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Space of initial conditions for the four-dimensional Fuji-Suzuki-Tsuda system

By

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Abstract

A geometric study for an integrable 4-dimensional dynamical system so called the Fuji-Suzuki-Tsuda system is given. By the resolution of indeterminacy, the group of its Bäklund transformations is lifted to a group of pseudo-isomorphisms between rational varieties obtained from $(\mathbb{P}^1)^4$ by blowing-up along eight 2-dimensional subvarieties and four 1-dimensional subvarieties. The root basis is realised in the Néron-Severi bilattices. A discrete Painlevé system with quadratic degree growth is also realised as its translational element.

§ 1. Introduction

§ 1.1. Background and the results

The Painlevé equations are nonlinear second-order ordinary differential equations whose solutions are meromorphic except some fixed points, but not reduced to known functions such as solutions of linear ordinary differential equations or Abel functions. In [7] Okamoto introduced the notion of space of initial conditions where the flow of each Painlevé equation is regularised on a family of rational algebraic surfaces (minus some subvarieties called vertical leaves) even around poles. In [9] Sakai extended this notion to the discrete case and used it to obtain symmetry group of the equations. A benefit of this approach is that the root systems can be realised geometrically in the Picard group on the surfaces.

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In recent years, research on four-dimensional Painlevé systems has been progressed mainly from the viewpoint of isomonodromic deformation of linear equations [10] and pointed out that there are 4 master equations in the sense that other equations can be obtained from them by limiting procedure [6]. The four-dimensional Fuji-Suzuki-Tsuda system is one of these 4 master equations. In [5, 12] Suzuki and Fuji obtained the $2N$-dimensional ($N = 1, 2, 3, \cdots$) system by a reduction from so called the Drinfeld-Sokolov hierarchies of type A and in [14] Tsuda obtained it from so called the UC-hierarchies.

In this paper, starting from known Bäclund transformations, we construct the space of initial conditions for the 4D Fuji-Suzuki-Tsuada system. By resolution of indeterminacy, the Bäklund transformations are lifted to pseudo-isomorphisms between rational varieties obtained from $(\mathbb{P}^1)^4$ by blowing-up along eight 2-dimensional subvarieties and four 1-dimensional subvarieties[1]. The root basis is realised in the Néron-Severi bilattice. A discrete Painlevé system with quadratic degree growth is also realised as its translational element.

§ 1.2. Fuji-Suzuki-Tsuda system and its Bäcklund transformations

The 4D Fuji-Suzuki-Tsuada system is a Hamiltonian system

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad (i = 1, 2) \tag{1.1}$$

with

$$\begin{aligned} t(t-1)H =& H_{\mathrm{VI}}(q_1, p_1; a_2, a_0 + a_4, a_3 + a_5 - \eta, \eta a_1) \\ &+ H_{\mathrm{VI}}(q_2, p_2; a_0 + a_2, a_4, a_1 + a_3 - \eta, \eta a_5) \\ &+ (q_1 - t)(q_2 - 1)\{(q_1 p_1 + a_1)p_2 + p_1(p_2 q_2 + a_5)\} \end{aligned} \tag{1.2}$$

and $a_0 + a_1 + \cdots + a_5 = 1$, where H_{VI} is the polynomial Hamiltonian of the sixth Painlevé equation introduced by Okamoto in [8]:[2]

$$\begin{aligned} H_{\mathrm{VI}}(q, p; a, b, c, d) =& q(q-1)(q-t)p^2 - \{(a-1)q(q-1) \\ &+ bq(q-t) + c(q-1)(q-t)\}p + dq. \end{aligned} \tag{1.3}$$

This equation has Bäcklund transformations, i.e. transformations of variables which keep the equation except parameters as in the following two tables[3].

[1] Sasano has also constructed the space of initial conditions in [11], where he started from a four-dimensional analog of the Hirzebruch surface. In our study, the structure of the Picard group becomes simple, since we start from the direct product of $\mathbb{P}^1$ instead of the analog of the Hirzebruch surface.

[2] Precisely saying, H_{VI} is the Hamiltonian introduced by Okamoto multiplied by $t(t-1)$.

[3] s_i's are reported in [5], while π is in [12], where a misprint has been corrected, and ρ is in [14].

Actions on parameters

	$\bar{a}_0$	$\bar{a}_1$	$\bar{a}_2$	$\bar{a}_3$	$\bar{a}_4$	$\bar{a}_5$	$\bar{\eta}$	$\bar{t}$
s_0	$-a_0$	a_0+a_1	a_2	a_3	a_4	a_0+a_5	$\eta+a_0$	t
s_1	a_0+a_1	$-a_1$	a_1+a_2	a_3	a_4	a_5	$\eta-a_1$	t
s_2	a_0	a_1+a_2	$-a_2$	a_2+a_3	a_4	a_5	$\eta+a_2$	t
s_3	a_0	a_1	a_2+a_3	$-a_3$	a_3+a_4	a_5	$\eta-a_3$	t
s_4	a_0	a_1	a_2	a_3+a_4	$-a_4$	a_4+a_5	$\eta+a_4$	t
s_5	a_0+a_5	a_1	a_2	a_3	a_4+a_5	$-a_5$	$\eta-a_5$	t
π	a_1	a_2	a_3	a_4	a_5	a_0	$-\eta$	t^{-1}
ρ	a_0	a_5	a_4	a_3	a_2	a_1	η	t^{-1}

