QUALITATIVE DIFFERENCES BETWEEN THE REAL LINE AND NONSEPARABLE LINEARLY ORDERED TOPOLOGICAL SPACES

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Abstract. We shall discuss qualitative differences between the real line and nonseparable LOTS.

1. Introduction

The set $\mathbb{R}$ of real numbers is used everywhere in mathematics and science to describe quantities. It is well-known that as a linearly ordered topological space (LOTS in short), $\mathbb{R}$ is characterized as a complete self-dense separable LOTS without end points. In this paper, we shall discuss what happens if we replace it with a nonseparable LOTS or more generally a nonseparable GO-space. In fact, we shall observe that many phenomena that we take for granted cannot occur when we use a nonseparable LOTS.

The structure of this article is as follows. In Section 2, we shall define some notions that are used in Sections 3 and 4. In Section 3, we shall outline the proof that for every nonnegative integer n, every continuous injection from the product of n-many connected nowhere separable LOTS to the product of n-many connected nowhere separable LOTS is coordinate-wise. In Section 4, we shall give a new proof of G. I. Čertanov's Theorem that if X and Y are infinite Hausdorff spaces such that $X \times Y$ is a continuous image of a countably compact GO space, then both X and Y are metrizable. The proof uses countable elementary submodels, too. In Section 5, we shall discuss J. Aczel's Theorem that every cancellative connected linearly ordered topological semigroup is order- and semigroup-isomorphic to a subsemigroup of $(\mathbb{R}, +, \leq)$. In Section 6, we consider a positively ordered semigroup. In particular, we present the author's theorem that if S is a positively ordered archimedean semigroup with no maximal element, then there is an order and semigroup-homomorphism f from S onto $([0,\infty), +, \leq)$ such that $f^{-1}\{0\}$ is either empty of the singleton of an identity. We shall also discuss how to use this result to an analogue of metrizable spaces by using positively ordered semigroups.

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2. Key definitions and observations

2.1. **Definitions.** In Sections 3 and 4, we rely on the notions introduced by the author in [10] and [11]. In this section, we shall present some of them.

Throughout this section, let K be a GO-space and M a countable elementary submodel of $H(\theta)$ where θ is a sufficiently large regular cardinal so that $K \in M$.

Definition 2.1. Let K be a GO-space and M a countable elementary submodel of $H(\theta)$ where θ is a sufficiently large regular cardinal so that $K \in M$.

Let $J_0(K, M)$ be the set of all $x \in K$ such that there exist $a, b \in K \cap M$ such that $a \leq x \leq b$. Let $J(K, M) = J_0(K, M) \setminus \mathrm{cl}(K \cap M)$.

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For every $p \in K$, let

$$\begin{aligned}\eta(K,M,p) &= \sup\{\, x \in \mathrm{cl}(K \cap M) : x \le p \,\}\\ \zeta(K,M,p) &= \inf\{\, x \in \mathrm{cl}(K \cap M) : x \ge p \,\}\\ I(K,M,p) &= [\eta(K,M,p), \zeta(K,M,p)]\end{aligned}$$

if they exist.

If K is nowhere separable and U is an open subset of K, then $U \setminus \mathrm{cl}(K \cap M) \neq \varnothing$. So, if there are a lower bound and an upper bound of U that belong to M, then $U \cap J(K,M) \neq \varnothing$.

2.2. **Observations.** If $p \in J(K,M)$, then $(\eta(K,M,p), \zeta(K,M,p))$ is an open neighborhood of p that is disjoint from M. Thus, $[\eta(K,M,p), \zeta(K,M,p)]$, i.e. $I(K,M,p)$ cannot be controlled well by M.

It is easy to prove the following lemma.

Lemma 2.2. *Let $p, q \in J(K,M)$ with $p < q$ and $I(K,M,p) \neq I(K,M,q)$. Then,*

(1) $\zeta(K,M,p) \le \eta(K,M,q)$, and
(2) for all $x \in I(K,M,p)$ and $y \in I(K,M,q)$, we have $x \le y$. In particular, if $I(K,M,p) \cap I(K,M,q) \neq \varnothing$, then $\zeta(K,M,p) = \eta(K,M,q)$ and it is the only element of $I(K,M,p) \cap I(K,M,q)$.

