RIMS Kôkyûroku Bessatsu
B87 (2021), 039–056

The solution to the initial value problem for the ultradiscrete Somos-4 and 5 equations

By

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Abstract

We propose a method to solve the initial value problem for the ultradiscrete Somos-4 and Somos-5 equations by expressing terms in the equations as convex polygons and regarding max-plus algebras as those on polygons.

§ 1. Introduction

It is still a difficult problem to define the integrability of discrete equations in a way that does not rely on the properties differently from that of differential ones. Several criteria have been proposed for solving this problem by observing the behavior of the solutions to discrete equations which are considered as integrable ones. For example, in the singularity confinement test [1], the property that the singularities due to an initial value are resolved after several time steps and that the information on the initial value is finally restored, is considered to be a discrete analogue of the Painlevé property, which is an indication of integrability. The algebraic entropy [2] focuses on the growth of the degree of the solution as a rational expression of the initial values. It is considered that the system is integrable if the degree grows in at most polynomial order and is non-integrable if the order is exponential. These criteria are also related to the structure of discrete equations such as co-primeness and irreducibility [3].

Over the past decade, it was discovered that cluster algebras, introduced by Fomin and Zelevinsky [4], are strongly related with discrete integrable equations [5, 6]. The time evolution of many integrable discrete equation can be expressed as the mutation of

Received July 1, 2020. Revised November 1, 2020.
2020 Mathematics Subject Classification(s): 37K10, 14T05
Key Words: *Integrable Systems, Discrete Systems, Ultradiscrete Systems, Somos sequence;*
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cluster variables, where these cluster variables are expressed not as rational expressions but in the form of Laurent polynomials of the initial values by properly performing fractional reduction in the recursive application of the equation (which includes divisions [7]). Furthermore, recent studies discovered that such polynomials are irreducible and co-prime for known integrable discrete equations and these properties correspond to the criteria described above [3].

Ultradiscrete systems are difference equations in which only max and $\pm$ operators appear. These equations are obtained from minus-free canonical difference equations by a limiting procedure called "ultradiscretization" [8], which is defined as follows:

1. Transform the dependent variables and parameters by exponential functions, upon introduction of a positive parameter ε, for example $a = e^{A/\varepsilon}$ where a is the dependent variable or the parameter in the discrete system.

2. Take the logarithm and multiply ε for each side of the equation and take the limit $\varepsilon \to +0$. Then, by means of the identity

$$\lim_{\varepsilon \to +0} \varepsilon \log(e^{A/\varepsilon} + e^{B/\varepsilon}) = \max(A, B) \tag{1.1}$$

and the exponential law, the operators $+$ and $\times$ in canonical difference equations are replaced with max and $+$ respectively.

The remarkable point of this procedure is that it preserves the good properties of integrable systems, although the dependent variables only take discrete values. The most famous example is the Box and Ball system (BBS) [9], which is a cellular automaton consisting of an infinite sequence of boxes and a finite amount of balls. The BBS has solitons and an infinite amount of conserved quantities and is obtained by the ultradiscretization of the KdV equation.

For the ultradiscrete equations, we can obtain solutions by ultradiscretizing those of discrete equations. However, it still remains the problem how to interpret good properties of the equation, for example, the Laurent phenomenon in the ultradiscrete systems. By ultradiscretizing Laurent polynomials naively, one expects that the form of solutions should be expressed as $\max_{i=1,\dots,N}(F_i(\mathbf{A}))$, where F_i is a linear function of $\mathbf{A} \in \mathbb{R}^n$. However, a mechanism corresponding to the reduction of the fraction is required in the operation to keep such a form even if the evolution equation contains minus terms. We believe that such a mechanism can be explained by using combinatorics and we finally conclude that it can be interpreted as the inverse of the Minkowski sum between convex polygons. Applying this idea to several known integrable ordinary difference equations, we obtain the exact solution to their initial value problems.

In this paper, we first explain this key idea by a simple ultradiscrete equation in Section 2. By virtue of this idea, we introduce the solution of the initial value problem to the ultradiscrete Somos-4 equation and discuss properties of its solutions and the relation with an ultradiscrete QRT map in Section 3 and that to the ultradiscrete Somos-5 equation in Section 4. We also discuss a numerical result for higher order equations in Section 5.

§ 2. Key idea

Let us consider the following equation, which arises as the mutation of cluster variables in an $A_1^{(1)}$-type cluster algebra

$$f_n f_{n-2} = f_{n-1}^2 + 1 \quad (n \geq 2). \tag{2.1}$$

Here, as an evolution equation (2.1) contains a division. However, f_n is always a Laurent polynomial of f_0 and f_1 with positive coefficients [10]. Therefore, if the initial values f_0 and f_1 are positive, all f_n take positive values and ultradiscretizable in the sense of [8]. Applying the ultradiscretization procedure to (2.1), we obtain:

$$F_n + F_{n-2} = 2\max(F_{n-1}, 0) \quad (n \geq 2). \tag{2.2}$$

Due to the Laurent phenomenon for the discrete system, the solution to the ultradiscrete system (2.2) should be expressible as:

$$F_n = \max_{(\alpha,\beta)\in V_n} (\alpha A + \beta B), \tag{2.3}$$

where $V_n \subset \mathbb{Z}^2$ is a finite set, $A = F_0$ and $B = F_1$. On the other hand, the equation (2.2) can behave a evolution equation, that is, we can obtain F_n uniquely by recurrence.