Actions on dependent variables

	$\bar{q}_1$	$\bar{q}_2$	$\bar{p}_1$	$\bar{p}_2$
s_0	q_1	q_2	$p_1-a_0(q_1-q_2)^{-1}$	$p_2+a_0(q_1-q_2)^{-1}$
s_1	$q_1+a_1p_1^{-1}$	q_2	p_1	p_2
s_2	q_1	q_2	$p_1-a_2(q_1-t)^{-1}$	p_2
s_3	$q_1+a_3q_1B^{-1}$	$q_2+a_3q_2B^{-1}$	$p_1-a_3p_1A^{-1}$	$p_2-a_3p_2A^{-1}$
s_4	q_1	q_2	p_1	$p_2-a_4(q_2-1)^{-1}$
s_5	q_1	$q_2+a_5p_2^{-1}$	p_1	p_2
π	$C_1C_t^{-1}$	$C_2C_t^{-1}$	$-(q_1-t)C_t(t-1)^{-1}$	$-(q_2-q_1)C_t(t-1)^{-1}$
ρ	q_2t^{-1}	q_1t^{-1}	p_2t	p_1t

where $A = q_1p_1 + q_2p_2 + \eta$, $B = A - a_3$, $C_1 = A - p_1 - p_2$, $C_2 = A - tp_1 - p_2$ and $C_t = A - tp_1 - tp_2$.

These transformations constitute so called the extended affine Weyl group of type $A_5^{(1)}$, whose fundamental relations are

$$s_i^2 = \rho^2 = \pi^6 = \text{identity}$$
$$s_i \circ s_{i+1} \circ s_i = s_{i+1} \circ s_i \circ s_{i+1}, \qquad s_i \circ s_j = s_j \circ s_i \quad (|i-j| > 1)$$
$$s_i \circ \pi = \pi \circ s_{i+1}, \qquad s_i \circ \rho = \rho \circ s_{6-i}, \qquad (\pi \circ \rho)^2 = \text{identity},$$

where indices are considered to be cyclic as $\alpha_{i+6} = \alpha_i$.

In order to construct its space of initial conditions without blowing-downs, we introduce a new coordinate system $(q_i, r_i) = (q_i, q_ip_i)$ $(i = 1, 2)$, exactly the same manner with the case of two-dimensional Painlevé equations.

The action on dependent variables become

Actions on new dependent variables

	$\bar{q}_1$	$\bar{q}_2$	$\bar{r}_1$	$\bar{r}_2$
s_0	q_1	q_2	$r_1 - a_0 q_1 (q_1 - q_2)^{-1}$	$r_2 + a_0 q_2 (q_1 - q_2)^{-1}$
s_1	$q_1(1 + a_1 r_1^{-1})$	q_2	$r_1 + a_1$	r_2
s_2	q_1	q_2	$r_1 - a_2 q_1 (q_1 - t)^{-1}$	r_2
s_3	$q_1 + a_3 q_1 D^{-1}$	$q_2 + a_3 q_2 D^{-1}$	r_1	r_2
s_4	q_1	q_2	r_1	$r_2 - a_4 q_2 (q_2 - 1)^{-1}$
s_5	q_1	$q_2(1 + a_5 r_2^{-1})$	r_1	$r_2 + a_5$
π	$E_1 E_t^{-1}$	$E_2 E_t^{-1}$	$-(q_1 - t) E_1 F^{-1}$	$-(q_2 - q_1) E_2 F^{-1}$
ρ	$q_2 t^{-1}$	$q_1 t^{-1}$	r_2	r_1

where $D = r_1 + r_2 - a_3 + \eta$, $E_1 = q_1 q_2 (r_1 + r_2 + \eta) - q_2 r_1 - q_1 r_2$, $E_2 = q_1 q_2 (r_1 + r_2 + \eta) - t q_2 r_1 - q_1 r_2$, $E_t = q_1 q_2 (r_1 + r_2 + \eta) - t q_2 r_1 - t q_1 r_2$ and $F = q_1 q_2 (t - 1)$.

§ 1.3. Basic facts

In this paper, we use the following basic facts; see § 2 of [3] for details.

Let $\mathcal{X}$ and $\mathcal{Y}$ be smooth projective varieties. For a birational map $f : \mathcal{X} \to \mathcal{Y}$, let $I(f)$ denote the indeterminate set (i.e. the set of points where f is not defined) of f in $\mathcal{X}$.

We say a sequence of birational maps $\varphi_n : \mathcal{X}_n \to \mathcal{X}_{n+1}$ for smooth projective varieties $\mathcal{X}_n$ $(n \in \mathbb{Z})$ to be algebraically stable if

$$(\varphi_{n+k-1} \circ \cdots \circ \varphi_{n+1} \circ \varphi_n)^* = \varphi_n^* \circ \varphi_{n+1}^* \circ \cdots \circ \varphi_{n+k-1}^*$$

holds as a mapping from the Picard group of $\mathcal{X}_{n+k}$ to that of $\mathcal{X}_n$ for any integers n and $k \geq 1$.