We shall also prove the following lemma on functions from K into a Hausdorff space.

Lemma 2.3. *Let X be a Hausdorff space and $g : K \to X$ a continuous function with $X, g \in M$. Let $p \in J(K,M)$ and $g(p) \in M$. Then, either*

(1) $\{\, t \in K \cap M \mid t < p \,\} \neq \varnothing$ and $g(\eta(K,M,p)) = g(p)$ or
(2) $\{\, t \in K \cap M \mid t > p \,\} \neq \varnothing$ and $g(\zeta(K,M,p)) = g(p)$.

In particular, in this situation, there are only two candidates for the value of $g(p)$.

3. Coordinate-wise Theorem

3.1. **Introduction.** Extending the result of K. Eda and R. Kamijo in [6], the author proved the following theorem in [11]:

Theorem 3.1 (T. Ishiu [11]). *Let n be a non-zero natural number, $K_0, \ldots, K_{n-1}$, $L_0, \ldots, L_{n-1}$ connected nowhere separable LOTS, and $f : \prod_{i<n} K_i \to \prod_{j<n} L_j$ a continuous injective function. Then, f is coordinate-wise, namely there exists a bijection $h : n \to n$ and a function $p_i : K_i \to L_{h(i)}$ for each $i < n$ such that for all $x \in \prod_{i<n} K_i$ and $i < n$,*

$$f(x)(h(i)) = p_i(x(i))$$

This theorem distinguishes $\mathbb{R}$ and a connected nowhere separable LOTS. For example, it is easy to see the following corollary.

Corollary 3.2. *Let L be a connected LOTS. Then, the following are equivalent.*

(1) L is nowhere separable.
(2) Every injection from $L \times L$ to $L \times L$ is coordinate-wise.
(3) For every integer $n \ge 1$, every injection from L^n to L^n is coordinate-wise.

The proof of Theorem 3.1 uses countable elementary submodels. In the next several subsections, we shall sketch the main idea of the proof for the case $n = 2$.

3.2. **Connected nowhere separable LOTS.** When we consider a continuous function from a connected nowhere separable LOTS into a connected nowhere separable LOTS, we can prove the following stronger lemma than the ones we discussed in Subsection 2.2.

Lemma 3.3. *Let K, L be connected nowhere separable LOTS and $g : K \to L$ a continuous function. Let M be a countable elementary submodel of $H(\theta)$ where θ is a sufficiently large regular cardinal such that $K, L, g \in M$. Let $p \in J(K, M)$. Then,*

(1) $g \restriction I(K, M, p)$ has a maximum and a minimum at the endpoints.
(2) If $g(p) \in M$, then $g \restriction I(K, M, p)$ is constant.
(3) If $g \restriction I(K, M, p)$ is not constant, then $g(p) \in J(L, M)$ and $g^{\to}I(K, M, p) = I(L, M, g(p))$.

By using the previous lemma, we can prove the following lemma.

Lemma 3.4. *Let K_0, K_1 and L be connected nowhere separable LOTS and $f : K_0 \times K_1 \to L$ a continuous function. Let M be a countable elementary submodel of $H(\theta)$ where θ is a sufficiently large regular cardinal such that $K_0, K_1, L, f \in M$. If $p \in J(K_0, M)$, $q \in \mathrm{cl}(K_1 \cap M)$, and $f(p, q) \in J(L, M)$, then*

$$\{ f(\eta(K_0, M, p), q), f(\zeta(K_0, M, p), q) \} = C(L, M, f(p, q))$$

and

$$f^{\to}(I(K_0, M, p) \times \{ q \}) = I(L, M, f(p, q))$$

Note that if $q \in M$, then this lemma is an easy consequence of Lemma 3.3. The significance of this lemma is that this holds even when q is a limit point of $K_1 \cap M$. By using it together with countable elementary submodel arguments, we can show the following lemma.

Lemma 3.5. *Let K_0, K_1, L, f and M be as in Lemma 3.4. Let $p \in J(K_0, M)$ and $q \in J(K_1, M)$.*

(1) If f is not constant, then

$$f^{\to}(I(K_0, M, p) \times I(K_1, M, q)) = I(L, M, f(p, q)).$$

(2) If $f(p, q) \in M$, then f is constant on $I(K_0, M, p) \times I(K_1, M, q)$.

