UNITARY REPRESENTATIONS AND DIFFERENTIAL REPRESENTATIONS OF THE GROUP OF DIFFEOMORPHISMS AND ITS APPLICATIONS

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1. Introduction

Let M be a d-dimensional paracompact C^∞-manifold and Diff(M) be the group of all C^∞-diffeomorphisms on M. Among the subgroups of Diff(M), we take here the group $\mathrm{Diff}_0(M)$ which consists of all $g \in \mathrm{Diff}(M)$ with compact supports, that is the set $\{P \in M |\ \ g(P) \neq P\}$ is relatively compact. Up to the present time, unitary representations $(U, \mathcal{H})$ of $\mathrm{Diff}_0(M)$ or of its subgroups ($\mathcal{H}$ is the representation Hilbert space of U) are constructed and considered by many authors. A purpose of this report is a trial to construct some differential method to analyze these representations $(U, \mathcal{H})$ of $\mathrm{Diff}_0(M)$ or of its subgroups. Roughly speaking, we wish to consider a differential representation of a given one.

So the first step we should do is to define a suitable Lie algebra $\mathcal{G}_0$ of $\mathrm{Diff}_0(M)$, regarding it as an infinite dimensional Lie group. For the case of compact manifold, it is well known for a pretty long time ago that $\mathrm{Diff}(M) = \mathrm{Diff}_0(M)$ is an infinite dimensional Lie group whose modeled space is a nuclear Fréchet space called strong inductive limit of Hilbert spaces by a few authors, especially by H.Omori.(cf.[12]) So after them, we are naturally derived that we should take a set $\Gamma_0(M)$ of all C^∞-vector fields X with compact supports as the Lie algebra $\mathcal{G}_0$, and it is appropriate to take a map $\mathrm{Exp}(X)$ as the exponential map from $\Gamma_0(M)$ to $\mathrm{Diff}_0(M)$, where $\{\mathrm{Exp}(tX)\}_{t\in\mathbf{R}}$ is an integral curve along a vector field $X \in \Gamma_0(M)$.

Thus formally we have self adjoint operators $dU(X)$ on $\mathcal{H}$ by Stone theorem,

$$U(\mathrm{Exp}(tX)) = \exp(\sqrt{-1}t dU(X)) \qquad \text{for all} \quad t \in \mathbf{R},$$

and simaltaneouly there arise many problems for such $dU(X)$ and for $\mathrm{Exp}(X)$. Among them the following questions are fundamental.

(1) Is a common domain of $\{dU(X)\}_{X\in\Gamma_0(M)}$ rich such one like Gårding space ?
(2) Does $\sqrt{-1}dU$ become a linear representation under suitable restriction of the domain of each $dU(X)$?
(3) Is a subgroup generated by $\mathrm{Exp}(X)$, $X \in \Gamma_0(M)$ dense in $\mathrm{Diff}_0(M)$?

It is easily expected that the linearity of $\sqrt{-1}dU$ mostly depends on a formula which is similar with one derived from usual Campbell-Hausdorff formula, listed as the following theorem and actually it was made sure in [19]. (2) is affirmative.

Theorem 1.1. *Let $X, Y \in \Gamma_0(M)$. Then as n tends to $+\infty$,*

(1) $\{\mathrm{Exp}(\frac{tX}{n}) \circ \mathrm{Exp}(\frac{tY}{n})\}^n$ *converges to* $\mathrm{Exp}(t(X+Y))$, *and*

(2) $\{\mathrm{Exp}(-\frac{tX}{\sqrt{n}}) \circ \mathrm{Exp}(-\frac{tY}{\sqrt{n}}) \circ \mathrm{Exp}(\frac{tX}{\sqrt{n}}) \circ \mathrm{Exp}(\frac{tY}{\sqrt{n}})\}^n$ *converges to* $\mathrm{Exp}(-t^2[X,Y])$

in τ_K uniformly on every compact interval of t, respectively, where K is any compact set containig supp$X$ *and* supp$Y$, *and τ_K is a toplogy of uniform convergence on K together with its every derivative.*

Proof. It is carried out by using C^1-hair theory on regular Fréchet group. For details see [13] and [19].

Now the theory of product integral works so effectively on (3). It turns that the above subgroup is dense in the connected component $\mathrm{Diff}_0^*(M)$ of id, where id is the identity map and the topology τ on $\mathrm{Diff}_0(M)$ is the inductive limit topology of $\{\mathrm{Diff}(K), \tau_K\}_{K:\mathrm{cpt}}$, where $\mathrm{Diff}(K) := \{g \in \mathrm{Diff}_0(M) | \ \mathrm{supp} g \subseteq K\}$. It is noteworthy that τ never gives a group topology, unless M is compact (cf. [21], [22]), so we must take care of topological group operations on $\mathrm{Diff}_0(M)$. Nevertheless $\mathrm{Diff}_0^*(M)$ is normal and it is also arcwise connected. In other words, any element in $\mathrm{Diff}_0^*(M)$ is homotopic to the identity as a map, and vice versa.

Theorem 1.2. *A subgroup generated by* $\mathrm{Exp}(X), X \in \Gamma_0(M)$ *is dense in the arcwise connected subgroup* $\mathrm{Diff}_0^*(M)$.

Proof is omitted. (cf.[19])

Now as a direct cosequence of (2) and (3), for example, we have that there is no continuous finite dimensional representations of $\mathrm{Diff}_0^*(M)$ except for a trivial one.

However for almost all parts concerning the questions (2) and (3), I have already reported at several places (cf. [19] and [20]). What I wish to discuss in this paper are problems for the first question. Thus in what follows I will write this report fully placing the focus on the matters for the first question. The last section is briefly devoted to an application of these reults to 1-cocycles.

2. C^∞-VECTORS AND QUASI-INVARIANT MEASURES ON THE GROUP OF DIFFEOMORPHISMS

Now to the first question the following is a partial answer which is a main theorem of this issue.

Main theorem. *Assume that*

(1) *M is a compact Riemannian manifold and*

(2) *$(U, \mathcal{H})$, which is a unitary representation of* $\mathrm{Diff}_0^*(M)$ *at first, has a continuous extension to a larger group* $\mathrm{Diff}^{*K}(M)$, *which consists of all C^K-diffeomorphisms g being homotopic to* id. *Then a set of C^∞-vectors is dense in $\mathcal{H}$.*

Let us show first an idea of the proof and next follow the proof itself. The idea comes from the usual locally compact Lie group theory.

Idea of the proof. For any $h \in \mathcal{H}$, put

$$w_h := \int_{\xi(V)} Q(f)U(f)h\, \mu(df),$$

where $\xi(V)$ is a neighbourhood of id in $\mathrm{Diff}^{*k+\gamma}(M)$ $(0 < \gamma < 1)$ (later, this group will be explained exactly), Q is a non negative function such that

$$\mathrm{supp}Q \subset \xi(V) \quad \text{and} \quad \int_{\xi(V)} Q(f)\mu(df) = 1,$$

and finally μ is a $\mathrm{Diff}^{*k+m}(M)$-quasi-invariant measure on $\mathrm{Diff}^{*k+\gamma}(M)$ which was first considered by Shavgulidze. Of course m must be taken so largely. In the papers [14], [15] and [16], Shavgulidze constructed such a measure. His idea is nice, but there needs some corrections to his proofs. So a definite proof of the existence of such a measure is desired. Now I'll justify it by the following successive 8-steps.