Proposition 1.1 ([2, 1]). *A sequence of birational maps $\varphi_n : \mathcal{X}_n \to \mathcal{X}_{n+1}$ for smooth projective varieties $\mathcal{X}_n$ $(n \in \mathbb{Z})$ is algebraically stable if and only if there do not exist integers n and $k \geq 1$ and a divisor D on $\mathcal{X}_{n-1}$ such that $\varphi(D \setminus I(\varphi_{n-1})) \subset I(\varphi_{n+k-1} \circ \cdots \circ \varphi_{n+1} \circ \varphi_n)$.*[4]

We call a birational mapping $f : \mathcal{X} \to \mathcal{Y}$ a *pseudo-isomorphism* if f is isomorphic except on finite number of subvarieties of codimension two at least. This condition is equivalent to that there is no prime divisor pulled back to the zero divisor by f or f^{-1}. Hence, if φ_n is a pseudo-isomorphism for each n, then $\{\varphi_n\}_{n \in \mathbb{Z}}$ and $\{\varphi_n^{-1}\}_{n \in \mathbb{Z}}$ are algebraically stable.

[4] This statement is a non-autonomous analog of a proposition shown in [2, 1]. The proof does not change except in notations.

Proposition 1.2 ([4]). *Let $\mathcal{X}$ and $\mathcal{Y}$ be smooth projective varieties and φ a pseudo-isomorphism from $\mathcal{X}$ to $\mathcal{Y}$. Then φ acts on the Néron-Severi bi-lattice as an automorphism preserving the intersections.*

The Néron-Severi bi-lattice of a smooth rational variety $\mathcal{X}$ is isomorphic to $H^2(\mathcal{X},\mathbb{Z})\times H_2(\mathcal{X},\mathbb{Z})$ which is explicitly given in the following.

Blowup of a direct product of $\mathbb{P}^1$
In accordance with [15], we take the basis of the Néron-Severi bi-lattice as follows.

Let $\mathcal{X}$ be a rational variety obtained by K successive blowups from $(\mathbb{P}^1)^N$ and

$$(\mathbf{x}_1,\mathbf{x_2},\ldots,\mathbf{x}_N)=(x_{10}:x_{11},\ x_{20}:x_{21},\ \cdots,\ x_{N0}:x_{N1})$$

the direct product of homogeneous coordinate chart. Let $\mathcal{H}_i$ denote the total transform of the class of a hyper-plane $\mathbf{c}_i\mathbf{x}_i=c_{i0}x_{i0}+c_{i1}x_{i1}=0$, where $\mathbf{c}_i=(c_{i0}:c_{i1})$ is a constant vector in $\mathbb{P}^1$, and $\mathcal{E}_k$ the total transform of the k-th exceptional divisor class. Let h_i denote the total transforms of the class of a line

$$\{\mathbf{x}\mid\forall j\neq i,(x_{j0}:x_{j1})=(c_{j0}:c_{j1})\},$$

where $\mathbf{c}_j=(c_{j0}:c_{j1})$'s $(j\neq i)$ are constant vectors in $\mathbb{P}^1$, and e_k the class of a line in a fiber of the k-th blow-up. Note that the exceptional divisor for a blowing-up along a d-dimensional subvariety V is isomorphic to $V\times\mathbb{P}^{N-d-1}$, where $\mathbb{P}^{N-d-1}$ is a fiber.

Then the Picard group $\simeq H^2(\mathcal{X},\mathbb{Z})$ and its Poincaré dual $\simeq H_2(\mathcal{X},\mathbb{Z})$ are lattices

$$H^2(\mathcal{X},\mathbb{Z})=\bigoplus_{i=1}^{n}\mathbb{Z}\,\mathcal{H}_i\oplus\bigoplus_{k=1}^{K}\mathbb{Z}\,\mathcal{E}_k,\quad H_2(\mathcal{X},\mathbb{Z})=\bigoplus_{i=1}^{n}\mathbb{Z}\,h_i\oplus\bigoplus_{k=1}^{K}\mathbb{Z}\,e_k \tag{1.4}$$

and the intersection form is given by

$$\langle\mathcal{H}_i,h_j\rangle=\delta_{ij},\quad\langle\mathcal{E}_k,e_l\rangle=-\delta_{kl},\quad\langle\mathcal{H}_i,e_k\rangle=0,\quad\langle\mathcal{E}_k,h_i\rangle=0. \tag{1.5}$$

Degree of a mapping
Let ψ be a rational mapping from $\mathbb{C}^N$ to itself:

$$\psi:(\bar{x}_1,\ldots,\bar{x}_N)=(\psi_1(x_1,\cdots,x_N),\ldots,\psi_N(x_1,\cdots,x_N)).$$

The degree of $\bar{x}_i$ of ψ with respect to x_j is defined as the degree of ψ_i as a rational function of x_j, i.e. the maximum of degrees of numerator and denominator. If $\mathbb{C}^N$ is compactified as $(\mathbb{P}^1)^N$, the degree of $\bar{x}_i$ of ψ with respect to x_j is given by the coefficient of $\mathcal{H}_j$ in $\psi^*(\mathcal{H}_i)$. This formula also holds when $(\mathbb{P}^1)^N$ is blown-up if $\mathcal{H}_i$ denotes the total transform with respect to blowing-up as the above settings.

