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On the expansion coefficients of Tau-functions of the KP hierarchy

By

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Abstract

We study the coefficients of the tau function of the KP hierarchy. If the tau function does not vanish at the origin, it is known that the coefficients are given by Giambelli formula. We introduce a generalization of Giambelli formula to the case when the tau function vanishes at the origin. This paper is a summary of [6].

§ 1. KP hierarchy

Recently formulas like Giambelli formula in [3] play an important role in connecting quantum integrable systems to classical integrable hierarchies. They also have an application to the study of higher genus theta functions . In the latter case, it is necessary to consider the generalization of Giambelli formula [2, 4, 5]. In this summary we show the generalization of Giambelli formula with several examples.

§ 1.1. partition

A partition is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ of nonnegative integers such that $|\lambda| = \sum_{i\geq 1} \lambda_i$ is finite. We identify a partition λ with its Young diagram, which is a left-justified array of $|\lambda|$ cells with λ_i cells in the ith row.

Example 1.1. If $\lambda = (4, 3, 3, 1)$, its Young diagram is

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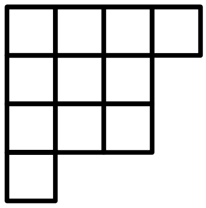
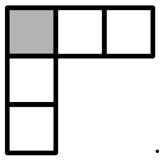


Given a partition λ, we put

$$p(\lambda) = \#\{i : \lambda_i \geq i\}, \quad \alpha_i = \lambda_i - i, \quad \beta_i = \lambda'_i - i \quad (1 \leq i \leq p(\lambda)),$$

where λ'_i is the number of cells in the jth column of the Young diagram of λ. Then we write $\lambda = (\alpha_1, \dots, \alpha_{p(\lambda)}|\beta_1, \dots, \beta_{p(\lambda)})$ and call it the Frobenius notation of λ.

Example 1.2. If $\lambda = (3, 1, 1)$ the Frobenius notation of λ is $(2|2)$ with $p(\lambda) = 1$. Its Young diagram is



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§ 1.2. KP hierarchy

For the function $\tau(x)$ of $x = (x_1, x_2, \dots)$ the KP hierarchy [1] is the bilinear equation given by

$$\int \tau(x - y - [k^{-1}])\tau(x + y + [k^{-1}]) \exp\left(-2\sum_{j=1}^{\infty} y_j k^j\right) dk = 0, \tag{1.1}$$

where $[k^{-1}] = (k^{-1}, k^{-2}/2, k^{-3}/3, \dots)$, $y = (y_1, y_2, \dots)$. The integral denotes taking the coefficient of k^{-1} in the Laurent expansion.

We consider formal power series solution of the KP hierarchy. It is known that any formal power series in x can be expanded by Schur functions as

$$\tau(x) = \sum_{\lambda} \xi_\lambda s_\lambda(x),$$

where λ runs over all partitions. The variables x is so-called Sato variables. Set $x_k = \frac{t_1^k + t_2^k + \cdots}{k}$ Schur functions $s_\lambda(x)$ become the symmetric functions of $t_1, t_2, \cdots$. The KP hierarchy reduces to the Plücker relations for the coefficients $\{\xi_\lambda\}$.

We denote the indices of the coefficients $\xi_{(\alpha_1,\cdots,\alpha_r|\beta_1,\cdots,\beta_r)}$ as $\xi\begin{pmatrix}\alpha_1, \cdots, \alpha_r \\ \beta_1, \cdots, \beta_r\end{pmatrix}$.

