' \in \mathfrak{t}, U, U' \in \mathfrak{n}.$$

The Lie algebra of $(\mathfrak{s}, \cdot)$ is of course isomorphic to $\mathfrak{g}$ and so every simply connected Lie group G with a Lie algebra isomorphic to $\mathfrak{g}$ is itself isomorphic to $(\mathfrak{s}, \cdot)$. In particular

$$G = \exp(\mathfrak{t})\exp(\mathfrak{n})$$

and

$$\exp(T)\exp(U)\exp(T')\exp(U') = \exp(T+T')\exp(Q(T,T'))\exp(^{T'}U)\exp(U')$$

$(T, T' \in \mathfrak{t}, U, U' \in \mathfrak{n})$ (see [Le.Lu.]).

1.10. Let us now consider closed connected subgroups $H = \exp(\mathfrak{h})$ of the simply connected solvable Lie group G. The quotient space G/H is then diffeomorphic to the space $\mathfrak{g}/\mathfrak{h}$. We obtain coordinates on G/H in the following way:

Consider a J.H.-sequence $S = (\mathfrak{a}_j)_j$ of $\mathfrak{g}$, which passes through $\mathfrak{n}$, i.e. such that $\mathfrak{a}_{j_0} = \mathfrak{n}$ for some j_0. For every j, we take a subspace $\mathfrak{w}_j$ of $\mathfrak{a}_j$, such that $\mathfrak{a}_j + \mathfrak{h} = (\mathfrak{a}_{j+1} + \mathfrak{h}) \oplus \mathfrak{w}_j$. The mapping $E_S^{G/H} : \mathfrak{w} = \sum_j \mathfrak{w}_j \to G/H$

$$E_S^{G/H}(\sum_j w_j) = \left(\prod_{j=1}^{m} \exp(w_j) \right) H$$

is then a diffeomorphism. In particular if $\mathfrak{h} = (0)$, then $E_S^G : \mathfrak{w} = \sum_j \mathfrak{w}_j \to G$ is a diffeomorphism (see [Le.Lu.]).

1.11. We can use the mapping E_S^G to describe the left Haar measure on G. Indeed the left Haar measure dx is given by

$$\int_G \varphi(x)dx = \int_{\mathfrak{w}} \varphi(E_S^G(w))dw$$

for φ in the space $C_c(G)$ of the continuous functions with compact support on G. Associated to the Haar measure is the modular function Δ_G of G. The uniqueness of the Haar measure implies that for any $s \in G$ the left invariant measure $\varphi \mapsto \int_G \varphi(xs^{-1})dx$ is a positive multiple, denoted by $\Delta_G(s)$, of our Haar measure and so

$$\int_G \varphi(xs^{-1})dx = \Delta_G(s) \int_G \varphi(x)dx, \varphi \in C_c(G)$$

The function Δ_G is easy to compute. In fact $\Delta_G(\exp(U)) = e^{-tr\, ad(U)}, U \in \mathfrak{g}$, where $tr\, ad(U)$ denotes the trace of the operator $ad(U)$ on $\mathfrak{g}$.

1.12. We realize many of our representations on function spaces, for instance on spaces of functions which satisfy certain covariance conditions.

Let $H = \exp(\mathfrak{h})$ be a closed connected subgroup of G and let

$$\mathcal{E}(G,H) = \{\xi : G \to \mathbb{C}; \xi \text{ continuous with compact support modulo } H,$$

$$\xi(xh) = \frac{\Delta_H(h)}{\Delta_G(h)}\xi(x), x \in G, h \in H\}.$$

This space is left translation invariant and the linear mapping

$$p_{G/H} : C_c(G) \to \mathcal{E}(G,H); \quad \psi \mapsto \left(x \mapsto \int_H \psi(xh)\frac{\Delta_G(h)}{\Delta_H(h)}dh\right)$$

is surjective. The space $\mathcal{E}(G,H)$ admits a left invariant linear form, namely

$$\oint_{G/H} du : \mathcal{E}(G,H) \to \mathbb{C}, \quad \xi \mapsto \int_{\mathfrak{w}} \xi(E_S^{G/H}(w))dw.$$

Hence the linear form

$$C_c(G) \to \mathbb{C}, \quad \psi \mapsto \oint_{G/H} p_{G/H}(\psi)(u)du$$

is left translation invariant and positive and so is a multiple of our Haar measure. The uniqueness of the Haar measure implies that the positive linear form $\oint_{G/H} du$ is unique (up to a positive multiple) and so it does not depend on the choice of the J.H. sequence and not on the complementary spaces $\mathfrak{w}_j$.

1.13. The convolution algebra $L^1(G)$ of the integrable functions on G with respect to Haar measure plays a fundamental role in the theory of representations of G. The convolution of two functions φ and ψ is defined by

$$\varphi * \psi(x) = \int_G \varphi(u)\psi(u^{-1}x)du, \quad x \in G.$$

The L^1-norm on $L^1(G)$ is given by

$$\|\varphi\|_1 = \int_G |\varphi(x)|dx, \quad \varphi \in L^1(G).$$

There exists an isometric involution * on $L^1(G)$:

$$\varphi^*(x) = \Delta_G(x)^{-1}\overline{\varphi(x^{-1})}, x \in G, \varphi \in L^1(G).$$

The connection between left translation λ and convolution is the following:

$$\lambda(x)(\varphi * \psi) = (\lambda(x)\varphi) * \psi, x \in G, \varphi, \psi \in L^1(G).$$

2. The Dual Space of Exponential Solvable Lie Groups

2.1. We begin with the definitions of the different types of irreducible bounded representations.

Let G be a locally compact group. A representation (T, V) of G on a Banach space V is a strongly continuous homomorphism $T : G \to Gl(V)$ of the group G into the group $Gl(V)$ of the bounded invertible linear operators on V. Strongly continuous means that the mappings

$$G \to V, \quad x \mapsto T(x)v,$$

are continuous for every $v \in V$.