§ 2. Construction of the space of initial conditions

First of all, let us compactify the phase space $\{(q_1, q_2, r_1, r_2) \in \mathbb{C}^4\}$ to $(\mathbb{P}^1)^4$ by introducing new coordinates $Q_1 = q_1^{-1}$, $Q_2 = q_2^{-1}$, $R_1 = r_1^{-1}$, $R_2 = r_2^{-1}$ around $q_1 = \infty$, $q_2 = \infty$ and so on.

Next, we search $(\mathbb{P}^1)^4$ for hyper-surfaces which are contracted to lower dimensional subvarieties. Such hyper-surfaces appear as a factor of the numerator of Jacobian of a map $\partial(\bar{q}_1, \bar{q}_2, \bar{r}_1, \bar{r}_2)/\partial(q_1, q_2, r_1, r_2)$. However, note that $(\mathbb{P}^1)^4$ has essentially $2^4 = 16$ charts, and the Jacobian should be considered between all the pairs of charts.

For example, the Jacobian $\partial(\bar{q}_1, \bar{q}_2, \bar{r}_1, \bar{r}_2)/\partial(q_1, q_2, r_1, r_2)$ of s_0 is a constant of 1 in the original chart, while it becomes nontrivial on another chart as

$$\partial(\bar{q}_1, \bar{q}_2, \bar{R}_1, \bar{R}_2)/\partial(q_1, q_2, r_1, r_2) = \frac{(q_1 - q_2)^4}{(a_0 q_1 - q_1 r_1 + q_2 r_1)^2 (a_0 q_2 + q_1 r_2 - q_2 r_2)^2}.$$

Thus, the image of the generic part of hyper-surface $q_1 - q_2 = 0$ is contracted to some lower dimensional subvariety. Indeed, substituting $q_2 = q_1 + \varepsilon$ to $(\bar{q}_1, \bar{q}_2, \bar{R}_1, \bar{R}_2)$, we have

$$(\bar{q}_1, \bar{q}_2, \bar{R}_1, \bar{R}_2) = \left(q_1, q_1 + \varepsilon, \frac{\varepsilon}{a_0 q_1} + O(\varepsilon^2), -\frac{\varepsilon}{a_0 q_1} + O(\varepsilon^2) \right),$$

where O is the big O asymptotic notation, Hence, we can see that the generic part of hyper-surface $q_1 - q_2 = 0$ is contracted to a two-dimsensional variety

$$\{(\bar{q}_1, \bar{q}_2, \bar{R}_1, \bar{R}_2) \in \mathbb{C}^4 \mid \bar{q}_1 - \bar{q}_2 = 0,\ \bar{R}_1 = \bar{R}_2 = 0\}. \tag{2.1}$$

Similarly, for s_1, from

$$\partial(\bar{q}_1, \bar{q}_2, \bar{r}_1, \bar{r}_2)/\partial(q_1, q_2, r_1, r_2) = \frac{a_1 + r_1}{r_1}$$

and

$$\partial(\bar{Q}_1, \bar{q}_2, \bar{r}_1, \bar{r}_2)/\partial(q_1, q_2, r_1, r_2) = -\frac{r_1}{q_1^2 (a_1 + r_1)},$$

we see that $r_1 + a_1 = 0$ is contracted to

$$\{(\bar{q}_1, \bar{q}_2, \bar{r}_1, \bar{r}_2) \in \mathbb{C}^4 \mid \bar{q}_1 = 0, \bar{r}_1 = 0\} \tag{2.2}$$

and $r_1 = 0$ is contracted to

$$\{(\bar{Q}_1, \bar{q}_2, \bar{r}_1, \bar{r}_2) \in \mathbb{C}^4 \mid \bar{Q}_1 = 0, \bar{r}_1 + \bar{a}_1 = 0\}. \tag{2.3}$$

For s_2, from

$$\partial(\bar{q}_1, \bar{q}_2, \bar{R}_1, \bar{r}_2)/\partial(q_1, q_2, r_1, r_2) = -\frac{(q_1 - t)^2}{(a_2 q_1 - q_1 r_1 + r_1 t)^2},$$

we see $q_1 - t = 0$ is contracted to

$$\{(\bar{q}_1, \bar{q}_2, \bar{R}_1, \bar{r}_2) \in \mathbb{C}^4 \mid \bar{q}_1 - \bar{t} = 0, \bar{R}_1 = 0\}. \tag{2.4}$$

For s_3, from

$$\partial(\bar{q}_1, \bar{q}_2, \bar{r}_1, \bar{r}_2)/\partial(q_1, q_2, r_1, r_2) = \frac{(r_1 + r_2 + \eta)^2}{(r_1 + r_2 - a_3 + \eta)^2}$$

and

$$\partial(\bar{Q}_1, \bar{Q}_2, \bar{r}_1, \bar{r}_2)/\partial(q_1, q_2, r_1, r_2) = \frac{(r_1 + r_2 - a_3 + \eta)^2}{q_1^2 q_2^2 (r_1 + r_2 + \eta)^2},$$

we see that $r_1 + r_2 + \eta = 0$ is contracted to

$$\{(\bar{q}_1, \bar{q}_2, \bar{r}_1, \bar{r}_2) \in \mathbb{C}^4 \mid \bar{q}_1 = \bar{q}_2 = 0, \bar{r}_1 + \bar{r}_2 - \bar{a}_3 + \bar{\eta} = 0\} \tag{2.5}$$

and $r_1 + r_2 - a_3 + \eta = 0$ is contracted to

$$\{(\bar{Q}_1, \bar{Q}_2, \bar{r}_1, \bar{r}_2) \in \mathbb{C}^4 \mid \bar{Q}_1 = \bar{Q}_2 = 0, \bar{r}_1 + \bar{r}_2 + \bar{\eta} = 0\}. \tag{2.6}$$

