FREE PROBABILITY THEORY AND FREE DIFFUSION

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1. Introduction

Free probability theory was introduced and developed by Dan Voiculescu in an operator algebraic context, but has since then turned out to possess links to a lot of quite different fields of mathematics and physics. I will give a short general introduction into the basics of free probability and illuminate certain aspects of that theory (in particular, the analogy between classical and free probability theory) by a closer look at free diffusion.

An extensive presentation of the basic theory of free probability is given in the monograph [VDN], whereas for getting an impression of the diversity of this field one should consult [V2, V3].

2. Free probability theory

Free probability theory was introduced by Dan Voiculescu around 1985 as a tool for investigating the structure of special von Neumann algebras. Voiculescu separated from that concrete context the following abstract concept of 'freeness' and found it worth to be investigated on its own sake. The definition and the main properties of freeness do not require an operator algebraic frame, but can be formulated on the level of unital algebras and unital linear functionals.

Definition 2.1. Let $\mathcal{A}$ be a unital algebra and $\varphi : \mathcal{A} \to \mathbb{C}$ a linear functional with $\varphi(1) = 1$.
1) Let $\mathcal{A}_1, \ldots, \mathcal{A}_m \subset \mathcal{A}$ be unital subalgebras. The subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_m$ are called **free**, if $\varphi(a_1 \cdots a_k) = 0$ for all $k \in \mathbb{N}$ and all $a_i \in \mathcal{A}_{j(i)}$ $(1 \leq j(i) \leq m)$ whenever $\varphi(a_i) = 0$ for all $i = 1, \ldots, k$, and neighbouring elements are from different subalgebras, i.e., $j(1) \neq j(2) \neq \cdots \neq j(k)$.
2) Elements $a_1, \ldots, a_m \in \mathcal{A}$ are called free, if $\mathcal{A}_1, \ldots, \mathcal{A}_m$ are free, where, for $i = 1, \ldots, m$, $\mathcal{A}_i := \text{alg}(1, a_i)$ is the unital algebra generated by a_i.

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Voiculescu chose the name 'free' because the basic example where such situations occur are von Neumann algebras which are constructed from free groups (the so-called free group factors).

The basic philosophy for the investigation of the concept 'freeness' is to consider it as an analogue of the concept 'independence' from classical probability theory. Hence we are using a probabilistic kind of language and are usually guided by concepts and ideas from classical probability theory. In this sense, the theory of freeness can be considered as a part of non-commutative probability theory and it is usually referred to as 'free probability theory'.

Let us first introduce some general notions from non-commutative probability theory.

Notations 2.2. A pair $(\mathcal{A}, \varphi)$ consisting of a unital algebra $\mathcal{A}$ and a unital linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ is called a **(non-commutative) probability space**, elements $a_1, \dots, a_m$ from the given algebra $\mathcal{A}$ are called **random variables** and expressions like $\varphi(a_{j(1)} \cdots a_{j(k)})$ are called **moments**. The collection of all moments, for all $k \in \mathbb{N}$ and all $1 \leq j(1), \dots, j(k) \leq m$, is called the **(joint) distribution** of the random variables $a_1, \dots, a_m$.

Remark 2.3. One should note that in the case of one self-adjoint bounded random variable $a = a^* \in B(\mathcal{H})$, one can identify the so-defined distribution of a indeed with a probability measure μ on $\mathbb{R}$ by the requirement that the moments of a coincide with the moments of μ, i.e.

$$(1) \qquad \varphi(a^n) = \int x^n d\mu(x) \qquad \text{for all } n \in \mathbb{N}.$$

In that case we will denote this probability measure also with $\mathrm{distr}(a)$. In general, the distribution of random variables cannot be identified with some kind of probability measure, but is just a collection of numbers.

Examples 2.4. Let us now give some examples of probability spaces and distributions in this general algebraic sense – in order to become familiar with this kind of notations and to introduce some basic frame for our later investigations.