Proposition 1.3. *The function $\tau(x)$ is a solution of the KP hierarchy if and only if the coefficients ξ_λ satisfy the following Plücker relations:*

$$\sum_{i=1}^{p+1}(-1)^i \xi\begin{pmatrix} m_1,\dots,\widehat{m_i},\dots,m_{p+1} \\ m'_1,\dots,m'_p \end{pmatrix} \xi\begin{pmatrix} m_i,n_1,\dots,n_q \\ n'_1,\dots,n'_{q+1} \end{pmatrix} = \sum_{j=1}^{q+1}(-1)^{p+j} \xi\begin{pmatrix} m_1,\dots,m_{p+1} \\ m'_1,\dots,m'_p,n'_j \end{pmatrix} \xi\begin{pmatrix} n_1,\dots,n_q \\ n'_1,\dots,\widehat{n'_j},\dots,n'_{q+1} \end{pmatrix}, \tag{1.2}$$

for any sequences $m_1,\dots,m_{p+1},m'_1,\dots,m'_p,n_1,\dots,n_q,n'_1,\dots,n'_{q+1}$ of nonnegative integers.

Corollary 1.4. *The function $\tau(x)$ is a solution of the KP hierarchy if and only if the coefficients ξ_λ satisfy the following Plücker relations:*

$$\begin{aligned} \xi\begin{pmatrix} a_1,\dots,a_r \\ b_1,\dots,b_r \end{pmatrix} \xi\begin{pmatrix} c_1,\dots,c_s \\ d_1,\dots,d_s \end{pmatrix} &= \sum_{k=1}^{r}(-1)^{r-k} \xi\begin{pmatrix} a_1,\dots,\widehat{a_k},\dots,a_r \\ b_1,\dots,b_{r-1} \end{pmatrix} \xi\begin{pmatrix} a_k,c_1,\dots,c_s \\ b_r,d_1,\dots,d_s \end{pmatrix} \\ &\quad + \sum_{l=1}^{s}(-1)^{l-1} \xi\begin{pmatrix} a_1,\dots,a_r \\ b_1,\dots,b_{r-1},d_l \end{pmatrix} \xi\begin{pmatrix} c_1,\dots,c_s \\ b_r,d_1,\dots,\widehat{d_l},\dots,d_s \end{pmatrix}, \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} \xi\begin{pmatrix} a_1,\dots,a_r \\ b_1,\dots,b_r \end{pmatrix} \xi\begin{pmatrix} c_1,\dots,c_s \\ d_1,\dots,d_s \end{pmatrix} &= \sum_{k=1}^{r}(-1)^{r-k} \xi\begin{pmatrix} a_1,\dots,a_{r-1} \\ b_1,\dots,\widehat{b_k},\dots,b_r \end{pmatrix} \xi\begin{pmatrix} a_r,c_1,\dots,c_s \\ b_k,d_1,\dots,d_s \end{pmatrix} \\ &\quad + \sum_{l=1}^{s}(-1)^{l-1} \xi\begin{pmatrix} a_1,\dots,a_{r-1},c_l \\ b_1,\dots,b_r \end{pmatrix} \xi\begin{pmatrix} a_r,c_1,\dots,\widehat{c_l},\dots,c_s \\ d_1,\dots,d_s \end{pmatrix}, \end{aligned} \tag{1.4}$$

for any sequence of nonnegative integers $(a_1,\dots,a_r)$, $(b_1,\dots,b_r)$, $(c_1,\dots,c_s)$ and $(d_1,\dots,d_s)$.

Example 1.5. In case of $a_1=\alpha$, $b_1=\beta$, $c_1=\gamma$ and $d_1=\delta$ the first term of the right hand side of (1.3) becomes zero. Then we have

$$\xi\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \xi\begin{pmatrix} \gamma \\ \delta \end{pmatrix} = \xi\begin{pmatrix} \alpha \\ \delta \end{pmatrix} \xi\begin{pmatrix} \gamma \\ \beta \end{pmatrix}. \tag{1.5}$$

Example 1.6. In case of $(a_1,a_2)=(\alpha_1,\alpha_2)$, $(b_1,b_2)=(\beta_1,\beta_2)$, $c_1=\gamma$ and