For example, the solution F_n for the first several n is obtained as

$$F_2 = 2\max(B, 0) - A = \max(-A + 2B, -A) \tag{2.4}$$

$$F_3 = 2\max(2B, 0, A) - 2A - B = \max(-2A + B, -2A - B, -A - B) \tag{2.5}$$

$$F_4 = 2\max(4B, 0, 2A, 2A + B) - 2\max(B, 0) - 3A - 2B. \tag{2.6}$$

Here, by virtue of the rules of the max-plus algebra, one has

$$\begin{aligned} \max(4B, 0, 2A, 2A + B) &= \max(4\max(B, 0), 2A + \max(B, 0)) \\ &= \max(3\max(B, 0), 2A) + \max(B, 0) \\ &= \max(3B, 0, 2A) + \max(B, 0). \end{aligned} \tag{2.7}$$

Then, $-\max$ in (2.6) is cancelled and it is finally simplified into

$$F_4 = 2\max(3B, 0, 2A) - 3A - 2B = \max(-3A + 4B, -3A - 2B, A - 2B). \tag{2.8}$$

Continuing the calculation, we obtain

$$F_5 = 2\max(6B, 0, 4A, 3A + 2B) - 2\max(2B, 0, A) - 4A - 3B. \tag{2.9}$$

However, in this case there is no immediately apparent way to put formula (2.9) in the form (2.3), which should nonetheless be feasible because of the uniqueness of the solution to the evolution equation (2.2). Analyzing the right-hand side of (2.9) case by case, we can simplify F_5 into

$$F_5 = 2\max(4B, 0, 3A) - 4A - 3B. \tag{2.10}$$

Therefore, the following identity should hold in general:

$$\max(4B, 0, 3A) + \max(2B, 0, A) = \max(6B, 0, 4A, 3A + 2B). \tag{2.11}$$

Our goal is to explain this identity by means of a general procedure. By naively expanding the left hand side, we obtain

$$\max(4B, 0, 3A) + \max(2B, 0, A) = \max(6B, 4B, A + 4B, 2B, 0, A, 3A + 2B, 3A, 4A). \tag{2.12}$$

Therefore, to prove the identity one has to show that $4B$, $A + 4B$, $2B$, A, $3A$ are less than $\max(6B, 0, 4A, 3A + 2B)$. Here, $A + 4B$ can be expressed as

$$A + 4B = \frac{1}{4} \times 4A + \frac{2}{3} \times 6B + \frac{1}{12} \times 0 \tag{2.13}$$

and the summation of coefficients of $4A$, $6B$ and 0 is 1, that is, $A + 4B$ is written in a convex combination of $4A$, $6B$ and 0. It is trivial to see that other terms are also written as convex combinations. We can evaluate the magnitude relationship for such convex combined terms by the following proposition.

Proposition 2.1. *For the finite set of points $\{(\alpha_i, \beta_i)\}_{i=1}^{M} \subset \mathbb{R}^2$, if there exists $j \in \{1,\dots, M\}$ satisifying*

$$(\alpha_j, \beta_j) = \sum_{\substack{i=1\\ i\neq j}}^{M} \lambda_i(\alpha_i, \beta_i) \tag{2.14}$$

for some $\sum_{i=1, i\neq j}^{M} \lambda_i = 1$, $\lambda_i \geq 0$, one has

$$\max_{i=1,\dots,M}(\alpha_i A + \beta_i B) = \max_{\substack{i=1,\dots,M\\ i\neq j}}(\alpha_i A + \beta_i B). \tag{2.15}$$

Proof. By virtue of equation (2.14), one has

$$\alpha_j A + \beta_j B = \sum_{\substack{i=1 \\ i\neq j}}^{M} \lambda_i(\alpha_i A + \beta_i B), \tag{2.16}$$

which means (α_j, β_j) is expressed as the weighted average of other (α_i, β_i), i.e., it is less than the maximum of others and more than the minimum. □

With this proposition, it is easily confirmed that (2.12) holds. Now, let us proceed further with this proposition.

Corollary 2.2. *Let $V = \{(\alpha_i, \beta_i)\}_{i=1}^{N}$ and let $V_e \subset V$ be the set of extreme points of V (the vertices of the convex hull of V), then*

$$\max_{(\alpha,\beta)\in V}(\alpha A + \beta B) = \max_{(\alpha,\beta)\in V_e}(\alpha A + \beta B). \tag{2.17}$$

Proposition 2.3. *For all $(\alpha', \beta') \in V_e$, there exists $(A, B) \in \mathbb{R}^2$ such that $\alpha' A + \beta' B > \alpha A + \beta B$ for $\forall(\alpha, \beta) \in V_e \backslash \{(\alpha', \beta')\}$, that is,* $\max_{(\alpha,\beta)\in V_e\backslash\{(\alpha',\beta')\}}(\alpha A + \beta B) \neq \max_{(\alpha,\beta)\in V_e}(\alpha A + \beta B)$.

Proof. Let $\boldsymbol{q} = (t, s) \in \mathbb{R}^2$ be an arbitrary internal point of the polygon whose vertices are V_e. By denoting $\boldsymbol{p} = (\alpha, \beta)$ and $\boldsymbol{p}' = (\alpha', \beta')$, then, the inequality to be proven is equivalent to $(\alpha' - t)A + (\beta' - s)B > (\alpha - t)A + (\beta - s)B \Leftrightarrow (\alpha - \alpha')A + (\beta' - \beta)B = \langle \boldsymbol{p} - \boldsymbol{p}', (A, B)\rangle > 0$, where $\langle \cdot, \cdot \rangle$ is the canonical inner product of $\mathbb{R}^2$. Here, $(A, B) = (\alpha - t, \beta - s) = \boldsymbol{p} - \boldsymbol{q}$ satisfies this inequality because the angle of two vectors whose origins are the same vertex and the destinations are in the polygon, respectively, is acute due to convexity. □