1) Classical probability spaces. Classical probability spaces $(\Omega, \mathcal{Q}, P)$ – consisting of a set Ω, a σ-algebra $\mathcal{Q}$ of measurable subsets of Ω and a probability measure P on Ω – can be treated in this frame by setting, e.g., $\mathcal{A} = L^{\infty-}(\Omega) := \cup_{p=1}^{\infty} L^p(\Omega)$ and where $\varphi = E$ is the expectation

$$\varphi(X) = \int_\Omega X(\omega)dP(\omega) \qquad (X \in \mathcal{A}). \tag{2}$$

2) Matrices. Let, for $n \in \mathbb{N}$, $\mathcal{A} = M_n$ be equal to the $n \times n$-matrices. A canonical state on this is given by the normalized trace $\varphi = \text{tr}$, i.e., for $a = (a_{ij})_{i,j=1}^n \in \mathcal{A}$ we have

$$\varphi(a) = \frac{1}{n}\sum_{i=1}^n a_{ii}. \tag{3}$$

One should note that for self-adjoint matrices $a = a^*$ the distribution $\text{distr}(a)$ is nothing but the eigenvalue distribution of a, i.e., if $\lambda_1, \dots, \lambda_n$ are the (real) eigenvalues of a, then $\text{distr}(a)$ is that probability measure on $\mathbb{R}$ which puts mass $1/n$ on each of the eigenvalues, i.e.

$$\text{distr}(a) = \frac{1}{n}\sum_{i=1}^n \delta_{\lambda_i}. \tag{4}$$

3) Random matrices. Random matrices are a combination of (1) and (2), namely matrices whose entries are classical random variables: $\mathcal{A} = M_n \otimes L^{\infty-}(\Omega)$ and $\varphi = \text{tr} \otimes E$, i.e., $a \in \mathcal{A}$ are of the form $a = (a_{ij})_{i,j=1}^n$, where the entries $a_{ij} \in L^{\infty-}(\Omega)$, and

$$\varphi(a) = E[\frac{1}{n}\sum_{i=1}^n a_{ii}] = \frac{1}{n}\sum_{i=1}^n \int_\Omega a_{ii}(\omega)dP(\omega). \tag{5}$$

In the case of a self-adjoint random matrix $a = a^*$, the distribution $\text{distr}(a)$ is the averaged eigenvalue distribution of a.

To enrich the general frame of non-commutative probability theory by some substance one has to add additional structure. In free probability theory this is the concept of 'freeness'. In analogy with the concept 'independence' it should be considered as a rule for calculating mixed moments in free random variables. This might not be directly clear from the definition, so let us present some examples to get familiar with the concept of freeness.

Examples 2.5. Let x and y be free random variables (with respect to a given unital functional φ). We want to calculate some mixed moments in x and y.
1) The simplest mixed moment is $\varphi(xy)$. The definition of freeness tells us immediately that $\varphi(xy) = 0$, if $\varphi(x) = 0$ and $\varphi(y) = 0$. But we can also reduce the general case to the definition by going over to centered variables: since $\hat{x} := x - \varphi(x)1$ is an element from the unital

algebra generated by x with the property $\varphi(\hat{x}) = 0$, and similarly for $\hat{y} := y - \varphi(y)1$, we have that $\varphi(\hat{x}\hat{y}) = 0$; however, by linearity, we also have

$$0 = \varphi(\hat{x}\hat{y}) = \varphi\big((x - \varphi(x))(y - \varphi(y))\big) = \varphi(xy) - \varphi(x)\varphi(y).$$

Hence we have in general for free variables x and y that

$$\varphi(xy) = \varphi(x)\varphi(y). \tag{6}$$

2) The mixed moment $\varphi(xxyy)$ calculates in the same way by going over to the centered variables:

$$\varphi\big((x^2 - \varphi(x^2))(y^2 - \varphi(y^2))\big) = 0$$

yields

$$\varphi(xxyy) = \varphi(xx)\varphi(yy). \tag{7}$$

3) Let us also consider a more complicated mixed moment:

$$\varphi\big((x - \varphi(x))(y - \varphi(y))(x - \varphi(x))(y - \varphi(y))\big) = 0$$

leads to

$$\varphi(xyxy) = \varphi(xx)\varphi(y)\varphi(y) + \varphi(x)\varphi(x)\varphi(yy) - \varphi(x)\varphi(y)\varphi(x)\varphi(y). \tag{8}$$