2.1. Construction of quasi-invariant measures on the group of diffeomorphisms.

1-step Let $U \subseteq \mathbf{R}^d$ be an open set and f be a C^k-diffeomorphism defined on U. Take $m, \ell, k \in \mathrm{N}$ such that $3m \leq \ell \leq k$. Shavgulidze defined a map $A_{U,\ell,m}(f)$ for each $h_1, \cdots, h_\ell \in \mathbf{R}^d$ as follows.

$$A_{U,\ell,m}(f)(x)(h_1, \cdots, h_\ell) := \frac{1}{\ell!} \sum_{\sigma \in \mathfrak{S}_\ell} \sum_{i=0}^{m} \alpha_i \partial_{h_{\sigma(1)}} \cdots \partial_{h_{\sigma(i)}} df_x^{-1}(\partial_{h_{\sigma(i+1)}} \cdots \partial_{h_{\sigma(\ell)}} f(x)),$$

where α_i $(i = 0, \cdots, m)$ is a real number which satisfies the following equations,

$$\sum_{i=0}^{m} \alpha_i = 1, \qquad \sum_{i=0}^{m} {}_{\ell-i}C_{p-j}\, {}_iC_j \alpha_i = 0 \quad (0 \leq {}^\forall j < p, \;\; 1 \leq {}^\forall p \leq m).$$

Of course ${}_iC_j$ is the combinatorial number, if $i \geq j$ and it is equal to 0, if $j > i$. Further ∂_h is a directional derivative along h and df_x is a differential of the map f at x. Needless to say, here all tangent spaces are identified with each other. Note that $A_{U,\ell,m}(f)(x)(h_1, \cdots, h_\ell)$ defines a $C^{k-\ell}$-vector field on U for each fixed $h_1, \cdots, h_\ell$.

Theorem 2.1. *If ϕ is a C^{k+m}-diffeomorphism on $f(U)$,*

$$A_{U,\ell,m}(\phi \circ f)(x)(h_1, \cdots, h_\ell) - A_{U,\ell,m}(f)(x)(h_1, \cdots, h_\ell)$$

is a vector field of $C^{k+m-\ell}$-class.

Proof is derived from the usual chain rure and Leibniz formula. (cf. [16])

2-step Let us consider a group $\mathrm{Diff}^{k+\gamma}(M)$ $(k \in \mathrm{N}, \;\; 0 < \gamma < 1)$. The definition is as follows : $g \in \mathrm{Diff}^{k+\gamma}(M)$ if and only if $g \in \mathrm{Diff}^k(M)$ and every derivative of order k is Lipshitz continuous of order γ. Making a parallel definition of the vector field, we obtain a Banach space space $\Gamma^{k+\gamma}(M)$ with the natural norm and a Banach manifold $\mathrm{Diff}^{k+\gamma}(M)$ via a coordinate map ξ on an open neighbourhood U of $0 \in \Gamma^{k+\gamma}(M)$ given by Omori,

$$\xi(u)(x) := \exp_x u(x) \qquad (u \in \Gamma^{k+\gamma}(M)),$$

where $\exp_x u(x)$ is a terminal point of a unit geodisic starting at x along the direction $u(x)$.

3-step In what follows we always assume that

$$3m \leq 2\ell \leq k.$$

According to Shavgulidze, we extend the previous map $A_{U,2\ell.m}$ to a global one as $A_{2\ell,m}$ from $\mathrm{Diff}^{k+\gamma}(M)$ to $\Gamma^{k+\gamma-2\ell}(M)$ such that

$$A_{2\ell,m}(f)(x) = \sum_{i_1=1}^{d} \cdots \sum_{i_\ell=1}^{d} \sum_{i,j=1}^{n} \rho_j(f(x))\rho_i(x)(d\psi_i)_{\psi_i^{-1}(x)} A_{U_i\cap\psi_i^{-1}(f^{-1}(V_j)),2\ell,m}(\psi_j^{-1}\circ f\circ\psi_i)$$
$$(\psi_i^{-1}(x))(h_{i.i_1}, h_{i.i_1}, h_{i.i_2}, h_{i.i_2}, \cdots, h_{i.i_\ell}, h_{i.i_\ell}),$$

where $\{(V_i,\psi_i)\}_{i=1}^n$ is an atras of M, $\{\rho_i\}_{i=1}^n$ is a partition of unity such that $\mathrm{supp}\rho_i \subset V_i$, $U_i := \psi_i^{-1}(V_i)$ and finally $(d\psi_i)_{\psi_i^{-1}(x)}(h_{i,1}), \cdots, (d\psi_i)_{\psi_i^{-1}(x)}(h_{i,d})$ is a linear base in a tangent space $T_x(M)$.

Theorem 2.2. (1) *$A_{2\ell,m}$ is a C^∞-map from $\mathrm{Diff}^{k+\gamma}(M)$ to $\Gamma^{k+\gamma-2\ell}(M)$.*
(2) *$A_{2\ell,m}(\phi\circ f) - A_{2\ell,m}(f) \in \Gamma^{k+m-2\ell}(M)$, whenever $\phi \in \mathrm{Diff}^{k+m}(M)$.*
(3) *Put $L := dA_{2\ell,m}|_{f=\mathrm{id}}$. Then L is a differential operator of elliptic type with C^∞-coefficient on the vector field.*

Proof. It is not hard to see the properties (1) and (2). Let us check the third property. Set

$$L(u) := dA_{2\ell,m}|_{f=\mathrm{id}}(u) \qquad (u \in \Gamma^{k+\gamma}(M)).$$

In a little while let us use notations as below for simplicity.

$$y := \psi_i^{-1}(x), f_t(y) := \psi_j^{-1}\circ\xi(tu)\circ\psi_i(y), \quad U := U_i\cap\psi_i^{-1}(V_j), \quad \text{and} \quad k_s := h_{i,i_s}(1 \leq s \leq \ell).$$

Then it is easy to see that $\left.\dfrac{d}{dt}\right|_{t=0} A_{U,2\ell,m}(f_t)(y)(k_1,k_1,\ldots,k_\ell,k_\ell)$ is a differential operator with respect to u with C^∞-coefficients and the term of order 2ℓ, which is the highest part, is given by

$$\frac{1}{(2\ell)!}\sum_{\sigma\in\mathfrak{S}_{2\ell}}\sum_{s=0}^{m}\alpha_s(df_0^{-1})_y(\partial_{k_{\sigma(1)}}\cdots\partial_{k_{\sigma(2\ell)}}d\psi_j^{-1}(u(x))) = d\psi_i^{-1}\circ d\psi_j(\partial_{k_1}\partial_{k_1}\cdots\partial_{k_\ell}\partial_{k_\ell}d\psi_j^{-1}(u(x))$$

Hence

$$dA_{2\ell,m}(u)(x) = \sum_{i_1=1}^{d}\cdots\sum_{i_\ell=1}^{d}\sum_{i,j=1}^{n}\rho_j(x)\rho_i(x)(d\psi_j)_{\psi_j^{-1}(x)}\partial_{k_1}\partial_{k_1}\cdots\partial_{k_\ell}\partial_{k_\ell}d\psi_j^{-1}(u(x))$$
$$+ \text{ terms of order less than } 2\ell.$$

