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Limit theorem of the max-plus walk

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Abstract

The max-plus algebra is a semiring on $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ with addition $\oplus$ and multiplication $\otimes$ defined by $\oplus = \max$ and $\otimes = +$, respectively. It is known that eigenvalues of max-plus matrices are equivalent to the maximal average weight of the corresponding directed graph. In [9], authors introduced the max-plus walk which is a walk model on one dimensional lattice on $\mathbb{Z}$ over max-plus algebra, and discussed its properties such as the conserved quantities and the steady state. In this paper, we will discuss the limit measure of the max-plus walk.

§ 1. Introduction

Studies on a spatial discretization of the Schrödinger equation is known as the discrete-Schrödinger equation. Quantum walk can be regarded as a temporal discretization of a discrete-Schroëdinger equation [6, 7]. This is a reason that quantum walk is recently known as a quantum simulator as envisioned by Feynman [2]. On the other hand, there is a further discretization; so called the ultradiscretization, which is introduced in [8]. Ultradiscretization is a technique which transform a difference equation

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into piecewise linear equation and appears in the context of integrable systems. It is based on the following formula.

$$\lim_{\epsilon\to+0} \epsilon \log(e^{A/\epsilon} + e^{B/\epsilon}) = \max\{A, B\}.$$

Since ultradiscrete equations can be written by using operations "max" and "+", they can be considered over max-plus algebra. In max-plus algebra, the sum of two elements is their maximum and the product of two elements is their sum. This algebraic structure is known as idempotent semiring. In [9], a new walk model on one-dimensional lattice over max-plus algebra is introduced. Such a walk is called the max-plus walk.

In the usual quantum walk (resp. random walk) on one dimensional lattice, the limit theorem can be characterized as follows [4]: (1) the scaling order is linearly proportional (resp. proportional to square root) to time steps and (2) the limit distribution is described by the Konno distribution (resp. the Gaussian distribution). In [9], we find the conserved quantities which are independent of the time step, such as the ℓ^2-norm conservation in the unitary quantum walk, and a useful necessary and sufficient condition of the setting of the max-plus walk for the conservation. The conserved quantities are given by the summation of eigenvalues of the state decision matrices over all the lattice points. In this paper, we define a measure on the lattice points under the condition. We also obtain a limit theorem on this measure in the long-time limit. This corresponds to the original unitary quantum walk.

This paper is organized as follows. In sections 2 and 3, we give a short review on the max-plus algebra and max-plus walk, respectively. In section 4, under a certain conserved condition, we obtain the limit theorem corresponding to the usual quantum walk. Finally, in section 5, we give concluding remarks.

§ 2. Max-plus algebra

Max-plus algebra is defined in a set $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ with two binary operations

$$a \oplus b = \max\{a, b\}$$

and

$$a \otimes b = a + b$$

for $a, b \in \mathbb{R}_{\max}$. Addition $\oplus$ is commutative, associative and has the identity element $\varepsilon := -\infty$. Multiplication $\otimes$ is also commutative, associative and has the identity element $e := 0$. $\otimes$ is distributive with respect to $\oplus$. There exists an inverse element with respect to $\otimes$ for any elements in $\mathbb{R}_{\max} \setminus \{\varepsilon\}$. Note that there is not exist inverse elements with respect to $\oplus$. $\oplus$ is idempotent, namely, $a \oplus a = a$ for $a \in \mathbb{R}_{\max}$.

As in the conventional algebra, we extend the operations $\oplus$ and $\otimes$ to matrices. Let $\mathbb{R}_{\max}^{m\times n}$ be the set of $m \times n$ matrices whose entries are in $\mathbb{R}_{\max}$ and $[X]_{ij}$ be the (i,j) entry of the matrix X. For $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}_{\max}^{m\times n}$, their sum $A \oplus B \in \mathbb{R}_{\max}^{m\times n}$ is defined by

$$[A \oplus B]_{ij} = a_{ij} \oplus b_{ij}.$$

For $A = (a_{ij}) \in \mathbb{R}_{\max}^{m\times k}$ and $B = (b_{ij}) \in \mathbb{R}_{\max}^{k\times n}$, their product $A \otimes B \in \mathbb{R}_{\max}^{m\times n}$ is defined by

$$[A \otimes B]_{ij} = \bigoplus_{l=1}^{k} a_{il} \otimes b_{lj}.$$

We denote $A^{\otimes k}$ as the k-th power of A, namely,

$$A^{\otimes k} := \underbrace{A \otimes A \otimes \cdots \otimes A}_{k \text{ times}}.$$

For a matrix $A = (a_{ij}) \in \mathbb{R}_{\max}^{n\times n}$, we define the tropical determinant of A by

$$\operatorname{tropdet}(A) = \bigoplus_{\sigma\in S_n} a_{1\sigma(1)} \otimes a_{2\sigma(2)} \otimes \cdots \otimes a_{n\sigma(n)}$$

where S_n denotes the symmetric group of order n.