Remarks 2.6. 1) The last example shows that freeness gives a different result than independence. Although both concepts are analogous, they provide different rules for calculating mixed moments. In particular, freeness is not a non-commutative generalization of independence.
2) If x and y are classical random variables, then, in particular, they commute, i.e. we have in this case that $\varphi(xxyy) = \varphi(xyxy)$. However, for x and y free we have quite different expressions for these two mixed moments and one can easily see that they can only agree if at least one of the two variables is a constant. Thus classical random variables are, apart from trivial cases, never free. Freeness is really a concept for non-commuting variables.
3) As the last example above indicates the formulas for mixed moments in free variables are more complicated than the corresponding formulas for independent variables and it is not clear from the definition of freeness how the structure of a general mixed moment can be described. However, there is a nice combinatorial structure behind these formulas. I have shown that their structure is (via so-called free cumulants) governed by the lattice of non-crossing partitions (see, e.g., the survey [Sp2]). This description is totally analogous to the description in classical probability theory via cumulants and the lattice of all partitions and it provides an alternative approach (compared to the analytical approach of Voiculescu) to the theory of free random variables.

Let me end this short introduction into the generalities of free probability theory by pointing out that there are two fundamental types of examples for free variables: The definition of freeness is modeled according to the situation occurring in free group factors, thus it is not very surprising that special operators in free group factors (or more concretely, special operators on full Fock spaces) are free. But there is also a totally different context where free variables arise, namely it is one of basic results of Voiculescu [V1] that special $n \times n$-random matrices become free in the limit $n \to \infty$. I will be more concrete on such types of examples when I present the free Brownian motion.

3. Free Diffusion

As pointed out before one of the basic philosophies in free probability theory is to consider freeness as an analogue of independence. Thus one tries to develop a free theory which goes parallel to classical probability theory. Astonishingly, this analogy is very far reaching and there exist a lot of (non-trivial) free counterparts of classical results.

In the following I want to illuminate this general statement by a recent joint work [BSp1, BSp2] with Philippe Biane on free diffusion.

3.1. **Classical diffusion.** Let me first explain what I mean with the corresponding classical notion. If $V : \mathbb{R} \to \mathbb{R}$ is a sufficiently nice function (called potential in the following), one can consider the classical diffusion in this potential. On one side there is a probabilistic construction of this object, namely it is a stochastic process $(X_t)_{t\geq 0}$ which is given as the solution of a special stochastic differential equation. What I call here 'diffusion in the potential V' is the solution of

$$dX_t = -\frac{1}{2}V'(X_t)dt + dB_t, \tag{9}$$

where B_t is classical Brownian motion.

There exists also an analyical aspect of this diffusion, namely if we denote, for fixed $t \geq 0$, by $\mathrm{distr}(X_t)$ the distribution of the random variable X_t, then this is a probability measure on $\mathbb{R}$ which has a density with respect to Lebesgue measure. Denote this density by p_t. Then one can write down a differential equation for the time evolution of this density, namely

$$\frac{\partial p_t(x)}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x}\big[(\frac{\partial}{\partial x} + V'(x))p_t(x)\big]. \tag{10}$$

This linear partial differential equation is usually called the Fokker-Planck equation of the corresponding diffusion and, from an analytical point of view, one can consider the diffusion also as a solution of that equation. Furthermore, there exist also connections between such diffusions and classical entropy.

The problem which I want to address in the following is whether there exist a free counterpart of these statements, i.e., can we define a free diffusion as a solution of a free stochastic differential equation and is there a corresponding free Fokker-Planck equation. In order to speak about free stochastic differential equations, we first have to introduce free Brownian motion.

3.2. **Free Brownian motion.** In analogy with classical Brownian motion one could define free Brownian motion [Sp1] abstractly as a (non-commutative) stochastic process, i.e. a collection $(S_t)_{t\geq 0}$ of random variables, which have the properties that their increments are free and that the distribution of the increments is given by the free analogue of the Gaussian distribution (which is what one gets as the limit distribution in a free central limit theorem). It is easy to verify that, by abstract reasons, such an object exists and that its distribution is uniquely determined. Fortunately, there are also nice concrete realizations of free Brownian motion.