By using Lemma 3.5, we can prove the following lemma.

Lemma 3.6. *Let K_0, K_1, L_0, L_1 be connected nowhere separable LOTS and $f : K_0 \times K_1 \to L_0 \times L_1$ be an injective function. Let M be a countable elementary submodel of $H(\theta)$ where θ is a sufficiently large regular cardinal such that $K_0, K_1, L_0, L_1, f \in M$. Let $x_0 \in J(K_0, M)$ and $x_1 \in J(K_1, M)$. Define $\langle y_0, y_1 \rangle = f(x_0, x_1)$. Then,*

$$f^{\to}(I(K_0, M, x_0) \times I(K_1, M, x_1)) = I(L_0, M, y_0) \times I(L_0, M, y_1)$$

So, the behavior of the function f is severely restricted by M. By extending this local result to the entire space, we can prove Theorem 3.1.

4. A proof of Čertanov's Theorem by using countable elementary submodels

4.1. **History.** A well-known theorem of G. Peano says that there is a continuous surjection from $[0, 1]$ onto $[0, 1] \times [0, 1]$. D. Kurepa conjectured that for every nondegenerate connected compact LOTS K, if there exists a continuous surjection from K onto $K \times K$, then K is isomorphic to $[0, 1]$. In fact, he proved this theorem in the case that K has countable cellularity.

S. Mardešić and P. Papić [14] extended the result and showed that if X and Y are nondegenerate connected compact LOTS and $X \times Y$ is a continuous image of a connected compact LOTS, then both X and Y are metrizable. Then, L. B. Treybig showed the following theorem, which strengthens these results even further:

Theorem 4.1 (L.B. Treybig [15])**.** *If X and Y are infinite Hausdorff spaces and $X \times Y$ is a continuous image of a compact LOTS, then both X and Y are metrizable.*

G. I. Čertanov [4] showed that this theorem holds even when a compact LOTS is replaced by a countably compact GO space, namely:

Theorem 4.2 (G. I. Čertanov [4])**.** *If X and Y are infinite Hausdorff spaces and $X \times Y$ is a continuous image of a countably compact GO space, then X and Y are compact and metrizable.*

Note that in the conclusion of both Theorem 4.1 and Theorem 4.2, X and Y are compact and metrizable, so they are separable.

In the next subsection, we shall give a proof of Theorem 4.2 by using countable elementary submodels.

4.2. **The use of countable elementary submodels.** First, we shall prove the following lemma. As we mentioned, this is an easy corollary of Theorem 4.2.

Lemma 4.3. *If X and Y are infinite Hausdorff spaces and $X \times Y$ is an image of a countably compact GO space. Then X and Y are separable.*

Proof.(Sketch) Let X and Y be infinite Hausdorff space, K a countably compact GO space, and f a continuous surjection from K onto $X \times Y$. Let M be a countable elementary submodel of $H(\theta)$ where θ is a sufficiently large regular cardinal with $X, Y, K, f \in M$. Let g_1, g_2 be the coordinate functions of f, i.e. $f(t) = \langle g_1(t), g_2(t) \rangle$ for every $t \in K$.

Suppose that at least one of X and Y is nonseparable. Without loss of generality, suppose that X is nonseparable. So, there exists $x_0 \in X \setminus \mathrm{cl}(X \cap M)$. Since Y is infinite, $Y \cap M$ is also infinite. By a little argument, we can build a monotone sequence $\langle t_n | n < \omega \rangle$ such that for every $n < m < \omega$, $g_1(t_n) = x_0$, $g_2(t_n) \in M$, and $g_2(t_n) \neq g_2(t_m)$.

By using Lemma 2.3, we can prove that for every $n < \omega$, $t_{n+2} \notin I(K, M, t_n)$. Let $t_\omega = \sup \{ t_n \mid n < \omega \}$. Then, $g_1(t_\omega) = x_0$ since $g_1(t_n) = x_0$ for all $n < \omega$.

However, t_ω is also a limit point of $\{ \zeta(K, M, t_n) \mid n < \omega \}$. Since $\zeta(K, M, t_n) \in \mathrm{cl}(K \cap M)$, we have $g_1(t_\omega) \in \mathrm{cl}(Y \cap M)$. This is a contradiction. □(Lemma 4.3)

Theorem 4.2 can be obtained as a corollary of the previous lemma as follows.