Although it has different operators to the conventional algebra, the eigenvalue problem for matrices is fundamental in both algebra.

Definition 2.1. For a max-plus matrix $A \in \mathbb{R}_{\max}^{n\times n}$, a scalar $\lambda \in \mathbb{R}_{\max}$ is called an eigenvalue of A if there is a vector $\boldsymbol{x} \neq (\varepsilon, \dots, \varepsilon)^\top \in \mathbb{R}_{\max}^{n}$ satisfying

$$A \otimes \boldsymbol{x} = \lambda \otimes \boldsymbol{x}.$$

Such vector $\boldsymbol{x}$ is called an eigenvector of A with respect to λ.

The max-plus eigenvalues were shown, for example, in Baccelli et al. [1] to have a close relationship with the weighted directed graphs.

For a max-plus matrix $A = (a_{ij}) \in \mathbb{R}_{\max}^{n\times n}$, we define a weighted directed graph (briefly digraph) $G(A) = (V, E, w)$ as follows. The vertex set and the edge set are $V = \{1, 2, \dots, n\}$ and $E = \{(i,j) | a_{ij} \neq \varepsilon\} \subset V \times V$, respectively, and the weight function $w : E \to \mathbb{R}$ is defined by $w((i,j)) = a_{ij}$ for $(i,j) \in E$. A sequence $C = (i_1, i_2, \dots, i_s, i_1)$ of vertices is called a circuit if $(i_k, i_{k+1}) \in E$ for all $k = 1, 2, \dots, s$ where $i_{s+1} = i_1$ and $i_p \neq i_q$ for $1 \leq p < q \leq s$. The number $\ell(C) := s$ is called the length of C and $w(C) := w((i_1,i_2)) + w((i_2,i_3)) + \cdots + w((i_s,i_1))$ is called the weight of C. We define the average weight of C by $\operatorname{ave}(C) := w(C)/\ell(C)$.

Proposition 2.2 (cf. F. Baccelli et al. [1]). For a max-plus matrix $A \in \mathbb{R}_{\max}^{n\times n}$, $\lambda \neq \varepsilon$ is an eigenvalue of A if there exists a circuit in the weighted digraph $G(A)$ whose average weight is equal to λ. In particular, if a max-plus matrix A is irreducible, A has precisely one eigenvalue and is equal to the maximum average weight of circuits in $G(A)$.

§ 3. Max-plus walk [9]

In this section, we give a brief review of the max-plus walk which is introduced in [9]. We consider the one dimensional lattice on $\mathbb{Z}$ and a max-plus vector $\psi_k^n \in \mathbb{R}_{\max}^2$ given on a lattice point. The vector ψ_k^n is called the *state* at a position $k \in \mathbb{Z}$ and a discrete time n. We set the initial state ψ_k^0 as

$$\psi_k^0 = \begin{cases} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, & k = 0, \\ \begin{bmatrix} \varepsilon \\ \varepsilon \end{bmatrix}, & k \neq 0. \end{cases}$$

In the max-plus walk, the state ψ_k^n changes according to two max-plus matrices $P, Q \in \mathbb{R}_{\max}^{2\times 2}$ given as

$$P = \begin{bmatrix} a\, b \\ \varepsilon\, \varepsilon \end{bmatrix}, \quad Q = \begin{bmatrix} \varepsilon\, \varepsilon \\ c\, d \end{bmatrix}.$$

Then, the state ψ_k^n is determined by the following evolution equation.

$$\psi_k^n = (P \otimes \psi_{k+1}^{n-1}) \oplus (Q \otimes \psi_{k-1}^{n-1}) = A_k^n \otimes \psi_0^0,$$

where $A_k^n \in \mathbb{R}_{\max}^{2\times 2}$ is called the state decision matrix. It is easy to see that, at discrete time n, $A_k^n = \begin{bmatrix} \varepsilon\, \varepsilon \\ \varepsilon\, \varepsilon \end{bmatrix}$ for $k = -n+1, -n+3, \ldots, n-3, n-1$. We define a set of positions whose state vector has at least one finite element as $\mathcal{K} = \{-n, -n+2, \ldots, n-2, n\}$. Time evolution of the max-plus walk is illustrated in Figure 1. For example, $A_{-3}^3 = P^{\otimes 3}$ and $A_1^3 = Q \otimes (Q \otimes P \oplus P \otimes Q) \oplus P \otimes Q^{\otimes 2}$. Here we introduce two matrices $R, S \in \mathbb{R}_{\max}^{2\times 2}$ as follows.