Examples 3.2.1. In the spirit of the last statement in Sect. 2 there exist two such realizations, a functional analytic one by concrete operators on Fock spaces and a probabilistic one by random matrices.
1) Realization on full Fock space. Denote by $\mathcal{H}$ the Hilbert space $\mathcal{H} := L^2(\mathbb{R}_+)$ and let

$$\mathcal{F}(\mathcal{H}) := \mathcal{H}^{\otimes 0} \oplus \mathcal{H}^{\otimes 1} \oplus \mathcal{H}^{\otimes 2} \oplus \dots \tag{11}$$

be the full Fock space over $\mathcal{H}$, where $\mathcal{H}^{\otimes 0}$ is a one-dimensional Hilbert space which we write in the form $\mathcal{H}^{\otimes 0} = \mathbb{C}\Omega$ for a distinguished vector Ω of norm 1. Ω is also called vacuum. For each vector $f \in \mathcal{H}$, we define on $\mathcal{F}(\mathcal{H})$ a creation operator $l(f)$ and an annihilation operator $l^*(f)$ by linear extension of

$$l(f) f_1 \otimes \dots \otimes f_n = f \otimes f_1 \otimes \dots \otimes f_n \tag{12}$$

and

$$l^*(f) f_1 \otimes \dots \otimes f_n = \langle f, f_1 \rangle f_2 \otimes \dots \otimes f_n \tag{13}$$

$$l^*(f)\Omega = 0. \tag{14}$$

The operators $l(f)$ and $l^*(f)$ are bounded and adjoints of each other. Now put

$$S_t := l(1_{[0,t)}) + l^*(1_{[0,t)}), \tag{15}$$

where $1_{[0,t)}$ is the characteristic function of the interval $[0, t)$. Then it is quite easy to check that $(S_t)_{t\geq 0}$ is with respect to the vacuum expectation state φ, given by

$$\varphi(a) := \langle \Omega, a\Omega \rangle, \tag{16}$$

indeed a free Brownian motion.

The von Neumann algebra generated by all S_t $(t \geq 0)$ is isomorphic to a free group factor, and this example comes from the original context of Voiculescu's investigations on the free group factors. Thus the appearance of freeness in this context is not very surprising.

2) Realization by random matrices. Let, for $1 \leq i \leq j < \infty$, $B_{ij}(t)$ be independent classical real-valued Brownian motions, and put $B_{ji}(t) = B_{ij}(t)$ for $j > i$. We put now these Brownian motions as entries in a matrix, i.e. we consider the selfadjoint random matrices

$$X_t^{(n)} := \frac{1}{\sqrt{n}} \big(B_{ij}(t)\big)_{i,j=1}^{n} \tag{17}$$

in the probability space $(M_n \otimes L^{\infty-}(\Omega), \varphi^{(n)} = \mathrm{tr} \otimes E)$. (These special random matrices are usually called Gaussian random matrices.) Then the basic result of Voiculescu [V1] on the connection between freeness and random matrices tells us that the processes $(X_t^{(n)})_{t\geq 0}$ converge in distribution, for $n \to \infty$, towards the free Brownian motion $(S_t)_{t\geq 0}$. This means that

$$\lim_{n\to\infty} \varphi^{(n)}(X_{t_1}^{(n)} \cdots X_{t_k}^{(n)}) = \varphi(S_{t_1} \cdots S_{t_k}) \tag{18}$$

for all $k \in \mathbb{N}$ and all $t_1, \dots, t_k \geq 0$. Thus, in a sense, free Brownian motion can be considered as an $\infty \times \infty$-random matrix. However, one should note that this is not just an infinite array of entries, but the crucial information lies in the state. There exists no normalized trace on infinite arrays, and freeness is the mathematical structure which survives under taking this limit.