Proof of Theorem 4.2. Let X and Y be infinite Hausdorff space, K a countably compact GO space, and f a continuous surjection from K onto $X \times Y$. By Lemma 4.3, X and Y are separable. So, there exists a countable subset D of K such that $f^{\rightarrow} \mathrm{cl}(D) = X \times Y$. Since $\mathrm{cl}(D)$ is a closure of a countable subset of countably compact GO space, it is a compact LOTS. By Theorem 4.1 applied to $f \restriction \mathrm{cl}(D)$, we can see X and Y are metrizable. □

However, we can prove Theorem 4.2 without using Theorem 4.1 by using the following lemma.

Lemma 4.4 (T .Ishiu [13])**.** *Let X be a Hausdorff space that is a continuous image of a separable countably compact LOTS. Then, X is first countable.*

There are many results in this line of research. The author will consider applications of this technique in these situations to find new proofs and theorems.

5. Linearly Ordered Topological Semigroups

5.1. **Aczel's Theorem.** We shall consider topological semigroups on a LOTS, i.e., the following structures.

Definition 5.1. A *linearly ordered topological semigroup* is a triple $\langle S, \cdot, \leq \rangle$ such that $\langle S, \leq \rangle$ is a linearly ordered set, and $\langle S, \cdot \rangle$ is a topological semigroup when the topology on S is given by the order topology.

Recall the following notion.

Definition 5.2. We say that a semigroup S is *cancellative* if and only if for every $a, x, y \in S$, $ax = ay$ implies $x = y$ and $xa = ya$ implies $x = y$.

The following corollary can be obtained from Theorem 3.1.

Corollary 5.3. *Let K be a connected nowhere separable LOTS which has at least two elements. Then, there is no cancellative linearly ordered topological semigroup on K.*

Proof. Let $\langle K, \cdot, \leq \rangle$ be a cancellative linearly ordered topological semigroup where K is connected and nowhere separable. Define

$$f(x, y) = \langle x, x \cdot y \rangle$$

Since K is cancellative, f is not coordinate-wise and injective. By Theorem 3.1, f is coordinate-wise. This is a contradiction. □(Corollary 5.3)

In fact, J. Aczel [1] proved the following stronger result. R. Craigen and Z. Páles [5] gave a simpler proof.

Theorem 5.4 (J. Aczel [1])**.** *Let S be a cancellative connected linearly ordered topological semigroup. Then, S is order- and semigroup-isomorphic to a subsemigroup of* $(\mathbb{R}, +)$*.*

By using this theorem, it is easy to observe that every connected linearly ordered topological semigroup is order- and semigroup-isomorphic to one of $(-\infty, \infty)$, $[0, \infty)$, $(0, \infty)$, $[1, \infty)$, $(1, \infty)$, $(-\infty, 0]$, $(-\infty, 0)$, $(-\infty, -1]$, and $(-\infty, -1)$.

As a corollary, we can see the following theorem proved by E .Cartan [3].

Corollary 5.5 (E. Cartan [3])**.** *Every connected linearly ordered topological group is order- and semigroup-isomorphic to* $(\mathbb{R}, +)$*.*

We may wonder if these results can be extended to a broader class of connected topological groups and semigroups, not necessarily topologized by the order topology. It is related to the question of whether Theorem 3.1 can be extended to more topological spaces.

6. Linearly ordered semigroups

6.1. **History.** We may also consider another class of semigroups on LOTS.

Definition 6.1. We say that a triple $\langle S, \cdot, \leq \rangle$ is a *linearly ordered semigroup* if and only if $\langle S, \cdot \rangle$ is a semigroup, $\langle S, \leq \rangle$ is a linearly ordered set, and for all $a, b, c \in S$, $a \leq b$ implies $ac \leq bc$ and $ca \leq cb$.

A trivial but motivating example of a linearly ordered semigroup is $(\mathbb{R}, +, \leq)$. Linearly ordered semigroups and linearly ordered topological semigroups are independent notions. Namely, there exists a linearly ordered semigroup that is not a linearly ordered topological semigroup and vice versa. (See [8]).