$d_1 = \delta$ (1.3) becomes

$$\xi\begin{pmatrix}\alpha_1,\alpha_2\\ \beta_1,\beta_2\end{pmatrix}\xi\begin{pmatrix}\gamma\\ \delta\end{pmatrix} = -\xi\begin{pmatrix}\alpha_2\\ \beta_1\end{pmatrix}\xi\begin{pmatrix}\alpha_1,\gamma\\ \beta_2,\delta\end{pmatrix} + \xi\begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix}\xi\begin{pmatrix}\alpha_2,\gamma\\ \beta_2,\delta\end{pmatrix}$$
$$+\xi\begin{pmatrix}\alpha_1,\alpha_2\\ \beta_1,\delta\end{pmatrix}\xi\begin{pmatrix}\gamma\\ \beta_2\end{pmatrix}. \tag{1.6}$$

Example 1.7. In case of $(a_1, a_2) = (\alpha_1, \alpha_2)$, $(b_1, b_2) = (\beta, \delta)$, $c_1 = \gamma$ and $d_1 = \delta$ (1.4) becomes

$$\xi\begin{pmatrix}\alpha_1,\alpha_2\\ \beta,\delta\end{pmatrix}\xi\begin{pmatrix}\gamma\\ \delta\end{pmatrix} = -\xi\begin{pmatrix}\alpha_1\\ \delta\end{pmatrix}\xi\begin{pmatrix}\alpha_2,\gamma\\ \beta,\delta\end{pmatrix} + \xi\begin{pmatrix}\alpha_1,\gamma\\ \beta,\delta\end{pmatrix}\xi\begin{pmatrix}\alpha_2\\ \delta\end{pmatrix}. \tag{1.7}$$

We use (1.5), (1.6) and (1.7) in Example 6.

§ 2. Main theorem

Fix a partition $\mu = (\gamma_1, \ldots, \gamma_s|\delta_1, \ldots, \delta_s)$. We assume that $\tau(x)$ has the following expansion:

$$\tau(x) = s_\mu(x) + \sum_{\lambda \supsetneq \mu} \xi_\lambda s_\lambda(x). \tag{2.1}$$

Theorem 2.1. *[6] The function $\tau(x)$ gievn by (2.1) is a solution of the KP hierarchy if and only if the expansion coefficitnes $\{\xi_\lambda\}_\lambda$ satisfy the following formulae for a partition $\lambda = (\alpha_1, \ldots, \alpha_r|\beta_1, \ldots, \beta_r)$:*

$$\xi_\lambda = (-1)^s \det\begin{pmatrix} \left(z_{\alpha_i,\beta_j}\right)_{1\le i,j\le r} & \left(u^{(j)}_{\alpha_i}\right)_{1\le i\le r, 1\le j\le s} \\ \left(v^{(i)}_{\beta_j}\right)_{1\le i\le s, 1\le j\le r} & O \end{pmatrix}, \tag{2.2}$$

where $z_{\alpha,\beta}$, $u^{(j)}_\alpha$, $v^{(i)}_\beta$ satisfy

$$\begin{cases} z_{\alpha,\beta} = \xi\begin{pmatrix}\alpha,\gamma_1,\ldots,\gamma_s\\ \beta,\delta_1,\ldots,\delta_s\end{pmatrix}, \\ u^{(j)}_\alpha = \xi\begin{pmatrix}\alpha,\gamma_1,\ldots,\hat{\gamma_j},\ldots,\gamma_s\\ \delta_1,\ldots,\delta_s\end{pmatrix}, \\ v^{(i)}_\beta = \xi\begin{pmatrix}\gamma_1,\ldots,\gamma_s\\ b,\delta_1,\ldots,\hat{\delta_i},\ldots,\delta_s\end{pmatrix}. \end{cases} \tag{2.3}$$

To derive the determinant formulae (2.2) we need the following lemma.