Now take any $u \in \Gamma^{k+\gamma}(M)$ and $\varphi \in C^\infty(M)$ with properties, $u(x) \neq 0, \varphi(x) = 0$ and $d\varphi(x) \neq 0$. Then it follows from an equality,

$$\partial_{k_1}\partial_{k_1}\cdots\partial_{k_\ell}\partial_{k_\ell}d\psi_j^{-1}((\varphi^{2\ell}u)(x)) = (2\ell)!\prod_{s=1}^{\ell}\{d\varphi_x\circ d\psi_i(k_s)\}^2(d\psi_j^{-1})_x(u(x)),$$

that we have

$$L(\varphi^{2\ell}u)(x) = (2\ell)!\sum_{i=1}^{n}\sum_{i_1=1}^{d}\cdots\sum_{i_\ell=1}^{d}\rho_i(x)\prod_{s=1}^{\ell}\{d\varphi_x\circ d\psi_i(h_{i,i_s})\}^2u(x).$$

The linear independence of $d\psi_i(h_{i,j})$ $(j = 1, \cdots d)$ and the choice of φ lead to that $d\varphi_x \circ d\psi_i(h_{i,i_0}) \neq 0$ for some i_0, and so a term corresponding to $i_1 = i_2 = \cdots = i_\ell = i_0$ is positive. Thus, we get $L(\varphi^{2\ell}u)(x) \neq 0$. □

4-step Generalized Hodge theorem. Let $E_p^{k+\gamma}(M)$ be a collection of all p-forms of class C^k together with all kth derivatives having Lipshitz continuity of order γ, and let L be a differential operator of elliptic type of order ℓ with C^∞-coefficients on the space of p-forms.

Theorem 2.3.

$$E_p^{k+\gamma}(M) = L(E_p^{k+\ell+\gamma}(M)) \oplus \ker L^*$$

$$E_p^{k+\gamma}(M) = L^*(E_p^{k+\ell+\gamma}(M)) \oplus \ker L,$$

where $\oplus$ means an orthgonal decomposition defined by the L^2-norm, in the orientable case, with respect to the volume form on the compact Riemannian manifold M. While in the non orientable case, it is defined by an inner product on $E_p^{k+\gamma}(M)$ defined by

$$< \omega_1, \omega_2 >_M := < \delta\pi\omega_1, \delta\pi\omega_2 >_{\tilde{M}},$$

where $(\tilde{M}, \pi)$ is the the double covering of M, π is a natural projection, and $< \cdot, \cdot >_{\tilde{M}}$ is an inner product which defines the L^2-structure on $\tilde{M}$. Further L^* is a formal adjoint operator of L with respect to these inner products.

Proof. It is derived from theorem 4.1 in p84 in [17] concerning with interior Shauder estimates.

Note that $\ker L$ and $\ker L^*$ have finite dimensions, respectively, so $L(E^{k+\ell+\gamma})$ is also a Banach space with the induced normed topology.

Remark 2.1. *According to an example 4.1 in p 85 in [17], the above theorem is no longer true, even for Laplace-Beltrami operator for the case $\gamma = 0$. This is the reason why the γ-factor is added to the regularity of diffeomorphisms.*

In what follows, I use the above result for the 1-form and identify $E_1^{k+\gamma}(M)$ with $\Gamma^{k+\gamma}(M)$ by the Riemannian metric on M.

5-step This step is devoted to a definition of a fundamental map A. So let

$$\pi_1^{k+\gamma-2\ell} : \Gamma^{k+\gamma-2\ell}(M) \longmapsto L(\Gamma^{k+\gamma}(M)), \qquad \pi_2^{k+\gamma} : \Gamma^{k+\gamma}(M) \longmapsto \ker L$$

be natural projections, respectively, and put

$$Z^{k+\gamma} := L(\Gamma^{k+\gamma}(M)) \times \ker L.$$

Now define $A : \xi(U) \longmapsto Z^{k+\gamma}$ by

$$A(f) := (\pi_1^{k+\gamma-2\ell}(A_{2\ell,m}(f)), \pi_2^{k+\gamma}(\xi^{-1}(f))).$$

Theorem 2.4. *There exists a neighbourhood $U_1 (\subseteq U)$ of 0 such that A is a C^∞-diffeomorphism from $\xi(U_1)$ to $Z^{k+\gamma}$.*

Proof. It is straightforward to check that $dA|_{f=\mathrm{id}}(u) = (Lu, \pi_2^{k+\gamma}(u))$, and that it is a continuous bijection from $\Gamma^{k+\gamma}(M)$ to $Z^{k+\gamma}$. So the inverse function theorem on Banach manifold assures its validity. □

Of course we may assume that the relations

$$\xi(U_1)\xi(U_1) \subseteq \xi(U), \qquad \xi(U_1)^{-1} = \xi(U_1)$$

holds good, if necessary, taking a sufficiently small neighbourhood of 0.

6-step Here we make preparations from a category of Sobolev spaces. Put $d* = \left[\frac{d}{2}\right] + 1$ and $m = 3d^* + 2$. So the relation of m, ℓ and k now becomes,

$$9d^* + 6 = 3m \leq 2\ell \leq k.$$

Consider a Sobolev space $H^s(M)$ of all vector fields with square summable derivatives of order less than or equal to s equipped with the natural Hilbertian norm. Then we have

$$\Gamma^{k+3d^*+1-2\ell}(M) \subset H^{k+3d^*+1-2\ell}(M) \subset H^{k+d^*+1-2\ell}(M) \subset \Gamma^{k+1-2\ell}(M) \subset \Gamma^{k+\gamma-2\ell}(M),$$

where the second inclusion map is nuclear and the third one is actually imbedding due to the choice of d^*. Next let us put

$$E^s(M) := Cl(L(\Gamma^{s+2\ell}(M)) \quad \text{in} \quad H^s(M).$$

Then

$$L(\Gamma^{k+3d^*+1}(M)) \subset E^{k+3d^*+1-2\ell}(M) \subset E^{k+d^*+1-2\ell}(M) \subset L(\Gamma^{k+\gamma}(M)),$$

where the third set is actually a subset of the last one. For, given any $f \in E^{k+d^*+1-2\ell}(M)$, choose $\{f_n\}_n \subset L((\Gamma^{k+d^*+1}(M))$ such that $f_n \longrightarrow f$ $(n \longrightarrow \infty)$ in $H^{k+d^*+1-2\ell}(M)$. Since $f_n \in (\ker L^*)^{\perp}$ for all n, the same holds for f, which together with Theorem 2.3 assures $f \in L(\Gamma^{k+\gamma}(M))$.