Remark 3.2.2. The realization of free Brownian motion by random matrices gives us an interesting connection with systems of interacting particles. Namely, for fixed t, we know that the distribution $\mathrm{distr}(X_t^{(n)})$ is the averaged eigenvalue distribution of these $n \times n$-random matrices and thus free Brownian motion describes in particular also the behaviour of the eigenvalues of Gaussian $n \times n$-random matrices in the limit $n \to \infty$. However, it is well known that the eigenvalues of such Gaussian random matrices are not independent, but they behave like electrically charged particles in two dimensions, i.e. like particles with a special type of pair-interaction. In a probabilistic language, the eigenvalues of the random matrices $X_t^{(n)}$ obey the stochastic differential equation

$$(19) \qquad d\lambda_i(t) = \frac{1}{\sqrt{n}} dB_i(t) + \frac{1}{n} \sum_{\substack{1 \le j \le n \\ j \ne i}} \frac{1}{\lambda_i - \lambda_j} dt \qquad (i = 1, \dots, n),$$

where $B_i(t)$ $(i = 1, \dots, n)$ are independent classical Brownian motions.

In the limit $n \to \infty$, the diffusive term can be neglected compared to the deterministic term and thus this limit corresponds to a system of infinitely many particles which interact with each other by a special type of pair interaction. Free Brownian motion provides thus in particular the description for such a system of infinitely many interacting particles.

3.3. **Free stochastic differential equations.** The next step is to develop a stochastic calculus with respect to free Brownian motion in order to be able to define and deal effectively with corresponding stochastic differential equations. By integration the meaning of a stochastic differential equation is reduced to the meaning of the corresponding stochastic integrals. In our case, this means that we have to define objects like $\int A_t dS_t B_t$, where dS_t is the increment of the free Brownian motion and where $(A_t)_{t \ge 0}$ and $(B_t)_{t \ge 0}$ are adapted processes. ($(A_t)_{t \ge 0}$ adapted means that, for each $t \ge 0$, A_t is an element of the von Neumann algebra generated by all S_s with $s \le t$.) In contrast to the classical case, our processes and the increments do not commute, so one should really consider this bilinear integral in (A_t, B_t) instead just a one-sided integral. Such stochastic integrals are defined as usual, namely for elementary processes, which are constant on time intervals I_i and take there a fixed value A_i or B_i, the integral is defined as

$$(20) \qquad \int A_t dS_t B_t := \sum_i A_i S(I_i) B_i,$$

where $S(I_i)$ is the increment of the free Brownian motion over the interval I_i. Then one has to prove estimates for such integrals in some suitable norms and extend the definiton of the integral to the closure of elementary functions under the involved norms. The easiest norm estimate is an L^2-estimate which works in the same way as for other stochastic calculi and which yields the usual Ito-isometry. Results of Pisier and Xu [PX] on non-commutative martingales can be used to obtain L^p-estimates for $p < \infty$. Whereas such kind of estimates are also true for other kind of stochastic calculi, a very specific feature of the free calculus is that one can also derive L^∞-estimates, i.e. one can estimate the integrals in operator norm.

Theorem 3.3.1. ([BSp1]) *Let* $(A_t)_{t\geq 0}$ *and* $(B_t)_{t\geq 0}$ *be adapted processes. Then we have*

$$\| \int A_t dS_t B_t \| \leq 2\sqrt{2} \Big(\int \|A_t\|^2 \cdot \|B_t\|^2 dt \Big)^{1/2}. \tag{21}$$

Having established the existence of the free stochastic integrals in nice topologies one can continue to investigate the corresponding stochastic calculus. There exists also a free Ito formula [KSp, BSp1], which, on a formal differential level, states that

$$dS_t A dS_t = \varphi(A)dt \qquad \text{for } A \text{ adapted.} \tag{22}$$

This should be compared to the classical Ito formula $dB_t A dB_t = Adt$. The differences between the usual stochastic calculus and the free stochastic calculus can, on a formal level, be reduced to this difference between the corresponding Ito formulas.

One can also derive free analogues of classical stochastic analysis. In [BSp1] we treated, e.g., iterated stochastic integrals, which give rise to a chaos decompositon of the L^2-space of the free Brownian motion and allow to prove a representation theorem for martingales or to extend the free Ito integral to a free Skorohod integral for non-adapted processes.

