A Girsanov-type Formula for Lévy Processes on Commutative Hypergroups

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Abstract

In this note we present a Girsanov-type formula which turns (central) Bessel processes on $[0,\infty[$ of arbitary indices into non-central ones. It will be shown that this result may be seen as a special case of a general Girsanov formula for Lévy processes on commutative hypergroups which connects Lévy processes on different hypergroup structures on the same ground space, where the associated convolutions are related by some deformation.

1 Introduction

In this paper we present some Girsanov formula for Lévy processes on commutative hypergroups. We first illustrate the main result with Bessel processes on $[0,\infty[$, as these processes may be regarded as Lévy processes on the so-called Bessel-Kingman hypergroups; the understanding of this example requires no knowledge about hypergroups.

We start with an n-dimensional Brownian motion $(B_t)_{t\geq 0}$ defined on the Wiener space $(\Omega,\mathcal{F},P)$ with

$$\Omega=\mathcal{C}(\mathbb{R}^n):=\{f:[0,\infty[\to\mathbb{R}^n,\ f \text{ continuous}\},$$

which carries the right-continuous, complete induced filtration $(\mathcal{F}_t)_{t\geq 0}$ as usually with $\mathcal{F}=\sigma(\mathcal{F}_t:t\geq 0)$. The classical formula of Girsanov then in particular implies that for any drift vector $c\in\mathbb{R}^n$ there is a unique probability measure Q_c on $(\Omega,\mathcal{F})$ with

$$Q_c|_{\mathcal{F}_t}=e^{<c,B_t>-t\|c\|_2^2/2}P|_{\mathcal{F}_t} \qquad \text{for} \quad t\geq 0,$$

and with respect to Q_c, the process $(B_t)_{t\geq 0}$ is a Brownian motion on $\mathbb{R}^n$ with drift c. Moreover, for

$$\Phi:\mathbb{R}^n\longrightarrow[0,\infty[,\quad x\longmapsto|x|=(x_1^2+\ldots+x_n^2)^{1/2},$$

the process $(\Phi(B_t))_{t\geq 0}$ is a Bessel process of dimension n; see [RY] for details. This process may be regarded as coordinate process $(X_t)_{t\geq 0}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ with

$$\tilde{\Omega} := \{f : [0,\infty[\to [0,\infty[,\ f \text{ continuous}\},$$

with the canonical σ-algebras, and with $\tilde{P}$ as image of P under the projection $\Psi : \Omega \to \tilde{\Omega}$ which is uniquely determined by

$$\Psi(\omega)_t = \Phi(\omega_t) \qquad \text{for} \quad t \geq 0.$$

Using the rotation invariance of $(B_t)_{t\geq 0}$ and the integral representation

$$j_{n/2-1}(x) := \int_{S^{n-1}} e^{i<x,y>}\, dU_{n-1}(y) \quad (x \in \mathbb{C})$$

of the spherical Bessel function $j_{n/2-1}$ (with U_{n-1} the uniform distribution on the unit sphere $S^{n-1} \subset \mathbb{R}^n$; see 9.1.20 of [AS]), we obtain for any drift $c \in \mathbb{R}^n$ that the distribution $\tilde{Q}_c := \Psi(Q_c) \in M^1(\tilde{\Omega}, \tilde{\mathcal{F}})$ satisfies

$$\tilde{Q}_c|_{\tilde{\mathcal{F}}_t} = e^{-t\|c\|_2^2/2} j_{n/2-1}(i\|c\|_2 X_t)\tilde{P}|_{\tilde{\mathcal{F}}_t} \qquad for \quad t \geq 0.$$

Moreover, as for a Brownian motion $(B_t)_{t\geq 0}$ on $\mathbb{R}^n$ with drift c the process $(\Phi(B_t))_{t\geq 0}$ is a non-central Bessel process with dimension n and non-centrality parameter $\|c\|_2$, it can be derived from the classical Girsanov formula that, with respect to $\tilde{Q}_c$, the coordinate process $(X_t)_{t\geq 0}$ is such a process. As there exist central and non-central Bessel processes also for "fractional dimensions" $n \in \mathbb{R}$, $n \geq 1$, it is natural to ask whether the change of measure above here also turns central Bessel processes into non-central ones. We shall give a positive answer in Theorem 3.8 below.

We shall show below how this result may be regarded as a special case of a Girsanov-type formula for Lévy processes on commutative hypergroups of the following kind: Let $(X_t)_{t\geq 0}$ be a Lévy process on some commutative hypergroup $(K, *)$ that is associated with some convolution semigroup $(\mu_t)_{t\geq 0}$. Then, for any positive semicharacter α of $(K, *)$, the hypergroup convolution $*$ can be deformed into some new hypergroup convolution, say $\bullet$ (see [BH, V1, V2]). We shall show that under some growth condition, $(\mu_t)_{t\geq 0}$ can be transformed into some convolution semigroup $(\tilde{\mu}_t)_{t\geq 0}$ on $(K, \bullet)$, and that some Girsanov-type change of measure transforms $(X_t)_{t\geq 0}$ into a Lévy process on $(K, *)$ associated with $(\tilde{\mu}_t)_{t\geq 0}$. The proof of this result will be based on a martingale characterization of Lévy processes in terms of hypergroup characters; see [RV]. This main result will be discussed in Section 2 of this paper. Section 3 will be devoted to several examples and includes, in particular, a discussion of Bessel processes.