$$R = \begin{bmatrix} c\, d \\ \varepsilon\, \varepsilon \end{bmatrix}, \quad S = \begin{bmatrix} \varepsilon\, \varepsilon \\ a\, b \end{bmatrix}.$$

Let ℓ and m be the number of moving left (P) and right (Q), respectively, and $r = \lfloor \tilde{r}/2 \rfloor$, where $\tilde{r}$ is the summation of the number of making a turn from left to right and from

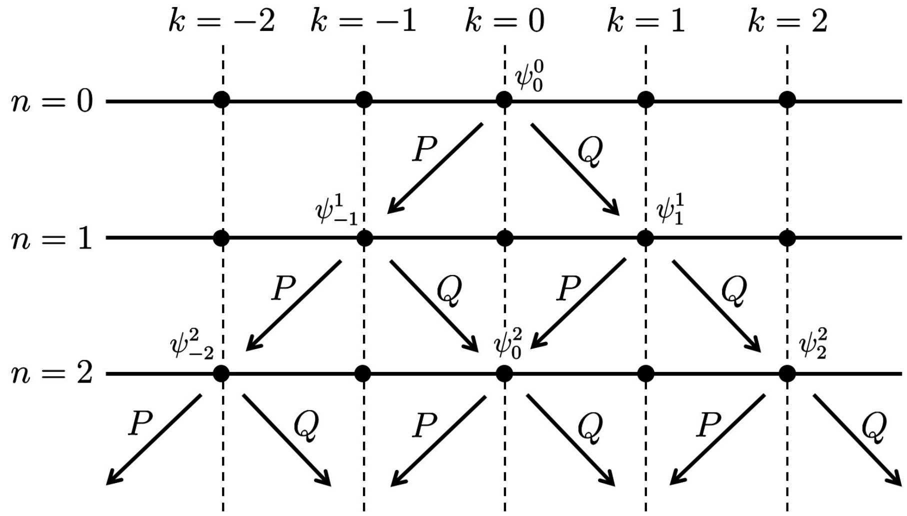

Figure 1. Max-plus walk.

right to left. Note here that $\ell = \dfrac{n-k}{2}$ and $m = \dfrac{n+k}{2}$. Then, the state decision matrices A_k^n at position $k \in \mathcal{K}$ and discrete time n can be given as

$$\begin{aligned} A_k^n = &\bigoplus_{r=1}^{(\ell-1)\wedge m} \{(\ell-r-1)a + rb + rc + (m-r)d\} \otimes P \\ &\oplus \bigoplus_{r=1}^{\ell\wedge(m-1)} \{(\ell-r)a + rb + rc + (m-r-1)d\} \otimes Q \\ &\oplus \bigoplus_{r=1}^{\ell\wedge m} \{(\ell-r)a + rb + (r-1)c + (m-r)d\} \otimes R \\ &\oplus \bigoplus_{r=1}^{\ell\wedge m} \{(\ell-r)a + (r-1)b + rc + (m-r)d\} \otimes S, \end{aligned}$$

where $a \wedge b = \min\{a, b\}$. Let $U = P \oplus Q = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and impose the following condition to the entries $a, b, c, d \in \mathbb{R}_{\max}$ of U.

$$\text{(C1)} \qquad \begin{cases} a \otimes d = e, \\ b \otimes c = e. \end{cases}$$

Then the condition (C1) is an analogue of the property that the absolute value of the determinant of unitary matrices is 1 in conventional algebra, namely,

$$\operatorname{tropdet}(U) = (a \otimes d) \oplus (b \otimes c) = e.$$

Under the condition (C1), A_k^n can be rewritten as

$$\begin{aligned} A_k^n &= \{(\ell a + md - a) \otimes P\} \oplus \{(\ell a + md - d) \otimes Q\} \\ &\quad \oplus \{(\ell a + md - c) \otimes R\} \oplus \{(\ell a + md - b) \otimes S\} \\ &= \begin{bmatrix} \ell a + md & \max\{\ell a + md - a + b, \ell a + md - c + d\} \\ \max\{\ell a + md + a - b, \ell a + md + c - d\} & \ell a + md \end{bmatrix}, \\ &\quad n = 0, 1, \ldots, \quad k = -n + 2, -n + 4, \ldots, n - 4, n + 2. \end{aligned}$$