There are several results about sufficient conditions for a linearly ordered semigroup to be order- and semigroup-isomorphic to a subsemigroup of $(\mathbb{R}, +, \leq)$. To state them, we shall define several notions on linearly ordered semigroups.

Definition 6.2. Let S be a linearly ordered semigroup. We say that S is *positively ordered* if and only if for all $a, b \in S$, $ab \geq a$ and $ab \geq b$. We say that S is *strictly positively ordered* if and only if for all $a, b \in S$, $ab > a$ and $ab > b$.

Note that $(\mathbb{R}, +, \leq)$ is not positively ordered, but $([0, \infty), +, \leq)$ is positively ordered. For simplicity, we say S is a positively ordered semigroup to mean a positively ordered linearly ordered semigroup.

Definition 6.3. We say that S is a *naturally totally ordered semigroup* if and only if S is a positively ordered semigroup and for all $a, b \in S$ with $a < b$, there exist $x, y \in S$ such that $ax = ya = b$. If S is also strictly positively ordered, we say that S is *strictly naturally totally ordered.*

Namely, a positively ordered semigroup S is naturally totally ordered, for all distinct $a, b \in S$, $a < b$ is equivalent to the existence of x and y such that $ax = ya = b$. So, the linear ordering can be determined by the semigroup operation.

Definition 6.4. Let S be a positively ordered semigroup. We say that S is *archimedean* if and only if for all $a, b \in S$, whenever a is not an identity, there exists $n \in \mathbb{N}$ such that $a^n \geq b$.

For example, $([0, \infty), +, \leq)$ is archimedean.

O. Hölder [9] proved the following theorem, which gives the first sufficient condition for a positively ordered semigroup to be order- and semigroup-isomorphic to a subsemigroup of $(\mathbb{R}, +, \leq)$.

Theorem 6.5 (O. Hölder [9])**.** *Let $(S, \cdot, \leq)$ be a strictly naturally totally ordered semigroup with no least element.*

(1) If S is complete as a linearly ordered set, then S is order- and semigroup-isomorphic to $((0, \infty), +, \leq)$.
(2) S is archimedean if and only if S is order- and semigroup-isomorphic to a subsemigroup of $((0, \infty), +, \leq)$.

N. G. Alimov and L. Fuchs strengthened this theorem to give equivalent conditions. To state their result, we shall give the following definition.

Definition 6.6. Let S be a positively ordered semigroup. We say that $a, b \in S$ *form an anomalous pair* if and only if $a \neq b$ and for all positive natural number n, $a^n < b^{n+1}$ and $b^n < a^{n+1}$.

For example, $((0, \infty), +, \leq)$ has no anomalous pair. The following example demonstrates what an anomalous pair is.

Example 6.7. Let $S = (0, \infty) \times [0, \infty)$. Define a semigroup operator $\cdot$ by

$$\langle a, x \rangle \cdot \langle b, y \rangle = \langle a + b, x + y \rangle$$

and let S be ordered by the lexicographical ordering. It is easy to observe that S is a positively ordered semigroup that is archimedean.

Let $a \in (0, \infty)$ and x and y be distinct elements of $[0, \infty)$. It is easy to see that $\langle a, x \rangle$ and $\langle a, y \rangle$ form an anomalous pair.

Then, N. G. Alimov's theorem can be stated as follows.

Theorem 6.8 (N. G. Alimov [2])**.** *Let S be a positively ordered semigroup. Then, S is order- and semigroup-isomorphic to a subsemigroup of $([0, \infty), +, \leq)$ if and only if S is cancellative and has no anomalous pair.*

L. Fuchs gave the following equivalent condition.

Theorem 6.9 (L. Fuchs [7])**.** *Let S be a positively ordered semigroup. Then, S is order- and semigroup-isomorphic to a subsemigroup of $([0, \infty), +, \leq)$ if and only if S is archimedean, has no anomalous pair, and has no maximal element unless S is a singleton.*

So, the seemingly benign assumptions in the previous theorems are sufficient to show that S is separable and even further.

6.2. **With anomalous pairs.** Given Theorem 6.8 and Theorem 6.9, we may wonder what if S has an anomalous pair. To consider this problem, the author defined the following equivalence relation in [12].