We finally mention that the results of this paper are completely disjoint to Girsanov formulas for Brownian motions on Lie groups (see [I, Kar]), as groups do not admit nontrivial positive semicharacters and hypergroup deformations. On the other hand, we hope that martingale characterizations of Lévy processes on locally compact groups in [V3, V4] in terms of group representations may be used to generalize the results of [Kar].

2 Renormalization of commutative hypergroups and a Girsanov-type formula

We first recapitulate some notations and facts about Lévy processes on commutative hypergroups. For details on hypergroups we refer to the monograph [BH] and to [J].

2.1. Commutative hypergroups. A commutative hypergroup $(K,*)$ consists of a locally compact space K together with a commutative, weakly continuous, probability preserving convolution $*$ on the Banach space $M_b(K)$ of all bounded regular Borel measures on K satisfying certain axioms which are well known from convolutions of measures on locally compact abelian groups. We denote the identity of $(K,*)$ by e, and the hypergroup involution by $.^-$. It is well known (see [S]) that each commutative hypergroup $(K,*)$ admits a Haar measure $\omega_{(K,*)}$ which is unique up to some multiplicative constant. The dual space

$$\widehat{K}^* := \{\alpha \in C_b(K): \ \alpha \not\equiv 0, \ \int \alpha \, d(\delta_x * \delta_{\bar{y}}) = \alpha(x)\overline{\alpha(y)} \quad \text{for all } x, y \in K\}$$

is a locally compact space w.r.t. the topology of compact-uniform convergence. Elements of $\widehat{K}^*$ are called characters.

The Fourier transforms of $f \in L^1(K, \omega_{(K,*)})$ and $\mu \in M_b(K)$ are given by

$$\widehat{f}^*(\alpha) = \int_K \overline{\alpha(x)}\, f(x)\, d\omega_{(K,*)}(x) \qquad \text{and} \qquad \widehat{\mu}^*(\alpha) = \int_K \overline{\alpha}(x)\, d\mu(x) \qquad (\alpha \in \widehat{K}^*)$$

respectively. It is also well-known (Jewett [J]) that $\widehat{K}^*$ carries a unique Plancherel measure $\pi_{(K,*)}$ such that the Fourier transform on $L^1(K, \omega_{(K,*)}) \cap L^2(K, \omega_{(K,*)})$ extends uniquely to an isometric isomorphism between $L^2(K, \omega_{(K,*)})$ and $L^2(\widehat{K}, \pi_{(K,*)})$. Notice that $supp\, \pi_{(K,*)}$ may be a proper subset of $\widehat{K}^*$. We here notice that the Fourier transform

$$\widehat{.}^* : M_b(K) \longrightarrow C_b(supp\, \pi_{(K,*)}), \quad \mu \longmapsto \widehat{\mu}^*|_{supp\, \pi_{(K,*)}}$$

is injective (see Theorem 2.2.4 of [BH]).

2.2. Convolution semigroups and Lévy processes. A family $(\mu_t)_{t\geq 0} \subset M^1(K)$ of probability measures on a commutative hypergroup $(K,*)$ is called a convolution semigroup, if

$\mu_s * \mu_t = \mu_{s+t}$ for all $s, t \geq 0$ with $\mu_0 = \delta_e$, and if $[0, \infty[\to M^1(K)$, $t \mapsto \mu_t$ is weakly continuous.

Let $(\mu_t)_{t\geq 0}$ be a convolution semigroup on $(K, *)$. A K-valued Markov process $X = (X_t)_{t\geq 0}$ with filtration $(\mathcal{F}_t)_{t\geq 0}$ (and defined on some probability space $(\Omega, \mathcal{F}, P)$) is called a Lévy process on $(K, *)$ associated with $(\mu_t)_{t\geq 0}$ and $(\mathcal{F}_t)_{t\geq 0}$, if its transition probabilities satisfy

$$P(X_t \in A |\, X_s = x) = (\mu_{t-s} * \delta_x)(A) \quad (0 \leq s \leq t,\ x \in K,\ A \subset K \text{ a Borel set}).$$

If the process X above is defined on a time interval $[0, T]$ only and has the properties above there, then it is called a restriction of a Lévy process on $(K, *)$ associated with $(\mu_t)_{t\geq 0}$ and $(\mathcal{F}_t)_{t\geq 0}$.

It can be easily checked that all (restricted) Lévy processes on $(K, *)$ are Feller processes and hence admit càdlàg versions; see [RV]. Moreover, one can construct martingales from Lévy processes on $(K, *)$ by using hypergroup characters. The following version of a martingale characterization of Lévy processes on commutative hypergroups was derived in [RV]; it is closely related with other versions for general (homogeneous) Markov processes as discussed, for instance, in Ch. 4 of [EK].

2.3. Lemma. *Let $(\mu_t)_{t\geq 0}$ be a convolution semigroup on the commutative hypergroup $(K, *)$. Then for each stochastic process X on K, which is adapted w.r.t. some filtration $(\mathcal{F}_t)_{t\geq 0}$, the following statements are equivalent:*

(1) X is a Lévy process on $(K, *)$ associated with $(\mu_t)_{t\geq 0}$ and $(\mathcal{F}_t)_{t\geq 0}$.