We say that the representation (T, V) is bounded, if

$$C_T = \sup_{x \in G} \|T(x)\|_{op} < \infty.$$

Here $\|a\|_{op}$ denotes the operator norm of a bounded operator a on V. Since a solvable group G is amenable, every bounded representation (T, V) on a Banach space $(V, \|\cdot\|_V)$ is in fact isometric, there exists another norm $\|\cdot\|'$ on V, which is equivalent to $\|\cdot\|_V$, such that $\|T(x)v\|' = \|v\|'$ for every $v \in V$ and $x \in G$ (see [Pi.]).

2.2. Bounded representations can be integrated to bounded representations of the Banach algebra $L^1(G)$. Indeed, for $\varphi \in L^1(G)$, the operator

$$T(\varphi) = \int_G \varphi(x)T(x)dx$$

on V is bounded and $\|T(v)\|_{op} \leq C_T \|\varphi\|_1$. We have the relations

$$T(\varphi * \psi) = T(\varphi) \circ T(\psi), T(\lambda(x)\varphi) = T(x) \circ T(\varphi), \quad x \in G, \varphi, \psi \in L^1(G).$$

Conversely, given a bounded representation (T, V) of the algebra $L^1(G)$ on a Banach space V, we have at the same time a bounded representation (T, V) of G, such that

$$T(x) \circ T(\varphi) = T(\lambda(x)\varphi)$$

for every $x \in G$ and $\varphi \in L^1(G)$ (see [Di.4]).

2.3. A closed subspace W of V is said to be G-invariant, if for every $x \in G, w \in W$, $T(x)w \in W$. The same type of definitions is valid for representations of the Banach algebra $L^1(G)$. If T is bounded, a closed subspace W of V is G-invariant if and only if it is $L^1(G)$-invariant.

2.4. We say that a representation (T, V) is (topologically) irreducible, if the two trivial spaces (0) and V are the only closed G- invariant subspaces of V.

A Banach module (T, V) of $L^1(G)$ is said to be simple or algebraically irreducible if the trivial spaces (0) and V are the only $L^1(G)$-invariant subspaces of V.

2.5. We say that a representation $(\pi, \mathcal{H})$ is unitary if the Banach space $\mathcal{H}$ is in fact a Hilbert space (with scalar product $\langle , \rangle$) and if $\pi(x)$ is a unitary operator for any $x \in G$. A unitary operator being isometric, every unitary representation of G is bounded and the corresponding representation of $L^1(G)$ has the property that $\pi(\varphi)^* = \pi(\varphi^*)$ for any $\varphi \in L^1(G)$.

2.6. Two representations (T, V) and (T', V') are called equivalent if there exists a bounded linear bijection $u : V \to V'$, which intertwines T and T', i.e. such that

$$T'(x) \circ u = u \circ T(x), \forall x \in G.$$

We write $T \simeq T'$ for two equivalent representations. In particular if $T \simeq T'$, then T is irreducible if and only T' is.

2.7. By Schur's lemma, we know that a unitary representation $(\pi, \mathcal{H})$ is irreducible if and only if every bounded operator $a \in L(\mathcal{H})$, which commutes with π, i.e. for which $\pi(x) \circ a = a \circ \pi(x)$ for every $x \in G$, is a multiple of the identity operator $\mathbb{I}_{\mathcal{H}}$. Hence for two equivalent irreducible unitary representations $(\pi, \mathcal{H})$ and $(\pi', \mathcal{H}')$ there exists a unique (up to scalar multiple) interwining operator $u : \mathcal{H} \to \mathcal{H}'$, which is even unitary .

We write $[\pi]$ for the equivalence class of the representation π, i.e. for the set $\{(\pi', \mathcal{H}'), \pi \simeq \pi'\}$.

We denote by $\widehat{G}$ the family of all the equivalence classes of irreducible unitary representations of G.

By the theorem of Gelfand-Naimark, the irreducible unitary representations separate the points of G (see [Di.4]).

2.8. In 1931 Stone and von Neumann determined the unitary dual of the Heisenberg group. In the late fourties Mackey proved his imprimitivity theorem, the fundamental tool to compute irreducible unitary representations in the solvable case. Dixmier proved in 1957 (see [Di.5]), that every irreducible unitary representation of a connected nilpotent Lie group is monomial, i.e. is induced from a unitary character. The breakthrough came with Kirillov's orbit picture of the dual space of nilpotent Lie groups in 1962 (see [Ki.]). Kirillov's orbit method also works for exponential groups. Bernat, Pukanszky and Vergne determined the dual space of these groups in the years 1965-1970 with the orbit method (see [Ber.], [Puk.1,2], [Ve.1,2,3]).