Lemma 2.2. *Fix a partition μ. Suppose that $\tau(x)$ given by (2.1) is a solution of the KP hierarchy. Then ξ_λ can be expressed as a polynomial in*

$$\begin{aligned} I_\mu = &\left\{ \xi \begin{pmatrix} a, \gamma_1, \ldots, \gamma_s \\ b, \delta_1, \ldots, \delta_s \end{pmatrix} : a, b \in \mathbb{Z}_{\geq 0} \right\} \\ &\cup \left\{ \xi \begin{pmatrix} a, \gamma_1, \ldots, \hat{\gamma_j}, \ldots, \gamma_s \\ \delta_1, \ldots, \delta_s \end{pmatrix} : a \in \mathbb{Z}_{\geq 0}, 1 \leq j \leq s \right\} \\ &\cup \left\{ \xi \begin{pmatrix} \gamma_1, \ldots, \gamma_s \\ b, \delta_1, \ldots, \hat{\delta_i}, \ldots, \delta_s \end{pmatrix} : b \in \mathbb{Z}_{\geq 0}, 1 \leq i \leq s \right\}. \end{aligned}$$

Example 2.3. We consider the case of $\mu = (\gamma|\delta)$. The set I_μ becomes

$$I_\mu = \left\{ \xi \begin{pmatrix} a, \gamma \\ b, \delta \end{pmatrix} \right\} \cup \left\{ \xi \begin{pmatrix} a \\ \delta \end{pmatrix} \right\} \cup \left\{ \xi \begin{pmatrix} \gamma \\ b \end{pmatrix} \right\}.$$

In this case $\xi \begin{pmatrix} \gamma \\ \delta \end{pmatrix} = 1$. We derive the coefficients in case of $\lambda = (\alpha_1, \alpha_2 | \beta_1, \beta_2)$.

Using(1.6) we have

$$\xi \begin{pmatrix} \alpha_1, \alpha_2 \\ \beta_1, \beta_2 \end{pmatrix} = -\xi \begin{pmatrix} \alpha_2 \\ \beta_1 \end{pmatrix} \xi \begin{pmatrix} \alpha_1, \gamma \\ \beta_2, \delta \end{pmatrix} + \xi \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \xi \begin{pmatrix} \alpha_2, \gamma \\ \beta_2, \delta \end{pmatrix} + \xi \begin{pmatrix} \alpha_1, \alpha_2 \\ \beta_1, \delta \end{pmatrix} \xi \begin{pmatrix} \gamma \\ \beta_2 \end{pmatrix}.$$

Similarly using (1.5) and (1.7) we have

$$\begin{aligned} \xi \begin{pmatrix} \alpha_i \\ \beta_j \end{pmatrix} &= \xi \begin{pmatrix} \alpha_i \\ \delta \end{pmatrix} \xi \begin{pmatrix} \gamma \\ \beta_j \end{pmatrix}, \\ \xi \begin{pmatrix} \alpha_1, \alpha_2 \\ \beta_1, \delta \end{pmatrix} &= -\xi \begin{pmatrix} \alpha_1 \\ \delta \end{pmatrix} \xi \begin{pmatrix} \alpha_2, \gamma \\ \beta_1, \delta \end{pmatrix} + \xi \begin{pmatrix} \alpha_1, \gamma \\ \beta_1, \delta \end{pmatrix} \xi \begin{pmatrix} \alpha_2 \\ \delta \end{pmatrix} \end{aligned}$$

Then we have

$$\xi \begin{pmatrix} \alpha_1, \alpha_2 \\ \beta_1, \beta_2 \end{pmatrix} = -\det \begin{pmatrix} \xi \begin{pmatrix} \alpha_1, \gamma \\ \beta_1, \delta \end{pmatrix} & \xi \begin{pmatrix} \alpha_1, \gamma \\ \beta_2, \delta \end{pmatrix} & \xi \begin{pmatrix} \alpha_1 \\ \delta \end{pmatrix} \\ \xi \begin{pmatrix} \alpha_2, \gamma \\ \beta_1, \delta \end{pmatrix} & \xi \begin{pmatrix} \alpha_2, \gamma \\ \beta_2, \delta \end{pmatrix} & \xi \begin{pmatrix} \alpha_2 \\ \delta \end{pmatrix} \\ \xi \begin{pmatrix} \gamma \\ \beta_1 \end{pmatrix} & \xi \begin{pmatrix} \gamma \\ \beta_2 \end{pmatrix} & O \end{pmatrix}.$$

This equation is the case of $r = 2$ and $s = 1$ in (2.2).

We also introduce the generalization of Giambelli formula in case of BKP hierarchy [7].

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