Corollary 2.4 ([11]). *Let $\mathcal{F} = \{f : \mathbb{R}^2 \to \mathbb{R} \mid f(A, B) = \max_{(\alpha,\beta)\in V}(\alpha A + \beta B)$, $V \subset \mathbb{R}^2$ is finite set.$\}$. Then, there exists one-to-one correspondence between convex polygons on $\mathbb{R}^2$ and elements of $\mathcal{F}$.*

By this corollary, we can regard formulae for max as convex polygons. Next we want to interpret the algebra for max formulae as polygon operations. By the relations

$$\max(\max_i(\alpha_i A + \beta_i B), \max_j(\gamma_j A + \delta_j B)) = \max_{i,j}(\alpha_i A + \beta_i B, \gamma_j A + \delta_j B) \tag{2.18}$$

$$\max_i(\alpha_i A + \beta_i B) + \max_j(\gamma_j A + \delta_j B) = \max_{i,j}((\alpha_i + \gamma_j)A + (\beta_i + \delta_j)B), \tag{2.19}$$

we obtain that max operation gives the convex hull of the union of two polygons and + operation gives the Minkowski sum of two polygons, where the Minkowski sum of two subsets is defined as $U + V := \{u + v \mid u \in U, v \in V\}$.

From these dicussions, it is found that the expressions of max correspond to convex polygons and the max-plus algebra for these expressions can be replaced with calculations on convex polygons. In general, however, it is very difficult to determine the extreme points of the Minkowski sum. Fortunately, by virtue of the results of computational geometry, there is a simple method to calculate Minkowski sums for planar convex polygons, by focusing on their edges [12].

Proposition 2.5. *([12]) Let P and Q be convex polygons in $\mathbb{R}^2$ and let $E(X)$ be the set of edge vectors of polygon X. Then, the edges of their Minkowski sum $E(P+Q)$ are obtained by the following algorithm:*

- *Let $E(P) = \{e_i\}_{i=1}^n$, $E(Q) = \{\tilde{e}_j\}_{j=1}^m$, where indices are sorted by the argument.*
- *Start from $i = 1$ and $j = 1$ and apply the following until $i > n$ or $j > m$:*
- *Compare two arguments of e_i and $\tilde{e}_j$.*
  - *If $\arg e_i > \arg \tilde{e}_j$, append e_i to $E(P+Q)$ and let $i \mapsto i+1$.*
  - *If $\arg e_i < \arg \tilde{e}_j$, append $\tilde{e}_j$ to $E(P+Q)$ and let $j \mapsto j+1$.*
  - *If $\arg e_i = \arg \tilde{e}_j$, append $e_i + \tilde{e}_j$ to $E(P+Q)$ and let $i \mapsto i+1$ and $j \mapsto j+1$.*
- *If $i > n$, append $\tilde{e}_j, \dots, \tilde{e}_m$ to $E(P+Q)$.*
- *If $j > m$, append $e_i, \dots, e_n$ to $E(P+Q)$.*

We note that $\max(0, A)$ does not seems to be a polygon but a line segment. In this case, we consider this as a dihedral and its edge vectors are $\{(1,0), (-1,0)\}$. We also note that the sum of all edge vectors is 0.

Here, we demonstrate this algorithm by an example. Let us consider two polygons $P = \{(0,4), (0,0), (3,0)\}$ and $Q = \{(0,2), (0,0), (1,0)\}$ (we express polygons by their extreme points) . The edge vectors of each polygon are expressed as $E(P) = \{(0,-4), (3,0), (-3,4)\}$ and $E(Q) = \{(0,-2), (1,0), (-1,2)\}$. Then, the edge vectors of their Minkowski sum are $E(P+Q) = \{(0,-6), (4,0), (-1,2), (-3,4)\}$. By transforming this to extreme points, one has $P + Q = \{(0,6), (0,0), (4,0), (3,2)\}$, which is another proof of identity (2.12). We can confirm the result visually in Figure 1.

The remarkable point is that we can obtain the inverse of the Minkowski sum by executing this algorithm:

Corollary 2.6. *Let P and R be convex polygon in $\mathbb{R}^2$. Then, we can obtain a convex polygon Q which satisfies $R = P + Q$ by the following algorithm:*

- *Let $E(P) = \{e_i\}_{i=1}^n$, $E(R) = \{e'_k\}_{k=1}^l$, where indices are sorted by the argument.*

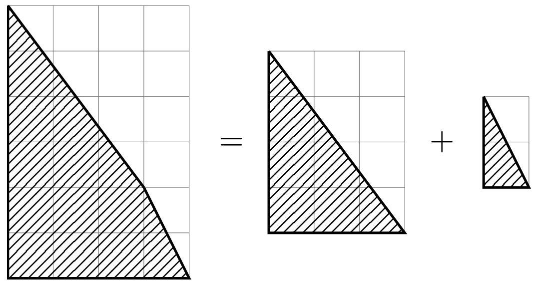

Figure 1. Polygon interpretation of equation (2.12). The + operator in the max-plus algebra corresponds to the Minkowski sum of polygons.

- *Start from $i = 1$ and $k = 1$ and apply the following until $i > n$ or $k > l$.*
- *Compare two arguments of e_i and e'_k.*
  - *If $\arg e_i < \arg e'_k$, append e_i to $E(Q)$ and let $i \to i + 1$.*
  - *If $\arg e_i = \arg e'_k$, next compare lengths of two vectors:*
    * *If $e_i = e'_k$, append e_i to $E(Q)$ and let $i \to i + 1$ and $k \to k + 1$.*
    * *If there exists $c > 1$ such that $e'_k = ce_i$, append $e'_k - e_i$ to $E(Q)$ and let let $i \to i + 1$ and $k \to k + 1$.*
    * *If there exists $c < 1$ such that $e'_k = ce_i$, such a polygon Q does not exist.*
  - *If $\arg e_i > \arg e'_k$, such a polygon Q does not exist.*
- *If $i > n$, append $e'_j, \dots, e'_l$ to $E(Q)$.*
- *If $k > l$, such a polygon Q does not exist.*

The key point is $E(P) \subset E(R)$ if $R = P + Q$. This algorithm also yields that the necessary and sufficient condition for calculating the inverse of the Minkowski sum.