For s_4, from

$$\partial(\bar{q}_1, \bar{q}_2, \bar{r}_1, \bar{R}_2)/\partial(q_1, q_2, r_1, r_2) = -\frac{(q_2 - 1)^2}{(a_4 q_2 + r_2 - q_2 r_2)^2},$$

we see that $q_2 - 1 = 0$ is contracted to

$$\{(\bar{q}_1, \bar{q}_2, \bar{r}_1, \bar{R}_2) \in \mathbb{C}^4 \mid \bar{q}_2 - 1 = 0, \bar{R}_2 = 0\}. \tag{2.7}$$

For s_5, from

$$\partial(\bar{q}_1, \bar{q}_2, \bar{r}_1, \bar{r}_2)/\partial(q_1, q_2, r_1, r_2) = \frac{a_5 + r_2}{r_2}$$

and

$$\partial(\bar{q}_1, \bar{Q}_2, \bar{r}_1, \bar{r}_2)/\partial(q_1, q_2, r_1, r_2) = -\frac{r_2}{q_2^2 (r_2 + a_5)},$$

we see that $r_2 + a_5 = 0$ is contracted to

$$\{(\bar{q}_1, \bar{q}_2, \bar{r}_1, \bar{r}_2) \in \mathbb{C}^4 \mid \bar{q}_2 = 0, \bar{r}_2 = 0\} \tag{2.8}$$

and $r_2 = 0$ is contracted to

$$\{(\bar{q}_1, \bar{Q}_2, \bar{r}_1, \bar{r}_2) \in \mathbb{C}^4 \mid \bar{Q}_2 = 0, \bar{r}_2 + \bar{a}_5 = 0\}. \tag{2.9}$$

Let us take a closer look at the image (2.1) of $q_1 - q_2 = 0$ by s_0 and introduce new coordinates (by blowing-up)

$$(w, q_2, u, v) = \left(\frac{q_1 - q_2}{R_1}, q_2, R_1, \frac{R_2}{R_1}\right). \tag{2.10}$$

Then, from

$$\partial(\bar{w},\bar{q}_2,\bar{u},\bar{v})/\partial(q_1,q_2,r_1,r_2)=\frac{(q_1-q_2)^2}{(a_0q_2+q_1r_2-q_2r_2)^2},$$

we see that $q_1-q_2=0$ is still contracted to

$$\{(\bar{w},\bar{q}_2,\bar{u},\bar{v})\in\mathbb{C}^4\ \mid \bar{w}-\bar{a}_0\bar{q}_2=0,\bar{u}=0,\bar{v}+1=0\}. \tag{2.11}$$

Let us introduce further new coordinates (by blowing-up)

$$(w',q_2,u',v')=\left(\frac{w-a_0q_2}{u},q_2,u,\frac{v+1}{u}\right). \tag{2.12}$$

Then, from

$$\partial(\bar{w}',\bar{q}_2,\bar{u}',\bar{v}')/\partial(q_1,q_2,r_1,r_2)=-1,$$

we see that there is no hyper-surface in the affine space of (q_1,q_2,r_1,r_2) chart contracted to a subvariety whose generic part is included in the affine space of $(\bar{w}',\bar{q}_2,\bar{u}',\bar{v}')$ chart. (Charts (2.10) and (2.12) are the same with U_{11} and U_{12} below.)

By investigating other transformations in the same way, we can see that, resolution of singularities requires two infinitesimally near blowing-ups (i.e. the center of the second blowing-up is included in the exceptional divisor of the first blowing up) for s_0, s_2 or s_4, while it also requires two blowing-ups but not infinitesimally near for s_1, s_3 or s_5.

These blowing-ups are given by the following list.

(2.13)

$C_1: q_1=r_1=0$ $\qquad U_1:(u_1,q_2,v_1,r_2)=(q_1,q_2,r_1q_1^{-1},r_2)$

$C_2: Q_1=r_1+a_1=0$ $\qquad U_2:(u_2,q_2,v_2,r_2)=(Q_1,q_2,(r_1+a_1)Q_1^{-1},r_2)$

$C_3: q_2=r_2=0$ $\qquad U_3:(q_1,u_3,r_1,v_3)=(q_1,q_2,r_1,r_2q_2^{-1})$

$C_4: Q_2=r_2+a_5=0$ $\qquad U_4:(q_1,u_4,r_1,v_4)=(q_1,Q_2,r_1,(r_2+a_5)Q_2^{-1})$