For the topologies on these spaces, we give the natural Banach topologies on the series of L-image of Γ-spaces and give the Hilbertian topologies on the series of E-spaces. Then the all injections are continuous. Now put,

$$X \equiv X^{k,\ell} := E^{k+d^*+1-2\ell}(M) \times \ker L,$$

which is a subspace of $Z^{k+\gamma}$, and consider a transformation

$$A_\phi := A \circ L_\phi \circ A^{-1} \quad \text{on} \quad X^{k,\ell}$$

for all $\phi \in \xi(U_1) \cap \mathrm{Diff}^{k+m}(M)$. For any $(\eta, r) \in X^{k,\ell} \cap A(\xi(U_1))$, let us write down A_ϕ explicitly using its components,

$$A_\phi(\eta, r) := (\eta + F_\phi^1(\eta, r), F_\phi^2(\eta, r)).$$

Theorem 2.5. (1) *For $(\eta, r) \in X^{k,\ell} \cap A(\xi(U_1))$, $F_\phi^1(\eta, r)$ belongs to $L(\Gamma^{k+3d^*+1}(M))$ and for the map F_ϕ^1, regarding it as $L(\Gamma^{k+3d^*+1}(M))$-valued map from $X^{k,\ell} \cap A(\xi(U_1))$, it is continuously differentiable.*
(2) *A_ϕ is a local C^1-diffeomorphism on $X^{k,\ell}$.*

Proof. The most part of them are derived from Theorem 2.2. □

7-step Now we shall introduce a basic measure for our arguments. As we have seen, $H_1 := E^{k+3d^*+1-2\ell}$ is nuclearly imbedded into $H := E^{k+d^*+1-2\ell}$. Let ι be the imbedding map and decompose it into T and U, $\iota = T \circ U$, where $U : H_1 \longmapsto H$ is an onto isometric operator and T is a strictly positive-definite nuclear operator on H. It is well known that there exists a Gaussian measure g_T with mean 0 and variance operator T on H,

$$\hat{g}_T(x) \left(:= \int_h \exp(\sqrt{-1} < x, y >_H) g_T(dy)\right) = \exp(-\frac{1}{2} < Tx, x >_H).$$

The following is a transformation formula for variable change.

Theorem 2.6. *Let $X := H \times \mathbf{R}^s$ $(\ni (\eta, r))$, where H is a real separable Hilbert space and $s \in \mathrm{N}$. Suppose that*

$$F(\eta, r) = (\eta + Tf_1(\eta, r), f_2(\eta, r))$$

is a C^1-diffeomorphism from an open set U in X to $F(U)$, where f_1 is a C^1-map from X to H and T is a strictly positive-definite nuclear operator on H. Then for any Borel set $B \subseteq U$,

$$g_T \otimes \lambda(F(B)) = \int_B \exp(- < \eta, f_1(\eta, r) >_H - \frac{1}{2} < Tf_1(\eta, r), f_1(\eta, r) >_H) \cdot \\ |\det(dF_{(\eta,r)})| g_T \otimes \lambda(d\eta, dr),$$

where λ is Lebesgue measure on $\mathbf{R}^s$ and

$$\det(dF_{(\eta,r)}) := \lim_{n\to\infty} \det(P_n dF_{(\eta,r)}|X_n) \quad (\textit{the limit surely exists at every point in } U),$$

P_n is a natural projection from X to $X_n := Sp\{\eta_k \times \mathbf{R}^s \ (k = 1, \cdots, n)\}$ and finally η_k is an eigen-vector of T

$$T\eta = \sum_{k=1}^{\infty} \tau_k < \eta, \eta_k >_H \eta_k, \quad (\tau_1 \geq \tau_2 \geq \cdots \geq \tau_n \geq \cdots > 0).$$

Of course there are more fundamental formulas for variable change, namely without finite dimensional component λ. They are also actively now studied by many mathematicians. A particular one of these theorems is due to [16]. The above theorem is a simple version of this result.

Now let us return to our case. That is,

$$H = E^{k+d^*+1-2\ell}(M), \quad \mathbf{R}^s = \ker L \quad \text{and} \quad F = A_\phi.$$

Then settling the above arguments, we find that

Theorem 2.7. *For any Borel set $B \subseteq X \cap A(\xi(U_1))$,*

$$g_T \otimes \lambda(A_\phi(B)) = \int_B \exp(- < \eta, UF^1_\phi(\eta, r) >_H - \frac{1}{2} < F^1_\phi(\eta, r), UF^1_\phi(\eta, r) >_H) \cdot \\ |\det((dA_\phi)_{(\eta,r)})| g_T \otimes \lambda(d\eta, dr). \tag{2.1}$$

8-step Now we are in a position to define a desired mesure. Define

$$\mu(E) := g_T \otimes \lambda(A(E) \cap X) \qquad (E \subseteq \xi(U_1)).$$

Theorem 2.8. *There exists a neighbourhood $U_2(\subseteq U_1)$ of 0 in $\Gamma^{k+\gamma}(M)$ such that for any Borel set $E \subseteq \xi(U_2)$,*

$$\mu(E \ominus L_\phi(E)) \longrightarrow 0, \quad \textit{whenever} \quad \phi \longrightarrow \mathrm{id} \quad \textit{in} \quad \mathrm{Diff}^{k+3d^*+2}(M).$$

Proof. It is done by long but elementary calculations, using standard techniques in measure theory and subgaussian poperty described , for example, in p79 in [5].

The detailed proof is as follows. First we state the following lemma which is an immediate consequence of Theorm 2.6 without finite dimensional component λ.

Lemma 2.1. *Let H be a real separable Hilbert space, B be a bounded operator on H, and T be a strictly positive-definite nuclear operator on H, which has a form,*

$$Tx = \sum_{n=1}^{\infty} \tau_n < x, h_n > h_n, \quad \tau_1 \geq \cdots \geq \tau_n \geq \cdots > 0, \quad \textit{and} \quad \sum_{n=1}^{\infty} \tau_n < \infty.$$

Further let us assume that $\mathrm{Id} + TB$ *is invertible. Then a limit*

$$\det(\mathrm{Id} + TB) := \lim_{n\to\infty} \det(\mathrm{Id} + P_n TB | H_n)$$

exists, where $H_n := \mathrm{Sp}\{h_1, \cdots, h_n\}$ and $P_n : H \longmapsto H_n$ is the natural projection. Moreover the following formula holds good for Gaussian measure g_T on H and for any but fixed continuous non negative bounded function $s \not\equiv 0$ with bounded support.

$$(2.2) \quad \int_H s((\mathrm{Id} + TB)^{-1}x) g_T(dx) = |\det(\mathrm{Id} + TB)| \cdot \int_H s(x) \exp(- < Bx, x >_H - \frac{1}{2} < TBx, Bx >_H) g_T(dx).$$

Lemma 2.2. *Under the same notation as in* Lemma 2.1,
(1) $\det(\mathrm{Id} + TB)$ *is bounded on a domain $\|B\| \leq r$ for any but fixed $r > 0$.*
(2) $|\det(\mathrm{Id} + TB)|$ *is a continuous function of B with respect to the operator norm.*

Proof. They follow easily from (2.2).