By virtue of these discussions, we can regard the max-plus algebra as polygon calculus and apply this result to ultradiscrete equations which correspond to discrete ones that have the Laurent property.

For example, let us go back to obtain the solution F_n of (2.2). By virtue of Corollary 2.6, we obtain the solution to the initival value problem:

$$F_n = 2\max\Big((n-1)B, 0, (n-2)A\Big) - (n-1)A - (n-2)B \quad (n \geq 2). \tag{2.20}$$

The proof is derived by the identity

$$\max\Big(0, nA, (n+1)B\Big) + \max\Big(0, (n-2)A, (n-1)B\Big) = \max\Big(0, 2(n-1)A, 2nB, nA + (n-1)B\Big) \tag{2.21}$$

for $n \geq 2$, which is also proved by Proposition 2.5.

Equation (2.1) is equivalent to the linear equation [13]:

$$f_n + f_{n-2} = \frac{f_0^2 + f_1^2 + 1}{f_0 f_1} f_{n-1}. \tag{2.22}$$

By ultradiscretizing this relation, we find

$$\max(F_n, F_{n-2}) = F_{n-1} + 2\max(A, B, 0) - A - B. \tag{2.23}$$

Substituting (2.20), we have another identity

$$\begin{aligned} &\max\Big((n-1)B, 0, (n-2)A, (n-2)B + A, A + B, (n-3)A + B\Big) \\ &\qquad = \max\Big((n-2)B, 0, (n-3)A\Big) + 2\max(B, 0, A). \end{aligned} \tag{2.24}$$

for $n \geq 2$ which can be proven using polygon calculus. It is an interesting point that we cannot obtain F_n recursively from (2.23) because it does not form an evolution equation although the corresponding discrete equation (2.22) is linear (which is generally considered to be easier to solve than a non-linear one).

The polygon corresponding to the max formula is nothing but the Newton Polygon of the polynomial before the ultradiscretization. It is known that the Newton Polygon behaves as a lattice for the union and Minkowski sum operation. One can obtain that the necessary condition to factorize a polynomial is that its Newton polygon is decomposable. However, this is not sufficient. For example, $a^2 + 3ab + b^2 \neq (a + b)^2$ but their ultradiscretizations are equal. Furthermore, the polygon decomposition is not unique. For example, $\max(2E, 2D, 3D, 3D + E, 3D + E, D + 2E)$ is decomposed into $\max(E, D, D + E) + \max(E, D, 2D)$ or $\max(D, 0) + \max(2E, 2D, 2D + E)$ and both of these cannot be decomposed further. The reason for such phenomena is that the Newton polygon ignores the terms except those corresponding to the extreme points.

Finally, let us note that the polygon we dealt with in this section is considered to be dual to a tropical curve, and that operations between polygons can therefore be also interpreted as operations on tropical curves.

§ 3. Ultradiscrete Somos-4 equation

The Somos sequences are the difference equations expressed as

$$f_n f_{n-q} = \sum_{i=1}^{\lfloor \frac{q}{2} \rfloor} f_{n-i} f_{n-q+i}, \tag{3.1}$$

where q is an integer more than 4. This equation is also called Somos-q equation for some specific value of q.

In these sequences, the case where q satisfies $4 \leq q \leq 7$ is related to integrable systems. For these value of q, it has been proven that f_n is a Laurent polynomial of $f_0, \ldots, f_{q-1}$ with positive coefficients [7]. Furthermore, these sequences are derived as reductions of some integrable partial difference equations. For example, the Somos-4 and 5 equations are derived from the discrete KP equation and the Somos-6 and 7 equations are from the discrete BKP equation [14]. We also note that Somos-6 and 7 are not obtained from cluster algebras rather from Laurent Phenomenon algebras [15], which are analogues of the cluster algebras [16].

By applying the ultradiscretization procedure to the Somos-4 equation, we obtain

$$F_n + F_{n-4} = \max(F_{n-1} + F_{n-3}, 2F_{n-2}) \quad (n \geq 4). \tag{3.2}$$

We call this the ultradiscrete Somos-4 equation. This equation is a fourth order difference equation and solutions are expressed in terms of F_0, F_1, F_2, F_3. However, since this equation is invariant under the gauge transformation $F_n \mapsto F_n + a + bn$ $(a, b \in \mathbb{R})$, we can set $F_0 = F_1 = 0$ without loss of generality by taking the proper gauge. Therefore, the solution should be expressed by planar polygons. Due to Corollary 2.6, we can calculate the evolution of this equation as the polygon representation and finally obtain the solution of its initial value problem.