2.9. The irreducible representations of exponential groups are induced from characters. Let us describe briefly induced representations. Let H be a closed subgroup of the group G and let $(\rho, \mathcal{F})$ be a unitary representation of H. We realize the induced representation $\tau = \tau_\rho$ of ρ by left translation on a space of mappings $\mathcal{E}(\rho)$ from G into $\mathcal{H}$. The space $\mathcal{E}(\rho)$ is the space

$$\mathcal{E}(\rho) = \{\xi : G \to \mathcal{F}; \xi \text{ continuous with compact support modulo } H,$$

$$\xi(xh) = (\frac{\Delta_H(h)}{\Delta_G(h)})^{1/2}\rho(h)^{-1}\xi(x), x \in G, h \in H\}.$$

This space of mappings is left translation invariant and we observe that for $\xi \in \mathcal{E}(\rho)$, the function $x \to \|\xi(x)\|^2$ is contained in $\mathcal{E}(G, H)$. Hence the scalar product

$$(\xi, \eta) \to \langle \xi, \eta \rangle_{\mathcal{H}} = \oint_{G/H} \langle \xi(x), \eta(x) \rangle_{\mathcal{F}} dx$$

is G-invariant, positive and hermitian and so left translation is isometric on the prehilbert space $(\mathcal{E}(\rho), \langle , \rangle)$. The completion $\mathcal{H}$ of the space $\mathcal{E}(\rho)$ with respect to the norm $\|\cdot\|_{\mathcal{H}}$ is a Hilbert space on which the group G acts by left translation, i.e.

$$\tau(x)\xi(s) = \xi(x^{-1}s), x, s \in G, \xi \in \mathcal{H}.$$

We take now the special case where ρ is a unitary character of H. Then $\mathcal{H}$ is a space of complex valued functions and we see that the operators $\tau(\varphi), \varphi \in C_c(G)$, are kernel operators with continuous kernels. Indeed, for $\xi \in \mathcal{E}(\rho)$,

$$\tau(\varphi)\xi(s) = \int_G \varphi(x)\xi(x^{-1}s)dx = \int_G \varphi(sx^{-1})\Delta_G(x)^{-1}\xi(x)dx$$

$$= \oint_{G/H}\int_H \varphi(sh^{-1}x^{-1})\Delta_G(xh)^{-1}\frac{\Delta_G(h)}{\Delta_H(h)}(\frac{\Delta_H(h)}{\Delta_G(h)})^{1/2}\overline{\chi(h)}\xi(x)dhdx$$

$$= \oint_{G/H} \Delta_G(x)^{-1}\left(\int_H \varphi(shx^{-1})(\frac{\Delta_G(h)}{\Delta_H(h)})^{1/2}\chi(h)dh\right)\xi(x)dx.$$

Hence the kernel $\varphi_{H,\chi}$ of the operator $\tau(\varphi)$ is the function

$$(s, x) \to \Delta_G(x)^{-1}\int_H \varphi(shx^{-1})\chi(h)(\frac{\Delta_G(h)}{\Delta_H(h)})^{1/2}dh.$$

2.10. Let $H = \exp(\mathfrak{h})$ be a closed connected subgroup of G. Every unitary character χ of H is of the form

$$\chi(\exp(T)) = \chi_f(\exp(T)) = e^{-if(T)}, T \in \mathfrak{h},$$

where f is a real linear functional on $\mathfrak{g}$, such that

$$f([\mathfrak{h}, \mathfrak{h}]) = (0).$$

We remark that for every $t \in G$, the representations $\tau_{H,\chi}$ and $\tau_{tHt^{-1},{}^t\chi}$ are equivalent. Here ${}^t\chi$ is the unitary character of the group tHt^{-1} defined by ${}^t\chi(p) = \chi(t^{-1}pt), p \in tHt^{-1}$. An intertwining operator u between these two representations is given by right translation:

$$u(\xi)(s) = \xi(st), \quad \xi \in \mathcal{E}(\chi), s \in G.$$

We define the coadjoint representation Ad^* of G on the dual vector space $\mathfrak{g}^*$ of $\mathfrak{g}$ by:

$$Ad^*(x)f(U) = f(Ad(x^{-1})U), U \in \mathfrak{g}, x \in G, f \in \mathfrak{g}^*.$$

Hence the induced representations τ_{H,χ_f} and $\tau_{tHt^{-1},\chi_{Ad^*(t)f}}$ are equivalent, since $\chi_{Ad^*(t)f} = {}^t\chi, t \in G$.

2.11. A subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is called a polarisation at $f \in \mathfrak{g}^*$, if $\mathfrak{p}$ is subordinated to f (i.e. if $f([\mathfrak{p}, \mathfrak{p}]) = (0)$), and if $\mathfrak{p}$ has maximal dimension with this property. This maximal dimension is equal to $\frac{1}{2}(\dim \mathfrak{g} + \dim \mathfrak{g}(f))$. Here $\mathfrak{g}(f)$ denotes the stabilizer of f in $\mathfrak{g}$, i.e. $\mathfrak{g}(f) = \{U \in \mathfrak{g}; f([U, \mathfrak{g}]) = (0)\}$. For a polarisation $\mathfrak{p}$ at f we always have that $Ad^*(H)f$ is open in $f + \mathfrak{p}^\perp$. We say that $\mathfrak{p}$ is a Pukanszky polarisation, if $Ad^*(H)f = f + \mathfrak{p}^\perp$.

2.12. We can now describe the unitary dual of an exponential group G. The theory of Kirillov-Bernat-Vergne-Pukanszky says that the induced representation τ_{H,χ_f} is irreducible if and only if $\mathfrak{h}$ is a Pukanszky polarisation at f. Furthermore, given $f \in \mathfrak{g}^*$, there always exists a Pukanszky polarisation $\mathfrak{p}$ at f and for two Pukanszky polarisations $\mathfrak{p}$, resp. $\mathfrak{p}'$ at f, resp. at f', the representations τ_{P,χ_f} and $\tau_{P',\chi_{f'}}$ are unitarily equivalent, if and only if the coadjoint orbits of f and f' are the same. Finally, by Mackey's imprimitivity theorem, every irreducible unitary representation π of G is equivalent to some induced representation τ_{P,χ_f}. We obtain in this way a bijection (the orbit picture) between the space of the coadjoint orbits $\mathfrak{g}^*/G$ and the dual space of G:

$$\mathcal{K} : \mathfrak{g}^*/G \to \widehat{G}, \quad Ad^*(G)f \to [\tau_{P,\chi_f}], (P = \exp(\mathfrak{p}), \mathfrak{p} \text{ any Pukanszky polarisation at } f).$$