$C_5: q_1=q_2=r_1+r_2-a_3+\eta=0$

$\qquad U_5:(u_5,v_5,w_5,r_2)=(q_1,q_2q_1^{-1},(r_1+r_2-a_3+\eta)q_1^{-1},r_2)$

$C_6: Q_1=Q_2=r_1+r_2+\eta=0$ $\qquad U_6:(u_6,v_6,w_6,r_2)=(Q_1,Q_2Q_1^{-1},(r_1+r_2+\eta)Q_1^{-1},r_2)$

$C_7: q_1-t=R_1=0$ $\qquad U_7:(v_7,q_2,u_7,r_2)=((q_1-t)R_1^{-1},q_2,R_1,r_2)$

$C_8: u_7=v_7-a_2t=0$ $\qquad U_8:(v_8,q_2,u_8,r_2)=((v_7-a_2t)u_7^{-1},q_2,u_7,r_2)$

$C_9: q_2-1=R_2=0$ $\qquad U_9:(q_1,v_9,r_1,u_9)=(q_1,(q_2-1)R_2^{-1},r_1,R_2)$

$C_{10}: u_9=v_9-a_4=0$ $\qquad U_{10}:(q_1,v_{10},r_1,u_{10})=(q_1,(v_9-a_4)u_9^{-1},r_1,u_9)$

$C_{11}: q_1-q_2=R_1=R_2=0$

$\qquad U_{11}:(w_{11},q_2,u_{11},v_{11})=((q_1-q_2)R_1^{-1},q_2,R_1,R_2R_1^{-1})$

$C_{12}: u_{11}=v_{11}+1=w_{11}-a_0q_2=0$

$\qquad U_{12}:(w_{12},q_2,u_{12},v_{12})=((w_{11}-a_0q_2)u_{11}^{-1},q_2,u_{11},(v_{11}+1)u_{11}^{-1}),$

where we write only one of new coordinate systems where the exceptional divisor is given by $u_i = 0$ $(i = 1, 2, \cdots, 12)$. The other coordinate systems are automatically determined from the above data. For example, the other two coordinate systems for blowing up along C_5 are

$$U_5' : (u_5', v_5', w_5', r_2) = (q_1 q_2^{-1}, q_2, (r_1 + r_2 - a_3 + \eta) q_2^{-1}, r_2)$$

and

$$U_5'' : (u_5'', v_5'', w_5'', r_2) = (q_1(r_1+r_2-a_3+\eta)^{-1}, q_2(r_1+r_2-a_3+\eta)^{-1}, r_1+r_2-a_3+\eta, r_2),$$

where the exceptional divisor is given by $v_5' = 0$ and $w_5'' = 0$ respectively. More precisely, above coordinate systems are obtained only by blow-ups along open subset of C_i's, but the other systems are also determined automatically by algebraic continuation (we assume C_i's are irreducible).

Theorem 2.1. *Let $\mathcal{X}_A$ $(A = (a_0, a_1, \cdots, a_5, \eta, t))$ be a rational variety obtained by blowing-ups along $C_1, \cdots, C_{12}$ above. Each Bäclund transformation $s_0, \cdots, s_5$ or ρ is lifted to a pseudo-isomorphism from a rational variety $\mathcal{X}_A$ to $\mathcal{X}_{\bar{A}}$, where $\bar{A} = (\bar{a}_0, \bar{a}_1, \cdots, \bar{a}_5, \bar{\eta}, \bar{t})$ is given by Table 'Actions on parameters".*

Proof. It is confirmed by direct computation that the Jacobians of lifted mappings do not vanish. □

Remark. Transformation π is not lifted to a pseudo-isomorphism from $\mathcal{X}_A$ to $\mathcal{X}_{\bar{A}}$. Indeed, $q_2 = 0$ (a hyper-surface) is mapped to

$$(\bar{w}_{11}, \bar{q}_2, \bar{u}_{11}, \bar{v}_{11}) = \left(\frac{r_1}{q_1}, \frac{1}{t}, 0, -1 + \frac{t}{q_1}\right)$$

with $\bar{q}_1 - \bar{q}_2 = \bar{R}_1 = \bar{R}_2 = 0$. Hence the image of $q_2 = 0$ is included in a two-dimentional subvariety of C_{11} such that the subvariety is different from C_{12}. Hence $q_2 = 0$ is contracted to lower dimensional variety in $\mathcal{X}_{\bar{A}}$. The same thing happens also for $Q_1 = 0$: which is contracted to a two-dimentional subvariety of C_{11} different from C_{12}.

Of course, there is possibility to be able to construct the space of initial conditions that allow π, but it needs several more blowing-ups, and the action of the root system would become more complicated.

Let us denote $\mathcal{E}_i$ as the class of total transform of the exceptional divisor obtained by the blowing-up along C_i. Then, since C_5, C_6, C_{11}, C_{12} are subvarieties of codimension 3, while the others are of codimension 2, the anti-canonical divisor class is

$$-K_{\mathcal{X}_A} = 2\,\mathcal{H}_{q_1} + 2\,\mathcal{H}_{q_2} + 2\,\mathcal{H}_{r_1} + 2\,\mathcal{H}_{r_2} - \sum_{i=1,2,3,4,7,8,9,10} \mathcal{E}_i - 2 \sum_{i=5,6,11,12} \mathcal{E}_i\,.$$