Noting that $\ell a + md = (\ell - m)a = -ka$, then the state decision matrices A_k^n is given as follows.

$$A_k^n = \begin{cases} \begin{bmatrix} -ka & (-k-1)a + b \\ (-k+1)a - b & -ka \end{bmatrix} & \text{if } k = -n+2, -n+4, \ldots, n-4, n-2, \\ \begin{bmatrix} na & (n-1)a + b \\ \varepsilon & \varepsilon \end{bmatrix} & \text{if } k = -n, \\ \begin{bmatrix} \varepsilon & \varepsilon \\ (-n+1)a - b & -na \end{bmatrix} & \text{if } k = n, \\ \begin{bmatrix} \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \end{bmatrix} & \text{otherwise.} \end{cases}$$

Let $\lambda(A_k^n)$ be the eigenvalue of A_k^n. From Proposition 2.2, we have

$$\lambda(A_k^n) = -ak, \quad k \in \mathcal{K}.$$

It is remarkable here that the eigenvalues of A_k^n is not dependent of the discrete time n. Moreover, let Φ^n be the summation of eigenvalues $\lambda(A_k^n)$ of all the position $k \in \mathcal{K}$ at arbitrary discrete time n, namely

$$\Phi^n = \sum_{k \in \mathcal{K}} \lambda(A_k^n) = \sum_{k \in \mathcal{K}} -ak.$$

Then, we have the following theorem.

Theorem 3.1 (S. Watanabe, A. Fukuda, E. Segawa, I. Sato [9])**.** Under the condition (C1), the summation of all the eigenvalues Φ^n are invariant under the time evolution of the max-plus walk and is given as

$$\Phi^n = 0.$$

Namely, the summation of the eigenvalues are the conserved quantities of the max-plus walk.

In the quantum walk, the ℓ_2-norm of the state vectors are probability and their summation is 1, which is conserved quantities of the quantum walk. Noting that, in the max-plus algebra, the identity element with respect to $\otimes$ is 0, Theorem 3.1 is a max-plus analogue of the conserved quantities of the quantum walk. In the next section, using the analogue of the probability, we will discuss the limit measure of the max-plus walk. For the readers' reference, the following are also conserved quantities of the max-plus walk:

$$\bigoplus_{k\in\mathcal{K}}\left\{\frac{1}{n}\lambda(A_k^n)\right\}=\bigoplus_{k\in\mathcal{K}}\left\{\frac{1}{k}\lambda(A_k^n)\right\}=|a|,$$

which is written by using the max-plus analogue of the summation. This quantity corresponds to the pseudo velocity of traditional quantum walks [3].

§ 4. Limit measure of max-plus walk

First let us give a short review on the limit theorem for the traditional quantum walk on $\mathbb{Z}$ [4]. Let $\mu_n : \mathbb{Z} \to [0,1]$ be the probability distribution of this quantum walk at time n with the mixed initial state at the origin. Then μ_n converges in law as follows:

For any $x \in \mathbb{R}$,

$$\lim_{n\to\infty}\sum_{k\le nx}\mu_n(k)=\int_{-\infty}^{x} f_K(u;|a|)du$$

where $f_K(u;|a|)$ is the Konno distribution with the parameter $|a|$

$$f_K(u;|a|)=\begin{cases}\dfrac{\sqrt{1-|a|^2}}{\pi(1-u^2)\sqrt{|a|^2-u^2}}, & \text{if } -|a|\le u\le |a|,\\ 0, & \text{otherwise}\end{cases}$$

In the following, let us consider the corresponding limit theorem for the max-plus walk. Since the counterpart of the probability distribution $\mu_n(k)$ is the eigenvalues of the state decision matrix A_k^n, then we compute

$$\sum_{k\in\mathcal{K},k\le nx}\lambda(A_k^n)=\sum_{k\in\mathcal{K},k\le nx}(-ak).$$

Putting $p \in \mathbb{Z}$ by $nx-2 < -n+2p \le nx$, we have

$$\begin{aligned}\sum_{k\in\mathcal{K},k\le nx}(-ak)&=-a\{-n+(-n+2)+\cdots+(-n+2p)\}\\ &=a(p+1)(n-p)\end{aligned} \tag{4.1}$$