(2) For each $\alpha \in \widehat{K}^*$, the $\mathbb{C}$-valued process $(\widehat{\mu}_t^*(\overline{\alpha})^{-1} \cdot \alpha(X_t))_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-martingale.

(3) For each $\alpha \in supp\, \pi$, the process $(\widehat{\mu}_t(\overline{\alpha})^{-1} \cdot \alpha(X_t))_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-martingale.

An inspection of the proof of this lemma in [RV] shows that a corresponding result also holds for restricted Lévy processes.

2.4. Renormalization of commutative hypergroups. For commutative hypergroups $(K, *)$, the support $supp\, \pi_{(K,*)}$ of the Plancherel measure may be a proper subset of $\widehat{K}^*$. It was observed in [V1] that this property is closely related with the fact that commutative hypergroups $(K, *)$ may admit positive semicharacters, i.e., positive functions $\alpha_0 \in C(K)$ that admit all properties of characters except that they may be unbounded. It was shown in [V1] that each positive semicharacter α_0 on a commutatice hypergroup $(K, *)$ induces a new hypergroup structure $(K, \bullet)$ (where, by convention, the underlying positive semicharacter α_0 as index will be suppressed); the convolution $\bullet$ is determined uniquely by the convolution of point measures:

$$\delta_x \bullet \delta_y = \frac{1}{\int \alpha_0 \, d(\delta_x * \delta_y)} \cdot \alpha_0 \cdot (\delta_x * \delta_y) \qquad (x, y \in K).$$

Identity and involution of $(K, \bullet)$ are the same as of $(K, *)$. We next give a list of further connections between the data of the hypergroups $(K, *)$ and $(K, \bullet)$; for details see [V1]:

(1) If $\mu, \nu \in M_b(K)$ satisfy $\alpha_0\mu, \alpha_0\nu \in M_b(K)$, then $\alpha_0\mu \bullet \alpha_0\nu = \alpha_0(\mu * \nu)$.

(2) $\omega_{(K,\bullet)} := \alpha_0^2 \omega_{(K,*)}$ is "the" Haar measure of $(K, \bullet)$.

(3) The dual space of $(K, \bullet)$ is given by

$$\widehat{K}^\bullet := \{\alpha/\alpha_0 : \alpha \text{ a semicharacter of } (K, *) \text{ with } |\alpha| \leq \alpha_0\}.$$

(4) If $\pi_{(K,\bullet)}$ denotes the Plancherel measure of $(K, \bullet)$ on $\widehat{K}^\bullet$, then the mapping $\widehat{K}^* \longrightarrow \widehat{K}^\bullet$, $\alpha \longmapsto \alpha/\alpha_0$ is a homeomorphism that maps $\pi_{(K,*)}$ into $\pi_{(K,\bullet)}$.

(5) The hypergroups $(K, *)$ and $(K, \bullet)$ may be interchanged above by using the fact that $1/\alpha_0$ is a positive semicharacter of $(K, \bullet)$, and that the associated renormalized hypergroup structure is just the original hypergroup $(K, *)$.

Let α_0 be a positive semicharacter on a commutative hypergroup $(K, *)$. We now show how convolution semigroups on $(K, *)$ can be transformed into convolution semigroups on $(K, \bullet)$. For this we say that a convolution semigroup $(\mu_t)_{t \geq 0}$ on $(K, *)$ is α_0-continuous whenever

$$[0, \infty[\to [0, \infty[, \quad t \longmapsto h(t) := \int_K \alpha_0 \, d\mu_t$$

is finite and continuous. If $\alpha_0 \in \widehat{K}^*$ is a positive character, then clearly each convolution semigroup on $(K, *)$ is α_0-continuous.

2.5. Lemma. *Let α_0 be a positive semicharacter and $(\mu_t)_{t \geq 0}$ an α_0-continuous convolution semigroup on $(K, *)$ Then, for all $s, t \geq 0$, $h(s) \cdot h(t) = h(s + t)$, and $\big(\mu_t^{\alpha_0} := \frac{1}{h(t)} \cdot \alpha_0 \mu_t\big)_{t \geq 0}$ is a convolution semigroup on $(K, \bullet)$.*

Proof. Clearly, $\mu_t^{\alpha_0} \in M^1(K)$ for all $t \geq 0$. Hence, for all $s, t \geq 0$, $\mu_s^{\alpha_0} \bullet \mu_t^{\alpha_0} \in M^1(K)$. Moreover, by Section 2.4,

$$\mu_s^{\alpha_0} \bullet \mu_t^{\alpha_0} = \frac{1}{h(s)h(t)} (\alpha_0 \mu_s) \bullet (\alpha_0 \mu_t) = \frac{1}{h(s)h(t)} \alpha_0 (\mu_s * \mu_t) = \frac{h(s+t)}{h(s)h(t)} \frac{1}{h(s+t)} \alpha_0 \mu_{s+t}.$$

As $\frac{1}{h(s+t)} \alpha_0 \mu_{s+t} \in M^1(K)$, it follows that $h(s) \cdot h(t) = h(s+t)$ and $\mu_s^{\alpha_0} \bullet \mu_t^{\alpha_0} = \mu_{s+t}^{\alpha_0}$. The continuity of h finally ensures that $t \mapsto \mu_t^{\alpha_0}$ is vaguely and hence weakly continuous. $\square$

The following Girsanov formula connects Lévy processes associated with $(\mu_t)_{t\geq 0}$ and $(\mu_t^{\alpha_0})_{t\geq 0}$.