Moreover, since C_8, C_{10} and C_{12} are subvarieties of $\mathcal{E}_7$, $\mathcal{E}_9$ and $\mathcal{E}_{11}$ respectively, $\mathcal{E}_7-\mathcal{E}_8$, $\mathcal{E}_9-\mathcal{E}_{10}$ and $\mathcal{E}_{11}-\mathcal{E}_{12}$ are effective divisor class in $\mathcal{X}_A$. Hence, $-K_{\mathcal{X}_A}$ is decomposed into irreducible divisors as

$$\begin{aligned} -K_{\mathcal{X}_A} =&(\mathcal{H}_{q_1}-\mathcal{E}_1-\mathcal{E}_5)+(\mathcal{H}_{q_1}-\mathcal{E}_2-\mathcal{E}_6)+(\mathcal{H}_{q_2}-\mathcal{E}_3-\mathcal{E}_5)+(\mathcal{H}_{q_2}-\mathcal{E}_4-\mathcal{E}_6)\\ &+2(\mathcal{H}_{r_1}-\mathcal{E}_7-\mathcal{E}_{11})+2(\mathcal{H}_{r_2}-\mathcal{E}_9-\mathcal{E}_{11})\\ &+(\mathcal{E}_7-\mathcal{E}_8)+(\mathcal{E}_9-\mathcal{E}_{10})+2(\mathcal{E}_{11}-\mathcal{E}_{12}), \end{aligned} \tag{2.14}$$

where each irreducible divisor is explicitly written by coordinates as

$$\begin{aligned} \mathcal{H}_{q_1}-\mathcal{E}_1-\mathcal{E}_5 &: q_1=0\\ \mathcal{H}_{q_1}-\mathcal{E}_2-\mathcal{E}_6 &: Q_1=0\\ \mathcal{H}_{q_2}-\mathcal{E}_3-\mathcal{E}_5 &: q_2=0\\ \mathcal{H}_{q_2}-\mathcal{E}_4-\mathcal{E}_6 &: Q_2=0\\ \mathcal{H}_{r_1}-\mathcal{E}_7-\mathcal{E}_{11} &: R_1=0\\ \mathcal{H}_{r_2}-\mathcal{E}_9-\mathcal{E}_{11} &: R_2=0. \end{aligned}$$

Note that these subvarieties correspond to vertical leaves, i.e. the ordinary differential equations are not defined on these subvarieties.

Remark. Through the natural identification between exceptional divisors for different values of parameter A's, we use the same symbol $\mathcal{E}_i$'s for all A's.

§ 3. The root system and the actions on the bilattice

We can directly compute the actions s_i's on the Picard group. However, in order to see a geometric way of construction of the root system explicitly, let us reconstruct the Bäcklund transformations from a root system defined in the Neron-Severi bilattice.

We use a higher dimensional analog of the notion of Cremona isometry introduced in [3].

Definition 3.1. An automorphism s of the Néron-Severi bilattice is called a Cremona isometry if the following three properties are satisfied:
(a) s preserves the intersection form;
(b) s leaves the decomposition of $-K_{\mathcal{X}}$ fixed;
(c) s leaves the semigroup of effective classes of divisors invariant.

Our aim is to realise the group of Cremona isometries as a root system, though it is the most heuristic part of this procedure. Different to the two-dimensional case, we merely know the decomposition of the null-root (the anti-canonical divisor) in one of the dual spaces as (2.14), and hence, we can collect the vectors orthogonal to the elements of the decomposition only in the homology space.

Let us set the roots and co-roots as

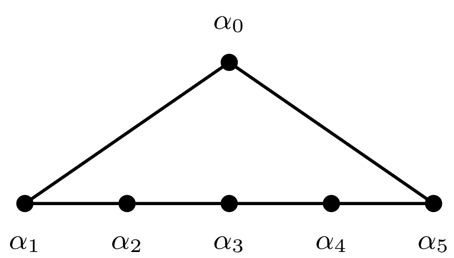
$$\alpha_0 = \mathcal{H}_{q_1} + \mathcal{H}_{q_2} - \mathcal{E}_{5,6,11,12}, \quad \alpha_1 = \mathcal{H}_{r_1} - \mathcal{E}_{1,2}, \quad \alpha_2 = \mathcal{H}_{q_1} - \mathcal{E}_{7,8},$$
$$\alpha_3 = \mathcal{H}_{r_1} + \mathcal{H}_{r_2} - \mathcal{E}_{5,6,11,12}, \quad \alpha_4 = \mathcal{H}_{q_2} - \mathcal{E}_{9,10}, \quad \alpha_5 = \mathcal{H}_{r_2} - \mathcal{E}_{3,4}$$

and

$$\check{\alpha}_0 = h_{r_1} + h_{r_2} - e_{11,12}, \quad \check{\alpha}_1 = h_{q_1} - e_{1,2}, \quad \check{\alpha}_2 = h_{r_1} - e_{7,8},$$
$$\check{\alpha}_3 = h_{q_1} + h_{q_2} - e_{5,6}, \quad \check{\alpha}_4 = h_{r_2} - e_{9,10}, \quad \check{\alpha}_5 = h_{q_2} - e_{3,4},$$

where $\mathcal{E}_{i_1,\dots,i_n}$ and $e_{i_1,\dots,i_n}$ are the abbreviations of $\mathcal{E}_{i_1} + \cdots + \mathcal{E}_{i_n}$ and $e_{i_1} + \cdots + e_{i_n}$ respectively. Then, they constitute the root basis of type $A_5^{(1)}$ whose Cartan matrix and the Dynkin diagram are as follows.