Theorem 3.1. *The solution of the initial value problem for (3.2) is expressed as:*

$$F_n = -\nu_{n+2}C - \nu_{n+1}D + \tilde{F}_n, \tag{3.3}$$

where $C = F_2$, $D = F_3$ and ν_n is the solution to the same equation (3.2) for the initial values $\nu_0 = 1$, $\nu_1 = \nu_2 = \nu_3 = 0$, represented as follows:

$$\nu_{8k} = 4k^2 - 4k + 1 \tag{3.4}$$
$$\nu_{8k+1} = 4k^2 - 3k \tag{3.5}$$
$$\nu_{8k+2} = 4k^2 - 2k \tag{3.6}$$
$$\nu_{8k+3} = 4k^2 - k \tag{3.7}$$
$$\nu_{8k+4} = 4k^2 - 1 \tag{3.8}$$
$$\nu_{8k+5} = 4k^2 + k \tag{3.9}$$
$$\nu_{8k+6} = 4k^2 + 2k \tag{3.10}$$
$$\nu_{8k+7} = 4k^2 + 3k. \tag{3.11}$$

$\tilde{F}_n$ *is*

$$\tilde{F}_{8k} = (4k^2 - k)Q + kP \tag{3.12}$$
$$\tilde{F}_{8k+1} = 4k^2Q + kP \tag{3.13}$$
$$\tilde{F}_{8k+2} = (4k^2 + k)Q + kP \tag{3.14}$$
$$\tilde{F}_{8k+3} = (4k^2 + 2k)Q + kP \tag{3.15}$$
$$\tilde{F}_{8k+4} = (4k^2 + 3k)Q + kP + \max(D, 2C) \tag{3.16}$$
$$\tilde{F}_{8k+5} = (4k^2 + 4k)Q + kP + \max(D, C + D, 3C) \tag{3.17}$$
$$\tilde{F}_{8k+6} = (4k^2 + 5k)Q + kP + \max(3D, 2D, 4C, 3C + D) \tag{3.18}$$
$$\tilde{F}_{8k+7} = (4k^2 + 6k)Q + kP + \max(4D, 3D, 6C, 7C, C + 4D), \tag{3.19}$$

and $Q = \max(2D, D, 2C, 3C)$ *and* $P = \max(4D, 3D, 6C)$.

Before starting the proof, we calculate the first several expressions by the recurrence and obtain

$$F_0 = F_1 = 0, F_2 = C, F_3 = D \tag{3.20}$$
$$F_4 = \max(D, 2C) \tag{3.21}$$
$$F_5 = \max(D, C + D, 3C) \tag{3.22}$$
$$F_6 = -C + \max(3D, 2D, 4C, 3C + D) \tag{3.23}$$
$$F_7 = -D + \max(4D, 3D, 6C, 7C, C + 4D), \tag{3.24}$$

which are consistent with the above result for $k = 0$.

Proof. We first prove the statement concerning ν_n. Substituting ν_n in both sides of (3.2), the terms depending on k are factored out from the max in the right hand side, such that the terms are the same on both sides. For example, $\nu_{8k+4} + \nu_{8k} = 8k^2 - 4k$ and $\max(\nu_{8k+3} + \nu_{8k+1}, 2\nu_{8k+2}) = 8k^2 - 4k$. Therefore, we should consider only the cases from $n = 4$ to 11 and prove these by simple calculations.

Next we consider $\tilde{F}_n$. Since ν_n satisfies (3.2), we transform (3.2) to an equation for $\tilde{F}_n$:

$$\tilde{F}_n + \tilde{F}_{n-2} = \max(\tilde{F}_{n-1} + \tilde{F}_{n-3} + (d_n)_-C + (d_{n-1})_-D, 2\tilde{F}_{n-2} + (d_n)_+C + (d_{n-1})_+D), \tag{3.25}$$

where $d_n := \nu_n + \nu_{n+2} - 2\nu_{n+1}$, $(a)_+ := \max(a, 0)$ and $(a)_- := \max(-a, 0)$. Here, due to (3.4)–(3.11), one obtains that d_n has period 8.

Therefore, by virtue of the same discussion as for ν_n solving (3.2), we have to consider only the cases from $n = 4$ to 11 and obtain that $\tilde{F}_n$ solves (3.25) by the using polygon calculus in the previous section. □

Next, we focus on the properties of the solution that we obtained. We first point out that the solution (3.3) is decomposable (actually already decomposed) and contains the same polygon in decomposed ones in contrast with the irreducibility and co-primeness of the solution which was proven for the (discrete) Somos-4 equations. The reason is the same as for the polygon expression described in the previous section. We also note that the growth of the coefficients of C, D in the solution for n is of square order, which follows the preceding studies [17].

We next discuss the relation to the QRT systems. By introducing the dependent variable $g_n = f_n f_{n+2}/f_{n+1}^2$, the Somos-4 is written as

$$g_n g_{n-2} = \frac{g_{n-1}+1}{g_{n-1}^2}, \tag{3.26}$$

which is one of the QRT maps [14]. The corresponding ultradiscrete dependent variable is

$$G_n = F_n + F_{n+2} - 2F_{n+1} \tag{3.27}$$

and the ultradiscrete Somos-4 (3.2) is transformed into

$$G_n + G_{n-2} = \max(G_{n-1}, 0) - 2G_{n-1}, \tag{3.28}$$

which is one of the ultradiscrete QRT maps.

By substituting (3.3) into relation (3.27) we obtain the following corollary.