2.6. Theorem. *Let α_0 be a positive semicharacter and $(\mu_t)_{t\geq 0}$ an α_0-continuous convolution semigroup on the commutative hypergroup $(K,*)$. Let $(X_t)_{t\geq 0}$ be a Lévy process on $(K,*)$ with filtration $(\mathcal{F}_t)_{t\geq 0}$ and with convolution semigroup $(\mu_t)_{t\geq 0}$ that is defined on some probability space $(\Omega,\mathcal{F},P)$. Then for each $T\geq 0$, the process $(X_t)_{t\in[0,T]}$ on the probability space $(\Omega,\mathcal{F}_T,\frac{1}{h(T)}\alpha_0(X_T)\cdot P)$ is the restriction of a Lévy process on $(K,\bullet)$ associated with $(\mu_t^{\alpha_0})_{t\geq 0}$.*

Proof. As $(X_t)_{t\geq 0}$ is a Lévy process on $(K,*)$ with filtration $(\mathcal{F}_t)_{t\geq 0}$ and with convolution semigroup $(\mu_t)_{t\geq 0}$, we see that for all $s,t\geq 0$ and P-almost all $\omega\in\Omega$,

$$E(\alpha_0(X_{s+t})|\mathcal{F}_s)(\omega) = E(\alpha_0(X_{s+t})|X_s)(\omega) = \int_K \alpha_0\, d(\mu_t * \delta_{X_s(\omega)}) = h(t)\cdot\alpha_0(X_s(\omega)).$$

Using $h(s+t) = h(s)h(t)$, we obtain that $(Z_t := \frac{1}{h(t)}\alpha_0(X_t))_{t\geq 0}$ is a positive $(\mathcal{F}_t)_{t\geq 0}$-martingale with $E(Z_t)=1$. In particular, $(Z_t\cdot P|_{\mathcal{F}_t})_{t\geq 0}$ is a family of probability measures with

$$(Z_t\cdot P|_{\mathcal{F}_t})_{\mathcal{F}_s} = Z_s\cdot P|\mathcal{F}_s \qquad \text{for} \quad s,t\geq 0.$$

Now let $\alpha\in supp\,\pi_{(K,\bullet)}$ be a character of $(K,\bullet)$ contained in the support of the Plancherel measure. Section 2.4 shows that $\tilde{\alpha} := \alpha\cdot\alpha_0$ is a character of $(K,*)$, and, by the definition of $\mu_t^{\alpha_0}$,

$$\widehat{\mu}_t^*(\overline{\tilde{\alpha}}) = \int_K \alpha(x)\alpha_0(x)\, d\mu_t(x) = h(t)\cdot(\mu_t^{\alpha_0})^{\wedge\bullet}(\overline{\alpha}) \qquad (t\geq 0)$$

where $.^{\wedge\bullet}$ denotes the Fourier transform w.r.t. $(K,\bullet)$. Lemma 2.3 now yields that

$$\left(\frac{1}{(\mu_t^{\alpha_0})^{\wedge\bullet}(\overline{\alpha})}\cdot Z_t\alpha(X_t) = \frac{1}{\widehat{\mu}^*(\overline{\tilde{\alpha}})}\tilde{\alpha}(X_t)\right)_{t\geq 0}$$

is an $(\mathcal{F}_t)_{t\geq 0}$-martingale on $(\Omega,\mathcal{F},P)$. Using the properties of $(Z_t)_{t\geq 0}$, we see that for $T>0$,

$$\left(\frac{1}{(\mu_t^{\alpha_0})^{\wedge\bullet}(\overline{\alpha})}\cdot\alpha(X_t)\right)_{t\in[0,T]}$$

is an $(\mathcal{F}_t)_{t\in[0,T]}$-martingale on the probability space $(\Omega,\mathcal{F},Z_TP)$. As this holds for all $\alpha\in supp\,\pi_{(K,\bullet)}$, Lemma 2.3 implies that the process $(X_t)_{t\in[0,T]}$ on $(\Omega,\mathcal{F},Z_TP)$ is the restriction of a Lévy process on $(K,\bullet)$ associated with $(\mu_t^{\alpha_0})_{t\geq 0}$. □

We now give an extension of the preceding result to the complete time interval $[0,\infty[$.

2.7. Theorem. *Let α_0 be a positive semicharacter and $(\mu_t)_{t\geq 0}$ an α_0-continuous convolution semigroup on the commutative Polish hypergroup $(K,*)$. Let $(X_t)_{t\geq 0}$ be a Lévy process on $(K,*)$ associated with $(\mu_t)_{t\geq 0}$ defined on the probability space $(\Omega,\mathcal{F},P)$ with*

$$\Omega = \mathcal{D}(K) := \{f : [0,\infty[\rightarrow K,\ f \text{ càdlàg}\}$$

and equipped with the right-continuous and complete induced filtration $(\mathcal{F}_t)_{t\geq 0}$. Then there exists a unique probability measure Q on $(\Omega, \sigma(\mathcal{F}_t : t\geq 0))$ with

$$Q|_{\mathcal{F}_t} = \frac{1}{h(t)}\alpha_0(X_t)P|_{\mathcal{F}_t} \qquad for \quad t \geq 0,$$

and with respect to Q, the process $(X_t)_{t\geq 0}$ is a Lévy process on $(K,\bullet)$ associated with $(\mu_t^{\alpha_0})_{t\geq 0}$.