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & 0 & -1 & -2 \end{bmatrix}$$



Remark.

1. The decomposition (2.14) constitutes of 9 irreducible divisors in rank 14 space (with null-vector). Hence, rank 6 of type $A_5^{(1)}$ is maximum.

2. The number of free parameters of the space of initial conditions having the decomposition (2.14) is 19: 2 for C_i $(i = 1, 2, 3, 4, 7, 9, 12)$, and 1 for C_i $(i = 5, 6, 8, 10, 11)$. We can reduce it using Möbius transformation with respect to each coordinate to $19 - 3 \times 4 = 7$. We leave one of 12 variables (able to be fixed by Möbius transformations) to be free for the continuous time variable t, while the sum of a_i's is fixed to be 1. Hence we have 7 variables in total as $a_0, \cdots, a_5$, η and t in the formulation of Fuji-Suzuki [5].

The actions of the roots on the Néron-Severi bilattice are given by the formulae

$$s_i(\mathcal{D}) = \mathcal{D} - 2\frac{\langle \mathcal{D}, \check{\alpha}_i \rangle}{\langle \alpha_i, \check{\alpha}_i \rangle}\alpha_i, \quad s_i(d) = d - 2\frac{\langle \alpha_i, d \rangle}{\langle \alpha_i, \check{\alpha}_i \rangle}\check{\alpha}_i$$

for any $\mathcal{D} \in H^2(\mathcal{X}_A, \mathbb{Z})$ and $d \in H_2(\mathcal{X}_A, \mathbb{Z})$, while ρ acts on $H^2(\mathcal{X}_A, \mathbb{Z})$ as

$$\begin{array}{ll} \mathcal{H}_{q_1} \leftrightarrow \mathcal{H}_{q_2}, \quad \mathcal{H}_{p_1} \leftrightarrow \mathcal{H}_{p_2} & h_{q_1} \leftrightarrow h_{q_2}, \quad h_{p_1} \leftrightarrow h_{p_2} \\ \mathcal{E}_1 \leftrightarrow \mathcal{E}_3, \quad \mathcal{E}_2 \leftrightarrow \mathcal{E}_4 & e_1 \leftrightarrow e_3, \quad e_2 \leftrightarrow e_4 \\ \mathcal{E}_7 \leftrightarrow \mathcal{E}_9, \quad \mathcal{E}_8 \leftrightarrow \mathcal{E}_{10} & e_7 \leftrightarrow e_9, \quad e_8 \leftrightarrow e_{10} \end{array} \tag{3.1}$$

Translation
A translation of $A_5^{(1)}$

$$T_{\alpha_i} : \bar{\alpha}_i = \alpha_i + 2\delta, \quad \bar{\alpha}_{i\pm1} = \alpha_{i\pm1} - \delta, \quad \bar{\alpha}_j = \alpha_j \quad (|i-j| > 1), \tag{3.2}$$

where $\delta = -K_{\mathcal{X}}$, is realised as

$$T_{\alpha_i} = s_{i+1} \circ s_{i+2} \circ s_{i+3} \circ s_{i+4} \circ s_{i+5} \circ s_{i+4} \circ s_{i+3} \circ s_{i+2} \circ s_{i+1} \circ s_i.$$

The action of T_{α_i} on the Picard group is given by Kac's formula

$$T_{\alpha_i}^*(\mathcal{D}) = \mathcal{D} + \langle \mathcal{D}, \check{\delta} \rangle \alpha_i + \left(\langle \mathcal{D}, \check{\delta} \rangle + \frac{1}{2} \langle \alpha_i, \check{\alpha}_i \rangle \langle \mathcal{D}, \check{\alpha}_i \rangle \right) \delta, \tag{3.3}$$

for any $\mathcal{D} \in H^2(\mathcal{X}_A, \mathbb{Z})$, where

$$\check{\delta} = \sum_{i=0}^{5} \check{\alpha}_i = 2h_{q_1} + 2h_{q_2} + 2h_{r_1} + 2h_{r_2} - \sum_{i=1}^{12} e_i. \tag{3.4}$$

Since the degrees of the iteration of this mapping with respect to q_1, q_2, r_1, r_2 are given by the coefficients of $\mathcal{H}_{q_1}$, $\mathcal{H}_{q_2}$, $\mathcal{H}_{r_1}$, $\mathcal{H}_{r_1}$ in

$$\left((T_{\alpha_i})^n \right)^* (\mathcal{D}) = \mathcal{D} + n \langle \mathcal{D}, \check{\delta} \rangle \alpha_i + \left(n^2 \langle \mathcal{D}, \check{\delta} \rangle + \frac{n}{2} \langle \alpha_i, \check{\alpha}_i \rangle \langle \mathcal{D}, \check{\alpha}_i \rangle \right) \delta \tag{3.5}$$

with $\mathcal{D} = \mathcal{H}_{q_1}, \mathcal{H}_{q_2}, \mathcal{H}_{p_1}, \mathcal{H}_{p_2}$ (as explained in § 1.3), they increase quadratically with respect to n.

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