2.13. We can construct Pukanszky polarisations at $f \in \mathfrak{g}^*$ in the following way. Let as before $\mathfrak{n}$ denote the nilradical or any nilpotent ideal of $\mathfrak{g}$ containing $[\mathfrak{g}, \mathfrak{g}]$. Take a J.H. sequence $(\mathfrak{a}_j)_{j=s}^m$ for the action of $\mathfrak{g}$ on $\mathfrak{n}$ and let $\mathfrak{g}(q)$ be the stabilizer of $q = f_{|\mathfrak{n}}$ in $\mathfrak{g}$. The subspace

$$\mathfrak{p}_0 = \sum_{j=m}^{s} \mathfrak{a}_j(f_{|\mathfrak{a}_j})$$

is then a polarisation at q in $\mathfrak{n}$ (see [Ve.1,2]. The stabilizer $\mathfrak{g}(q)$ of q in $\mathfrak{g}$ is a subalgebra of $\mathfrak{g}$ containing $\mathfrak{g}(f)$ and the quotient algebra $\mathfrak{g}(q)/\ker(f)\cap\mathfrak{n}(q)$ is either abelian or isomorphic to a Heisenberg algebra. Furthermore we have that $[\mathfrak{g}(q),\mathfrak{p}_0]\subset\mathfrak{p}_0$. Let $\mathfrak{p}_1$ be a polarisation at $f_{|\mathfrak{g}(q)}$. Then $\mathfrak{p}=\mathfrak{p}_1+\mathfrak{p}_0$ is a Pukanszky polarisation at f (see also [Le.Lu]).

The Heisenberg algebra

$$\mathfrak{h}_n = \text{ span } \{X_1\cdots,X_n,Y_1,\cdots,Y_n,Z\},(n\in\mathbb{N})$$

has the bracket relations:

$$[X_i,Y_j]=\delta_{i,j}Z,[X_i,X_j]=[Y_i,Y_j]=0=[U,Z],\quad 1\leq i,j\leq n,U\in\mathfrak{h}_n.$$

For a linear functional f on $\mathfrak{h}_n$, we see that the stabilizer $\mathfrak{h}_n(f)$ at f is equal to $\mathfrak{h}_n$ if $f(Z)=0$ and $\mathfrak{h}_n(f)=\mathbb{R}Z$ if $f(Z)\neq 0$. In the latter case we have many polarisations. For instance the subspaces $\text{span}\{X_1+\alpha_1Y_1,\cdots,X_n+\alpha_nY_n,Z\}$, where $\alpha_1,\cdots,\alpha_n$ are any real numbers, give us an infinity of polarisations at f.

2.14. The irreducible representations $\pi=\tau_{p,\chi_f}$ of an exponential group G have the following property. The subspace $\mathcal{H}^1$ of all the vectors ξ in the space $\mathcal{H}$ of π, for which there exists an element $\varphi=\varphi_\xi\in L^1(G)$, such that the operator $\pi(\varphi)$ is the orthogonal projection P_ξ onto $\mathbb{C}\xi$ is different from (0), and hence is dense in $\mathcal{H}$ since π is irreducible. There exist even non zero elements ξ in $\mathcal{H}^1$, such that φ_ξ is rapidly decreasing, which means that $\nu\varphi_\xi$ is also in $L^1(G)$ for every real character ν of G. This was proved by Howe (see [Ho.]) in the nilpotent case, by Ludwig (see [Lu.2]) and by Poguntke (see [Po.1]) in the exponential case.

3. Algebraically and topologically irreducible Representations.

3.1. Let A be a Banach algebra and (T,V) an algebraically irreducible A-module. For any $v\in V, v\neq 0$, the annihilator $A_v=\{a\in A;T(a)v=0\}$ is a maximal modular left ideal, which is automatically closed, and so the representation (T,V) is equivalent to the left module $(\lambda,A/A_v)$. In particular (T,V) is a Banach module of A. (see [Bo. Du.])

3.2. Let now (T,V) be a topologically irreducible representation of A. We can again fix a non zero vector of V and consider the annihilator A_v of v in A, which is a closed left ideal. We have an injection

$$i:A/A_v\to V, i(a \bmod A_v)=T(a)v,$$

and the image of the mapping i is dense in V since T is irreducible. We transfer the norm $\|\cdot\|_V$of V to the space A/A_v via i and so we can replace the Banach space V by the completion of A/A_v and realize T by left translation on the space A/A_v and on its completion. In this way, the module (T,V) is determined by the closed left ideal A_v and a certain module norm $\|\cdot\|$ on A/A_v which satisfies the following inequality:

$$\|ab \bmod A_v\|\leq\|a\|_A\|b \bmod A_v\|, a,b\in A.$$

3.3. Let A^f be the ideal in A, consisting of all the a's in A, such that $T(a)$ is an operator of finite rank. Suppose that $A^f\neq(0)$. Then the submodule $V^1=$ span $\{T(a)v,a\in A^f,v\in V\}$ is dense in V and defines a simple A-module.