3.4. Free diffusion.

Definition 3.4.1. We will consider the free stochastic differential equation

$$dX_t = -\frac{1}{2} V'(X_t)dt + dS_t \tag{23}$$

We call the solution of (23), if it exists, the **free diffusion** in the potential V.

Remark 3.4.2. In the same way as free Brownian motion describes the behaviour of infinitely many particles which interact with a special pair-interaction, the free diffusion in the potential V describes the behaviour of such particles if we put them in addition into a potential V.

Theorem 3.4.3. ([BSp2]) *Let* X_0 *be free from the free Brownian motion* $(S_t)_{t\geq 0}$ *and* V' *be sufficiently smooth (e.g.,* $V' \in C^2$*).*
1) Then there exists a unique solution $(X_t)_{t\geq 0}$ *of the equation (23). Furthermore, we have that* X_t *lies in the* C^**-algebra generated by* X_0 *and all* S_s *with* $s \leq t$ *and that the mapping* $t \mapsto X_t$ *is* $\|\cdot\|$*-continuous.*
2) The distribution of X_t *is absolutely continuous with respect to Lebesgue measure,* $distr(X_t) = p_t(x)dx$*, where the density* p_t *is bounded*

(but not smooth in general) and a weak solution of the following **free Fokker-Planck equation**

$$\frac{\partial p_t(x)}{\partial t} = -\frac{\partial}{\partial x}\left[(Hp_t(x) - \frac{1}{2}V'(x))p_t(x)\right], \tag{24}$$

where H is (up to a constant) the Hilbert transform, i.e.

$$Hp(x) := \int \frac{p(y)}{x-y} dy. \tag{25}$$

Remarks 3.4.4. 1) Note that the free Fokker-Planck equation (24) is compatible with the picture of infinitely many interacting particles in the potential V: the particles at position x feel a force coming via the pair-interaction from the other particles at all possible positions y and in addition the force $V'(x)$ coming from the potential.
2) The structure of the free Fokker-Planck equation is on a formal level very similar to the classical Fokker-Planck equation (10); the only difference is that the second derivative is replaced by the Hilbert transform Hp_t; however, this changes of course totally the nature of the considered equation; instead of a second-order linear we have now a first-order non-linear partial differential equation. The non-linearity reflects the fact that we are dealing with interacting particles; in contrast, classical free diffusion can be thought of as infinitely many diffusing particles in the potential V without any interaction.

3.5. **Free diffusion and free entropy.** The above mentioned results show a formal analogy between classical diffusion and free diffusion. But this analogy goes much further. As mentioned in Sect. 2, there exists a relation between classical diffusion and classical entropy. There is also a free counterpart of that. Voiculescu introduced free analogues of the classical notions of entropy and Fisher information [V4, V5]. A relative version (with respect to V) of these are as follows. ($V = 0$ corresponds to the original definition of Voiculescu).

Notations 3.5.1. The **relative free entropy** and the **relative free Fisher information** are given by

$$\Sigma_V(\mu) := \int\int \log|x-y|\mu(dx)\mu(dy) - \int V(x)\mu(dx) \tag{26}$$

and (for $\mu(dx) = p(x)dx$)

$$I_V(\mu) := 4\int \left(Hp(x) - \frac{1}{2}V'(x)\right)^2 p(x)dx, \tag{27}$$

respectively.

With these notations we have the following theorem.

Theorem 3.5.2. ([BSp2]) *Let $(X_t)_{t\geq 0}$ be the solution of the free diffusion equation (23). Then we have*

$$\frac{d}{dt}\Sigma_V(X_t) = \frac{1}{2}I_V(X_t). \tag{28}$$

In particular, $\Sigma_V(X_t)$ is increasing with t.

If we replace Σ_V and I_V by their classical counterparts then the same theorem is true for classical diffusion.

3.6. **Conclusion.** Formally there exists a very far reaching analogy between the theory of free diffusion and the theory of classical diffusion. However, free diffusion and classical diffusion describe quite different situations. Whereas the latter provides a theory for diffusing particles without interaction the former describes particles with a special type of pair-interaction. It is very surprising (but also exciting and promising) that a special type of interaction behaves in a very probabilistic way. Free probability theory seems to be the right tool for dealing with this kind of interaction.

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