Definition 6.10. Let S be a positively ordered archimedean semigroup with no maximal element. Define an equivalence relation $\sim$ on S by $a \sim b$ if and only if either $a = b$ or a and b form an anomalous pair.

Let $S/{\sim}$ be the set of all equivalence classes. We shall define a semigroup operation $\cdot$ by $[a] \cdot [b] = [ab]$ and define a linear ordering $\leq$ by $[a] \leq [b]$ if and only if $a \leq b$.

It was shown in [12] that the definition of $\cdot$ and $\leq$ on $S/{\sim}$ is well-defined and $(S/{\sim}, \cdot, \leq)$ is a positively ordered semigroup. Then, the author proved the following theorem, which extends 6.9 to the case with anomalous pairs.

Theorem 6.11 (T. Ishiu [12])**.** *Let S be a positively ordered archimedean semigroup with no maximal element. Then, $(S/{\sim}, \cdot, \leq)$ is order- and semigroup-isomorphic to a subsemigroup of $([0,\infty), +, \leq)$.*

We can easily prove the following corollary. Note that if e is an identity of S, then it is easy to see that $[e] = \{ e \}$.

Corollary 6.12 (T. Ishiu [12])**.** *Let S be a positively ordered archimedean semigroup with no maximal element. Then, there exists an order- and semigroup-homomorphism f from S onto $([0,\infty), +, \leq)$ such that $f^{-1}\{ 0 \}$ is either empty or the singleton of an identity.*

Namely, if we remove the noise made by anomalous pairs, then we can apply Theorem 6.9.

6.3. **S-metrizability.** In this subsection, we shall consider an application of Theorem 6.11

O. Hölder's motivation in [9] to start the research on linearly ordered semigroups is the axiomatization of the theory of magnitude. So, it is natural to wonder what happens if we replace $[0,\infty)$ with a positively ordered semigroup in the definition of metrizable spaces.

First, we shall define an S-metric.

Definition 6.13. Let X be any set and S a positively ordered semigroup with identity e. We say that a function $d : X \times X \to S$ is a *S-metric on X* if and only if for all $x, y, z \in X$,

(1) $d(x,y) = e$ if and only if $x = y$,
(2) $d(x,y) = d(y,x)$, and
(3) $d(x,y) \leq d(x,z)d(z,y)$.

Note that by (3) with x and y switched, $d(y,x) \leq d(y,z)d(z,x)$. By applying (2), we get $d(x,y) \leq d(y,z)d(z,x)$. So, we have $d(x,y) \leq \min\{ d(x,z)d(z,y), d(z,y)d(x,z) \}$.

Definition 6.14. Let X be any set, S a positively ordered semigroup with identity e, and d an S-metric on X. For all $x \in X$ and $s \in S$, the *s-ball centered at x* is defined by

$$B_d(x,s) = \{ y \in X \mid d(x,y) < s \}$$

Let $\mathcal{B}_d$ be the set of all subsets of X of the form $B_d(x,s)$ where $x \in X$ and $s \in S$ with $s > e$.

If $\mathcal{B}_d$ is a basis for a topology on X, then we say that the topology $\mathcal{T}_d$ generated by $\mathcal{B}$ is the *topology induced by d*.

Unfortunately, not all S-metric induces a topology. But when it does, we may introduce the following notion.

Definition 6.15. Let X be a topological space and S a positively ordered semigroup with identity e. We say that X is *S-metrizable* if and only if there exists an S-metric d on X such that the topology of X coincides with the topology induced by d.

A natural question is whether every S-metrizable space is metrizable. Actually, there is an easy counterexample.

Example 6.16. Let δ be any limit ordinal and $S = \delta + 1$. We shall define the semigroup operator $\cdot$ by $\alpha \cdot \beta = \min\{ \alpha, \beta \}$, and let S be ordered by the reverse of the ordinary order, i.e. $\alpha \leq_S \beta$ if and only if $\beta \leq \alpha$. Then, S is a positively ordered semigroup with maximal element δ. Note that δ is an identity.

Let $X = {}^{<\delta}2$ be ordered by the lexicographical ordering and topologized by the order topology.