3.4. The simple $L^1(G)$-modules in the nilpotent case have been determined by Dixmier (see [Di.1]), Leptin (see [Le.2]), Poguntke (see [Po.4]), Jenkins (see [Je.]) and Ludwig (see [Lu.3]) from 70 to 77 and Leptin and Poguntke studied the exponential case in some papers from 76-81 (see for instance [Le.Po.]) and finally Poguntke (see [Po.2]) gave a complete description of these modules in 1983. It turns out that every simple $L^1(G)$-module is of the form (T, V^1) for some topologically irreducible Banach representation (T, V) of $L^1(G)$. We will describe them in (3.14).

3.5. Let us analyse such a topologically irreducible $L^1(G)$-module (T, V), for an exponential group G. Then T is also a G-irreducible module and we can restrict T to the nilradical $N = \exp(\mathfrak{n})$ of G. The group G acts on N by conjugation and so also on the functions of N and in particular on the elements of $L^1(N)$. Whence an ideal $I \subset L^1(N)$ is G-invariant if for every $\varphi \in I$ the function

$$n \mapsto \Delta_G(t)\varphi(t^{-1}nt) = {}^t\varphi(n), n \in N,$$

is also in I for every $t \in G$. The restriction of T to N is no longer irreducible, but the kernel $\ker_{L^1(N)}(T)$ of T in $L^1(N)$ is a closed G-prime ideal. A G- prime ideal I in $L^1(N)$ is by definition a twosided G-invariant ideal, which has the property that for every pair I_1, I_2 of twosided G-invariant ideals in $L^1(N)$, such that $I_1 * I_2 \subset I$, necessarily one of the two ideals I_1 and I_2 is contained in I. It has been shown by Molitor-Braun in 1996 (see [Mo.1] and [Lu.Mo.3]), that every closed G-prime ideal I in $L^1(N)$ is the kernel of a G-orbit in $\widehat{N}$, i.e.

$$I = \bigcap_{t \in G} \ker_{L^1(N)}({}^t\tau) = \ker({}^G\tau)$$

for some $\tau \in \widehat{N}$. The representation τ of N is associated to its Kirillov-orbit $Ad^*(N)q$ for some $q \in \mathfrak{n}^*$. Let $f \in \mathfrak{g}^*$ be an extension of q. We take a subspace $\mathfrak{t}$ of $\mathfrak{g}(f)$, such that $\mathfrak{g}(f) = \mathfrak{t} \oplus (\mathfrak{g}(f) \cap \mathfrak{n})$. Let $\mathfrak{h}$ be a subspace of $\mathfrak{g}$ containing $\mathfrak{n}$, such that $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{h}$. Then $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{n} \subset \mathfrak{h}$ and so $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. Let $p = f_{|\mathfrak{h}} \in \mathfrak{h}^*$. Let us choose a Pukanszky polarisation $\mathfrak{p}$ at f, such that $\mathfrak{p}_0 = \mathfrak{p} \cap \mathfrak{n}$ is a polarisation at q as in (2.13). Then $\mathfrak{p} \cap \mathfrak{h}$ is a Pukansky polarisation at p and the restriction of the representation $\pi = \tau_{P,\chi_f}$ of G to $H = \exp(\mathfrak{h})$ is irreducible and equivalent to $\sigma = \tau_{P\cap H,\chi_p}$. Our choice of $\mathfrak{h}$ implies that the H-orbit of p is saturated with respect to $\mathfrak{n}$, i.e. $Ad^*(H)p + \mathfrak{n}^{\perp} = Ad^*(H)p$. As a consequence, (see [Ha.Lu] and [Lu.Mo.3]),

$$\ker_{L^1(H)}(\sigma) = \ker_{L^1(H)}(T).$$

Hence the representation T annihilates the twosided ideal

$$I_T = \overline{\text{span } (L^1(G) * \ker_{L^1(H)}(\sigma))} = \overline{\text{span } (L^1(G) * \ker_{L^1(N)}(\tau))}$$

of $L^1(G)$ (here $\overline{(--)}$ denotes closure in $L^1(G)$). Thus the representation T factorizes through I_T and defines an irreducible representation $\tilde{T}$ of $A = L^1(G)/I_T$. The algebra A is itself a generalized L^1-algebra. As Banach space A is isometrically isomorhic to $L^1(\mathcal{T}, (L^1(H)/\ker_{L^1(H)}(\sigma)))$, where

$$\mathcal{T} = \exp(\mathfrak{t}) \simeq G(f)/G(f) \cap N \simeq G/H$$

and the algebra $L^1(H)$ acts by convolution on the right and on the left on A and so A has many idempotent multipliers (see [Po.2]). Indeed, we can choose exponentially decreasing elements $\varphi = \varphi_\lambda$ in $L^1(H)$, such that $\sigma(\varphi)$ is the orthogonal projector P_λ onto $\mathbb{C}\lambda$. Hence $\alpha \mapsto \varphi * \alpha \mod I_T$ defines an idempotent multiplier on A, since $\varphi * \varphi = \varphi$ modulo $\ker_{L^1(H)}(T)$. We take for every $t \in \mathcal{T} \subset G(f)$ the element $v(t) \in L^1(H)/\ker_{L^1(H)}(\sigma)$ for which $\sigma(v(t)) = \pi(t)^{-1} \circ P_\lambda$. The norm $\omega(t)$ of $v(t)$ in the quotient space $L^1(H)/\ker_{L^1(H)}(\sigma)$ is a measurable submultiplicative function which is constant on $G(f) \cap N$ and defines a weight on $G(f)/G(f) \cap N$. It follows from this that the subspace $B = B_\varphi = \varphi * A * \varphi$ is a closed subalgebra of A. Furthermore we have for $a \in A$ that

$$\varphi * a * \varphi(t) = h(t)v(t) \in L^1(H)/\ker_{L^1(H)}(\sigma), t \in \mathcal{T},$$

where $t \mapsto h(t)$ is a measurable function defined on $\mathcal{T}$ and in fact on $G(f)/G(f) \cap N$, such that

$$\|\varphi * a * \varphi\|_A = \int_{\mathcal{T}} |h(t)| \|v(t)\| dt.$$

It turns out that the mapping $\varphi * a * \varphi \mapsto h$ is even an isometric isomorphism of the algebra B onto the weighted convolution Banach algebra $L^1(G(f)/G(f) \cap N, \omega)$ (see [Po.2]). Since $G(f)/G(f) \cap N$ is commutative, it follows that B itself is commutative. Let now $W = T(\varphi_\lambda)V \subset V$. Since $T(\varphi_\lambda)$ is a projector we have that W is a closed subspace of V and W is an irreducible B-submodule of V. Let us denote by S the restriction of $\tilde{T}$ to W.