Returning to our case, we find that by Lemma 2.1, $\det((dA_\phi)_{(\eta,r)})$ has the following explicit form, using Gaussian measure $g_{\tilde{T}}$ on X, where $\tilde{T}$ is a nuclear operator defined by $\tilde{T}(\eta, r) := (T\eta, r)$, and using a continuous non negative bounded function $s \not\equiv 0$ on X with bounded support.

$$(2.3) \qquad |\det((dA_\phi)_{(\eta,r)})| = I_1 \cdot I_2^{-1},$$

$$I_1 := \int_X s((dA_\phi)^{-1}_{(\eta,r)})(\eta', r')) g_{\tilde{T}}(d\eta', dr'),$$

$$I_2 := \int_X s(\eta', r') \exp(- < U(dF^1_\phi)_{(\eta,r)}(\eta', r'), \eta' >_H - < (dF^2_\phi)_{(\eta,r)}(\eta', r') - r', r' >_{\ker L}) \cdot \exp(-\frac{1}{2} < (dF^1_\phi)_{(\eta,r)}(\eta', r'), U(dF^1_\phi)_{(\eta,r)}(\eta', r') >_H - \frac{1}{2} \|(dF^2_\phi)_{(\eta,r)}(\eta', r') - r'\|^2_{\ker L}) g_{\tilde{T}}(d\eta', dr').$$

Hereafter we will denote the integrand in (2.1) by $\rho_\phi(\eta, r)$. Note that $F^1_\phi(\eta, r)$ is a map of C^1 class from $X \times \mathrm{Diff}^{k+3d^*+2}(M)$ to $L(\Gamma^{k+3d^*+1}(M))$ and that $F^2_\phi(\eta, r)$ is a map of C^{3d^*+1} class on $X \times \mathrm{Diff}^{k+3d^*+2}(M)$. Hence there exists a neighbourhood $\xi(W)$ of id in

$\mathrm{Diff}^{k+3d^*+2}(M)$ $(W \subset U_1)$ and a neighbourhood $\xi(U_1^{(1)})$ of id in $\mathrm{Diff}^{k+\gamma}(M)$ $(U_1^{(1)} \subseteq U_1)$ such that the followings hold good with a positive constant K_1,

$$\|F_\phi^1(\eta,r)\|_{E^{k+3d^*+1-2\ell}} \leq K_1, \quad \|(dF_\phi^1)_{(\eta,r)}\|_{op} \leq K_1 \quad \text{and} \quad \|(dF_\phi^2)_{(\eta,r)}\|_{op} \leq K_1,$$

for all $\phi \in \xi(W)$ and $(\eta,r) \in A(\xi(U_1^{(1)}))$. Thus it follows from Lemma 2.2 and (2.3) that the second term in the integrand in (2.1), that is, $|\det(dA_\phi)_{(\eta,r)}|$ is bounded on $A(\xi(U_1^{(1)})) \times \xi(W)$. Further by the following elementary estimate of the first term,

$$\exp(-<\eta, UF_\phi^1(\eta,r)>_H - \frac{1}{2} < F_\phi^1(\eta,r), UF_\phi^1(\eta,r)>_H) \leq \exp(\frac{1}{2}K_1^2)\exp(K_1\|\eta\|_{E^{k+d^*+1-2\ell}}),$$

we get

$$|\rho_\phi(\eta,r)| \leq {}^\exists M \exp(K_1\|\eta\|_{E^{k+d^*+1-2\ell}})$$

on this region, and the later function is summable with respect to $g_T(d\eta)$. (cf.[5]) As (2) in Lemma 2.2 leads us to

$$\rho_\phi(\eta,r) \longrightarrow 1, \qquad \text{whenever} \quad \phi \longrightarrow \mathrm{id} \quad \text{in} \quad \mathrm{Diff}^{k+3d^*+2}(M),$$

it follows from the bounded convergence theorem that

$$\int_{X\cap A(\xi(U_1^{(1)}))} |\rho_\phi(\eta,r) - 1| g_T(d\eta)\lambda(dr) \longrightarrow 0,$$

whenever $\phi \longrightarrow \mathrm{id}$ in $\mathrm{Diff}^{k+3d^*+2}(M)$.

Next we take a sufficiently small neighbourhoods $U_1^{(2)}, U_1^{(3)}$ of 0 in $\Gamma^{k+\gamma}(M)$ such that

$$U_1^{(3)} \subseteq U_1^{(2)} \subseteq U_1^{(1)}, \ \xi(U_1^{(2)})\xi(U_1^{(2)}) \subseteq \xi(U_1^{(1)}), \ \xi(U_1^{(3)})\xi(U_1^{(3)}) \subseteq \xi(U_1^{(2)}), \ \xi(U_1^{(3)})^{-1} = \xi(U_1^{(3)}).$$

Moreover from now on till the end of this proof, let us assume that ϕ belongs to $\mathrm{Diff}^{k+3d^*+2}(M) \cap \xi(U_1^{(3)})$. Then for any Borel set $E \subseteq \xi(U_1^{(3)})$,

$$\begin{aligned} \mu(E \ominus L_\phi(E)) &= g_T \otimes \lambda(A(E \ominus L_\phi(E)) \cap X) \\ &= g_T \otimes \lambda(A(E) \cap X \ominus A_\phi A(E) \cap X) \\ &= g_T \otimes \lambda(A(E) \cap X \ominus A_\phi(A(E) \cap X)). \end{aligned}$$

Given $\epsilon > 0$, take a closed set F and an open set G in X which fulfills,

$$F \subseteq A(E) \cap X \subseteq G \subseteq A(\xi(U_1^{(3)})) \cap X \quad \text{and} \quad g_T \otimes \lambda(G \backslash F) < \epsilon,$$

and take a continuous function σ on X such that

$$0 \leq \sigma \leq 1, \quad \sigma = 1 \ \text{ on } \ F \ \text{ and } \ \sigma = 0 \ \text{ on } \ G^c.$$

Then

(2.4)

$$\mu(E \ominus L_\phi(E)) \leq \int_X |\chi_{A(E)\cap X}(\eta,r) - \sigma(\eta,r)| g_T \otimes \lambda(d\eta,dr) + \int_X |\sigma(\eta,r) - \sigma_\phi(\eta,r)| g_T \otimes \lambda(d\eta,dr) + \int |\sigma_\phi(\eta,r) - \chi_{A_\phi(A(E)\cap X)}(\eta,r)| g_T \otimes \lambda(d\eta,dr),$$

where a function σ_ϕ is defined by

$$\sigma_\phi(\eta,r) = \begin{cases} \sigma(A_{\phi^{-1}}(\eta,r)), & \text{if} \quad (\eta,r) \in A_\phi(A(\xi(U_1^{(2)})) \cap X) \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that

$$|\sigma(\eta,r)-\sigma_\phi(\eta,r)| \leq \chi_{A(\xi(U_1^{(2)}))\cap X}(\eta,r)|\sigma(\eta,r)-\sigma(A_{\phi^{-1}}(\eta,r))|,$$

so the second term in the right hand side in (2.4) converges to 0 according to $\phi \longrightarrow \mathrm{id}$ in $\mathrm{Diff}^{k+3d^*+2}(M)$. Further a sum of the remainder terms in that inequality is dominated by

$$\epsilon + \int_{A(\xi(U_1^{(1)}))\cap X} |\rho_\phi(\eta,r)-1| g_T \otimes \lambda(d\eta,dr),$$

by virtue of an obvious inequality,

$$|\chi_{A(E)\cap X}(\eta,r)-\sigma(\eta,r)| \leq \chi_G(\eta,r)-\chi_F(\eta,r).$$

Consequently, for any Borel set E in $\xi(U_2)$, where $U_2 := U_1^{(3)}$, we see that $\mu(E\ominus L_\phi(E)) \longrightarrow 0$, whenever $\phi \longrightarrow \mathrm{id}$ in $\mathrm{Diff}^{k+3d^*+2}(M)$. □

Next take a countable dense set $\{\phi_i\}_i$ from $\mathrm{Diff}^{*k+3d^*+2}(M)$ and define

$$\tilde{\mu}(B) := \sum_{i=1}^{\infty} \alpha_i \mu(L_{\phi_i}(B)\cap\xi(U_2)) \qquad (B \subseteq \mathrm{Diff}^{*k+\gamma}(M)),$$

where $\alpha_i > 0$ $(i = 1, 2, \cdots)$, and $\sum_{i=1}^{\infty}\alpha_i = 1$.