Corollary 3.2. *The solution to the equation (3.28) for the initial values $G_0 = C$ and $G_1 = -2C + D$ is expressed as*

$$G_{8k} = C \tag{3.29}$$

$$G_{8k+1} = -2C + D \tag{3.30}$$

$$G_{8k+2} = C - 2D + \max(D, 2C) \tag{3.31}$$

$$G_{8k+3} = D + \max(2D, C+D, 3C) - 2\max(D, 2C) \tag{3.32}$$

$$\begin{aligned} G_{8k+4} = {} & -C + \max(3D, 2D, 4C, 3C+D) + \max(D, 2C) \\ & -2\max(2D, C+D, 3C) \end{aligned} \tag{3.33}$$

$$\begin{aligned} G_{8k+5} = {} & C - D + \max(4D, 3D, 6C) + \max(0, C) \\ & +\max(2D, C+D, 3C) - 2\max(3D, 2D, 4C, 3C+D) \end{aligned} \tag{3.34}$$

$$\begin{aligned} G_{8k+6} = {} & -C + D + \max(3D, 2D, 4C, 3C+D) + \max(2D, D, 2C, 3C) \\ & -\max(4D, 3D, 6C) - 2\max(0, C) \end{aligned} \tag{3.35}$$

$$G_{8k+7} = -D + \max(0, C). \tag{3.36}$$

Therefore, the period of the solution is 8 *for arbitrary initial values.*

This corollary can be also proved by directly calculating G_n from equation (3.28) recurrently.

Nobe solved the ultradiscrete QRT maps including (3.28) by regarding the systems as additions on Tropical Elliptic Curves and obtained the same result [18]. In [18] it is pointed out that the solution to the discrete equation (3.26) has no periodicity, although that to the ultradiscrete equation (3.28) is periodic. The reason why the discrete equation has no periodicity is explained by the irreducibility and co-primeness of the solution [3] and by due to the discussion in the previous section, we must conclude that the ultradiscrete solution has periodicity because the information of non-dominant terms in the solution to the Somos-4 equation, which correspond to the irreducibility, is dropped by the ultradiscretization. We note that these preceding studies [18, 19] also employ the polygon geometry. However, in their approach, the solution is expressed as a point on polygon facets and our approach considers the solution as a polygon itself, which is a major difference. Finally, we note that Fordy and Hone also obtained the solution of the ultradiscrete Somos-4 equation before [20] by employing Nobe's result. We stress again that our solution is obtained without property of (3.28).

§ 4. Ultradiscrete Somos-5 equation

By ultradiscretizing the Somos-5 equation, one obtains the ultradiscrete Somos-5

$$F_n + F_{n-5} = \max(F_{n-1} + F_{n-4}, F_{n-2} + F_{n-3}). \quad (n \geq 5) \tag{4.1}$$

This equation is a fifth order difference equation. However, by employing the same approach to solving the ultradiscrete Somos-4, this equation is invariant under the gauge $F_n \mapsto F_n + a + bn + c(-1)^n$ $(a, b, c \in \mathbb{R})$ and we can set $F_0 = F_1 = F_2 = 0$ without loss of generality. Therefore, the solution is also expressed as a planar polygon. Since the approach of the proof is the same as that for the ultradiscrete Somos-4 equation, we omit the details and show only results.

Theorem 4.1. *The solution is written as*

$$F_n = -\nu_{n+3}D - \nu_{n+2}E + \tilde{F}_n, \tag{4.2}$$

where $D = F_3$*,* $E = F_4$*,* ν_n *is the solution to the same equation (4.1) for the initial* *value* $\nu_0 = 1$, $\nu_1 = \nu_2 = \nu_3 = 0 = \nu_4 = 0$ *and represented as follows:*

$$\nu_{7k} = 1 + \tfrac{1}{8}(-\phi_k - 6k + \psi_k) \tag{4.3}$$

$$\nu_{7k+1} = \tfrac{1}{8}(\phi_k - 2k + \psi_k) \tag{4.4}$$

$$\nu_{7k+2} = \tfrac{1}{8}(-\phi_k + 2k + \psi_k) \tag{4.5}$$

$$\nu_{7k+3} = \tfrac{1}{8}(\phi_k + 6k + \psi_k) \tag{4.6}$$

$$\nu_{7k+4} = \tfrac{1}{8}(-\phi_k + 10k + \psi_k) \tag{4.7}$$

$$\nu_{7k+5} = -1 + \tfrac{1}{8}(\phi_k + 14k + \psi_k) \tag{4.8}$$

$$\nu_{7k+6} = \tfrac{1}{8}(-\phi_k + 18k + \psi_k). \tag{4.9}$$

$\tilde{F}_n$ *is*

$$\tilde{F}_{7k} = \frac{1}{8}(-\phi_k - 2k + \psi_k)Q + \frac{-\phi_k + 2k}{4}R + kP \tag{4.10}$$

$$\tilde{F}_{7k+1} = \frac{1}{8}(\phi_k + 2k + \psi_k)Q + \frac{\phi_k + 2k}{4}R + kP \tag{4.11}$$

$$\tilde{F}_{7k+2} = \frac{1}{8}(-\phi_k + 6k + \psi_k)Q + \frac{-\phi_k + 2k}{4}R + kP \tag{4.12}$$

$$\tilde{F}_{7k+3} = \frac{1}{8}(\phi_k + 10k + \psi_k)Q + \frac{\phi_k + 2k}{4}R + kP \tag{4.13}$$

$$\tilde{F}_{7k+4} = \frac{1}{8}(-\phi_k + 14k + \psi_k)Q + \frac{-\phi_k + 2k}{4}R + kP \tag{4.14}$$

$$\tilde{F}_{7k+5} = \frac{1}{8}(\phi_k + 18k + \psi_k)Q + \frac{\phi_k + 2k}{4}R + kP + \max(E, D) \tag{4.15}$$

$$\tilde{F}_{7k+6} = \frac{1}{8}(-\phi_k + 22k + \psi_k)Q + \frac{-\phi_k + 2k}{4}R + kP + \max(E, D, D + E), \tag{4.16}$$

$\phi_k = 1 - (-1)^k$, $\psi_k = 14k(k-1)$, $Q = \max(E, 0) + \max(E, D, 2D)$, $R = \max(E, 0)$ *and* $P = \max(2E, 2D, 2D + E)$.