3.7. If T is algebraically irreducible, then (S, W) is also a simple B-module and B being abelian, it follows that W is one dimensional and S is a character of the algebra B, which we denote by χ_ν. We can describe this character by a linear form (denoted by ν) on $\mathfrak{g}(f)$:

$$\chi_\nu(\varphi * a * \varphi) = \int_{G(f)/G(f) \cap N} h(t) e^{-i\nu(\log(t))} dt, a \in A.$$

3.8. If T is only topologically irreducible, the space W need not be one dimensional. The commutative algebra $L^1(G(f)/G(f) \cap N, \omega)$ has infinite dimensional irreducible representations, if the weight ω is exponential. It suffices in that case for instance to take a real linear functional ν on $\mathfrak{g}(f)/\mathfrak{g}(f) \cap \mathfrak{n}$, such that $e^{\nu(T)} \leq \omega(\log(T)), T \in \mathfrak{g}(f)$, and to choose any infinite dimensional Banach space W, which admits a bounded operator u, which has no closed invariant subspaces except the trivial ones (see [Be.]). The representation S defined by

$$S(\varphi * a * \varphi) = \int_{G(f)/G(f) \cap N} h(t) e^{-\nu(\log(t)) u} dt, a \in A,$$

is then irreducible on W.

3.9. Conversely, every irreducible Banach space representation (S, W) of the algebra B allows us to define a family of topologically irreducible representations of G in the following way. Choose a non-zero vector $w \in W$ and let

$$B_w = \{b \in B; S(b)w = 0\}, \quad A_w = \{a \in A, \quad S(\varphi * b * a * \varphi)w = 0, \forall b \in A.\}$$

Define the function $\|\cdot\|_{min}$ on A/A_w by

$$\|a \bmod A_w\|_{min} = \inf_{\|b\|_A=1} \|S(\varphi * b * a * \varphi)w\|_W, a \in A.$$

It turns out that $\|\cdot\|_{min}$ is a norm on A/A_w for which

$$\|\psi * a \bmod A_w\|_{min} \leq \|\psi\|_1 \|a \bmod A_w\|_{min}, a \in A, \psi \in L^1(G).$$

Furthermore the restriction of $\|\cdot\|_{min}$ to $(B+A_w)/A_w \simeq B/B_w$ is equivalent to the norm $b \mapsto \|S(b)w\|_W$ of B. Hence we obtain a Banach space V^{min}, the completion of A/A_w with respect to $\|\cdot\|_{min}$ of A, such that convolution on the left on A/A_w extends to a bounded representation T^{min} of $L^1(G)$ on V^{min}. Furthermore the subspace $W^{min} = T^{min}(\varphi_\lambda)V^{min}$ is isomorphic to W and the representation S^{min} of B is equivalent to the representation (S, W). We say that (T^{min}, V^{min}) is an extension of (S, W). It is easy to show that T^{min} is even irreducible (see [Lu.Mo.3]).

3.10. There may be other extensions. For instance if S is character of B, then we may take as extension norm the quotient norm on A/A_w, since now B/B_w is one dimensional. The left ideal A_w is now modular and a modular left unit is given by any element of B, on which S has the value 1. It is not difficult to see that A_w is even maximal and so A/A_w is an algebraically irreducible submodule of the module V^{min}. We see also that two simple modules $\tilde{T}$ and $\tilde{T}'$ of A are equivalent, if and only if the corresponding characters of the algebra B coincide (see [Po.2]).

3.11. We say that a norm $\|\cdot\|$ on A/A_w is an extension norm, if

$$\|\psi * a \bmod A_w\| \leq C_{\|\cdot\|} \|\psi\|_1 \|a \bmod A_w\|$$

for any $a \in A$ and $\psi \in L^1(G)$ (for some constant $C_{\|\cdot\|}$) and if the restriction of $\|\cdot\|$ to $B/B_w \simeq (B+A_w)/A_w$ is equivalent to the norm $b \mapsto \|S(b)w\|$ of B. It turns out that every extension norm $\|\cdot\|$ dominates the minimal norm, i.e. we have that $\|a\|_{min} \leq C\|a\|, a \in A$, (for some constant C) and that the completion of A/A_w with respect to the norm $\|\cdot\|$, considered as a subspace of the Banach space V^{min}, is also an irreducible $L^1(G)$ module. Hence there are as many equivalence classes of irreducible extensions of a given (S, W) module as there are equivalence classes of extension norms (see [Lu.Mo.3]).