Proof. In view of the proof of the preceding result it suffices to check existence and uniqueness of Q. Uniqueness, however, is clear, and the existence follows from Lemma 16.18 of [Kal]. □

2.8. Remark. Lemmas 2.3 and 2.5 as well as Theorems 2.6 and 2.7 can easily be adapted to the setting of time-homogeneous random walks $(X_n)_{n\geq 0}$ on commutative hypergroups.

2.9. Remark. Theorems 2.6 and 2.7 may be regarded as special cases of more general Girsanov-type formulas for Feller processes which satisfy certain technical restrictions. We shall present details of this generalization elsewhere and include some ideas here only:

Assume that α_0 is a positive semicharacter and $(\mu_t)_{t\geq 0}$ an α_0-continuous convolution semigroup on some commutative hypergroup $(K,*)$. The associated Lévy processes are Feller, and the generator G of the associated Feller semigroup on $C_0(K)$ is given by

$$Gf(x) = \lim_{t\to 0}\frac{1}{t}(\mu_t^- * f(x) - f(x)) \qquad (x\in K,\ f\in D(G))$$

where the domain $D(G)$ of G is $\|.\|_\infty$-dense in $C_0(K)$; see [RV]. Now consider the generator G^{α_0} of the Feller semigroup on $C_0(K)$ that is associated with the renormalized convolution semigroup $(\mu_t^{\alpha_0})_{t\geq 0}$ on $(K,\bullet)$. Then, using the notation above, we have

$$((\mu_t^{\alpha_0})^- \bullet f)(x) = \frac{1}{h(t)}((\alpha_0\mu_t)^- \bullet f)(x) = \frac{1}{h(t)\alpha_0(x)}(\mu_t * \alpha_0 f)(x)$$

(see p. 408 of [V1]). Moreover, by Lemma 2.5 we have $h(t) = e^{ct}$ for some $c\in\mathbb{R}$, and hence

$\lim_{t\to 0}\frac{1}{t}(1/h(t)-1) = -c$. Hence,

$$\begin{aligned} G^{\alpha_0}f(x) &= \lim_{t\to 0}\frac{1}{t}(((\mu_t^{\alpha_0})^- \bullet f)(x) - f(x)) = \lim_{t\to 0}\frac{1}{t}\Big(\frac{1}{h(t)\alpha_0(x)}(\mu_t * \alpha_0 f)(x) - f(x)\Big) \\ &= \frac{1}{\alpha_0(x)}\lim_{t\to 0}\frac{1}{t}\Big(\frac{1}{h(t)}(\mu_t * \alpha_0 f)(x) - (\alpha_0 f)(x)\Big) \\ &= \frac{1}{\alpha_0(x)}G(\alpha_0 f)(x) + \frac{1}{\alpha_0(x)}\lim_{t\to 0}\Big(\frac{1}{t}(1/h(t)-1)(\mu_t * \alpha_0 f)(x)\Big) \\ &= \frac{1}{\alpha_0(x)}G(\alpha_0 f)(x) - cf(x). \end{aligned}$$

Therefore, if M_g is the multiplication operator with some function $g \in C(K)$, then formally

$$G^{\alpha_0} = M_{1/\alpha_0} \circ G \circ M_{\alpha_0} - c \tag{2.1}$$

where α_0 is an eigenfunction of G with eigenvalue c.

We expect that Theorems 2.6 and 2.7 can be extended in this way to arbitrary generators G of Feller semigroups on locally compact spaces K and arbitrary "eigenfunctions" $\alpha_0 \in C(K)$ of G with eigenvalue c under certain restrictions concerning the domain of G. We mention that a related result for Feller processes on finite state spaces is given in Section IV.22 of [RW].

Lemma 2.5 admits the following converse statement:

2.10. Lemma. *Let α_0 be a positive semicharacter on $(K,*)$ with $\alpha_0 \geq 1$, and let $(\mu_t)_{t\geq 0}$ a convolution semigroup on $(K,*)$ with generator G. Assume that*

$$G^{\alpha_0} := M_{1/\alpha_0} \circ G \circ M_{\alpha_0} - c$$

(where c satisfies $G\alpha_0 = c\alpha_0$, and M is given as in 2.9) is the generator of a convolution semigroup $(\tilde{\mu}_t^{\alpha_0})_{t\geq 0}$ on the modified hypergroup $(K,\bullet)$. Then $(\mu_t)_{t\geq 0}$ is α_0-continuous, and $(\tilde{\mu}_t^{\alpha_0})_{t\geq 0}$ is equal to the convolution semigroup $(\mu_t^{\alpha_0})_{t\geq 0}$ of Lemma 2.5.