Theorem 2.9. *$\tilde{\mu}$ is a $\mathrm{Diff}^{*k+m}(M)$-quasi-invariant and continuous measure on $\mathrm{Diff}^{*k+\gamma}(M)$, where $m = 3d^* + 2$, and $3m \leq (2\ell) \leq k$.*

Proof. It is evident that $\tilde{\mu}(B) = 0$ if and only if $\mu(L_{\phi_i}(B)\cap\xi(U_2)) = 0$ for all i. Now given any $\phi \in \mathrm{Diff}^{*k+m}(M)$, take a sequence ϕ_{i_j} converging to ϕ and put $\phi_{i_j} = \varphi_j\phi$. Then,

$$\begin{aligned}
|\mu(L_{\phi_{i_j}}(B)\cap\xi(U_2)) - \mu(L_\phi(B)\cap\xi(U_2))| &\leq \mu((L_{\phi_{i_j}}(B)\ominus L_\phi(B))\cap\xi(U_2)) \\
&= \mu((L_{\varphi_j}(L_\phi(B))\ominus L_\phi(B))\cap\xi(U_2)) \\
&\leq \mu(L_{\varphi_j}(L_\phi(B)\cap\xi(U_2))\ominus L_\phi(B)\cap\xi(U_2)) \\
&\quad + \mu(L_{\varphi_j}(\xi(U_2))\ominus\xi(U_2)) \longrightarrow 0, \quad (j\longrightarrow\infty),
\end{aligned}$$

due to Theorem 2.8. Therefore $\mu(L_\phi(B)\cap\xi(U_2)) = 0$, whenever $\tilde{\mu}(B) = 0$. This shows the quasi-invariance. For the continuity it is enough to show that

$$^\forall B,\ ^\forall i, \quad \mu(L_{\phi_i}(B\ominus L_\phi(B))\cap\xi(U_2)) \longrightarrow 0,$$

whenever $\phi \longrightarrow \mathrm{id}$ in $\mathrm{Diff}^{k+m}(M)$. Put

$$E := L_{\phi_i}(B) \quad \text{and} \quad \psi := \phi_i\phi\phi_i^{-1}.$$

Then,

$$\begin{aligned}
\mu(L_{\phi_i}(B\ominus L_\phi(B))\cap\xi(U_2)) &= \mu((E\ominus L_\psi(E))\cap\xi(U_2)) \\
&\leq \mu(L_\psi(E\cap\xi(U_2))\ominus E\cap\xi(U_2)) \\
&\quad + \mu(L_\psi(E\cap\xi(U_2))\ominus L_\psi(E)\cap\xi(U_2)) \\
&\leq \mu(L_\psi(E\cap\xi(U_2))\ominus E\cap\xi(U_2)) \\
&\quad + \mu(L_\psi(\xi(U_2))\ominus\xi(U_2)) \longrightarrow 0 \quad (\phi\longrightarrow\mathrm{id}).
\end{aligned}$$

□

2.2. Existence and denseness of C^∞-vectors. Let $(U, \mathcal{H})$ be a unitary representation of $\mathrm{Diff}^*(M)$ on a compact Riemannian manifold M. Suppose that our unitary representation $(U,\mathcal{H})$ has a continuous extension to a larger group $\mathrm{Diff}^{*K}(M)$. Take k so large that $k \geq K$.

Further take a C^∞-function $\rho \equiv \rho_{a,b}$ $(0 < a < b)$ on $[0,\infty)$ such that

$$0 \leq \rho \leq 1, \quad \rho = 1 \quad \text{on} \quad [0,a], \quad \rho = 0 \quad \text{on} \quad [b,\infty),$$

and define a function $\tilde{Q}$ on $Z^{k+\gamma}$ by

$$\tilde{Q}(\eta,r) := \rho(\|(\eta,r) - A(\mathrm{id})\|_X^2)\chi_X(\eta,r)/C,$$

where C is a normalizing constant such that $\int_X \tilde{Q}(\eta,r)g_T(d\eta)\lambda(dr) = 1$, and χ_X is an indicator function of X, and $\|\cdot\|_X$ is the natural norm. Finally put

$$Q(f) \equiv Q_{a,b}(f) := \tilde{Q}(Af) \qquad\qquad (f \in \xi(U_2)).$$

Then after long calculations we have the following announced result.

Theorem 2.10. *For any $h \in \mathcal{H}$ define*

$$w_h \equiv w_h^{a,b} := \int_{\xi(U_2)} Q_{a,b}(f)U(f)h\,\mu(df).$$

Then $w_h^{a,b}$ is a C^∞-vector and $w_h^{a,b}$ converges to h, whenever a,b tend to 0.

Proof. Needless to say,

$$dU(X)h = \frac{d}{d\tau}\bigg|_{\tau=0} U(\mathrm{Exp}(tX))h \qquad (X \in \Gamma(M) \text{ and } h \in \mathcal{H}),$$

and h is said to be a C^∞-vector of $(U,\mathcal{H})$, if and only if $dU(X_1)(\cdots(dU(X_n)h))$ exists for every n and $X_1,\cdots,X_n \in \Gamma(M)$. Thus for the proof it is enough to see that for any n and any $s(\leq n)$, $U(\mathrm{Exp}(t_1X_1)\cdots\mathrm{Exp}(t_nX_n))$ is s-times continuously differentiable on a neighbourhood of $t := (t_1,\cdots,t_n) = (0.\cdots,0)$. Hereafter we always assume that $\mathrm{supp}Q_{a,b} \subset \xi(U_2)$. Put

$$\phi_t := \mathrm{Exp}(t_1X_1)\circ\cdots\circ\mathrm{Exp}(t_nX_n), \quad \text{and} \quad \psi_t := \mathrm{Exp}(-t_nX_n)\circ\cdots\circ\mathrm{Exp}(-t_1X_1).$$

Then

$$U(\psi_t)w_h = \int_{A(\xi(U_2))\cap X} Q(A^{-1}(\eta,r))U(\psi_t \circ A^{-1}(\eta,r))h\,g_T\otimes\lambda(d\eta,dr),$$

and for sufficiently small $|t| := |t_1| + \cdots + |t_n|$,

$$(2.5)\quad U(\psi_t)w_h = \int_{A(\xi(U_3))\cap X} Q(A^{-1}A_{\phi_t}(\eta,r))\rho_{\phi_t}(\eta,r)U(A^{-1}(\eta,r))h\,g_T\otimes\lambda(d\eta,dr),$$

where $U_3 := U_1^{(2)}$ which was already given in the proof of Theorem 2.8. Thus for the proof we must check differentials of $Q(A^{-1}A_{\phi_t}(\eta,r))$ and $\rho_{\phi_t}(\eta,r)$ with respect to t.