3.12. In the case where S is a character, there are in general an infinity of such extensions. For instance, if G is nilpotent every closed prime ideal I of $L^1(G)$ is the kernel of an element π of $\widehat{G}$. Hence every irreducible bounded irreducible module (T, V) with $\ker_{L^1(G)}(T) = \ker_{L^1(G)}(\pi)$ contains as simple submodule a copy of $(\pi, \mathcal{H}^1)$. Let us realise π as $\mathrm{ind}_P^G \chi_f$ for a polarisation $P = \exp(\mathfrak{p})$ at f. Instead of taking the Hilbert space $\mathcal{H}$ we may take the Banach spaces

$$L^p(G/P, \chi_f) = \{\xi : G \to \mathbb{C}; \xi \text{ measurable }, \xi(xp) = \chi_f(p)^{-1}\xi(x), x \in G, p \in P,$$
$$\int_{G/H} |\xi(x)|^p dx = \|\xi\|_p^p < \infty, \}$$

$(1 \leq p < \infty)$. For $p = \infty$, we can take the space

$$C_\infty(G/P, \chi_f) = \{\xi : G \to \mathbb{C}; \xi(xp) = \chi_f(p)^{-1}\xi(x), x \in G, p \in P,$$
$$\xi \text{ continuous, tending to 0 at } \infty\}.$$

The group G acts by left translation on all these spaces and we write $\tau_{(P,\chi_f,p)}$ for these representations. Since the spaces $L^p(G/P, \chi_f)$ are not isomorphic, the representations $\tau_{(P,\chi_f,p)}$ cannot be equivalent. The operators $\tau_{(P,\chi_f,p)}(\varphi), \varphi \in L^1(G)$, are kernel operators whose kernels $\varphi_{(p,\chi_f,p)}$ do not depend on p. In fact

$$\varphi_{(P,\chi_f,p)}(u,v) = \int_P \varphi(upv^{-1})\chi_f(p)dp = \varphi_{P,\chi_f,2}(u,v), u, v \in G,$$

and so $\ker_{L^1(G)}(\tau_{(P,\chi_f,p)}) = \ker_{L^1(G)}(\pi)$ and the representations $\tau_{(P,\chi_f,p)}$ are irreducible and all contained in the corresponding V^{min}.

3.13. Let us sum up what has been said above. For every G-orbite $Ad^*(G)q$ in $\mathfrak{n}^*$, we have the commutative subalgebras $B_{\varphi_\lambda} \simeq \varphi_\lambda * L^1(G)/\overline{L^1(G) * \ker_{L^1(N)}(\tau_q)} * \varphi_\lambda$ which are all isomorhic to $L^1(\mathcal{T}, \omega) \simeq L^1(G(l)/G(l) \cap N, \omega)$, for some weight independent of λ. Having fixed one of the φ_λ's, every irreducible bounded module (T, V) defines an irreducible bounded module (S, W) of B, where for $h \in L^1(\mathcal{T}, \omega)$,

$$S(h) = \int_{\mathcal{T}} h(t)T(t)T(v_\lambda(t))dt.$$

The representations (T, V) and (T', V') are equivalent if and only if their $Ad^*(G)$ orbits in $\mathfrak{n}^*$ coincide, if the modules (S, W) and (S', W') are equivalent and if the extension norms on $A/A_w = A/A'_w$ are equivalent.

3.14. Let us finish this exposition with a characterisation of the simple modules of $L^1(G)$. We have seen that every simple module is determined by its orbit $Ad^*(G)q$ in $\mathfrak{n}^*$ and a character $\chi_T = \chi_\nu$ of $B = L^1(G(l)/G(l) \cap N, \omega) \simeq L^1(\mathcal{T}, \omega)$.

Poguntke has given a description of the weight ω (see [Po.2]). Choose a J.H. sequence $(\mathfrak{b}_j)_{j=1}^m$ of the $\mathfrak{g}(f)$-module $\mathfrak{n}/\mathfrak{p}_0$, where $\mathfrak{p}_0$ is a $\mathfrak{g}(f)$-invariant polarisation of q (see 2.13). Let for $T \in \mathfrak{g}(f)$,

$$\mu(T) = \mu_q(T) = \frac{1}{2}\sum_{j=1}^m |tr\, ad_{\mathfrak{b}_j/\mathfrak{b}_{j+1}}(T)|.$$

Then the weight ω satisfies the following inequalities:

$$e^{\mu(T)} \leq \omega(exp(T)) \leq e^{\mu(T)}R(T), T \in \mathfrak{g}(f),$$

for some polynomially bounded expression R of T. Hence the characters χ_ν of B are of the following form:

$$\chi_\nu(h) = \int_{\mathcal{T}} h(t)e^{-i(\nu(log(t)))}dt, h \in L^1(\mathcal{T}, \omega),$$

where ν is any complex linear functional of $\mathfrak{g}(f)$, for which $|Im(\nu)| \leq \mu$. We see thus that B has exponentially increasing characters, if and only if one of the modules $\mathfrak{b}_j/\mathfrak{b}_{j+1}$ is not trivial. In that case the group G is not *-regular in the sense of Boïdol (see [Boi.]).

3.15. We shall show now that for a simple module (T, V) of $L^1(G)$, there exists a topologically irreducible module $(T_{\overline{p}}, V_{\overline{p}})$ of G such that (T, V) is equivalent to $(T_{\overline{p}}, V_{\overline{p}}^1)$. Let $q \in \mathfrak{n}^*$ and let $f \in \mathfrak{g}^*$ be an extension of q. Let $\mathfrak{b} = \mathfrak{g}(q) + \mathfrak{n}$, which is an ideal of $\mathfrak{g}$ and which contains our Pukanszky polarisation $\mathfrak{p} = \mathfrak{p}_1 + \mathfrak{p}_0$ at f.