Proof. By our assumption, $1/\alpha_0$ is a positive character on $(K,\bullet)$. Now apply Lemma 2.5 and Remark 2.9 to $1/\alpha_0$ and the $1/\alpha_0$-continuous convolution semigroup $(\tilde{\mu}_t^{\alpha_0})_{t\geq 0}$ on $(K,\bullet)$. Then the renormalization of $\bullet$ is just $*$, and the generator of the convolution semigroup on $(K,*)$, which is the deformation of $(\tilde{\mu}_t^{\alpha_0})_{t\geq 0}$ according to 2.5, is given by G. Hence, for $t \geq 0$,

$$\mu_t = \frac{1}{\tilde{h}(t)\cdot\alpha_0}\tilde{\mu}_t^{\alpha_0} \quad \text{where} \quad t \longmapsto \tilde{h}(t) := \int_K 1/\alpha_0 \, d\tilde{\mu}_t^{\alpha_0} \quad \text{is continuous.}$$

This shows that the function h of Lemma 2.5 is equal to $1/\tilde{h}$ and hence continuous. The remaining assertions are now obvious. □

3 Examples

In this section we present a few examples to which the Girsanov-type formulas 2.6 and 2.7 may be applied. The most prominent examples will be Bessel processes which may be regarded as Lévy processes on the Bessel-Kingman hypergroups and their modifications. As a preparation we first discuss positive semicharacters on general Sturm-Liouville hypergroups on $[0,\infty[$.

3.1 Sturm-Liouville hypergroups on $[0,\infty[$

(1) A function $A \in C([0,\infty[) \cap C^1(]0,\infty[)$ is called admissible if $A(x) > 0$ for $x > 0$, and if there exist constants $\epsilon > 0$, $\alpha_0 \geq 0$ and $\alpha_1 \in C^\infty(]-\epsilon,\epsilon[)$ with

$$A'(x)/A(x) = \frac{\alpha_0}{x} + x \cdot \alpha_1(x) \quad \text{for all } x \in]0,\epsilon[.$$

In the singular case $\alpha_0 > 0$ we assume in addition that α_1 is even.

(2) The Sturm-Liouville operator associated with an admissible A is defined by

$$L^A f(x) := -\frac{1}{A(x)} \cdot (A(x) \cdot f'(x))' \quad \text{for } f \in C^2(]0,\infty[),\ x > 0.$$

(3) A hypergroup $([0,\infty[,*)$ is called a Sturm-Liouville hypergroup if there exists an admissible function A such that for each even $f \in C^\infty(\mathbb{R})$ the function $u_f(x,y) := \int_0^\infty f\, d(\delta_x * \delta_y)$ $(x,y \geq 0)$ satisfies $u_f \in C^2([0,\infty[^2)$ with

$$L_x^A u(x,y) - L_y^A u(x,y) = 0 \qquad \text{and} \qquad (u_f)_y(x,0) = 0 \qquad \text{for} \quad x,y \geq 0$$

where subscripts indicate variables with respect to which the operator L^A is applied.

3.1. Facts. Let $([0,\infty[,*)$ be a Sturm-Liouville hypergroup associated with some admissible function A that satisfies some further technical restriction; see [Z] and Ch. 3.5 of [BH]. Then the following statements hold:

(1) $\rho := \frac{1}{2}\lim_{x\to\infty} A'(x)/A(x)$ exists with $\rho \geq 0$; it is called the index of K.

(2) A function $\alpha \in C([0,\infty[)$ is multiplicative on K, i.e., $(\delta_x * \delta_y)(\alpha) = \alpha(x)\alpha(y)$ for all $x,y \geq 0$, if and only if $\alpha \in C^2([0,\infty[)$, and if α is the unique solution of the eigenvalue problem

$$L^A\alpha = s_\alpha \cdot \alpha \quad \text{with} \quad \alpha(0) = 1,\ \alpha'(0) = 0 \quad \text{for some } s_\alpha \in \mathbb{C}.$$

According to [BH, Z], we parametrize the eigenvalues by $\lambda_\alpha^2 + \rho^2 = s_\alpha$ with $\lambda_\alpha \in \mathbb{C}$. In this notation, the dual space $\widehat{K}$ and the support of the Plancherel measure are given by $\widehat{K} = \{\alpha \text{ multiplicative}: \lambda_\alpha \in [0,\infty[\cup i]0,\rho]\}$ and $supp\,\pi = \{\alpha \in \widehat{K} : \lambda_\alpha \in [0,\infty[\}$. Moreover, α is a positive semicharacter if and only if $\lambda_\alpha \in i\cdot[0,\infty[$ holds; see [V1, Z].

(3) If α is a positive character on $([0,\infty[,*)$ with $\lambda_\alpha \in i\cdot[0,\infty[$, then the associated modified hypergroup $([0,\infty[,\bullet)$ is the Sturm-Liouville hypergroup associated with the admissible function $A_\alpha(x) := \alpha(x)^2 A(x)$; see [V1].