For every $x, y \in X$ with $x \neq y$, let $d(x,y)$ be the least $\alpha < \delta$ such that $x(\alpha) \neq y(\alpha)$. We stipulate $d(x,x) = \delta$. We shall show that d is S-metric. It is easy to show that $d(x,y) = \delta$ if and only if $x = y$, and $d(x,y) = d(y,x)$. To show the triangle inequality, let $x, y, z \in X$. Without loss of generality, we may assume that they are distinct. If $d(x,y) < d(y,z)$, then clearly $d(x,z) = d(x,y) = \min\{d(x,y), d(y,z)\} = d(x,y) \cdot d(y,z)$. Similarly when $d(y,z) < d(x,y)$. Suppose $d(x,y) = d(y,z)$. Then, $d(x,z) > d(x,y) = \min\{d(x,y), d(y,z)\} = d(x,y) \cdot d(y,z)$. Thus, in all cases, we have $d(x,z) \geq d(x,y) \cdot d(y,z)$. Since the ordering of S is the reverse of the ordinary ordering, $d(x,z) \leq_S d(x,y) \cdot d(y,z)$.

We shall show that the topology of X coincides with the topology induced by d. It suffices to show that for all $x, y, z \in X$ with $x < y < z$, there exists $\alpha \in S$ such that $B_d(y, \alpha) \subseteq (x, z)$.

Let $\alpha = \max\{d(x,y), d(y,z)\}$. To show $B_d(y,\alpha) \subseteq (x,z)$, let $y' \in B_d(y,\alpha)$. Then, since $d(y,y') <_S \alpha$, we have $d(y,y') > \alpha$. So, $y' \restriction (\alpha+1) = y \restriction (\alpha+1)$. Since $\alpha = \max\{d(x,y), d(y,z)\}$ and $y' \restriction (\alpha+1) = y \restriction (\alpha+1)$, $x < y$ implies $x < y'$ and $y < z$ implies $y' < z$. Thus, $y' \in (x,z)$.

However, you may notice that the triangle inequality of the metric d in the last example can be replaced by $d(x,z) \leq_S \max\{d(x,y), d(y,z)\}$. Recall the following definition.

Definition 6.17. An *ultrametric on* X is a metric d on X such that for all $x, y, z \in X$,

$$d(x,z) \leq \max\{d(x,y), d(y,z)\}$$

Notice that in this definition, we only used $\mathbb{R}$ as a linearly ordered set. So, we may generalize this notion to an arbitrary linearly ordered set with the least element.

Definition 6.18. Let L be a linearly ordered set with the least element 0_L. An L*-ultrametric on a set* X is a function $d : X \times X \to L$ such that for all $x, y, z \in X$,

(1) $d(x,y) = 0_L$ if and only if $x = y$,
(2) $d(x,y) = d(y,x)$, and
(3) $d(x,z) \leq \max\{d(x,y), d(y,z)\}$.

For each $x \in X$ and $s \in L$, we define

$$B_d(x,s) = \{y \in X \mid d(x,y) < s\}$$

and $\mathcal{B}_d$ the set of all sets $B_d(x,s)$ of the form $x \in X$ and $s \in L$ with $s \neq 0_L$. If $\mathcal{B}_d$ is a basis for a topology on X, then the topology generated by $\mathcal{B}_d$ is called the topology *induced by* d and written as $\mathcal{T}_d$.

Note that not all L-ultrametrics induce a topology.

Definition 6.19. Let L be a linearly ordered set with the least element 0_L. A topological space X is an L*-ultrametrizable* if and only if there exists an L-ultrametric d such that the topology of X coincide with $\mathcal{T}_d$.

Now, the question is as follows:

Question 1. Let S be a positively ordered semigroup with an identity. For every topological space X, if X is S-metrizable, then is X either metrizable or L-metrizable for some LOTS L?

If the answer is yes, then S-metrizability is absorbed into metrizability and L-metrizability for a LOTS L.

By using Theorem 6.11, we can prove the following theorem.

Theorem 6.20. *Let S be an archimedean positively ordered semigroup with an identity but no maximum element. Then, every S-metrizable space is metrizable.*

If S does not have the least archimedean component, then it is easy to see that every S-metrizable space is L-ultrametrizable where L is the LOTS consisting of all archimedean components of S. If S has the least archimedean component that has no maximum element, then we may use Theorem 6.20. So, the only remaining case is when S has the least archimedean component that has a maximum element.

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