By introducing a new dependent variable $g_n = f_n f_{n+3}/f_{n+1}f_{n+2}$, the Somos-5 equation can be written as

$$g_n g_{n-2} = \frac{g_{n-1} + 1}{g_{n-1}}, \tag{4.17}$$

which is also a QRT map. The corresponding transformation of the dependent variable in the ultradiscrete system is

$$G_n = F_n + F_{n+3} - F_{n+1} - F_{n+2} \tag{4.18}$$

and we obtain its ultradiscretization:

$$G_n + G_{n-2} = \max(G_{n-1}, 0) - G_{n-1}. \tag{4.19}$$

Corollary 4.2. *The solution to the equation (3.28) for the initial values $G_0 = D$ and $G_1 = -D + E$ is expressed as*

$$G_{7k} = D \tag{4.20}$$

$$G_{7k+1} = -D + E \tag{4.21}$$

$$G_{7k+2} = -D - E + \max(E, D) \tag{4.22}$$

$$G_{7k+3} = D - E + \max(D, E, D + E) - \max(E, D) \tag{4.23}$$

$$G_{7k+4} = E + \max(2E, 2D, 2D + E) - \max(E, D) - \max(E, D, D + E) \tag{4.24}$$

$$G_{7k+5} = -D + \max(E, D) - \max(E, D, D + E) + \max(E, 0) \tag{4.25}$$

$$G_{7k+6} = -E + \max(E, D, D + E) + \max(E, D, 2D) - \max(2E, 2D, D + E). \tag{4.26}$$

Therefore, the period of the solution is 7 *for arbitrary initial values [18].*

§ 5. The higher degree of freedom cases

To apply this approach to higher order ultradiscrete ordinary differential equations, we have to treat higher dimensional polytopes because the degree of freedom becomes more. However, we can easily extend the discussion between polytopes and max-combined linear functions in the section 2 to the higher dimensional ones and the Minkowski sum operation for the higher dimensional polytopes are presented by Fukuda [21]. In this section, we introduce the result of numerical calculation for the ultradiscrete ODEs with Laurent property and 4 degree of freedom. First, let us consider this equation:

$$F_n + F_{n-4} = \max(F_{n-1} + F_{n-3}, F_{n-2}) \quad (n \geq 4) \tag{5.1}$$

with initial values $F_0 = A$, $F_1 = B$, $F_2 = C$, $F_3 = D$. For sufficiently large n, the solution is expressed as

$$F_n = \max_{(\alpha,\beta,\gamma,\delta)\in V_e} (\alpha A + \beta B + \gamma C + \delta D) - \nu_n A - \nu_{n-1} B - \nu_{n-2} C - \nu_{n-3} D, \tag{5.2}$$

where ν is a sequence $(-1, 0, 0, 0, 1, 1, 1, 2, 2, 2, \ldots)$ and V_e is expressed as in the case $n = 3k$, $V_e = \{(-13 + 2k, -8 + k, -8 + k, 3), (-13 + 2k, 0, 0, -6 + k), (-14 + 2k, -6 + k, -6 + k, 0), (-14 + 2k, 1, 1, -7 + k), (-15 + 2k, -4 + k, -8 + k, 1), (-15 + 2k, -8 + k, -3 + k, 1), (-6 + k, -7 + k, -7 + k, -5 + k), (-6 + k, 0, 0, -12 + 2k), (-7 + k, -12 + 2k, 0, 1), (-7 + k, -13 + 2k, 1, 0), (-7 + k, -5 + k, -7 + k, -6 + k), (-7 + k, -7 + k, -5 + k, -6 + k), (-7 + k, 1, -13 + 2k, 0), (-7 + k, 0, -3 + k, 1), (0, -12 + 2k, 0, -6 + k), (0, -6 + k, -6 + k, -12 + 2k), (0, -6 + k, -6 + k, 0), (0, 0, -12 + 2k, -6 + k)\}$,
in the case $n = 3k + 1$, $V_e = \{(-12 + 2k, -7 + k, -6 + k, 1), (-12 + 2k, 0, 1, -6 + k), (-13+$

$2k, -7+k, -4+k, 0), (-13+2k, -7+k, -8+k, 4), (-13+2k, 1, 0, -4+k), (-6+k, -3+k, -8+k, 2), (-5+k, -14+2k, 1, 1), (-6+k, -6+k, -5+k, -6+k), (-6+k, -6+k, -7+k, -4+k), (-6+k, 1, 0, -11+2k), (-6+k, 0, -11+2k, 0), (-6+k, 0, 1, -12+2k), (-7+k, -11+2k, 0, 2), (-7+k, -13+2k, 2, 0), (-7+k, -4+k, -7+k, -5+k), (0, -11+2k, 0, -5+k), (0, -5+k, -6+k, -11+2k), (0, -6+k, -5+k, 0), (0, 0, -11+2k, -6+k)\}$ and in the case $n = 3k+2$, $V_e = \{(-11+2k, -7+k, -7+k, 2), (-11+2k, 0, 0, -5+k), (-12+2k, -5+k, -7+k, 1), (-13+2k, -7+k, -3+k, 0), (-15+2k, -3+k, -7+k, 2), (-5+k, -12+2k, 0, 1), (-5+k, -6+k, -6+k, -5+k), (-5+k, 0, 0, -11+2k), (-6+k, -6+k, -4+k, -6+k), (-6+k, 0, -2+k, 0), (-7+k, -3+k, 1, 2), (-7+k, -13+2k, 3, 0), (-7+k, -4+k, -6+k, -5+k), (1, -12+2k, 0, -5+k), (1, -6+k, -6+k, 1), (0, -11+2k, 1, -5+k), (0, -5+k, -5+k, -11+2k), (0, -6+k, -4+k, 0), (0, 0, -10+2k, -6+k)\}$.