3.2. Diffusions on $[0,\infty[$ as Lévy processes. It is well known (see [C,RV]) that for each Sturm-Liouville hypergroup $([0,\infty[,*)$ with admissible A, the operator $-L^A$ is the generator of a convolution semigroup $(\mu_t)_{t\geq 0}$ on $([0,\infty[,*)$. Now let α is an arbitrary positive character on $([0,\infty[,*)$ with $\lambda_\alpha \in i\cdot[0,\infty[$. We now check that the assumptions of Theorems 2.6 and 2.7 are satisfied:

3.3. Lemma. *In the above setting, $(\mu_t)_{t\geq 0}$ is α-continuous with*

$$h(t) := \int_0^\infty \alpha\, d\mu_t = e^{-t(\lambda_\alpha^2+\rho^2)} \qquad (t \geq 0)$$

Proof. The lemma is obvious for $\alpha \in \widehat{K}$, i.e., $\lambda_\alpha^2 + \rho^2 \geq 0$. Otherwise we have $\alpha \geq 1$ on $[0,\infty[$ (see [BH] or [Z]) and we may consider the modified hypergroup $([0,\infty[,\bullet)$ with $A_\alpha := \alpha^2 A$ which is associated with α. A short computation yields

$$(M_{1/\alpha} \circ (-L^A) \circ M_\alpha) + \lambda_\alpha^2 + \rho^2 = -L^{\alpha^2 A}$$

where, by our considerations above, $-L^{\alpha^2 A}$ is the generator of a convolution semigroup on $([0,\infty[,\bullet)$. The lemma now follows from Lemma 2.10. □

Theorem 2.7 now reads as follows in our present case:

3.4. Theorem. *Let $([0,\infty[,*)$ be a Sturm-Liouville hypergroup with associated function A and index ρ. Then the operator*

$$-L^A = \frac{1}{A(x)} \cdot \frac{d}{dx}(A(x) \cdot \frac{d}{dx})$$

is the generator of a convolution semigroup $(\mu_t)_{t\geq 0}$ on $([0,\infty[,*)$. Let $(X_t)_{t\geq 0}$ be an associated Lévy process $([0,\infty[,*)$, i.e., $(X_t)_{t\geq 0}$ is a diffusion with generator $-L^A$. Assume that $(X_t)_{t\geq 0}$ is defined on the probability space $(\Omega,\mathcal{F},P)$ with

$$\Omega := \{f : [0,\infty[\to [0,\infty[,\ f \text{ continuous}\}$$

and is equipped with the right-continuous, complete induced filtration $(\mathcal{F}_t)_{t\geq 0}$. Then for each positive semicharacter α on $([0,\infty[,*)$, there exists a unique probability measure Q on $(\Omega, \sigma(\mathcal{F}_t : t \geq 0))$ with

$$Q|_{\mathcal{F}_t} = e^{t(\lambda_\alpha^2+\rho^2)}\alpha(X_t)P|_{\mathcal{F}_t} \qquad for \quad t \geq 0,$$

and with respect to Q, the process $(X_t)_{t\geq 0}$ is a diffusion with generator $-L^{\alpha^2 A}$.

We now investigate concrete examples, namely Bessel processes which are Lévy processes on the so-called Bessel-Kingman hypergroups.

3.2 Bessel-Kingman hypergroups and Bessel processes

3.5. Bessel-Kingman hypergroups (see [BH, J, Ki, RV]). For a first motivation, fix some integer $n \geq 1$ and consider the Banach spaces

$$M_b^{rad}(\mathbb{R}^n) := \{\mu \in M_b(\mathbb{R}^n) : A(\mu) = \mu \text{ for all rotations } A \in SO(n)\} \quad \text{for } n \geq 2$$

$$\text{and} \quad M_b^{rad}(\mathbb{R}^1) := \{\mu \in M_b(\mathbb{R}) : \mu(B) = \mu(-B) \text{ for all Borel sets } B \subset \mathbb{R}\}$$

consisting of all "radial" measures on $\mathbb{R}^n$. $M_b^{rad}(\mathbb{R}^n)$ is a Banach-$*$-subalgebra of $M_b(\mathbb{R}^n)$, and the extension of the projection $\Phi : \mathbb{R}^n \longrightarrow [0,\infty[,\ x \longmapsto |x| = (x_1^2 + \ldots + x_n^2)^{1/2}$ to measures is an isometric isomorphism between the Banach-$*$-algebras $M_b^{rad}(\mathbb{R}^n)$ and $M_b([0,\infty[)$ where the second space has to carry the corresponding convolution and involution. This leads to a symmetric hypergroup $([0,\infty[,*)$, the "Bessel-Kingman hypergroup of index $\alpha = n/2 - 1$".

The Bessel-Kingman hypergroup of arbitrary index $\alpha \geq -1/2$ is defined as the Sturm-Liouville hypergroup on $[0,\infty[$ with admissible function

$$A_\alpha(x) = x^{2\alpha+1} \qquad \text{for } x \geq 0.$$

The dual space is given by $\{\varphi_\lambda^\alpha : \lambda \geq 0\}$ where the φ_λ^α satisfy $\varphi_\lambda^\alpha(x) := j_\alpha(\lambda x)$ with the normalized Bessel functions

$$j_\alpha(z) = \sum_{k=0}^{\infty} \frac{(-1)^k\,\Gamma(\alpha+1)}{2^{2k} k!\,\Gamma(\alpha+k+1)}\, z^{2k} \quad (z \in \mathbb{C}).$$