We choose a J.H. sequence

$$\mathfrak{n} = \mathfrak{a}_s \supset \cdots \supset \mathfrak{a}_m \supset \mathfrak{a}_{m+1} = \mathfrak{p}_0$$

of the $\mathfrak{b}$-module $\mathfrak{n}/\mathfrak{p}_0$. Let $\mathfrak{x}$ be a subspace of $\mathfrak{p}$ such that $\mathfrak{b} = \mathfrak{x} \oplus (\mathfrak{p} + \mathfrak{n})$ and let $\mathfrak{s}$ be a subspace of $\mathfrak{g}$ such that $\mathfrak{s} \oplus \mathfrak{b} = \mathfrak{g}$. Let us also choose for every j a subspace $\mathfrak{w}_j$ of $\mathfrak{a}_j$ such that $\mathfrak{a}_j + \mathfrak{p}_0 = \mathfrak{w}_j \oplus (\mathfrak{a}_{j+1} + \mathfrak{p}_0)$. We let $\overline{p} = (p_1, \cdots, p_m) \in [1, \infty]^m$ and for $T \in \mathfrak{g}(q)$ we set

$$\delta_{\overline{p}}(T) = \sum_{j=1}^{m} \frac{tr(ad(T))_{\mathfrak{a}_j/\mathfrak{a}_{j+1}}}{p_j}.$$

Let $\Delta_{\overline{p}}(\exp(T)) = e^{\delta_{\overline{p}}(T)}, T \in \mathfrak{p}$, and let

$$L^{\overline{p}}(G/P, \chi_f) = \{\xi : G \to \mathbb{C}; \xi \text{ measurable }, \xi(xp) = \Delta_{\overline{p}}(h)\chi_f(p)^{-1}\xi(x), x \in G, p \in P,$$

$$\|\xi\|_{\overline{p}} = \Big(\int_{\mathfrak{x}}\Big(\int_{\mathfrak{s}}\Big(\int_{\mathfrak{w}_1}\Big(\cdots\Big(\int_{\mathfrak{w}_m} |\xi(\exp(S)\exp(X)\exp(U_1)\cdots\exp(U_m))|^{p_m} dU_m\Big)^{\frac{1}{p_m}}$$
$$\cdots\Big)^{p_1} dU_1\Big)^{\frac{1}{p_1}}\Big)^2 dX dS\Big)^{\frac{1}{2}} < \infty\}.$$

It is easy to verify that this norm $\|\cdot\|_{\overline{p}}$ is translation invariant and that for $\overline{p} = (2, \cdots, 2) = \overline{2}$, we obtain the Hilbert space of the induced representation $\mathrm{ind}_P^G \chi_f$. Left translation defines thus an isometric representation denoted by $\tau_{(P,\chi_f,\overline{p})}$ on $L^{\overline{p}(G/P,\chi_f)}$. For every $\varphi \in L^1(G)$, the operator $\tau_{(P,\chi_f,\overline{p})}(\varphi)$ is a kernel operator, whose kernel $\varphi_{(P,\chi_f,\overline{p})}$ is equal to the kernel of the operator $\tau_{H,\chi_f}(\Delta_{\overline{p}}\Delta_{\overline{2}}^{-1}\varphi)$, if φ is exponentially decreasing. This observation tells us that $\tau_{(P,\chi_f,\overline{p})}$ is irreducible and that there exist many $\varphi \in L^1(G)$, for which $\tau_{(P,\chi_f,\overline{p})}(\varphi)$ is of rank one. The character $\chi_{\nu_{f,\overline{p}}}$ of the commutative algebra B defined by the simple module $(\tau_{(P,\chi_f,\overline{p})}, L^{\overline{p}}(G/P, \chi_f)^1)$ is given by

$$\chi_{\nu_{f,\overline{p}}}(h) = \int_F h(t) e^{\sum_{j=1}^{m}(\frac{1}{p_j}-\frac{1}{2}) tr ad_{\mathfrak{a}_j/\mathfrak{a}_{j+1}}(\log t)} dt.$$

It turns out that every real linear functional $\nu = \nu_T$ on $\mathfrak{g}(f)$, for which $|\nu(T)| \leq \mu_q(T), T \in \mathfrak{g}(f)$, is of the form

$$\nu = \sum_{j=1}^{m}(\frac{1}{p_j} - \frac{1}{2}) tr\, ad_{\mathfrak{a}_j/\mathfrak{a}_{j+1}}$$

for some $(p_1, \cdots, p_m)$. This shows that any simple module (T, V) of $L^1(G)$ is equivalent to

$$(T_{\overline{p}}, V_{\overline{p}}) = (\tau_{(P,\chi_f,\overline{p})}, L^{\overline{p}}(G/H, \chi_f)^1)$$

for some $f \in \mathfrak{g}^*$ and some $\overline{p}$.

We obtain finally the following description of the space $\widetilde{G}$ of the equivalence classes of simple $L^1(G)$ modules.

Let $\mathfrak{g}^*_{prim}$ be the collection of all pairs $(f, \nu) \in \mathfrak{g}^* \times \mathfrak{g}(f)^*$, such that $|\nu| \leq \mu_{f_{|\mathfrak{n}}}$. The group G acts on $\mathfrak{g}^*_{prim}$ by Ad^*. Let $\mathfrak{g}^*_p/G$ be the corresponding quotient space. The mapping

$$\mathfrak{g}^*_{prim}/G \to \widetilde{G}, \quad [(f, \nu)] \mapsto [(\tau_{(P,\chi_f,\overline{p})}, L^{\overline{p}}(G/H, \chi_f)^1)], \quad \nu = \sum_{j=1}^{m} (\frac{1}{p_j} - \frac{1}{2}) tr\, ad_{\mathfrak{a}_j/\mathfrak{a}_{j+1}},$$

is a bijection (see [Po.2],[Lu.Mi.Mo.])

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