First note that by the definition of $\tilde{Q}$ and ρ, the integration in (2.5) is actually carried out over a set of (η,r) satisfying $\|A_{\phi_t}(\eta,r) - A(\mathrm{id})\|_X^2 \leq b$. Next, since the map $A_\phi(\eta,r) : \mathrm{Diff}^{k+3d^*+2}(M)\times X \longmapsto X$ is continuous, so for a sufficiently small $|t|$ and for such a b, the above inequality implies that $\|A_{\psi_t}(A_{\phi_t}(\eta,r)) - A(\mathrm{id})\|_X \leq 1$. In other words, an actual integral domain D in (2.5) may be assumed to be bounded.

Now let us consider first the differentials of $\rho_{\phi_t}(\eta,r)$, and so recall the definition of $F^1_{\phi_t}$ and $F^2_{\phi_t}$. Namely,

$$(2.6)\qquad F^1_{\phi_t}(\eta,r) = \pi_1^{k+3d^*+1-2\ell}(A_{2\ell,m}(\phi_t\circ f) - A_{2\ell,m}(f)),$$

$$(2.7)\qquad F^2_{\phi_t}(\eta, r) = \pi_2^{k+\gamma}\xi^{-1}(\phi_t \circ f),$$

where $f := A^{-1}(\eta, r)$. Further let us denote a terminal point of unit geodisic starting at x along a direction u by $K(x, u)$ and denote a tangent vector at x of unit geodisic with an initial point x and a terminal point y by $J(x, y)$. Then K and J are C^∞-maps on the tangent bundle on M and on $M \times M$, respectively. Since for the maps $f =: \xi(u)$ and $\phi_t =: \xi(v_{t_1,\cdots,t_n})$ we have,

$$\begin{aligned}
\phi_t \circ f(x) &= K(K(x, u(x)), v_{t_1,\cdots,t_n}(K(x, u(x))),\\
v_{t_1,\cdots,t_n}(x) &= J(x, \operatorname{Exp}(t_1X_1)\circ\cdots\circ\operatorname{Exp}(t_nX_n)(x)),\\
\xi^{-1}(\phi_t \circ f)(x) &= J(x, K(K(x, u(x)), v_{t_1,\cdots,t_n}(K(x, u(x)))),
\end{aligned}$$

so $F^1_{\phi_t}(\eta, r)$ and $F^2_{\phi_t}(\eta, r)$ are infinitely differentiable maps with respect to t. Further somewhat long and complicated calculations lead us to that there exists $\delta_1 > 0$ such that

$$(2.8)\qquad \|\partial_t^s F^1_{\phi_t}(\eta, r)\|_{E^{k+3d^*+1-2\ell}} \quad \text{and} \quad \|\partial_t^s F^2_{\phi_t}(\eta, r)\|_{\ker L} \quad \text{are bounded}$$

for any $|t| < \delta_1$ and $(\eta, r) \in D$. Thus the derivatives of the first term of $\rho_{\phi_t}(\eta, r)$, that is,

$$\exp(-<\eta, UF^1_{\phi_t}(\eta, r)>_{E^{k+d^*+1-2\ell}} - \frac{1}{2} < F^1_{\phi_t}(\eta, r), UF^1_{\phi_t}(\eta, r)>_{E^{k+d^*+1-2\ell}})$$

are also bounded and continuous. While for the second term in that function, namely $|\det((dA)_{\phi_t}(\eta, r))|$, we take, in the present case, $\sigma(\eta, r) := \rho(\|(\eta, r)\|_X^2)$ as the function s in (2.3) and write it down as follows.

$$I_1(t, \eta, r) := \int_X \sigma(\eta' + (dF^1_{\psi_t})_{(\eta,r)}(\eta', r'), (dF^2_{\psi_t})_{(\eta,r)}(\eta', r'))g_{\tilde{T}}(d\eta', dr'),$$

(Since $(\eta, r) \in A(\xi(U_3))$, we see that a support of the above integrand is bounded as far as $|t|$ is sufficiently small)

$$\begin{aligned}
I_2(t, \eta, r) := \int_X &\exp(-< U(dF^1_{\phi_t})_{(\eta,r)}(\eta', r'), \eta'>_{E^{k+d^*+1-2\ell}} - <(dF^2_{\phi_t})_{(\eta,r)}(\eta', r') - r', r'>_{\ker L})\cdot\\
&\exp(-\frac{1}{2} <(dF^1_{\phi_t})_{(\eta,r)}(\eta', r'), U(dF^1_{\phi_t})_{(\eta,r)}(\eta', r')>_{E^{k+d^*+1-2\ell}} - \frac{1}{2}\|(dF^2_{\phi_t})_{(\eta,r)}(\eta', r') - r'\|^2_{\ker L})\cdot\\
&\sigma(\eta', r')g_{\tilde{T}}(d\eta', dr').
\end{aligned}$$

Then by virtue of the previous arguments, $I_1(t, \eta, r)$ and $I_2(t, \eta, r)$ are bounded for any $(\eta, r) \in A(\xi(U_2)) \cap X$ and for any $|t| < {}^\exists\delta_2$.

Next let us observe $\partial_t^s(dF^1_{\phi_t})_{(\eta,r)}$ and $\partial_t^s(dF^2_{\phi_t})_{(\eta,r)}$. Since

$$(dF^1_{\phi_t})_{(\eta,r)}(\eta', r') = \frac{d}{d\tau}\bigg|_{\tau=0} \pi^{k+3d^*+1-2\ell}(A_{2\ell,m}(\phi_t\circ A^{-1}(\eta+\tau\eta', r+\tau r')) - A_{2\ell,m}(A^{-1}(\eta+\tau\eta', r+\tau r'))),$$

so changing $f = A^{-1}(\eta, r)$ to $f_\tau = A^{-1}(\eta + \tau\eta', r + \tau r')$, together changing $u := \xi^{-1}(f)$ to $u_\tau := \xi^{-1}(f_\tau)$, and proceeding in the same manner as before, we have

$$\|\partial_t^s(dF^1_{\phi_t})_{(\eta,r)}(\eta', r')\|_{E^{k+3d^*+1-2\ell}} \quad \text{is bounded}$$

for any $(\eta, r) \in A(\xi(U_3)) \cap X$ (if necessary, taking a smaller neighbourhood U_3' in place of U_3), for any $|t| < {}^\exists\delta_3$ and for any (η', r') in any but fixed bounded domain. The same estimate holds for $\|\partial_t^s(dF^2_{\phi_t})_{(\eta,r)}(\eta', r')\|_{\ker L}$. By the above, $\partial_t^s|\det(dA_{\phi_t}(\eta, r))|$ surely exists and it is bounded and continuos on the integral domain. Therefore the same conclusion for $\rho_{\phi_t}(\eta, r)$ follows directly.