3.6. Bessel processes. The convolution semigroup $(\rho_t^\alpha)_{t\geq 0}$ on the Bessel-Kingman hypergroup of index $\alpha \geq -1/2$ with generator

$$-L^A/2 = \frac{1}{2}\frac{d^2}{dx^2} + \frac{\alpha+1/2}{x}\frac{d}{dx}$$

is given by the Rayleigh distributions

$$d\rho_t^\alpha(x) = \frac{1}{\Gamma(\alpha+1)}\,\frac{2^\alpha}{t^{\alpha+1}}\,x^{2\alpha+1}\,e^{-x^2/(2t)}\,dx \quad \text{on } [0,\infty[\quad \text{for } t > 0; \tag{3.1}$$

see 7.3.18 of [BH]. Associated diffusions are called called Bessel processes of index α. Notice that in this notation, projections $(\Phi(B_t^n))_{t\geq 0}$ of n-dimensional Brownian motions $(B_t^n)_{t\geq 0}$ are Bessel processes of index $\alpha = n/2 - 1$.

We next consider the modification of Bessel-Kingman hypergroups.

3.7. Modified Bessel-Kingman hypergroups and non-central Bessel processes. For any $\alpha \geq -1/2$ and $\rho \geq 0$, the Bessel function $\varphi_{i\rho}^{\alpha}$ is a positive semicharacter on the Bessel-Kingman hypergroup of index α. The associated modified Sturm-Liouville hypergroup will be called modified Bessel-Kingman hypergroup of index α and non-centrality parameter ρ; the associated admissible function is

$$A_{\alpha,\rho}(x) := x^{2\alpha+1} \cdot (\varphi_{i\rho}^{\alpha}(x))^2 \quad (x \geq 0).$$

Diffusions on $[0, \infty[$ with the differential operator

$$-L^{A_{\alpha,\rho}}/2 = \frac{1}{2}\frac{d^2}{dx^2} + \Big(\frac{\alpha+1/2}{x} + \frac{\varphi_{i\rho}^{\alpha\prime}}{\varphi_{i\rho}^{\alpha}}\Big)\frac{d}{dx}$$

are called non-central Bessel processes with index α and non-centrality parameter ρ.

To motivate these notions, consider the n-dimensional Euclidean space $\mathbb{R}^n$ $(n \geq 1)$. Fix some non-centrality parameter $\rho \geq 0$ and consider the multiplicative mapping

$$h_\rho : \mathbb{R}^n \to]0, \infty[, \quad x \mapsto e^{<c_\rho, x>} \quad \text{with} \quad c_\rho := (\rho, 0, \ldots, 0) \in \mathbb{R}^n.$$

By [V2], the vector space

$$\{\mu \in M_b(\mathbb{R}^n) : \mu = h_\rho \cdot \nu, \ \nu \in M_b^{rad}(\mathbb{R}^n) \text{ with compact support}\}$$

is a subalgebra of $M_b(\mathbb{R}^n)$ whose total variation-closure $M_b^{rad,\rho}(\mathbb{R}^n)$ is a Banach subalgebra of $M_b(\mathbb{R}^n)$. Similar as in Section 3.5, the projection $\Phi : \mathbb{R}^n \to [0, \infty[$ leads to an isometric isomorphism between the Banach algebras $M_b^{rad,\rho}(\mathbb{R}^n)$ and $M_b([0, \infty[)$ where the latter has to be equipped with the corresponding "convolution". It can be easily verified (see [V2]) that $[0, \infty[$ with this convolution is the modified Bessel-Kingman hypergroup of index $\alpha = n/2 - 1$ and non-centrality parameter ρ. Moreover, if $(B_t^{n,\rho})_{t\geq 0}$ is an n-dimensional Brownian motion with drift c_ρ (i.e., $(B_t^{n,\rho} - tc_\rho)_{t\geq 0}$ is a Brownian motion), then $(\Phi(B_t^n))_{t\geq 0}$ is a non-central Bessel process with index $\alpha = n/2 - 1$ and non-centrality parameter ρ.

We now reformulate Theorem 3.4.

3.8. Theorem. *Let $(X_t)_{t\geq 0}$ be a Bessel process on $[0, \infty[$ of index $\alpha \geq -1/2$ which is defined on the probability space $(\Omega, \mathcal{F}, P)$ with*

$$\Omega := \{f : [0, \infty[\to [0, \infty[, \ f \ continuous\},$$

and which is equipped with the right-continuous, complete induced filtration $(\mathcal{F}_t)_{t\geq 0}$. Then for each $\rho \geq 0$, there exists a unique probability measure Q on $(\Omega, \sigma(\mathcal{F}_t : t \geq 0))$ with

$$Q|_{\mathcal{F}_t} = e^{-t\rho^2/2}\varphi_{i\rho}^{\alpha}(X_t) P|_{\mathcal{F}_t} \qquad for \quad t \geq 0,$$

and with respect to Q, the process $(X_t)_{t\geq 0}$ is a non-central Bessel process with index α and non-centrality parameter ρ.

3.9. Remark. In this section we obtained non-central Bessel processes from central ones via hypergroup deformations. On the other hand we used some change of drift argument in the introduction for $\alpha = n/2 - 1$, $n \in \mathbb{N}$, in order to obtain the same result. Both methods are, in fact, related from a more abstract point of view via deformations of orbit hypergroups; for the background and possible further examples we refer to [V2].

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