Since $-n+2p \leq nx < -n+2p+2$,

$$-\frac{1}{n}+\frac{x+1}{2} < \frac{p}{n} \leq \frac{x+1}{2}, \tag{4.2}$$

taking the limit of $n \to \infty$, we have

$$\lim_{n\to\infty}\frac{p}{n}=\frac{x+1}{2}.$$

Therefore

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n^2}\sum_{k\in\mathcal{K},k\leq nx}\lambda(A_k^n) &= \lim_{n\to\infty}\frac{1}{n^2}a(p+1)(n-p)\\ &= \lim_{n\to\infty} a\left(\frac{p}{n}+\frac{1}{n}\right)\left(1-\frac{p}{n}\right)\\ &= \frac{a}{4}(1-x^2).\end{aligned} \tag{4.3}$$

RHS of (4.3) is rewritten by

$$\frac{a}{4}(1-x^2)=\int_{-\infty}^{x}f(u)du.$$

Here

$$f(u)=\begin{cases}-\dfrac{au}{2} & : u\in[-1,1],\\ 0 & : \text{otherwise.}\end{cases}$$

We summarize the above statement in the following theorem.

Theorem 4.1. The limit measure of the max-plus walk is expressed by

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{k\in\mathcal{K},k\leq nx}\lambda(A_k^n)=\int_{-\infty}^{x}f(u)du.$$

Here

$$f(u)=\begin{cases}-\dfrac{au}{2} & : u\in[-1,1],\\ 0 & : \text{otherwise.}\end{cases}$$

§ 5. Concluding remarks

In this paper, we obtained the limit measure on the max-plus walk proposed by [9]. Theorem 3.1 implies that if the setting of the max-plus walk satisfies some conditions, then we obtain the conservation quantity with respect to the time step and we can define the sighed measure on $\mathbb{Z}$ at each time step. This corresponds to the ℓ^2-conservation property of the unitary time evolution of quantum walks. Let B_k^n be the state decision matrix of the usual quantum walk. (See for its explicit expression, for example, [4]). If we start the usual quantum walk from the mixed state at the origin, then the distribution at time n $\mu_n : \mathbb{Z} \to [0,1]$ is described by $\|B_k^n\|_F^2 = 2\sum_i(\sigma_i)^2$ which is quite similar to the measure of the max-plus walk. Here, $\|\cdot\|_F$ is the Frobenius norm and σ_i is a singular value of B_k^n.

By Theorem 4.1, the linearly spreading is also reflected in the max-plus walk since the scaling orders of both this measure of the max-plus walk and usual quantum walk are linearly proportional to the time steps. Moreover, it is also interesting that the limit measure of the max-plus walk depends only on the parameter a while the pseudo velocity of the usual quantum walk which is the parameter of the Konno distribution depends only on also "a". How usual quantum walk's behaviors are extracted in the max-plus walk's measure more clearly is one of the interesting future's problems.

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References

[1] F. Baccelli, G. Cohen, G.L. Olsder and J.P Quadrat, Syncronization and Linearity, Wiley, New York, 1992.

[2] R. P. Feynman, Simulating physics with computers, International Journal of Theoretical Physics, 21(1982), 467–588.

[3] M. Katori, S. Fujino, and N. Konno, Quantum walks and orbital states of a Weyl particle, Phys. Rev. A, 72(2005), 012316.

[4] N. Konno, Quantum Walk, Lecture Notes in Mathematics, 1954(2008), 309–452.

[5] D. Maclagan and B. Sturmfels, Introduction to Tropical Geometry, American Mathematical Society, 2015.

[6] I. Marquez-Martin, P. Arnault, G. D. Molfetta, and A. Perez, Electromagnetic lattice gauge invariance in two-dimensional discrete-time quantum walks, Phys. Rev. A, 98(2018), 032333.

[7] Y. Shikano, From discrete time quantum walk to continuous time quantum walk in limit distribution, J. Comput. Theor. Nanoscience, 10 (2013), 1558–1570.

[8] T. Tokihiro, D. Takahashi, J. Matsukidaira and J. Satsuma, From soliton equations to integrable cellular automata through a limiting procedure, Phys. Rev. Lett., 76 (1996), 3247–3250.

[9] S. Watanabe, A. Fukuda, E. Segawa, I. Sato, A walk on max-plus algebra, Linear Algebra and its Applications, 598 (2020), 29–48. (arXiv:1908.09051)