Apparently, the coefficients of A, B, C and D grow linear order of n. Here, (2.20) also grows linear order and solves second order equation (2.2) due to an n-dependent identity (2.24). Then, it is considered that (5.2) solves (5.1) by virtue of some n-dependent identity (which may contain about 400 arguments in max!).

The Somos-6 equation is an example of the equations whose solutions grow square order and its ultradiscretization is written in

$$F_n + F_{n-6} = \max(F_{n-1} + F_{n-5}, F_{n-2} + F_{n-4}, 2F_{n-3}) \quad (n \geq 6). \tag{5.3}$$

The order of this equation is sixth. However, similar as the ultradiscrete Somos-4 equation, this equation is invariant for the gauge transformation of $F_n \mapsto F_n + a + bn$. Then, the real degree of freedom is 4. Though the solution to the initial value problem of this equation is expressed as a polytope. However, differently from that for the ultradiscrete Somos-4 and 5 equations, we cannot obtain the explicit formula of this representation because the number of the vertices grow as n increases. By the dependent variable transformation

$$G_n = F_n + F_{n-2} - 2F_{n-1}, \tag{5.4}$$

the ultradiscrete Somos-6 equation (5.3) is transformed into a 4th order difference equation:

$$G_n + G_{n-4} = \max(G_{n-1} + 2G_{n-2} + G_{n-3}, G_{n-2}, 0) - (2G_{n-1} + 3G_{n-2} + 2G_{n-3}) \quad (n \geq 4). \tag{5.5}$$

If the solution is decomposed into several polytopes and n affects only the scale of polytopes similarly to that of the ultradiscrete Somos-4 and 5 equations, the solution to (5.5) should be periodic regardless of the initial value. However, for a given numerical initial value, the solution has a period, which depends on the initial value.

It is known that the solution of the (discrete) Somos-6 equation is expressed as the Wierstrass sigma function and the it evolves linearly on the corresponding Jacobian [22]. Since this approach is very similar to that for theta function solution to discrete Toda equation with periodic boundary condition, if we assume that it can be ultradiscretized by the similar method to that of theta function [23], the evolution of the solution is linearlized on the Jacobian of ultradiscrete sigma function, which is a double-torus. Then, one can consider that the orbit forms quasi-periodic and never come back to the initial point for general initial values. This is why the solution to the ultradiscrete Somos-6 equation becomes more complex as n grows.

Finally, we remark that such a phenomenon does not occur because the discrete Somos-6 equation is obtained from not the discrete KP equation but the discrete BKP one. We have the same problem for the generalization of the ultradiscrete Somos-4 equation:

$$F_n + F_{n-4} = \max(C + F_{n-1} + F_{n-3}, D + 2F_{n-2}). \tag{5.6}$$

The solution corresponds to a 4 dimensional convex polytope and its shape becomes more complex as n grows. We also note that such phenomenon does not occur just because the degree of freedom increases. We observed that the same phenomenon happens with simply dropping several informations (e.g. setting $A = B = 0$).

§ 6. Concluding Remarks

In this paper, we proposed a purely ultradiscrete calculus-based method to solve the initial value problem for the ultradiscrete Somos-4 and 5 equations by regarding max formulae as convex polygons. The solution can be written as a single max expression even if the evolution equations contain minus terms, which is an analogue of the Laurent property in ultradiscrete systems.

The idea discussed in Section 2 faithfully replaces the max-plus algebra with polygon operations. This means that problems arising in the max-plus algebra, are also present in polygon operations. For example, by setting $P = \max(B, A, B + 2A, A + 2B)$ and $Q = \max(0, 2A, 2A + 2B, 2B)$, $\max(P, Q)$ in fact no longer depends on P, which corresponds to the fact that a polygon included in other polygons, no longer influences their geometrical properties.

Our approach can be applied to equations with the Laurent property, even if they are higher order ones or have higher degree non-linearities. For example, the equation

$$f_n f_{n-2} = (f_{n-1})^m + 1 \tag{6.1}$$

for $m > 2$ has the Laurent property. The corresponding ultradiscrete equation is

$$F_n + F_{n-2} = m \max(F_{n-1}, 0) \tag{6.2}$$

and its solution is expressed as $F_n = \max(b_{n-1}F_1, b_{n-2}F_0, 0) - \alpha_n F_0 - \alpha_{n-1}F_1$ $(n \geq 2)$, where b_n is the solution of $b_n = mb_{n-1} - b_{n-2}$, $b_0 = 0$, $b_1 = m$ and α_n is the solution of the same difference equation $\alpha_n = m\alpha_{n-1} - \alpha_{n-2}$ with the different initial value $\alpha_0 = -1$, $\alpha_1 = 0$. Though the growth of b_n and α_n is of exponential order, the ultradiscretized solution is also expressible by means of the polygons. This solution also holds even when m is not integer. This result may indicate a suggestion on what is the Laurent property about difference equations with the non-integer degree non-linearity.

We finally note that the method used in the proofs of Theorems 6 and 8 to obtain the general solution F_n after finding a special solution ν_n, is very similar to the quadrature method for the general solution of the Riccati equation.

Acknowledgment

The author would like to thank Professors T. Tokihiro and R. Willox and Dr. T. Mase for helpful comments. This work was supported by Platform for Dynamic Approaches to Living System (the Platform Project for Supporting in Drug Discovery and Life Science Research) from the Ministry of Education, Culture, Sports, Science (MEXT) and Technology, Japan, and Japan Agency for Medical Research and Development (AMED). This work is also partially supported by the JPSJ KAKENHI Grant Number 17K14199.

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