Lastly for the function $Q(A^{-1}A_{\phi_t}(\eta, r))$, we have

$$Q(A^{-1}A_{\phi_t}(\eta, r)) = \tilde{Q}(A_{\phi_t}(\eta, r)) = C^{-1}\rho(\|(\eta + F^1_{\phi_t}(\eta, r), F^2_{\phi_t}(\eta, r)) - A(\mathrm{id})\|^2_X).$$

So there follows from (2.8) that $\partial_t^s Q(A^{-1}A_{\phi_t}(\eta, r))$ is continuous and bounded for the same region.

Consequently the s-th derivative of the integrand is continuous and bounded for any $|t| < \min(t_1, t_2, t_3)$ on the integral domain. Therefore $w_h^{a,b}$ is a C^∞-vector. The rest of the proof is obvious. □

3. Application to 1-cocycles on the group of diffeomorphisms

The rest of this issue is devoted to an application of these results to 1-cocycles. So let us introduce the notions of them briefly.

Assume that a subgroup G of $\mathrm{Diff}_0^s(M)$ acts on a measurable space $(X, \mathfrak{B})$ from left $(g, x) \in G \times X \longrightarrow gx \in X$.

A $U(H)$-valued function θ on $X \times G$, $U(H)$ is the unitary group of a complex Hilbert space H, is said to be 1-cocycle, if

$$^\forall g_1, g_2 \in G, \quad ^\forall x \in X, \qquad \theta(x, g_1)\theta(g_1^{-1}x, g_2) = \theta(x, g_1 g_2). \qquad \text{(cocycle equality)}$$

For regularity of 1-cocycles, several notions have been considered. Some of them are as follows.

Definition 3.1. (1) *θ is said to be precontinuous $\Longleftrightarrow$ $^\forall x_0$: fixed, $\theta(x_0, g)$ is continuous on a stabilizer group, $G(x_0) := \{g \in G |\ gx_0 = x_0\}$.*

(2) *θ is said to be continuous $\Longleftrightarrow$ $^\forall x_0$: fixed, $\theta(x_0, g)$ is continuous on the whole group, G.*

(3) *θ is said to be measurable $\Longleftrightarrow$ $^\forall g_0$: fixed, $\theta(x_0, g)$ is $\mathfrak{B}$-measurable.*

We remark that someimes (3) implies (1), for example, under an assumption of denseness of C^∞-vectors. (cf. p138-140 in [9])

Now for the present discussions, I pick up the following two spaces as X, since they are standard for the representation theory on the group of diffeomorphisms.

Finite configuration space B^n_M which is a collection of all n-point subsets in M. It is also a quotient space of $\hat{M}^n$, where $\hat{M}^n := \{\hat{P} = (P_1, \cdots, P_n) \in M^n |\ ^\forall P_i \neq P_j\}$, and the equivalence relation is defined in an obvious way.

Infinite configuration space Γ_M which is also a quotient space of $\hat{M}^\infty := \{\hat{P} = (P_1, \cdots, P_n, \cdots) \in M^\infty |\ ^\forall P_i \neq P_j,$ and $\{P_n\}_n$ has no accumulation points$\}$, and the equivalence relation is similar with the above one. In this case we should assume that M is non compact. Of course $\mathrm{Diff}_0^s(M)$ acts on these spaces diagonally as, $\hat{g}(\hat{P}) := (g(P_1), \cdots, g(P_n), \cdots)$.

Now let θ be a 1-cocycle on the finite or infinite configuration space. Then there correspondes one to one a symmetrical cocycle on the product space to θ. Thus it is reasonable to observe a cocycle form on the product space $\hat{M}^n$ or $\hat{M}^\infty$. Also for the sake of limit of pages and for simplicity, we will confine ourself to these situations.

Then the differential methods which we have seen lead us to the following theorem determining a local form of 1-cocycles.

Theorem 3.1. (*Local form of precontinuous* 1-*cocycle*)
Let θ be a $U(H)$-valued precontinuous 1-*cocycle on $\hat{M}^n \times \mathrm{Diff}_0^*(M)$, and assume that* $\dim(H) < \infty$. *Take an arbitrary finite Euclidean smooth measure μ on M. Then for any $\hat{Q} \in \hat{M}^n$ there exist a relatively compact open neighbourhood of $V(\hat{Q})$ of $\hat{Q}$, a $U(H)$-valued map C defined on $V(\hat{Q})$ and a commutative system of self-adjoint operators $\{H_k\}_{1\leq k\leq n}$ on H such that*

$$\theta(\hat{P}, g) = C(\hat{P})^{-1} \prod_{k=1}^{n} \left(\frac{d\mu_g}{d\mu}(P_k) \right)^{\sqrt{-1}H_k} C(\hat{g}^{-1}(\hat{P})), \tag{3.1}$$

provided that $(\hat{P}, g)$ satisfies the following condition.

$(*)$ *There exists a continuous path $\{g_t\}_{0\leq t\leq 1} \subset \mathrm{Diff}_0^*(M)$ such that $g_0 = \mathrm{id}, g_1 = g$ and $^\forall t, \hat{g}_t^{-1}(\hat{P}) \in V(\hat{Q})$.*

If moreover θ is continuous, then so is the map C.

Of course a global form of 1-cocycle will be obtained by patching up these local results. However difficulties arise because of non uniqueness of the above map C, which forms so called coboundary term. Roughly speaking we will meet a similar situation with many valuedness problem to analytic continuation. So some geometrical conditions on M are required in order to obtain a global result. One direction is as follows. (cf. [20])

Theorem 3.2. (*Global form of precontinuous* 1-*cocycle*)
Under the same notation in the above theorem and under the assumption that $\hat{M}^n$ is simply connected, (3.1) *gives a general form of precontinuous* 1-*cocycle.*

Remark 3.1. (1) *In oder that $\hat{M}^n$ is simply connected, it is sufficient that M is simply connected and* $\dim M \geq 3$, *thanks to dimension theory.*
(2) Theorem 3.2 *is no longer trure, unless $\hat{M}^n$ is simply connected.* (*cf.* [19], [20])

A cocycle form on $\hat{M}^\infty$, in a special case that M is simply connected, is described in the following last theorem.

Theorem 3.3. (1) *Suppose that M is simply connected,* $\dim(M) \geq 3$. *and* $\dim H < \infty$. *Then the general form of precontinuous $U(H)$-valued* 1-*cocycles on $\hat{M}^\infty \times \mathrm{Diff}_0^*(M)$ is as follows.*

$$\theta(\hat{P}, g) = C(\hat{P})^{-1} \prod_{k=1}^{\infty} \left(\frac{d\mu_g}{d\mu}(P_k) \right)^{\sqrt{-1}H_k^{[P]}} C(\hat{g}^{-1}(\hat{P})), \tag{3.2}$$

where C is a $U(H)$-valued map on $\hat{M}^\infty$, and $\{H_k^{[P]}\}_k$ is a commutative system of self-adjoint operators on H depending on the residue class $[P]$ defined by $[P] := \{\hat{Q} \in \hat{M}^\infty | \ Q_n = P_n$ except finite numbers of $n\}$.

Finally I wish to mension a few words about natural representations formed by measures and 1-cocycles. Their irreducibility and equivalence are also examined by similar methods established here and they are characterized by the above theorems.

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